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Using Ex Ante Payments in Self-Enforcing Risk-Sharing Contracts*

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Abstract / Résumé

In this paper we analyze a long-term risk-sharing contract between two risk-averse agents facing self-enforcing constraints. We enlarge the contracting space to allow for an ex ante transfer (at the beginning of the period) before the state of nature is realized. We analyze the trade-off between the self-enforcing constraints of the two agents by characterizing the optimal ex ante and ex post transfer payments. We show that optimal ex ante payments are non-stationary. They optimally depend on the surplus from the relationship each agent expects. The size of the ex ante payment an agent makes is inversely related to its expected surplus from the relationship. The introduction of ex ante payments generates interesting dynamic properties. In a two-state example with i.i.d. shocks, the dynamics of the optimal contract exhibit "experience rating" even though there is no private information or learning taking place.

Ce papier analyse les propriétés dynamiques d'un contrat de partage de risque entre deux agents riscophobes qui font face à des contraintes de faillite. L'espace des contrats est élargi pour permettre aux agents d'effectuer un transfert au début de chacune des périodes avant la réalisation de l'incertitude. Ces paiements ex ante ne sont pas stationnaires. Ils dépendent du surplus que chaque agent attend de la relation. Ce paiement est inversement proportionnel à ce surplus. Dans un environnement i.i.d. à deux états de la nature, les propriétés dynamiques de la consommation de chacun des agents démontrent un lissage qui ressemble à de la tarification *a posteriori*.

Mots-clés : partage de risque, relation dynamique, contrats auto-exécutoires

Key Words: Risk sharing, dynamic relationship, self-enforcing contracts

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1 Introduction

Long-term contracts are useful for the governance of long-term relationships. Such contracts can help improve incentives as well as risk-sharing between two agents. An optimal contract trades off between incentives and risk-sharing to attain an efficient allocation; however this efficient allocation is often time inconsistent. For example, an ex ante efficient allocation may not be ex post efficient (once certain actions have been undertaken or some information has been revealed). This type of time-consistency problems has led to the recent literature on renegotiation. Or, an ex ante profitable contract may not be ex post profitable following a given history. In this case, if enforcement costs are high (or mobility costs are low), agents may be tempted to renege on the contract to seek more profitable opportunities elsewhere. The literature on self-enforcing contracts studies this type of time-consistency problems.

This paper studies a dynamic risk-sharing relationship in which agents have commitment problems. Consider two agents that enter into a long-run relationship to share risk and for which enforcement costs are high. The risk-sharing problem analyzed in the literature usually has the following structure.[1] In every period a risk-averse agent receives a stochastic endowment. Risk-sharing is implemented by a contract specifying transfer payments between the two agents. These transfers take place at the end of the period once the state of nature has been observed. If the two agents can commit not to default on any prescribed transfer payment then the optimal contract achieves an efficient risk-sharing allocation; however, if an agent cannot commit not to default, efficient risk sharing may be impeded as the optimal contract is constrained by the possibility of ex post default. The contract should then prescribe payments that are self-enforcing, that is, payments that satisfy, for any realization of the state of nature and every period, a participation constraint for each agent. In any period the surplus one agent expects from the relationship conditions the transfer that it is willing to make in this period. An agent that expects a high surplus in the future has low incentives to break the relationship. It is therefore willing to make a high payment to continue the relationship. On the other hand a low expected surplus yields low incentives to maintain the relationship. The agent must then be induced to remain in the relationship by making a low (possibly negative) payment. Self-enforcing constraints generally limit transfer payments and therefore reduce the opportunity for efficient risk sharing.

In this paper we show that allowing for a more general contracting space may help

[1] For example, see Thomas and Worrall (1988).

relaxing self-enforcing constraints. Suppose that these constraints are quite stringent for one agent, say agent 1. This effectively limits the payments agent 1 can make to agent 2. In this case, agent 1 would like to make a transfer to agent 2 *before* the state of nature is realized. At this point, agent 1's self-enforcing constraints only have to hold in expectation over all states of nature. Such ex ante transfer would effectively relax agent 1's ex post self-enforcing constraints by reducing its ex post payments. However when the two agents face self-enforcing constraints, an ex ante payment made by one agent to relax its own self-enforcing constraints usually makes the other agent's self-enforcing constraints more stringent by leaving the ex post burden to that agent to make the necessary transfers for optimal risk-sharing. Consequently, the ex ante payment must trade off between the self-enforcing constraints of the two agents. We analyze the details of this trade-off in an optimal risk-sharing contract.

Our main results are that optimal ex ante payments are non-stationary. They depend on the surplus from the relationship each agent expects. This expected surplus evolves with the history of past realizations of states of nature. When an agent expects a low share of the surplus its ex post self-enforcing constraints are relatively stringent and it cannot be required to make a high ex post payment. In this case, the contract optimally requires that agent to pay an amount up front before the realization of the state of nature. This effectively relaxes its ex post self-enforcing constraints. In general, however, these constraints cannot be completely eliminated because a high ex ante payment by one agent increases the incentives of the other agent to break the relationship and run away with this payment. We show that the size of the ex ante payment an agent makes is inversely related to the surplus it expects to get from the relationship. We can also show that interesting dynamic properties emerge from our model even though shocks are independently and identically distributed across periods. For example, in a two-state example, we show that the dynamics of the optimal contract exhibit "experience rating" even though there is no private information or learning taking place.

Section 2 presents the basic model. In Section 3 we analyze the role of ex ante payments when only one agent faces self-enforcing constraints. Section 4 presents the main results of the paper when the two agents face self-enforcing constraints. Section 5 discusses some key assumptions about the economic environment. A conclusion follows.

2 The model

The environment we consider can be described by an infinite sequence of periods, $t = 1, 2, ..., \infty$, and for each period, a finite set of states of nature, $s \in \{1, 2, ..., S\}$, with $S \geq 2$. We assume that states are distributed independently and identically across all periods, and therefore, in each period, the state of nature s occurs with probability p^s where $\sum_{s=1}^{S} p^s = 1$. It is assumed that each period t is divided into three dates, t_0, t_1, and t_2, where t_1 is the date at which the state of nature is realized; the dates t_0 and t_2 denote respectively the dates preceding and following the realization of the state of nature.

Two infinitely-lived agents evolve in this environment. Both agents are risk averse. In each period, agent 1's preferences over consumption c are represented by $u(c)$ where u is a state-independent increasing and strictly concave utility function for $c \in [0, b]$. In each period, agent 1 obtains a state-contingent endowment y^s. We adopt the convention that $y^s > y^{s-1}$ for all states s. We assume that $0 < y^1 < y^S < b$. This endowment is observable to agent 2.[2] In each period, agent 2's preferences over consumption c are given by $v(c)$ where v is also a state-independent increasing and strictly concave function for $c \in [0, b]$. In each period, agent 2 obtains a state-independent endowment e.[3] To insure an interior solution we assume that $y^S + e < b$ and that $u'(0) = \infty$, $v'(0) = \infty$. Both agents discount the future by a common factor $\beta \in (0, 1)$.

We assume that there are no contingent markets that would allow the agents to diversify their risk and therefore the two agents enter into a risk-sharing relationship. For example, the reader can think of agent 1 as an insuree and agent 2 as an insuror. We call the governance of such relationship a contract where the term "contract" is interpreted in a broad sense, namely it can encompass implicit as well as explicit agreements. A contract then specifies various transfers between the two agents for all periods of the relationship. In each period t, a contract can specify the following structure of transfer payments.

1. A (positive or negative) ex ante transfer B_t from agent 2 to agent 1 at date t_0 (before the state of nature is realized).

[2] See Thomas and Worrall (1990), Phelan and Townsend (1991), and Wang (1994) for risk-sharing problems in private-information environments.

[3] The analysis can be easily generalized to the case in which the endowment of agent 2 is stochastic. See Kocherlakota (1994) for an example of this.

2. Ex post (positive or negative) transfers a_t^s from agent 1 to agent 2 at date t_2 (after the state of nature s is realized).

Consumption takes place at the end of the period. Agent 1's consumption in period t if state s is realized is $c_t^s = y^s + B_t - a_t^s$; agent 2's consumption is $e - B_t + a_t^s = e + y^s - c_t^s$.

In a typical relationship the prescribed transfers can potentially be contingent on the complete past history of the relationship. The history up to period t is the vector of all previous realizations of the state of nature. Let s_t denote the realized state of nature in period t. The history at the end of period $t-1$ (date $(t-1)_2$) or at the beginning of period t (date t_0) is denoted by $h_{t-1} = (s_1, s_2, ..., s_{t-1})$. We assume that $h_0 = \emptyset$. Denote by $\mathcal{H}_t$ the set of all possible histories up to the end of period t (date t_2). We can then define formally a contract between the two agents.

Definition 1 *A contract, δ, is a sequence of two functions: $\{\mathcal{B}_t, \mathcal{A}_t\}_{t=1}^{\infty}$ where $\mathcal{B}_t : \mathcal{H}_{t-1} \to \mathbf{R}$ and $\mathcal{A}_t : \mathcal{H}_t \to \mathbf{R}$. The variable $B_t = \mathcal{B}_t(h_{t-1})$ represents the transfer from agent 2 to agent 1 at the beginning of period t (date t_0) when history is h_{t-1}. The variable $a_t^s = \mathcal{A}_t(h_{t-1}, s)$ represents the transfer from agent 1 to agent 2 at the end of period t (date t_2) when the history is h_{t-1} up to period t and s is the realized state of nature in period t.*[4]

For any contract, δ, and any history, h_{t-1}, agent 1's expected surplus from the beginning of period t onwards is

$$U(\delta; h_{t-1}) \equiv E \sum_{\tau=t}^{\infty} \beta^{\tau-t} \{u(y_\tau^s + B_\tau - a_\tau^s) - u(y_\tau^s)\}$$

where E is the expectation operator taken over all possible histories starting with h_{t-1} and y_τ^s denotes that the endowment y^s is realized in period τ. Similarly, the expected surplus of agent 2 from the beginning of period t onwards is

$$V(\delta; h_{t-1}) \equiv E \sum_{\tau=t}^{\infty} \beta^{\tau-t} \{v(e - B_\tau + a_\tau^s) - v(e)\}.$$

The surplus of each agent is measured with respect to autarky where it would consume its endowment. The characterization of the implemented contract depends on the available

[4]Note that an equivalent description of a contract could be given by the functions $\{\mathcal{B}_t, \mathcal{C}_t\}_{t=1}^{\infty}$ where $\mathcal{C}_t(h_{t-1}, s) = y_t^s + \mathcal{B}_t(h_{t-1}) - \mathcal{A}_t(h_{t-1}, s)$.

technology to legally enforce the prescribed payments. The objective of the paper is to study the effects of limited enforceability of payments on optimal contracts.

We first establish a benchmark case in which the two agents sign a contract at the beginning of the first period and all prescribed transfers are legally enforceable. We refer to this case as the full-commitment case. In this case, the optimal contract, δ^{fc}, is the solution to the following maximization problem where, for simplicity, it is assumed that agent 1 has the bargaining power and agent 2's reservation utility is given by autarky.

$$\delta^{fc} = \arg\max_{\delta} \{U(\delta; h_0) \text{ s.t. } V(\delta; h_0) \geq 0\} \tag{1}$$

This maximization problem simply states that the optimal contract maximizes the discounted expected utility of agent 1 subject to agent 2's participation constraint. This constraint states that the contract must provide agent 2 with a nonnegative discounted expected surplus. A solution to this maximization problem exists and is characterized in the following proposition.[5]

Proposition 1 *When both agents can commit to the terms of the contract, the optimal contract, δ^{fc}, is characterized by the equalization of marginal rates of substitution of consumption of the two agents across all states and periods. Formally, for all periods t, τ, all states s, q, and all histories h_{t-1}, $\frac{u'(y_t^s+B_t-a_t^s)}{u'(y_\tau^q+B_\tau-a_\tau^q)} = \frac{v'(e-B_t+a_t^s)}{v'(e-B_\tau+a_\tau^q)}$.*

The optimal full-commitment contract specifies perfect risk-sharing with a stationary consumption rule. This consumption rule can be written as $c_t^s = c^*(c_{t-1}, y_{t-1}, s)$ where

$$\frac{u'(c^*(c_{t-1}, y_{t-1}, s))}{u'(c_{t-1})} = \frac{v'(e + y_t^s - c^*(c_{t-1}, y_{t-1}, s))}{v'(e + y_{t-1} - c_{t-1})}.$$

Two aspects of this characterization deserve mention. First, in problem (1), the functions U and V depend only on the net transfers $B_t - a_t^s$ and therefore, in each state only optimal net transfers are determined. This implies that the optimal value of B_t is arbitrary. With full commitment there is no role for the ex ante transfer B_t in the optimal contract. Second, in some states of nature net transfers from agent 1 to agent 2 are positive, and in other states the opposite is true. Complete legal enforcement of the contract is a sufficient condition to make these transfers feasible. In the next sections we relax the assumption of complete legal enforcement to study the characterization of optimal contracts under incomplete legal enforcement.

[5]The proof of this proposition is straightforward and is therefore omitted.

3 Contracting under one-sided commitment

In this section we consider an environment in which legal enforcement of all prescribed payments is limited. We examine the situation in which only agent 1 cannot commit to making all transfers prescribed by the contract.[6] We say that agent 1 faces self-enforcing constraints. These constraints impose that, at any point in time, agent 1 should always do at least as well obeying the contract as reneging on it. When the self-enforcing constraints are satisfied we say that the contract is self-enforcing.

When legal enforcement cannot provide a sufficient incentive for agent 1 to obey the contract it must be incited to do so differently. In a long-term relationship such incentive arises endogenously from the interaction of the two agents over time. One approach to study this incentive would be to model the relationship as a strategic game where each agent's strategy would be a sequence of payments for the complete history and following any history. In this case, the incentive for agent 1 to obey its equilibrium strategy would come from the anticipation of agent 2's response to a deviation. Any payment by agent 1 would therefore be enforced by the strategy of player 2. The more severe would be player 2's punishment, the higher would be cooperation between the two agents. In this case the Folk theorem states that given a high enough discount factor any individually rational feasible allocation can be sustained in equilibrium. For our purposes such an approach is unsatisfactory for two reasons. First, as is well known in the theory of supergames the multiplicity of equilibria creates significant coordination problems between the two agents. Second, we are interested here in characterizing allocations for any value of the discount factor and not just allocations for high values of the discount factor.

We therefore adopt the following approach. We assume that if agent 1 reneges on the contract it suffers the maximal punishment in that it must remain in autarky forever after. This punishment strategy by agent 2 allows us to characterize the best possible contract satisfying self-enforcing constraints.[7] The optimal contract is then the solution to a well-defined

[6]The analysis of the opposite case in which only agent 2 can renege on the contract is symmetric.

[7]In a labour market example MacLeod and Malcomson (1989) model a situation similar to ours as an explicit game and show that the maximal punishment is indeed subgame perfect. Any deviating agent is punished in the future by not being able to enter a successful relationship, all parties expecting the deviating agent to deviate again in the future. Furthermore, Kocherlakota (1994) shows that this maximal punishment is renegotiation-proof if it is interpreted as maintaining the contract but reverting to the point on the Pareto frontier that the deviating agent likes the least.

maximization problem. This approach resolves the coordination problem in effectively coordinating the two agents on a Pareto efficient allocation. Furthermore it allows us to characterize optimal allocations for any value of the discount factor.

Agent 1 will make a transfer to agent 2 if and only if it is in its interest to do so. Agent 1 will compare the benefit of making the transfer and obeying the contract with the payoff of reneging on the contract and staying in autarky thereafter. For example, suppose the two agents have signed a contract δ prescribing transfers $\{\mathcal{B}_t(h_{t-1}), \mathcal{A}_t(h_t)\}$ for all histories h_t. In period t agent 1 may decide to renege on the contract at date t_0 before receiving the (possibly negative) transfer B_t. Its surplus from staying in the contract is then $U(\delta; h_{t-1})$. Agent 1 may also decide to renege on the contract after the state of nature has been realized at date t_2. In this case its surplus from staying in the contract is $u(y_t^s + B_t - a_t^s) - u(y_t^s + B_t) + \beta U(\delta; h_{t-1}, s)$ where the first two terms represent its current surplus from the relationship and the last term, its discounted expected future surplus. We can now define a self-enforcing contract for agent 1.

Definition 2 *A contract δ is self-enforcing for agent 1 if and only if, for all histories h_{t-1}, periods t, and states s, the following constraints hold.*

$$\begin{aligned} (i) \quad & U(\delta; h_{t-1}) \geq 0 \\ (ii) \quad & u(y_t^s + B_t - a_t^s) - u(y_t^s + B_t) + \beta U(\delta; h_{t-1}, s) \geq 0 \end{aligned}$$

This definition states that a contract is self-enforcing for agent 1 if at all times during the relationship agent 1 prefers making the contractual transfer to reneging on the contract and being reduced to autarky from then on. Constraint (i) is an ex ante self-enforcing constraint in that it holds at date t_0; constraint (ii) is an ex post self-enforcing constraint in that it holds at date t_2 after the state of nature has been realized. It is important to note that even though all ex post self-enforcing constraints are satisfied, the ex ante self-enforcing constraints may not be so. For example, if B_t is negative, the ex ante constraint may bind while ex post constraints may not once the ex ante payment B_t has been paid. It is therefore necessary to consider these two sets of constraints.

When designing the optimal contract the two agents will take into account agent 1's incentive to renege. To solve for the optimal contract we must therefore add self-enforcing constraints to the maximization problem (1). The optimal contract with non-commitment by agent 1, δ^1, is then the solution to the following maximization problem.

$$\begin{aligned} \delta^1 = \quad & \arg\max_{\delta} U(\delta; h_0) \\ & \text{s.t.} \quad V(\delta; h_0) \geq 0 \\ & \qquad U(\delta; h_{t-1}) \geq 0 \quad \forall\, t, h_{t-1} \\ & \qquad u(y_t^s + B_t - a_t^s) - u(y_t^s + B_t) + \beta U(\delta; h_{t-1}, s) \geq 0 \quad \forall\, s, t, h_{t-1} \end{aligned} \tag{2}$$

The next proposition gives a characterization of the optimal contract δ^1.[8]

Proposition 2 *Suppose that the maximum ex ante payment agent 1 can make is $\underline{B}$.*[9]
(i) For all values of $\beta \in (0,1)$ the optimal contract with non-commitment by agent 1 is the optimal full-commitment contract, that is, $\delta^1 = \delta^{fc}$, if and only if $\underline{B} \geq y^S - c^{Sfc}$, where c^{sfc} is the optimal consumption in state s under the full-commitment contract.
Suppose that $\underline{B} < y^S - c^{Sfc}$.
(ii) There exists a β_1 which depends on $\underline{B}$ such that for $\beta \in [\beta_1, 1)$, the optimal contract with non-commitment by agent 1 is the optimal full-commitment contract, that is, $\delta^1 = \delta^{fc}$.
(iii) For all $\beta \in (0, \beta_1)$, the following characterization forms part of an optimal contract δ^1:

1. *Agent 1 makes the highest ex ante payment in every period, that is, $B_t = -\underline{B}$ for all periods t;*

2. *Agent 2's expected profit is non-increasing in time, that is, $V(\delta^1; h_{t-1}, s) \leq V(\delta^1; h_{t-1})$ for all histories h_{t-1}, time periods t, and states s.*

This proposition states that if agent 1 can make a high enough ex ante payment ($\underline{B} \geq y^S - c^{Sfc}$), then the optimal full-commitment contract satisfies agent 1's self-enforcing constraints. A large enough ex ante payment effectively allows all ex post transfers a_t^s to be negative which in turn implies that all ex post self-enforcing constraints are satisfied.[10] Because the optimal full-commitment contract yields agent 1 a surplus in every period its ex ante self-enforcing constraint is also satisfied.

[8] All proofs are relegated to the Appendix.

[9] The bound $\underline{B}$ is rather artificial and is introduced solely to study the case in which self-enforcing constraints can be binding. We remove this restriction in the next section.

[10] Note that the transfer $y^S - c^{Sfc}$ is the largest transfer agent 1 makes to agent 2 in the full-commitment contract δ^{fc}.

When the maximum ex ante payment agent 1 can make is not high enough, the optimal contract with full-commitment cannot be supported for all values of the discount factor. If the discount factor is high enough, that is, no lower than β_1 (where β_1 is defined in the Appendix), then agent 1's ex post self-enforcing constraints are not binding.[11] In this case the future benefits to player 1 of perfect risk-sharing exceed the short-run cost of making the prescribed transfer in any state s. Contrary to the full-commitment case however, the transfer B_t is not a matter of indifference. It will optimally be set to the maximum level agent 1 can pay. For given net transfers, a maximal ex ante payment reduces ex post transfers a_t^s from agent 1 to agent 2 and hence the incentive for the former to renege ex post on the contract. It therefore allows the optimal full-commitment contract δ^{fc} to be supported for the largest interval of discount factors.

When the discount factor is smaller than β_1, the optimal full-commitment contract cannot obtain if $\underline{B} < y^S - c^{Sfc}$. In this case a first property of an optimal contract is that agent 1 makes the maximum ex ante payment $B_t = -\underline{B}$ in all periods.[12] A substitution from ex post to ex ante transfers leaves the two agents' consumption unchanged and therefore does not change the value of agent 2's participation constraint, nor the value of ex ante self-enforcing constraints; however it does relax the ex post self-enforcing constraints of agent 1. When one of these constraints is binding this new contract (weakly) increases the utility of agent 1.

A second property of an optimal contract is that agent 2's expected profit is non-increasing in time. The optimal contract seeks two objectives: (1) to insure agent 1 against shocks to its endowment and (2) to smooth its consumption across periods. These objectives are impeded by the inability of agent 1 to commit. They can be improved upon by having agent 1 "save" in the early periods and good states of the world and withdraw these savings in later periods and bad states of the world. The optimal contract therefore prescribes using agent 2 as a savings account. This is possible given that agent 2 can commit not to "steal" agent 1's early savings. The objective of this savings account is precisely to insure future consumption against bad states of the world. This saving behavior implies that agent 2's expected profits are non-increasing in time as it will have to reimburse agent 1's savings in the future.

[11]This result is akin to results in the theory of supergames where any efficient outcome of a static game can be supported as an equilibrium of its associated supergame provided that the discount factor is high enough.

[12]Although making the largest ex ante payment is not necessary for an optimal contract for all values of the discount factor, it is clearly sufficient.

The following corollary gives a more precise characterization of the optimal consumption path.

Corollary 1 *Assume that $\underline{B} < y^S - c^{Sfc}$ and that $\beta < \beta_1$.*
(i) For each state s, there exists an optimal time-invariant consumption level $\underline{c}^s$ such that $c_t^s \geq \underline{c}^s$ for all time periods t.
(ii) The lower bounds of consumption, $\underline{c}^s$, are increasing in the state of the world, that is, $k > q \Rightarrow \underline{c}^k > \underline{c}^q$, and are decreasing in the maximal payment $\underline{B}$ that agent 1 can make.
(iii) For any history (h_{t-1}, s), optimal consumption at time t is such that:

$$c(h_{t-1}, s) = \begin{cases} \underline{c}^s & \text{if } c^*(c_{t-1}, y_{t-1}, s) < \underline{c}^s \\ c^*(c_{t-1}, y_{t-1}, s) & \text{otherwise} \end{cases}$$

where $c^*(c_{t-1}, y_{t-1}, s)$ is implicitly defined by $\frac{u'(c^*(c_{t-1}, y_{t-1}, s))}{u'(c_{t-1})} = \frac{v'(e + y_t^s - c^*(c_{t-1}, y_{t-1}, s))}{v'(e + y_{t-1} - c_{t-1})}$.

In each state, there exists an optimal time-invariant lower bound on agent 1's consumption. These bounds are defined by the ex post self-enforcing constraints and the optimal trade-off between current and future consumption. Efficient risk sharing requires agent 1 to save a large share of its endowment in the better states of the world; however when $\beta < \beta_1$ no consumption can take place below these bounds as such consumption would imply that agent 1 is saving too much compared with the future discounted surplus it expects from the relationship. These bounds are increasing with the state of the world. Agent 1's endowment is optimally shared between current and future consumption. As its endowment increases the lower bound on its current consumption increases as well. These bounds are also decreasing in the maximum ex ante payment that agent 1 can make. An increase in $\underline{B}$ increases agent 1's cost of reneging on the contract and thus relaxes its ex post self-enforcing constraint. This allows agent 1 to save more and consume less in the current period. This comparative statics result will be useful in the no-commitment case.

Consumption paths follow a simple rule. If, given consumption in period $t-1$ optimal consumption smoothing between periods $t-1$ and t satisfies agent 1's ex post self-enforcing constraints, then consumption in period t is equal to $c^*(c_{t-1}, y_{t-1}, s)$. If it does not satisfy agent 1's ex post self-enforcing constraints, then consumption in period t is equal to $\underline{c}^s$. Consumption then follows a stationary first-order Markov process where period t consumption depends on period $t-1$ consumption and the realized states in periods $t-1$ and t. The dynamics of consumption also imply that there is convergence to efficient risk sharing and consumption smoothing. In the steady state consumption only depends on the current state. Moreover, because optimal risk-sharing at actuarially fair prices is impossible when $\beta < \beta_1$, the steady-state consumption in every state must be higher than optimal consumption in the full-commitment case. This higher consumption is the result of agent 1's savings in the early periods and withdrawals in the later periods. This is acceptable to agent 2 because it gets a compensating surplus at the beginning of the relationship as it consumes agent 1's early savings.

The results of Proposition 2 and Corollary 1 are similar to results obtained by Harris and Holmström (1982) in a model of labor contracts. They showed that under the assumption of non-commitment by the employee wages are downward rigid as the risk-neutral employer fully insures the worker against bad states of the world. Our characterization is, first, a generalization to the case of two risk-averse agents. It shows that, in this case consumption can decrease in some states. Secondly, it shows that the non-committed party (agent 1) would like to make in each period ex ante transfers to relax its ex post self-enforcing constraints, that is, an optimal characterization sets $B_t = -\underline{B}$.

Having the non-committed agent making the maximal ex ante payment relaxes its ex post self-enforcing constraints. The possibility that the non-committed agent has of making an ex ante payment allows to shift (some or all) the burden of ex post transfers to the committed agent. However if both agents face self-enforcing constraints the above characterization may not be feasible. One agent may run away with the ex ante payment of the other agent as its ex post self-enforcing constraints would become too stringent. The optimal ex ante payment should therefore trade off between the two sets of self-enforcing constraints. The next section studies the details of that trade-off when the two agents face self-enforcing constraints.

4 Contracting under no commitment

We first define the concept of a self-enforcing contract under the non-commitment assumption.

Definition 3 *A contract δ is self-enforcing if and only if, for all histories h_{t-1}, periods t, and states s, the following constraints hold.*

$$(i) \quad U(\delta; h_{t-1}) \geq 0$$

$$(ii) \quad u(y_t^s + B_t - a_t^s) - u(y_t^s + B_t) + \beta U(\delta; h_{t-1}, s) \geq 0$$
$$(iii) \quad V(\delta; h_{t-1}) \geq 0$$
$$(iv) \quad v(e - B_t + a_t^s) - v(e - B_t) + \beta V(\delta; h_{t-1}, s) \geq 0$$

This definition simply states that a contract is self-enforcing if it is self-enforcing for agent 1 (constraints i and ii) as well as for agent 2 (constraints iii and iv).

Before proceeding with the analysis we assume that there are no exogenous bounds on the ex ante payment B_t. This assumption is motivated by the fact that we want to study how self-enforcing constraints rather than some exogenous bound limit the use of the ex ante payment.

The optimal contract without commitment, δ^{nc}, is the solution to the following maximization problem.

$$\begin{aligned} \delta^{nc} = \quad & \arg\max_{\delta} U(\delta; h_0) \\ & \text{s.t.} \quad U(\delta; h_t) \geq 0 \quad \forall\, t, h_t \\ & \qquad u(y_t^s + B_t - a_t^s) - u(y_t^s + B_t) + \beta U(\delta; h_{t-1}, s) \geq 0 \quad \forall\, s, t, h_{t-1} \\ & \qquad V(\delta; h_t) \geq 0 \quad \forall\, t, h_t \\ & \qquad v(e - B_t + a_t^s) - v(e - B_t) + \beta V(\delta; h_{t-1}, s) \geq 0 \quad \forall\, s, t, h_{t-1} \end{aligned} \tag{3}$$

It is difficult to characterize the optimal contract under this formulation. We therefore derive a more manageable recursive formulation. Following any time period and any history the optimal contract δ^{nc} will necessarily be efficient, since if it were not it would be possible to replace the nonefficient path by an efficient path thus (weakly) increasing the utility each agent derives from the contract and hence relaxing all previous self-enforcing constraints. This new contract would necessarily be self-enforcing and would dominate the old contract at the beginning of the relationship. This argument implies that the optimal contract from the start of period t onwards is the solution to the following maximization problem.

$$\begin{aligned} f(V_t) = \quad & \max_{B_t, (a_t^s)_s, (V_{t+1}^s)_s} E\left\{ u(y_t^s + B_t - a_t^s) - u(y_t^s) + \beta f(V_{t+1}^s) \right\} \\ & \text{s.t.} \quad f(V_{t+1}^s) \geq 0 \quad \forall\, s \\ & \qquad u(y_t^s + B_t - a_t^s) - u(y_t^s + B_t) + \beta f(V_{t+1}^s) \geq 0 \quad \forall\, s \\ & \qquad V_{t+1}^s \geq 0 \quad \forall\, s \\ & \qquad v(e - B_t + a_t^s) - v(e - B_t) + \beta V_{t+1}^s \geq 0 \quad \forall\, s \end{aligned} \tag{4}$$

$$V_t \leq E\left\{v(e - B_t + a_t^s) - v(e) + \beta V_{t+1}^s\right\}$$

where f represents the Pareto frontier that can be attained through an efficient self-enforcing contract after an arbitrary history h_{t-1}. This time-independent frontier is defined by

$$f(V_t) = \max_{\delta \in \Lambda(h_{t-1})} \{U(\delta; h_{t-1}) \text{ s.t. } V(\delta; h_{t-1}) \geq V_t\}$$

where $\Lambda(h_{t-1})$ is the set of contracts satisfying the self-enforcing constraints following the history h_{t-1}.

In problem (4), the variable V_{t+1}^s is to be interpreted as $V(\delta; h_{t-1}, s)$, that is, agent 2's expected surplus from period $t+1$ onwards when contract δ is signed and s is the realized state of nature in period t. The first two sets of constraints represent agent 1's ex ante and ex post self-enforcing constraints respectively. The next two sets represent agent 2's self-enforcing constraints. The last constraint of the problem ensures that the contract is dynamically consistent. Before characterizing the properties of the optimal contract we derive some useful technical results.

Lemma 1 *(i) The set of values of V_t for which a self-enforcing contract exists is a compact interval $[0, \bar{V}]$.*
(ii) The Pareto frontier f is decreasing, strictly concave, and continuously differentiable (almost everywhere) on $(0, \bar{V})$.
(iii) For each value of $V_t \in [0, \bar{V}]$ there exists a unique continuation of the contract δ at time t in which $V(\delta; h_{t-1}) = V_t$ and $U(\delta; h_{t-1}) = f(V_t)$.

To get a better understanding of the role of the ex ante payment in the no-commitment environment, we will first state the solution to problem (4) assuming that no ex ante payments are allowed.[13]

Proposition 3 *Set $B_t = 0$ for all time periods t.*
(i) For each state s, there exist optimal time-invariant consumption levels $\underline{c}^s$ and $\bar{c}^s$ such that $\underline{c}^s \leq c_t^s \leq \bar{c}^s$ for all time periods t.
(ii) The optimal lower bounds $\underline{c}^s$ and upper bounds $\bar{c}^s$ are increasing with the states of the world, that is, $k > q \Rightarrow \underline{c}^k > \underline{c}^q$ and $\bar{c}^k > \bar{c}^q$.

[13]This generalizes Thomas and Worrall (1988) to the case of bilateral risk aversion.

(iii) For any history (h_{t-1}, s), optimal consumption at time t is such that:

$$c(h_{t-1}, s) = \begin{cases} \underline{c}^s & \text{if } c^*(c_{t-1}, y_{t-1}, s) < \underline{c}^s \\ c^*(c_{t-1}, y_{t-1}, s) & \text{if } \underline{c}^s \leq c^*(c_{t-1}, y_{t-1}, s) \leq \bar{c}^s \\ \bar{c}^s & \text{otherwise} \end{cases}$$

(iv) There are no values of β such that the optimal contract with non-commitment, δ^{nc}, is the optimal contract with full commitment, δ^{fc}.

When no ex ante payments are allowed there are upper and lower bounds on the optimal consumption of agent 1. Lower (upper) bounds are determined by agent 1's (2's) ex post self-enforcing constraints and the optimal trade-off between current and future consumptions. Agent 1's consumption follows a simple stationary first-order Markov process. In period t consumption depends on period $t-1$ consumption and endowment and the state of the world realized in period t. This implies that the consumptions of the two agents between two adjacent periods are smoothed as much as possible subject to ex post self-enforcing constraints.

A second important property of this characterization is that the optimal risk-sharing contract, δ^{fc}, is not feasible under bilateral non-commitment. This contract yields zero expected utility to agent 2 in every period. Its ex post self-enforcing constraints then hold if and only if $a_t^s \geq 0$ for all states s. When $B_t = 0$ these restrictions are incompatible with optimal risk-sharing and hence there is no value of the discount factor for which the contract δ^{fc} is feasible.[14]

We now characterize the optimal solution when the ex ante payment is chosen optimally. An implication of Lemma 1 is that problem (4) is a concave program and therefore first-order conditions are both necessary and sufficient for a solution. Denote respectively by $\beta p^s \alpha^s$, $p^s \theta^s$, $\beta p^s \phi^s$, $p^s \lambda^s$, and ψ the multipliers of the five sets of constraints in problem (4). The first-order conditions are

$$B_t: \quad \sum_s p^s u'(y_t^s + B_t - a_t^s) + \sum_s p^s \theta^s \left(u'(y_t^s + B_t - a_t^s) - u'(y_t^s + B_t) \right) \\ - \sum_s p^s (\lambda^s + \psi) v'(e - B_t + a_t^s) + \sum_s p^s \lambda^s v'(e - B_t) = 0 \qquad (5)$$

[14] If agent 2 has some bargaining power over autarky there exist some values of β such that an optimal risk-sharing contract can be supported with no ex ante payments. See Kocherlakota (1994) for such characterization.

$$a_t^s: \quad -p^s(1+\theta^s)u'(y_t^s + B_t - a_t^s) + p^s(\lambda^s + \psi)v'(e - B_t + a_t^s) = 0 \quad \forall\, s \tag{6}$$

$$V_{t+1}^s: \quad (1 + \alpha^s + \theta^s)f'(V_{t+1}^s) + \lambda^s + \phi^s + \psi = 0 \quad \forall\, s \tag{7}$$

and the envelope condition is $f'(V_t) = -\psi$. Lemma 2 provides some basic properties of the solution.

Lemma 2 *(i) There exists a β_{nc} such that the optimal non-commitment contract, δ^{nc}, yields the same consumption as the optimal full-commitment contract, δ^{fc}, if and only if $\beta \in [\beta_{nc}, 1)$.*
Suppose that $\beta < \beta_{nc}$.
(ii) For $i = 1, 2$, there exists a state s_i in which agent i's ex post self-enforcing constraint is binding.

If the discount factor is high enough, the optimal full-commitment contract is feasible with non-commitment and is therefore optimal. Agent 2 pays up front a high enough payment ($B_t = c^{1fc} - y^1$) such that the resulting ex post payments, a_t^s, are all positive. These payments yield zero expected utility to agent 2 in every period and therefore its ex ante and ex post self-enforcing constraints are all satisfied. If the discount factor is high enough agent 1 prefers to make the ex post payments in all states of nature and be optimally insured in the future rather than keep the up-front payment, renege on the contract, and revert to autarky thereafter. The critical value of the discount factor β_{nc} (defined formally in the Appendix) is the lowest discount factor for which agent 1 does not renege on the contract in all states of nature. This result contrasts with the case $B_t = 0$ where the contract δ^{fc} is not feasible with non-commitment for any value of the discount factor. This is a first indication that the use of ex ante payments can strictly improve the utility of the two agents (at least for some values of the discount factor).

The second result of Lemma 2 states that, when the contract δ^{fc} is not feasible each agent always has at least one ex post self-enforcing constraint binding. Suppose only one agent was (ex post) constrained. This agent could then increase marginally its up-front payment and adjust its ex post payments to maintain its levels of consumption. This would relax its ex post self-enforcing constraints. At the margin, this would not violate the other agent's ex post self-enforcing constraints which were not binding before the increase in the ex ante payment. Such change would therefore increase the utility of a least one agent. An increase in the ex ante payment by one agent is possible until one of the other agent's self-enforcing constraint becomes binding, in which case further increases may not be self-enforcing anymore. Therefore, in the optimal contract each agent always has at least one ex post self-enforcing constraint binding.

The next proposition provides a characterization of the optimal ex ante payment when the contract δ^{fc} is not feasible.

Proposition 4 *Assume that $\beta < \beta_{nc}$.*
(i) The optimal value of the ex ante payment in period t is strictly decreasing in the expected surplus that agent 1 has to concede to agent 2 in period t, that is, $V_t' > V_t'' \Rightarrow B_t' < B_t''$ where B_t' (B_t'') is optimal if agent 2's expected surplus in period t is V_t' (V_t'').
(ii) The optimal ex ante payment is strictly positive when agent 2 has a zero expected surplus and negative when agent 2 has maximal expected surplus, that is, $B_t > 0$ if $V_t = 0$ and $B_t < 0$ if $V_t = \bar{V}$.

The ex ante payment is decreasing in the expected surplus of agent 2. Suppose that, following a given history the contract promises a low expected surplus to agent 2. This makes the contract not much more profitable than autarky to agent 2 and thus its ex post self-enforcing constraints are likely to be more constraining than those of agent 1. In this case agent 2 optimally pays out a relatively large ex ante payment to relax its ex post self-enforcing constraints. The size of the optimal ex ante payment is therefore inversely related to the expected surplus of agent 2. This logic can easily be extended to show that the optimal ex ante payment is negative when agent 2 expects a high surplus from the relationship, that is, agent 1 pays out to agent 2 a high ex ante transfer.

It is difficult to provide a more complete characterization of the solution in the general case given the number of inequality constraints; however, we can do so in a special case in which there are only two states. This simple example is sufficient to illustrate the role of the ex ante payment. We then compare our results with the case in which no ex ante payments are allowed. Suppose that $S = 2$. State 1 can represent a state in which an accident occurs and state 2, a state where no accident occurs. Also assume that the discount factor is such that full insurance at fair prices (contract δ^{fc}) is not feasible.

Proposition 5 *Suppose that $S = 2$ and $\beta < \beta^{nc}$.*
(i) The expected profit of agent 2 for period $t+1$ is larger (smaller) than that of period t if state 1 (2) occurs in period t, that is, $V(\delta^{nc}; h_{t-1}, 2) \leq V(\delta^{nc}; h_{t-1}) \leq V(\delta^{nc}; h_{t-1}, 1)$, with

strict inequality if $V(\delta^{nc}; h_{t-1}) \notin \left\{0, \bar{V}\right\}$.
Suppose that $0 < V_t < \bar{V}$.
(ii) Agent 1's consumption in period t *is smaller (larger) than* $c^*(c_{t-1}, y_{t-1}, s)$ *if* $s = 1$ *(if* $s = 2$*) in period* t*, that is,* $c_t^1 < c^*(c_{t-1}, y_{t-1}, 1)$ *and* $c_t^2 > c^*(c_{t-1}, y_{t-1}, 2)$.

If an accident occurs (state 1) agent 1 wants to smooth its impact across periods. It then borrows from agent 2, that is, $V_{t+1}^1 \geq V_t$. Alternatively agent 1 lends to (or reimburses) agent 2 if no accident takes place, that is, $V_{t+1}^2 \leq V_t$. In this case the good news of "no accident" is spread over many periods. The second result of the proposition states that the two agents bear more risk than they do in the full-commitment case. This shows that incomplete insurance need not be explained by the presence of asymmetric information. Non-commitment problems can also explain such occurrence.

These results may seem quite similar to those one would obtain when no ex ante payments are allowed. This may be a misleading conclusion. Consider first the case where no ex ante payments are allowed. The results of Proposition 3 imply that optimal consumption takes place at $\bar{c}^1$ ($\underline{c}^2$) if state 1 (2) occurs. These consumption levels are time-invariant and therefore consumption can only take one of these two values depending on the realized state. At any given period expected consumption for next period is the same regardless of the history. Now consider the case where ex ante payments are allowed. In any given period, for a given value B of the ex ante payment it is possible to define as in Proposition 3 consumption bounds $\bar{c}^1(B)$ and $\underline{c}^2(B)$. We know that these optimal bounds are increasing with the ex ante payment.[15] Furthermore Proposition 5 states that $V_{t+1}^2 \leq V_t \leq V_{t+1}^1$. Using Proposition 4 this implies that the ex ante payment from agent 2 to agent 1 will decrease (increase) in period $t+1$ compared to that of period t if state 1 (2) occurs. We can now characterize the optimal consumption paths. Suppose state 1 occurs in period t. Agent 2 is then promised a higher expected surplus for period $t+1$. This implies that it will make a lower ex ante payment in period $t+1$ thus reducing the two consumption bounds for that period. Expected consumption will then be lower in period $t+1$ than in period t. The opposite holds if state 2 occurs in period t, that is, expected consumption rises in period $t+1$ compared to that of period t. The presence of the ex ante payment therefore improves consumption smoothing. If a good state occurs, current and future expected consumptions are increased, while the opposite holds if the bad state is realized.

[15]This is a straightforward extension of Proposition 3 and Corollary 1.

In an insurance context these results can be interpreted as the insurance premium increasing when an accident occurs and decreasing when no accident occurs. Furthermore the dynamics of our model imply that the complete history is relevant for explaining contemporaneous insurance premium, that is, a sequence of accidents results in successive increases in insurance premia and therefore a drop in expected consumption. The consumption pattern with an ex ante payment looks like "experience rating", that is, average consumption in one period is positively related to the previous realizations of the state of nature. In this model experience rating arises from the desire of the insuree to smooth consumption over time. This is optimally achieved by having a premium increase when an accident occurs. In this case, the current marginal utility of the insuree is high and it promises higher premia in the future in exchange for a high current compensation. Our model therefore predicts that experience rating can take place in an insurance market even though information is symmetric and shocks are identically and independently distributed. Our model can therefore generate higher order correlation in consumption even though endowments are independently distributed. This shows that neither asymmetric information nor uncertainty and learning are necessary to explain experience rating in dynamic insurance contracts. This simple example shows that allowing for an ex ante payment yields predictions that are significantly different from those without ex ante payments.

5 Discussion

In this section we discuss two possible modifications to our economic environment, namely the introduction of savings and bonding.

One question that comes to mind is: would the contract still have value if, say agent 1 was allowed to save at a risk-free rate equal to its discount rate? The introduction of savings in the contracting framework developed here would have nontrivial effects on the patterns of consumption. Savings does not only modify the expected utility an agent gets from the contract but also its expected utility in autarky. As savings accumulate autarky becomes more attractive; but this does not imply that the contract may eventually play no role. Bewley (1977) and Schechtman (1976) show that even though an agent has accumulated important savings, still it does not fully insure itself against random shocks. Rather, good shocks are spread over many periods, as are bad shocks in an effort to smooth consumption. There is therefore some residual risk left in an agent's consumption. It is then presumed that some residual risk could be further insured by a self-enforcing contract with, for example a risk-neutral agent. Autarcic consumption would then correspond to consumption with savings (instead of consumption of the endowment). In any case the contract would be useful in the early periods where a bad shock increases the demand for a loan. The introduction of savings is worth investigating and is the subject of our current research.

Another interesting question that often comes to mind in non-commitment environments is: what if agents could post a bond? It is well known that posting a bond is a means for avoiding self-enforcing constraints, that is, the agent that cannot commit or that must be disciplined simply posts a bond that it loses if it does not perform satisfactorily. For example Williamson (1983) illustrates how the use of a bond can promote efficient trade. Posting a bond is equivalent to specifying a penalty for breach of contract. We can then provide a different interpretation for the ex ante payment. Suppose that all payments and consumption take place at the end of the period. The ex ante payment can now be interpreted as a penalty for breach of contract that is decided upon by the agents at the beginning of the period, that is paid only in case of default, and that is enforceable by the courts. Our results would therefore imply that the net penalty rests upon the shoulders of that agent that is the most likely to breach the contract, that is, the agent that expects the lower future surplus from the relationship.

Because B_t is the net penalty, this interpretation would imply that it cannot be made contingent on who breached the contract. This would be reasonable in an environment where the courts can observe whether a relationship is continuing or not, but if it is not they cannot determine why there has been a breach. This is a reasonable assumption if, for example, an agent can "force" the other agent to breach the contract by misbehaving. If courts can observe who breached the contract then an agent-specific bond would resolve all commitment problems.

One can argue that bonds could also be posted with third parties that would keep it if an agent ever breaches the contract. This solution also has some problems of its own. For one, it may encourage collusion between one of the agents and the third party to breach the contract and then share the bond. Or, faced with a possible breach the agents may renegotiate the contract to avoid losing the bond to the third party.

6 Conclusion

We develop a dynamic model of contracting for risk-sharing purposes. Complete insurance is impeded by ex post opportunism in that agents can break the relationship at any time if it is in their own interest to do so. However, agents can commit partially by making payments at the beginning of a period before the state of nature is realized. These payments can increase the potential gains from trade but cannot generally restore perfect risk sharing. These payments evolve inversely with the surplus an agent expects from the relationship.

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APPENDIX

Proof of Proposition 2 (i) Consider the optimal full-commitment contract δ^{fc} characterized in Proposition 1. The per-period surplus to agent 1 is $Eu(c^{qfc}) - Eu(y^q)$ which is positive. Hence $U(\delta^{fc}; h_{t-1}) > 0$ for all histories h_{t-1} and periods t. This implies that all ex ante self-enforcing constraints are satisfied. Suppose that $\underline{B} \geq y^S - c^{Sfc}$ and that agent 1 makes the maximum ex ante payment, namely, $B_t = -\underline{B}$. From Proposition 1 we know that $a_t^s = y_t^s + B_t - c^{sfc}$. If $B_t = -\underline{B}$, then $a_t^s \leq y_t^s - y^S + c^{Sfc} - c^{sfc}$. Because the transfer from agent 1 to agent 2 is largest when $s = S$, we have that $a_t^s \leq 0$ for all s, and hence no ex post self-enforcing constraints are binding. The contract δ^{fc} can then be supported as the optimal contract δ^1.

(ii) Assume that $\underline{B} < y^S - c^{Sfc}$. Consider the optimal full-commitment contract δ^{fc}. As argued above, all ex ante self-enforcing constraints are satisfied. Ex post self-enforcing constraints are satisfied if and only if

$$u(c^{sfc}) - u(y_t^s + B_t) + \frac{\beta}{1-\beta}\Big(Eu(c^{qfc}) - Eu(y^q)\Big) \geq 0 \quad \forall\, s.$$

These constraints become less binding if B_t is set at its lowest level, namely, $B_t = -\underline{B}$ and net transfers are adjusted such that agent 1's consumption be c^{sfc}. Setting $B_t = -\underline{B}$ and solving for β in the ex post self-enforcing constraint yields

$$\beta \geq \frac{u(y^s - \underline{B}) - u(c^{sfc})}{u(y^s - \underline{B}) - u(c^{sfc}) + Eu(c^{qfc}) - Eu(y^q)} \text{ for all } s.$$

The critical value β_1 above which all ex post self-enforcing constraints are satisfied is given by

$$0 < \beta_1 \equiv \frac{u(y^S - \underline{B}) - u(c^{Sfc})}{u(y^S - \underline{B}) - u(c^{Sfc}) + Eu(c^{qfc}) - Eu(y^q)} < 1.$$

Hence all (ex ante and ex post) self-enforcing constraints for agent 1 are satisfied for $\beta \in [\beta_1, 1)$. For values of β outside this interval, at least one of the ex post self-enforcing constraints for agent 1 is violated and thus the optimal full-commitment contract is not feasible when agent 1 can renege on the contract.

(iii) The proof of this part of the proposition is more involved and we need to introduce some notation. Define by $\Omega(h_{t-1})$ the set of contracts satisfying the self-enforcing constraints for agent 1 following history h_{t-1}. For convenience we define this set over the space of functions $\mathcal{C}_t$ and $\mathcal{B}_t$. This set is not convex due to the presence of the term $-u(y_t^s + B_t)$ in the self-enforcing constraints of agent 1. Because it is convenient to work with a convex set we convexify this set by allowing for lotteries over the fixed payment B_t.[16] Suppose that in every period there is a continuum of possible values of the fixed payment in the interval $\Phi \equiv [-\underline{B}, \infty)$. Define the function $\mathcal{Z}_t(\cdot; h_{t-1})$ as a density function over the support Φ. Agent 1's self-enforcing constraints then become for all histories, periods, and states:

$$\begin{aligned} &(i) \quad U(\delta; h_{t-1}) \geq 0 \\ &(ii) \quad u(c_t^s) - \int_\Phi u(y_t^s + B)\mathcal{Z}_t(B; h_{t-1})dB + \beta U(\delta; h_{t-1}, s) \geq 0 \end{aligned}$$

[16]We will show later that the optimal lottery is in fact degenerate.

The space of self-enforcing contracts is then $\widehat{\Omega}(h_{t-1}) \equiv \{\mathcal{C}_t, \mathcal{Z}_t(\cdot; h_{t-1})$ such that agent 1's self-enforcing constraints are satisfied$\}$. The Pareto frontier that is attainable by an efficient self-enforcing contract is given by

$$g(V_t) = \max_{\delta \in \widehat{\Omega}(h_{t-1})} \left\{ U(\delta; h_{t-1}) \text{ s.t. } V(\delta; h_{t-1}) \geq V_t \right\}.$$

The Pareto frontier is time-independent as all constraints defining $\widehat{\Omega}(h_{t-1})$ and the functions $U(\cdot; h_{t-1})$ and $V(\cdot; h_{t-1})$ are forward looking.

We now show that the Pareto frontier is strictly concave and continuously differentiable almost everywhere.[17] First we argue that the set $\widehat{\Omega}(h_{t-1})$ is convex. This follows directly form the concavity of the utility function u and linearity of the constraints in the terms $\mathcal{Z}_t(\cdot; h_{t-1})$.
Secondly, following history h_{t-1}, the set of V_t such that a self-enforcing contract for agent 1 exists is a compact interval $[-K_1, \bar{V}]$ where $-K_1$ is the discounted utility of agent 2 when it pays out to agent 1 its total endowment in every state and period.[18] Such transfers are obviously self-enforcing for agent 1. There exists an upper bound on the surplus agent 1 can concede to agent 2 in a self-enforcing contract. Denote this upper bound by $\bar{V}$. If $\bar{V}$ is attainable by a self-enforcing contract, then any $V_t \in [-K_1, \bar{V}]$ is also. The closedness of this interval can be shown by constructing a sequence of self-enforcing contracts yielding some utility level to agent 2 converging to $\bar{V}$. Because u is continuous and $\beta \in (0, 1)$, the Dominated Convergence Theorem implies that the limiting contract is also self-enforcing and hence $\bar{V}$ is included in the interval.
Finally, we show that the Pareto frontier is decreasing, strictly concave and continuously differentiable almost everywhere. It is obvious that the function g is decreasing. The strict concavity property follows from the strict concavity of u, the concavity of v, and the convexity of $\widehat{\Omega}$. The differentiability property follows from the continuity and differentiability of u. Consider an efficient self-enforcing contract δ such that $V(\delta; h_{t-1}) = V_t \in (-K_1, \bar{V})$. Construct a contract δ^γ which differs from the contract δ in that $a^\gamma(h_{t-1}, s) = a(h_{t-1}, s) + \gamma$. The state s is chosen such that agent 1's ex post self-enforcing constraint is not strictly binding. The contract δ^γ is therefore self-enforcing for γ small enough. Define the function $\hat{g}$ such that $U(\delta^\gamma; h_{t-1}) = \hat{g}(V(\delta^\gamma; h_{t-1})) \leq g(V(\delta; h_{t-1}))$ with equality if $\gamma = 0$. As γ is varied, it is easy to show that the function $\hat{g}$ is concave and differentiable at V_t. Therefore it satisfies Lemma 1 reported in Benveniste and Scheinkman (1979). The function g is then differentiable. Because it is monotonic, it is also continuously differentiable almost everywhere. This implies that for any value $V_t \in [-K_1, \bar{V}]$, there exists a unique efficient continuation of the contract δ at time t in which $V(\delta; h_{t-1}) = V_t$ and $U(\delta; h_{t-1}) = g(V_t)$. Existence is guaranteed by the compactness of the interval $[-K_1, \bar{V}]$; uniqueness is guaranteed by the convexity of $\widehat{\Omega}$ and the strict concavity of u.

This Pareto frontier can be used to characterize the optimal contract. Following any history the optimal contract δ^1 will necessarily be efficient, since if it were not it would be possible to replace the nonefficient path by an efficient path thus relaxing all previous self-enforcing constraints. This new contract would necessarily be self-enforcing and would dominate the old contract at the beginning of the relationship. This argument implies that the optimal contract from the start of period t

[17] Most of the arguments used here follow those of Lemma 1 of Thomas and Worrall (1988).

[18] Remember that agent 1's utility function is defined over the interval $[0, b]$.

onwards is the solution to the following maximization problem.

$$
\begin{aligned}
g(V_t) = \quad & \max_{z_t,(c_t^s)_s,(V_{t+1}^s)_s} E\left\{u(c_t^s) - u(y_t^s) + \beta g(V_{t+1}^s)\right\} \\
& \text{s.t.} \quad g(V_{t+1}^s) \geq 0 \quad \forall\, s \\
& \qquad\; u(c_t^s) - \int_\Phi u(y_t^s + B) z_t(B) dB + \beta g(V_{t+1}^s) \geq 0 \quad \forall\, s \\
& \qquad\; \int_\Phi z_t(B) dB = 1 \\
& \qquad\; z_t(B) \geq 0 \quad \forall\, B \\
& \qquad\; V_t \leq E\left\{v(e + y_t^s - c_t^s) - v(e) + \beta V_{t+1}^s\right\}
\end{aligned}
\tag{8}
$$

In this problem, the variable V_{t+1}^s is to be interpreted as $V(\delta; h_{t-1}, s)$, that is, agent 2's expected utility from period $t+1$ onwards when contract δ is signed, h_{t-1} is the history up to period t, and s is the realized state of nature in period t. The first two sets of constraints represent agent 1's self-enforcing constraints. The next two constraints ensures that z_t is a density function. The last constraint of the problem ensures that the contract is dynamically consistent.

The properties of the Pareto frontier g and the convexity of $\widehat{\Omega}$ imply that problem (8) is a concave program, and therefore its first-order conditions are both necessary and sufficient for a solution. Let $\beta p^s \alpha^s$, $p^s \theta^s$, η, $\mu(B)$, and ψ be the respective multipliers of the constraints in problem (8). The first-order conditions are then

$$
z_t(B): \quad \sum_s \left(-p^s \theta^s u(y_t^s + B)\right) + \mu(B) + \eta = 0 \quad \forall\, B \tag{9}
$$

$$
c_t^s: \quad p^s(1 + \theta^s) u'(c_t^s) - p^s \psi v'(e + y_t^s - c_t^s) = 0 \quad \forall\, s \tag{10}
$$

$$
V_{t+1}^s: \quad (1 + \alpha^s + \theta^s) g'(V_{t+1}^s) + \psi = 0 \quad \forall\, s \tag{11}
$$

and the envelope condition is $g'(V_t) = -\psi$.

1. If no self-enforcing constraint binds ($\theta^s = 0$ for all s), only net payments matter and hence $z_t(B)$ can be set arbitrarily such that it has a mass point at $\underline{B}$ and be zero elsewhere. Note that this density is the one for which ex post self-enforcing constraints are the least binding. If there is at least one self-enforcing constraint that binds, then there exists a state s such that $\theta^s > 0$. The expression $\sum_s \left(-p^s \theta^s u(y_t^s + B)\right)$ is decreasing and convex in B. This implies that there is at most one value of B which has positive density (i.e. $\mu(B) = 0$). Furthermore this value must be at the lower bound of Φ. This implies that $z_t(B)$ is such that it has a mass point at $\underline{B}$ and is zero elsewhere.

2. Condition (11) and the envelope condition jointly imply that $(1 + \alpha^s + \theta^s) g'(V_{t+1}^s) = g'(V_t)$. Because $\alpha^s + \theta^s \geq 0$ and the Pareto frontier g is decreasing and concave, it implies that $V_{t+1}^s \leq V_t$ with strict inequality when $\alpha^s + \theta^s > 0$. *Q.E.D.*

Proof of Corollary 1 We first show that $\alpha^s = 0$ for all s. First-order conditions and the envelope condition imply that

$$
(1 + \alpha^s + \theta^s) g'(V_{t+1}^s) = g'(V_t).
$$

Suppose that $\alpha^s > 0$, then $g(V_{t+1}^s) = 0$. Because agent 2's expected profits are non-increasing in time and g is decreasing we must have that $g(V_t) = 0$. But then the above expression cannot hold. This implies that $\alpha^s = 0$ for all states s.

(i) Denote by $\bar{V}^s$ and $\underline{c}^s$ the optimal maximum and minimum values for V^s_{t+1} and c^s_t respectively such that there exists a self-enforcing contract for agent 1. These values are implicitly defined by

$$\frac{u'(\underline{c}^s)}{v'(e+y^s-\underline{c}^s)} = -g'(\bar{V}^s) \tag{12}$$

$$u(\underline{c}^s) - u(y^s-\underline{B}) + \beta g(\bar{V}^s) = 0. \tag{13}$$

The first equation follows from first-order conditions to problem (8) and the fact that $\alpha^s = 0$ for all s while the second represents agent 1's ex post self-enforcing constraint in state s. Note that these equations are time-independent. After substituting for $\bar{V}^s$ in agent 1's ex post self-enforcing constraint, the optimal bound on consumption, $\underline{c}^s$, is then implicitly defined by

$$u(\underline{c}^s) - u(y^s-\underline{B}) + \beta g\left(g'^{-1}\left(-\frac{u'(\underline{c}^s)}{v'(e+y^s-\underline{c}^s)}\right)\right) = 0.$$

The left-hand side of the ex post self-enforcing constraint is increasing in $\underline{c}^s$ which implies that it is satisfied for $c^s_t \geq \underline{c}^s$ for all t and s.

(ii) Totally differentiating equations (12) and (13) with respect to $\underline{c}^s$, $\bar{V}^s$, and y^s yields $d\underline{c}^s/dy^s > 0$.[19] Hence, $\underline{c}^s$ is increasing in the states of the world, that is, $\underline{c}^k > \underline{c}^q$ if and only if $y^k > y^q$. Finally, totally differentiating the same equations yields $d\underline{c}^s/d\underline{B} < 0$.

(iii) From the first-order conditions in periods $t-1$ and t we know that

$$(1+\theta^s)\left(u'(c^s_t)/v'(e+y^s_t-c^s_t)\right) = u'(c_{t-1})/v'(e+y_{t-1}-c_{t-1}).$$

Suppose first that $c^*(c_{t-1}, y_{t-1}, s) \geq \underline{c}^s$. This expression is satisfied when $c^s_t = c^*(c_{t-1}, y_{t-1}, s) \geq \underline{c}^s$ and $\theta^s = 0$. Now suppose that $c^*(c_{t-1}, y_{t-1}, s) < \underline{c}^s$. The above expression is satisfied when $c^s_t = \underline{c}^s$ and $\theta^s > 0$. *Q.E.D.*

Proof of Lemma 1 As in the proof of Proposition 2 the set $\Lambda(h_{t-1})$ is not convex and we must convexify it to prove our results. Use the same notation as before, namely, $\mathcal{Z}_t(\cdot; h_{t-1})$ is a density function defined over all possible values of B (possibly the real line). The self-enforcing constraints then become for all histories h_{t-1}, periods t, and states s:

$$(i) \quad U(\delta; h_{t-1}) \geq 0$$

$$(ii) \quad u(c^s_t) - \int u(y^s_t + B)\mathcal{Z}_t(B; h_{t-1})dB + \beta U(\delta; h_{t-1}, s) \geq 0$$

$$(iii) \quad V(\delta; h_{t-1}) \geq 0$$

$$(iv) \quad v(e+y^s_t-c^s_t) - \int v(e-B)\mathcal{Z}_t(B; h_{t-1})dB + \beta V(\delta; h_{t-1}, s) \geq 0$$

The set of self-enforcing contracts $\widehat{\Lambda}(h_{t-1})$ is then $\{\mathcal{C}_t, \mathcal{Z}_t(\cdot; h_{t-1})$ such that both agents' self-enforcing constraints are satisfied$\}$.

[19]Because the function g is continuously differentiable almost everywhere and concave we know that g'' exists almost everywhere. Where it does not exist, we know that the right-hand and left-hand derivatives are negative, which is sufficient to prove the result.

Now consider the following modified version of problem (4) where we allow for random values of B. It will be used to show that the optimal $\mathcal{Z}_t(\cdot; h_{t-1})$ is degenerate at a single value.

$$\begin{aligned} f(V_t) = \quad & \max_{z_t,(c_t^s)_s,(V_{t+1}^s)_s} E\left\{u(c_t^s) - u(y_t^s) + \beta f(V_{t+1}^s)\right\} \\ & \text{s.t.} \quad f(V_{t+1}^s) \geq 0 \quad \forall\, s \\ & \qquad u(c_t^s) - \int u(y_t^s + B) z_t(B) dB + \beta f(V_{t+1}^s) \geq 0 \quad \forall\, s \\ & \qquad V_{t+1}^s \geq 0 \quad \forall\, s \\ & \qquad v(e + y_t^s - c_t^s) - \int v(e - B) z_t(B) dB + \beta V_{t+1}^s \geq 0 \quad \forall\, s \\ & \qquad \int z_t(B) dB = 1 \\ & \qquad z_t(B) \geq 0 \quad \forall\, B \\ & \qquad V_t \leq E\left\{v(e + y_t^s - c_t^s) - v(e) + \beta V_{t+1}^s\right\} \end{aligned} \tag{14}$$

Only consider the necessary first-order conditions for the density function to satisfy.

$$z_t(B): \quad \sum_s \left(-p^s \theta^s u(y_t^s + B) - p^s \lambda^s v(e - B)\right) + \mu(B) + \eta = 0 \quad \forall\, B$$

where the λ^s are the multipliers on agent 2's ex post self-enforcing constraints. If all multipliers θ^s and λ^s are zero then the ex ante payment is a matter of indifference and its density can be arbitrarily set at a single mass point. Now suppose some ex post self-enforcing constraints are binding. The expression $\sum_s \left(-p^s \theta^s u(y_t^s + B) - p^s \lambda^s v(e - B)\right)$ is convex in B. This implies that there are at most two values of B that can have positive density (i.e. $\mu(B) = 0$). Suppose that the density is positive for exactly two values. This contradicts the necessary first-order conditions as some multipliers $\mu(B)$ would have to be negative. Hence the optimal density is degenerate at a single value of the ex ante payment. There is therefore no loss in generality in restricting our attention to problem (4).
(i), (ii), (iii) The rest of the proof follows that of Proposition 2 with minor modifications. *Q.E.D.*

Proof of Proposition 3 Denote respectively by $\beta p^s \alpha^s$, $p^s \theta^s$, $\beta p^s \phi^s$, $p^s \lambda^s$, and ψ the multipliers of the five sets of constraints in problem (4). The first-order conditions when $B_t = 0$ for all time periods are

$$a_t^s: \quad -p^s(1 + \theta^s) u'(y_t^s + B_t - a_t^s) + p^s(\lambda^s + \psi) v'(e - B_t + a_t^s) = 0 \quad \forall\, s \tag{15}$$

$$V_{t+1}^s: \quad (1 + \alpha^s + \theta^s) f'(V_{t+1}^s) + \lambda^s + \phi^s + \psi = 0 \quad \forall\, s \tag{16}$$

and the envelope condition is $f'(V_t) = -\psi$. We first argue that $\alpha^s = \phi^s = 0$ for all states s. When $B_t = 0$ ex ante self-enforcing constraints cannot be binding when all ex post self-enforcing constraints are satisfied. There is therefore no loss in generality in supposing that $\alpha^s = \phi^s = 0$ for all states s.[20]
(i) First-order conditions imply that

$$\frac{u'(c_t^s)}{v'(e + y_t^s - c_t^s)} = -f'(V_{t+1}^s). \tag{17}$$

[20] A formal proof of this can be provided by the authors.

Condition (17) can be rewritten as $V_{t+1}^s = f'^{-1}\left(-u'(c_t^s)/v'(e + y_t^s - c_t^s)\right)$. The lower bound on consumption, $\underline{c}^s$, is defined by the intersection of this expression and agent 1's ex post self-enforcing constraint. More formally,

$$u(\underline{c}^s) - u(y^s) + \beta f\left(\min\left\{f'^{-1}\left(-u'(\underline{c}^s)/v'(e + y^s - \underline{c}^s)\right), \bar{V}\right\}\right) = 0 \tag{18}$$

where $f(\bar{V}) = 0$. This expression states that optimal lower bounds on consumption are defined by the intersection of first-order conditions and agent 1's ex post self-enforcing constraints to the extent that they respect the ex ante self-enforcing constraint; otherwise the expression reduces to $\underline{c}^s = y^s$. It is clear from expression (18) that $\underline{c}^s \leq y^s$. Note that these optimal lower bounds are time-independent.
We now define the optimal upper bounds on consumption. The expression

$$V_{t+1}^s = f'^{-1}\left(-u'(c_t^s)/v'(e + y_t^s - c_t^s)\right)$$

is substituted in agent 2's ex post self-enforcing constraint. More formally,

$$v(e + y^s - \bar{c}^s) - v(e) + \beta\left(\max\left\{f'^{-1}\left(-u'(\bar{c}^s)/v'(e + y^s - \bar{c}^s)\right), 0\right\}\right) = 0. \tag{19}$$

This expression states that optimal upper bounds on consumption are defined by the intersection of first-order conditions and agent 2's ex post self-enforcing constraint to the extent that they respect the ex ante self-enforcing constraints; otherwise the expression reduces to $\bar{c}^s = y^s$. It is clear from expression (19) that $\bar{c}^s \geq y^s$. Again, note that these optimal upper bounds are time-independent.
The preceding arguments show that in any time period t and state s, consumption c_t^s must be included in the interval $[\underline{c}^s, \bar{c}^s]$; otherwise one of the self-enforcing constraints or first-order conditions would be violated.

(ii) We now show that the optimal lower bounds are increasing in the states of the world. The optimal lower bounds are implicitly defined as a function of y^s in expression (18). This expression is continuous in $\underline{c}^s$ and y^s but is not differentiable at the point where the minimum switches from $f'^{-1}\left(-u'(\underline{c}^s)/v'(e + y^s - \underline{c}^s)\right)$ to $\bar{V}$. When the minimum equals $\bar{V}$, $\underline{c}^s = y^s$ and clearly the optimal bound is increasing in the state of the world. When the minimum equals the first expression, total differentiation of the implicit function yields

$$\frac{d\underline{c}^s}{dy^s} = -\frac{\beta f' f'^{-1'} * \left(\frac{u'(\underline{c}^s)v''(e+y^s-\underline{c}^s)}{v'(e+y^s-\underline{c}^s)^2}\right) - u'(y^s)}{u'(\underline{c}^s) + \beta f' f'^{-1'} * \left(-\frac{u''(\underline{c}^s)v'(e+y^s-\underline{c}^s)+u'(\underline{c}^s)v''(e+y^s-\underline{c}^s)}{v'(e+y^s-\underline{c}^s)^2}\right)}$$

which is positive because $f' f'^{-1'} > 0$.[21] Hence, because $\underline{c}^s$ is a continuous implicit function of y^s, these results imply that $\underline{c}^s$ is increasing in the states of the world, that is, $\underline{c}^k > \underline{c}^q$ if and only if $y^k > y^q$.
We now show that the optimal upper bounds are increasing in the states of the world. The optimal upper bounds are implicitly defined as a function of y^s in expression (19). This expression is continuous in $\bar{c}^s$ and y^s but is not differentiable at the point where the maximum switches from

[21] Because the function f is continuously differentiable almost everywhere and concave we know that $f'^{-1'}$ exists almost everywhere. Where it does not exist, we know that the right-hand and left-hand derivatives are negative, which is sufficient to prove the result.

$f'^{-1}\left(-u'(\bar{c}^s)/v'(e+y^s-\bar{c}^s)\right)$ to 0. When the maximum equals 0, $\bar{c}^s = y^s$ and clearly the optimal bound is increasing in the state of the world. When the maximum equals the first expression, total differentiation of the implicit function yields

$$\frac{d\bar{c}^s}{dy^s} = -\frac{v'(e+y^s-\bar{c}^s) + \beta f'^{-1\prime} * \left(\frac{u'(\bar{c}^s)v''(e+y^s-\bar{c}^s)}{v'(e+y^s-\bar{c}^s)^2}\right)}{-v'(e+y^s-\bar{c}^s) + \beta f'^{-1\prime} * \left(-\frac{u''(\bar{c}^s)v'(e+y^s-\bar{c}^s)+u'(\bar{c}^s)v''(e+y^s-\bar{c}^s)}{v'(e+y^s-\bar{c}^s)^2}\right)}$$

which is positive because $f'^{-1\prime} \le 0$.[22] Hence, because $\bar{c}^s$ is a continuous implicit function of y^s, these results imply that $\bar{c}^s$ is increasing in the states of the world, that is, $\bar{c}^k > \bar{c}^q$ if and only if $y^k > y^q$.

(iii) First-order conditions and the envelope condition imply that

$$\begin{aligned}(1+\theta^s)\frac{u'(c_t^s)}{v'(e+y_t^s-c_t^s)} - \lambda^s &= -f'(V_t)\\ \frac{u'(c_t^s)}{v'(e+y_t^s-c_t^s)} &= -f'(V_{t+1}^s)\end{aligned}$$

First-order conditions in period $t-1$ then imply that

$$(1+\theta^s)\frac{u'(c_t^s)}{v'(e+y_t^s-c_t^s)} - \lambda^s = \frac{u'(c_{t-1})}{v'(e+y_{t-1}-c_{t-1})}.$$

Suppose that $\underline{c}^s \le c^*(c_{t-1}, y_{t-1}, s) \le \bar{c}^s$. The solution must then be $c_t^s = c^*(c_{t-1}, y_{t-1}, s)$ with $\theta^s = \lambda^s = 0$. If $c^*(c_{t-1}, y_{t-1}, s) > \bar{c}^s$, then the solution must be $\lambda^s > 0$ and $c_t^s = \bar{c}^s$. If $c^*(c_{t-1}, y_{t-1}, s) < \underline{c}^s$, then the solution must be $\theta^s > 0$ and $c_t^s = \underline{c}^s$.

(iv) The contract δ^{fc} is such that $V_{t+1}^{sfc} = 0$ for all states and periods. This implies that agent 2's ex post self-enforcing constraints can be satisfied if and only if $a_t^s \ge 0$ in all states and periods. But this is inconsistent with the payments prescribed by the contract δ^{fc}, hence it cannot be self-enforcing when $B_t = 0$. *Q.E.D.*

Proof of Lemma 2 (i) We know that $V(\delta^{fc}; h_t) = 0$ for all histories h_t. The contract δ^{fc} is therefore self-enforcing if and only if $a_t^s \ge 0$ for all states s and periods t. Consider the following contract $\hat{\delta}$: the ex ante payment is set at $\hat{B}_t = c^{1fc} - y^1$ and contingent payments at $\hat{a}_t^s = y_t^s - y^1 + c^{1fc} - c^{sfc} \ge 0$ in all states and periods. This contract yields for both agents the same consumption as under the contract δ^{fc}. The contract $\hat{\delta}$ is self-enforcing for agent 2. Its ex ante self-enforcing constraints are trivially satisfied, that is, $\hat{V}_{t+1}^s = 0$ for all states and periods; its ex post self-enforcing constraints are also satisfied by construction because $\hat{a}_t^s \ge 0$ for all states and periods. It is self-enforcing for agent 1 if all its ex post self-enforcing constraints are satisfied, that is,

$$u(c^{sfc}) - u(y^s + c^{1fc} - y^1) + \frac{\beta}{1-\beta}\left\{Eu(c^{qfc}) - Eu(y^q)\right\} \ge 0 \text{ for all } s.$$

Furthermore its ex ante self-enforcing constraints are satisfied by $\hat{\delta}$. Define β_{nc} by

$$0 < \beta_{nc} \equiv \frac{u(y^S + c^{1fc} - y^1) - u(c^{Sfc})}{u(y^S + c^{1fc} - y^1) - u(c^{Sfc}) + Eu(c^{qfc}) - Eu(y^q)} < 1.$$

[22]The remark of the last footnote applies here as well.

It is the smallest discount factor that satisfies the self-enforcing constraints in all states. This shows that $\beta \geq \beta_{nc}$ is a sufficient condition for the contract $\hat{\delta}$ to be self-enforcing. It is also necessary since a contract with a smaller ex ante payment would not be self-enforcing for agent 2 as it would require at least one ex post payment to be negative; a contract with a larger ex ante payment would be self-enforcing for larger values of the discount factor than β_{nc}.
(ii) When $\beta < \beta_{nc}$, there is at least one ex post self-enforcing constraint that binds. Adding up all conditions (6) to condition (5) yields

$$\sum_s p^s \left\{\lambda^s v'(e - B_t) - \theta^s u'(y_t^s + B_t)\right\} = 0. \tag{20}$$

It therefore follows that there must be a s_1 for which $\theta^{s_1} > 0$ and a s_2 for which $\lambda^{s_2} > 0$. *Q.E.D.*

Proof of Proposition 4 Consider the optimal solution to maximization (4) as a function of the state variable V_t. By the theorem of the maximum we know that the solution is continuous in the state variable over the interval $[0, \bar{V}]$. Consider a marginal increase in the value of the state variable. We want to show that the optimal ex ante payment is strictly decreasing in the state variable. The proof goes by contradiction. Suppose that the optimal value of the ex ante payment is left unchanged following a marginal increase in the state variable. The envelope condition implies that $f''(V_t)dV_t = -d\psi < 0$. Consider all ex post constraints that are satisfied at equality before the increase in the state variable. Of these constraints, we choose all those that become strictly binding following the increase in the state variable. These are the only constraints that bind following the marginal increase in V_t. Consider the first-order conditions (6) for all states s for which one self-enforcing constraint becomes binding. In these states, consumption is left unchanged following the small increase in V_t.[23] For all those ex post self-enforcing constraints, for first-order conditions to continue to hold we have that $d\lambda^s = -d\psi$ if the binding constraint is that of agent 2, and $(u'(c_t^s)/v'(e + y_t^s - c_t^s))\, d\theta^s = d\psi$ if it is that of agent 1. If we substitute these changes in condition (20), we have

$$\sum_s p^s \left\{-u'(y_t^s + B_t) * \frac{v'(e + y_t^s - c_t^s)}{u'(c_t^s)} - v'(e - B_t)\right\} d\psi = 0$$

for this condition to continue to hold. Because $d\psi > 0$, this expression cannot be equal to 0 if B_t remains constant. This implies that for any marginal change in the state variable V_t, the ex ante payment B_t must change and therefore B_t is monotonic in the state variable V_t over the range $[0, \bar{V}]$.
Suppose that $V_t = \bar{V}$ and fix $B_t = 0$. In this case, only agent 1 has some ex post self-enforcing constraints that bind. We know from Proposition 2 that agent 1 pays the maximum ex ante payment. Because our problem is a concave problem, this implies that at $V_t = \bar{V}$, the optimal value of B_t is negative. A similar argument shows that the optimal B_t is positive at $V_t = 0$. Because B_t is monotonic in V_t the relationship between B_t and V_t must be decreasing. *Q.E.D.*

Proof of Proposition 5 We know from Lemma 2 that there exist states s_1 and s_2 such that $\theta^{s_1} > 0$ and $\lambda^{s_2} > 0$. Suppose that $\lambda^2 > 0$ and $\theta^1 > 0$. This implies that $\lambda^1 = 0$ and $\theta^2 = 0$.

[23]This follows from the results of Proposition 3 which show that for $B_t = 0$, the optimal lower (and upper) optimal consumption bounds that satisfy the ex post self-enforcing constraints are time-invariant. This result can easily be generalized to any fixed value of B_t.

First-order conditions then imply

$$\begin{aligned}(1+\theta^1)\frac{u'(c_t^1)}{v'(e+y_t^1-c_t^1)} &= -f'(V_t)\\ \frac{u'(c_t^2)}{v'(e+y_t^2-c_t^2)}-\lambda^2 &= -f'(V_t)\end{aligned}$$

which yields

$$(1+\theta^1)\frac{u'(c_t^1)}{v'(e+y_t^1-c_t^1)} = \frac{u'(c_t^2)}{v'(e+y_t^2-c_t^2)}-\lambda^2.$$

But this implies that $c_t^1 = \underline{c}^1 > c_t^2 = \bar{c}^2$ which is inconsistent with the results of Proposition 3.[24] Consequently it must be the case that $\lambda^1 > 0$, $\theta^2 > 0$, and $\lambda^2 = \theta^1 = 0$.

(i) First-order conditions imply that, for state 1,

$$(1+\alpha^1)f'(V_{t+1}^1)+\lambda^1+\phi^1 = f'(V_t).$$

If $\alpha^1 > 0$, then $V_{t+1}^1 = \bar{V} \geq V_t$. If $\alpha^1 = 0$, the above expression reduces to

$$f'(V_{t+1}^1)+\lambda^1+\phi^1 = f'(V_t)$$

which implies that $V_{t+1}^1 > V_t$ by the concavity of the Pareto frontier f. In state 2 we have that

$$(1+\alpha^2+\theta^2)f'(V_{t+1}^2)+\phi^2 = f'(V_t).$$

If $\phi^2 > 0$, then $V_{t+1}^2 = 0 \leq V_t$. If $\phi^2 = 0$, the above expression reduces to

$$(1+\alpha^2+\theta^2)f'(V_{t+1}^2) = f'(V_t)$$

which implies that $V_{t+1}^2 < V_t$ by the concavity of the Pareto frontier f. We then have $V_{t+1}^2 \leq V_t \leq V_{t+1}^1$ which proves the result. For future reference, note that these inequalities imply that $\phi^1 = \alpha^2 = 0$.

(ii) If $0 < V_t < \bar{V}$, first-order conditions in period $t-1$ imply that $u'(c_{t-1})/v'(e+y_{t-1}-c_{t-1}) = -f'(V_t)$. We then have

$$(1+\theta^2)\frac{u'(c_t^2)}{v'(e+y_t^2-c_t^2)} = \frac{u'(c_t^1)}{v'(e+y_t^1-c_t^1)}-\lambda^1 = \frac{u'(c_{t-1})}{v'(e+y_{t-1}-c_{t-1})}.$$

This yields the following inequalities.

$$\frac{u'(c_t^2)}{v'(e+y_t^2-c_t^2)} < \frac{u'(c_{t-1})}{v'(e+y_{t-1}-c_{t-1})} < \frac{u'(c_t^1)}{v'(e+y_t^1-c_t^1)}$$

These inequalities imply that $c_t^1 < c^*(c_{t-1}, y_{t-1}, 1)$ and $c_t^2 > c^*(c_{t-1}, y_{t-1}, 2)$. *Q.E.D.*

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[24] The results of Proposition 3 to the effect that the optimal bounds on consumption are increasing in the states of the world hold for $B_t = 0$. In any period this can be easily generalized to any value of the ex ante payment, namely the optimal value.