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Thibault GAJDOS, EUREQua
Jean-Marc TALLON, EUREQua
Jean-Christophe VERGNAUD, EUREQua

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On the Impossibility of Preference Aggregation under Uncertainty

Thibault Gajdos, Jean-Marc Tallon, Jean-Christophe Vergnaud *

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*All authors are at CNRS–EUREQua. e-mail: gajdos@univ-paris1.fr, jmtallon@univ-paris1.fr, vergnaud@univ-paris1.fr Address: EUREQua, Université Paris I, Maison des Sciences Economiques, 106-112 bd de l'Hôpital, 75647 Paris Cedex 13, France. We thank A. Chateauneuf, M. Cohen, J.Y. Jaffray, P. Mongin and B. Walliser for useful comments. Financial support from an ACI grant by the Ministry of Research is gratefully acknowledged.

Abstract

We provide a general theorem on the aggregation of preferences under uncertainty. We study, in the Anscombe-Aumann setting a wide class of preferences, that includes most known models of decision under uncertainty (and state-dependent versions of these models.) We prove that aggregation is possible and necessarily linear if (society's) preferences are "smooth". The latter means that society cannot have a non-neutral attitude towards uncertainty on a subclass of acts. A corollary to our theorem is that it is not possible to aggregate maxmin expected utility maximizers, even when they all have the same set of priors. We show that dropping a weak notion of monotonicity on society's preferences allows one to restore the possibility of aggregation of non-smooth preferences.

Keywords: Aggregation, Harsanyi, Uncertainty, Multiple Priors.

JEL Number: D70, D81.

Résumé

Nous établissons un résultat général donnant les conditions sous lesquelles l'agrégation des préférences dans l'incertain est possible. Dans un cadre à la Anscombe-Aummann, nous considérons une classe très large de préférences, incluant la plupart des modèles de décision (et leur version dépendant de l'état). Nous montrons que l'agrégation des préférences n'est possible, et nécessairement linéaire, que si les préférences agrégées possèdent une propriété assez restrictive pouvant s'interpréter comme de la neutralité face à l'incertitude. Un corollaire de notre théorème est qu'il n'est pas possible d'agréger des préférences du type maxmin utilité espérée. Nous montrons que relâcher une condition faible de monotonie sur les préférences de la société restaure la possibilité d'agréger des préférences non neutres par rapport à l'incertitude.

Mots clé: Agrégation, Harsanyi, Incertitude, Croyances Multiples.

Numéros JEL : D70, D81.

1 Introduction

Harsanyi (1955) celebrated result shows that it is possible to linearly aggregate von Neumann-Morgenstern (vNM) expected utility maximizers: the social utility is a convex combination of the agents' utilities. Extending this result to more general settings (for instance, staying within the expected utility framework but allowing for different beliefs) turns out to be difficult. In this paper, we take up this issue, widening considerably the set of preferences considered, that encompasses many well known models of decision under uncertainty (subjective expected utility, maxmin expected utility of Gilboa and Schmeidler (1989), Choquet expected utility of Schmeidler (1989) and more generally c-linear biseparable preferences of Ghirardato and Marinacci (2001), as well as state-dependent versions of these preferences).[1] We show a general impossibility result of the following form. Assume agents and society have preferences in this domain. Then, aggregating (some) agents' preferences is possible if and only if they possess a form of uncertainty neutrality, to be discussed momentarily, and leads to linear aggregation. In particular, if an agent has some kind of attitude towards uncertainty, then either he is a dictator (society's preferences place a zero weight on all other agents) or he gets a zero weight in the society's preferences. A particular case of interest is when agents follow the maxmin expected utility model (MMEU henceforth) of Gilboa and Schmeidler (1989), in which an agent evaluates an act by taking its minimal expected utility with respect to a *set* of priors. Then, a corollary of our result is that aggregation of such agents is impossible unless they are actually expected utility agents (in which case the set of priors is a singleton). Going beyond this impossibility result, our general approach enables us to identify which axioms are conflicting. In order to present this tension, we need to explain in more detail which class of preferences we deal with.

The class of preferences we consider can be described as follows. These are preferences for which there exists a set of acts, that we call regular acts, that have the following two properties. First, they cannot be used to hedge other acts (since we are in the Anscombe-Aumann setting, hedging is easily defined via mixture of acts). Second, the sure-thing principle applies when comparing binary acts in this class.[2] One can think of this set of acts as constant acts but, more generally, for instance for state-dependent preferences, constant acts need not have these two properties. Preferences of this type are shown to be represented on these binary acts by a functional V that is affine with respect to regular acts. We also show that V can be further decomposed on binary acts. For each event E, the evaluation of $f_E g$ can be decomposed additively: if f is preferred to g for instance, it is the sum of the evaluation of f on E, $V_E(f)$, and of the evaluation of g on E^c, $V_{E^c}(g)$. If the opposite preference holds between f and g, i.e., if now g is preferred to f, then the evaluation of $f_E g$ is still additive, but with respect to

[1] A limitation, on which we will comment in Section 3.2, is that we adopt Anscombe and Aumann (1963) approach.

[2] That is, if f, g, h, h' are acts that cannot be used to hedge other acts, and E an event, then, if $f_E h$, the act giving $f(s)$ if $s \in E$ and $h(s)$ otherwise is strictly preferred to $g_E h$ then $f_E h'$ is preferred to $g_E h'$.

two different functions V'_E and V'_{E^c}. In the particular case of subjective expected utility, V_E is simply the vNM utility functions times the subjective probability of E. For MMEU preferences, V_E is the vNM utility function times the lowest probability of E in the set of priors, while V'_E is that same vNM utility function multiplied this time by the highest probability of E in the set of priors. We proceed to show that $V(f_E g) + V(g_E f) = V(f) + V(g)$ if and only if the agent is uncertainty neutral with respect to binary acts (on E) in the following usual sense: combining probabilistically two indifferent acts $f_E g$ and $h_E \ell$ does not lead to an improvement.[3] In that case, we will say that preferences are smooth on event E. As soon as there is a wedge between $V(f_E g) + V(g_E f)$ and $V(f) + V(g)$, preferences exhibit a non neutral attitude towards uncertainty.

Our impossibility result can then be stated more precisely. Consider agents that have preferences in the class just described and assume they are sufficiently diverse (in a sense made precise in Section 3). Assume that society's preferences are also in this class. Then the Pareto axiom holds if and only if either there is an oligarchy of agents with smooth preferences, or there is a dictator. Specifically, if V_0 represents the society's preferences and V_i the agents', the only possible aggregation is that $V_0 = \sum_i \lambda_i V_i + \mu$ where $\lambda_i \geq 0$ and μ are real numbers. If two agents get a non zero weight, then they must have smooth preferences on any event and society has smooth preferences as well. Thus, any behavior that is non neutral towards uncertainty leads to the impossibility of linear aggregation. As a consequence, it is not possible that society's preferences be, say, uncertainty averse, unless there is a dictator. To reiterate, if one restricts attention to the class of MMEU preferences that do not reduce to subjective expected utility (SEU henceforth), then it is impossible to aggregate agents' preferences even if they happened to have the same set of priors.

The theorem also points to ways of restoring possibility of aggregating non-smooth preferences. One, which we discuss in some detail, is to abandon the sure-thing principle on binary acts. At a conceptual level, it shows that, in general, two forms of monotonicity of the society's preferences might be conflicting. The first one comes from Pareto, which is a monotonicity condition imposed on society's preferences when viewed as preferences defined on the agents' utilities. The second is a form of event-wise monotonicity that states that if an act is preferred to another one conditionally on an event and on the complement of that event, then it should be preferred globally. Under smoothness of the preferences, these two monotonicity conditions are compatible, as a direct consequence of the additivity of the functional representing society's preferences. However, as soon as one drops smoothness, the tension between these two forms of monotonicity creeps in and is responsible for the impossibility result we obtain. We argue in the final section of the paper that dropping the sure-thing principle on binary acts for the society's preferences might be compelling.

The result complements several previous results in the literature. Seidenfeld, Kadane, and

[3] This is how Gilboa and Schmeidler (1989) define uncertainty neutrality, for a richer set of acts.

Schervish (1989) and Mongin (1995) showed that aggregation of SEU agents was not possible as soon as they have different beliefs. Mongin (1998) showed that expanding the class of preferences to state-dependent preferences would yield a possibility result but argued against this way of restoring the possibility of aggregating preferences. He showed in particular that as soon as one pins down the beliefs of the agents then state-dependence is of no help. Chambers and Hayashi (2003) showed that eventwise monotonicity (P3) and weak comparative probability (P4) were incompatible with the Pareto axiom. Relaxing these axioms while keeping the sure-thing principle leads to state-dependent expected utility preferences, for which they show a possibility result in a Savage setting. Our setting allows for state-dependence preferences from the beginning, and our impossibility theorem applies to non-smooth state-dependent preferences as well. Gilboa, Samet, and Schmeidler (2004) showed in a SEU setting, that imposing the Pareto axiom on issues for which agents are unanimous (have identical beliefs) implies that the society's beliefs have to be an affine combination of agents' beliefs and, similarly, that the society's vNM utility function has to be a linear combination of agents' vNM utility functions (note that this does not imply that society's overall utility function is a convex combination of the agents'). Blackorby, Donaldson, and Mongin (2004) showed, in a somewhat different framework (that of ex ante-ex post aggregation), that aggregation was essentially impossible in the rank dependent expected utility model.

The rest of the paper is built as follows. Section 2 contains the decision theoretic material needed for our result; in particular it gives the characterization of a very wide class of preferences that include, to the best of our knowledge, all models of decision under uncertainty cast in the Anscombe-Aumann setup (as well as their state-dependent versions). Section 3 contains the main result of the paper, which establishes that the notion of smoothness identified in the previous section is key to draw the line between possibility and impossibility of aggregation. It also contains a discussion of the proof and how it could be extended to a more general setting than Anscombe-Aumann's. Finally, it contains a discussion of the tension between the Pareto principle and a weak form of monotonicity of society's preferences, giving directions to restore the possibility of aggregation of non smooth preferences. Proofs are gathered in two appendices; the first one contains all the material concerning preference representation while the second one contains the proof of our main theorem.

2 Setup and representation results

We consider a society made of a finite number of agents $N' = \{1, \ldots, n\}$. Let $N = \{0, 1, \ldots, n\}$ where 0 refers to society. Uncertainty is represented by a set S and an algebra of events Σ. Let X be a non-empty set of consequences and Y be the set of distributions over X with finite supports. Let $\mathcal{A}$ be the set of acts, that is, functions $f : S \to Y$ which are measurable with respect to Σ. Since Y is a mixture space, one can define for any $f, g \in \mathcal{A}$ and $\alpha \in [0, 1]$, the act

$\alpha f + (1-\alpha)g$ in $\mathcal{A}$ which yields $\alpha f(s) + (1-\alpha)g(s) \in Y$ for every state $s \in S$. Let $\mathcal{A}^c$ denote the set of constant acts. For an event E and two acts f, g, denote $f_E g$ the act giving $f(s)$ if $s \in E$ and $g(s)$ if not.

We model the preferences of an agent $i \in N'$ on $\mathcal{A}$ by a binary relation $\succsim_i$, and, as customary we denote by $\sim_i$ and $\succ_i$ its symmetric and asymmetric components. Society's preferences are denoted $\succsim_0$. We now introduce the structure we impose on the agents' and the society's preferences, dropping subscript for simplicity.

The first axiom is usual, will be maintained throughout and states that preferences are a complete, transitive and continuous relation.

Axiom 1 *For all $f, g, h \in \mathcal{A}$,*

1. *$f \succsim g$ or $g \succsim f$;*
2. *if $f \succsim g$ and $g \succsim h$ then $f \succsim h$;*
3. *If $f \succ g$ and $g \succ h$, then there exist $\alpha, \beta \in (0,1)$ such that $\alpha f + (1-\alpha)h \succ g$ and $g \succ \beta f + (1-\beta)h$.*

We next introduce a set of acts which will play a crucial role in the sequel. These are acts on which preferences have some linear structure.

Definition 1 *A set of acts $\mathcal{B} \subset \mathcal{A}$ is* regular *with respect to $\succsim$ if it satisfies the following conditions*

1. *$\mathcal{B}$ is a mixture set, that is for all $f, g \in \mathcal{B}$ and $\alpha \in (0,1)$, $\alpha f + (1-\alpha)g \in \mathcal{B}$;*
2. *For all $f \in \mathcal{B}$, for all $g, h \in \mathcal{A}$, $\alpha \in (0,1]$, $g \succsim h \Leftrightarrow \alpha g + (1-\alpha)f \succsim \alpha h + (1-\alpha)f$;*
3. *For all acts f, g, h, h' in $\mathcal{B}$ and events E, $f_E h \succ g_E h \Rightarrow f_E h' \succsim g_E h'$.*

A regular set of acts is thus a mixture set (condition 1) made of acts that cannot be used to hedge against other acts (condition 2) and that satisfy the sure-thing principle for binary acts (condition 3). It is included in the set of crisp acts as defined in Ghirardato, Maccheroni, and Marinacci (2004). Note that the whole set $\mathcal{A}$ is a regular set of acts for subjective expected utility (both state-independent and state-dependent).

Take now the MMEU model of Gilboa and Schmeidler (1989). In this model, $f \succsim g$ if and only if $\min_{\mathcal{C}} \int u \circ f dp \geq \min_{\mathcal{C}} \int u \circ g dp$, where $\mathcal{C}$ is a convex set of priors and $u : Y \to \mathbb{R}$ is a vNM utility function. For these preferences, the set of constant acts, $\mathcal{A}^c$, is a regular set. It is trivially a mixture set, and the two other properties can be easily checked on the functional.[4]

[4] For instance, the binary sure-thing principle is satisfied as $f_E h \succ g_E h$ is equivalent to saying that $\min_{\mathcal{C}} p(E)(u(f) - u(h)) > \min_{\mathcal{C}} p(E)(u(g) - u(h))$ which in turn implies that $u(f) > u(g)$, the latter implying that $\min_{\mathcal{C}} p(E)(u(f) - u(h')) \geq \min_{\mathcal{C}} p(E)(u(g) - u(h'))$.

From an axiomatic view point, the second property is equivalent to C-independence, and the third property is a consequence of Monotonicity.

$\mathcal{A}^c$ is also a regular set for Choquet Expected Utility (CEU henceforth) preferences as defined and axiomatized in Schmeidler (1989). In this model, $f \succsim g$ if and only if $\int_{Ch} u \circ f d\nu \geq \int_{Ch} u \circ g d\nu$, where u is a vNM function, ν is a *capacity*, i.e., a set function defined on (S, Σ) such that $\nu(\emptyset) = 0$ and $\nu(S) = 1$, and such that for two events E, F, $E \subset F$, $\nu(E) \leq \nu(F)$, and for any bounded and measurable function ϕ, $\int_{Ch} \phi d\nu$, the Choquet integral of ϕ with respect to ν is defined as follows

$$\int_{Ch} \phi d\nu = \int_0^\infty \nu(\{s \in S : \phi(s) \geq t\}) dt + \int_{-\infty}^0 [\nu(\{s \in S : \phi(s) \geq t\}) - 1] dt.$$

These two leading models, as well as the α-MMEU model,[5] are particular cases of a more general class of preferences, c-linear biseparable preferences, defined in Ghirardato and Marinacci (2001) and refined in Ghirardato, Maccheroni, and Marinacci (2005). These preferences can be represented by a function V, affine on $\mathcal{A}^c$, such that for $f, g \in \mathcal{A}^c$, $f \succ g$, $V(f_E g) = \rho(E)u(f) + (1 - \rho(E))u(g)$ where ρ is a capacity. It is easy to see that $\mathcal{A}^c$ is a regular set for any preferences of this type.

In all these models, constant acts have the feature that they cannot be used to possibly hedge other acts. $\mathcal{A}^c$ might also be a regular set for preferences that do not necessarily fall into the models mentioned above. For instance, consider a state-dependent MMEU representation with respect to $\mathcal{C} = \{p \in \Delta(\{1, 2, 3, 4\}) \,|p = (\alpha/2, (1 - \alpha)/2, \alpha/2, (1 - \alpha)/2), \alpha \in [0, 1]\}$, with state-dependent utilities u_s satisfying $u_1 = u_2$ and $u_3 = u_4$. One can easily check that $\mathcal{A}^c$ is regular for such a representation. More generally however, a regular set of acts does not have to include constant acts.

Next, we define the notion of a representation of preferences that is affine on a subset of acts.

Definition 2 *Let $\mathcal{B} \subset \mathcal{A}$. A function $V : \mathcal{A} \to \mathbb{R}$ is a $\mathcal{B}$-affine representation of $\succsim$, if*

1. *for all $f, g \in \mathcal{A}$, $f \succsim g$ if and only if $V(f) \geq V(g)$;*
2. *for all $f \in \mathcal{A}$, $h \in \mathcal{B}$, and $\alpha \in (0, 1)$, $V(\alpha f + (1 - \alpha)h) = \alpha V(f) + (1 - \alpha)V(h)$.*

It is well known that MMEU, α-MMEU and CEU preferences are $\mathcal{A}^c$-affine (or simply C-affine). The next result provides a representation of preferences that admit a set of regular acts and thus includes the models mentioned. This representation will be key to establish under which conditions linear aggregation is possible.

Proposition 1 *Let $\succsim$ be a binary relation on $\mathcal{A}$ that satisfies Axiom* 1*. Assume that there exists a set $\mathcal{B} \subset \mathcal{A}$ which is regular with respect to $\succsim$ and, furthermore, that $\succsim$ is not degenerate on $\mathcal{B}$ (i.e., there exist $f, g \in \mathcal{B}$ such that $f \succ g$.) Then,*

[5] These preferences are a generalization of MMEU preferences, in which the utility of an act f is given by $\alpha \min_{\mathcal{C}} \int u \circ f dp + (1 - \alpha) \max_{\mathcal{C}} \int u \circ f dp$. See Jaffray (1989) and Ghirardato, Maccheroni, and Marinacci (2004).

1. *there exists a $\mathcal{B}$-affine representation of $\succsim$, $V : \mathcal{A} \to \mathbb{R}$, which is unique up to a positive affine transformation;*

2. *for any event E, there exist four functions $\overline{V}_E, \underline{V}_{E^c}, \underline{V}_E, \overline{V}_{E^c}$ such that for all $f, g \in \mathcal{B}$*

$$\begin{aligned} V(f_E g) &= \overline{V}_E(f) + \underline{V}_{E^c}(g) \text{ if } f \succsim g \\ &= \underline{V}_E(f) + \overline{V}_{E^c}(g) \text{ if } f \precsim g \end{aligned}$$

3. *for any event E, there exists $k^E \in \mathbb{R}$ such that for all $f, g \in \mathcal{B}$, $V(f_E g) + V(g_E f) - V(f) - V(g) = k^E |V(f) - V(g)|$.*

Preferences that satisfy the requirements of Proposition 1 will be called pseudo-additive (with respect to $\mathcal{B}$) in the following. Most models of decision under uncertainty cast in the Anscombe-Aumann framework are pseudo-additive. α-MMEU preferences and more generally c-linear biseparable preferences are of this type. One could imagine state-dependent versions of these models that would fit our setting.[6] The following example illustrates its generality.

Example 1 *[state-dependent α-MMEU]* Consider the following functional form, representing $\alpha-$MMEU state-dependent preferences.

$$V(f) = \alpha \min_{p \in \mathcal{C}} E_p u_s(f(s)) + (1-\alpha) \max_{p \in \mathcal{C}} E_p u_s(f(s))$$

Let $\mathcal{B}$ be the set of constant utility acts, that is $\mathcal{B} = \{f \in \mathcal{A} \text{ s.th. } \forall s, t\ u_s(f(s)) = u_t(f(t))\}$. We now establish that $\mathcal{B}$ is regular with respect to the preferences represented by $\alpha-$MMEU state-dependent functional above. Notice first that $\mathcal{B}$ is a mixture set. Second, it is also easy to establish that $V(\alpha f + (1-\alpha) g) = \alpha V(f) + (1-\alpha) V(g)$ for all $f \in \mathcal{B}$ and $g \in \mathcal{A}$. Third, we check that condition 3 of Definition 1 holds as well.

Remark that for all $f, h \in \mathcal{B}$, one has:

$$V(f_E h) = \alpha \min\nolimits_{p \in \mathcal{C}} (p(E)V(f) + (1-p(E))V(h)) + (1-\alpha) \max\nolimits_{p \in \mathcal{C}} (p(E)V(f) + (1-p(E))V(h))$$

$$= \begin{cases} \left(\alpha \underline{p}(E) + (1-\alpha)\overline{p}(E)\right) V(f) + \left(\alpha(1-\underline{p}(E)) + (1-\alpha)(1-\overline{p}(E))\right) V(h) & \text{if} \quad V(f) \geq V(h) \\ \left(\alpha \overline{p}(E) + (1-\alpha)\underline{p}(E)\right) V(f) + \left(\alpha(1-\overline{p}(E)) + (1-\alpha)(1-\underline{p}(E))\right) V(h) & \text{if} \quad V(f) \leq V(h) \end{cases}$$

where $\underline{p}(E) = \min_{p \in \mathcal{C}} p(E)$ and $\overline{p}(E) = \min_{p \in \mathcal{C}} p(E)$.

Now, for all $f, g, h \in \mathcal{B}$, it is straightforward, using the expression obtained for $V(f_E h)$ and looking at all the possible ranking of $V(f), V(g), V(h)$, to check that $V(f_E h) \geq V(g_E h)$ if and only if $V(f) \geq V(g)$, thus establishing that condition 3 holds. Hence, one can conclude that state-dependent α-MMEU are pseudo-additive with respect to $\mathcal{B}$. The functions $\overline{V}_E$, $\underline{V}_{E^c}$, $\underline{V}_E$, and $\overline{V}_{E^c}$ can easily be identified by looking at the expression obtained for $V(f_E h)$.

[6] To the best of our knowledge such models have not been studied in the literature, although there is no difficulty imagining that they could be given sound axiomatic foundations.

We now comment on the properties of pseudo-additive preferences. The first item in the proposition is straightforward and well known (it follows from vNM like arguments). The second item establishes that the evaluation of acts of the form $f_E g$ for $f, g \in \mathcal{B}$ can be decomposed in a pseudo-additive manner, the decomposition being dependent on the ranking of the two acts. To illustrate this, consider MMEU preferences represented by $V(f) = \min_{p \in \mathcal{C}} \int u \circ f dp$. Let f, g be constant acts and assume $f \succ g$. Define for any event E, $\underline{p}(E) = \min_{p \in \mathcal{C}} p(E)$ and $\bar{p}(E) = \max_{p \in \mathcal{C}} p(E)$, $V(f_E g) = \underline{p}(E)u(f) + (1 - \underline{p}(E))u(g) = \underline{p}(E)u(f) + \bar{p}(E^c)u(g)$. Hence, defining $\overline{V}_E(f) = \underline{p}(E)u(f)$ and $\underline{V}_{E^c}(g) = \bar{p}(E^c)u(g)$, one gets $V(f_E g) = \overline{V}_E(f) + \underline{V}_{E^c}(g)$. Similarly, one can define $\underline{V}_E(f) = \bar{p}(E)u(f)$ and $\overline{V}_{E^c}(g) = \underline{p}(E^c)u(g)$.

The third item can be interpreted as defining a measure of the agent's attitude towards uncertainty attached to an event. We illustrate this for c-linear biseparable preferences: if $f \succsim g$, $V(f_E g) + V(g_E f) - V(f) - V(g) = \rho(E)u(f) + (1 - \rho(E))u(g) + \rho(E^c)u(f) + (1 - \rho(E^c))u(g) - u(f) - u(g) = (\rho(E) + \rho(E^c) - 1)(u(f) - u(g))$. Defining $k^E = \rho(E) + \rho(E^c) - 1$ yields the desired result. Intuitively, $k^E \neq 0$ means that preferences exhibit a "kink" on event E, reflecting some non-neutral attitude towards uncertainty: in the previous illustration, f and g are constant acts, whose evaluation is therefore independent from any attitude towards uncertainty. On the other hand, the evaluation of acts $f_E g$ and $g_E f$ is susceptible to be affected by the uncertainty introduced. If $k^E < 0$, the sum of the evaluations of these two acts is smaller than the sum of the evaluation of f and g. This downward bias reflects uncertainty aversion. A positive k^E would on the other hand reflect uncertainty loving. We now make this intuition more formal by defining a notion of "smoothness" of the preferences on an event for a given set of acts.

Definition 3 *Let $\mathcal{B} \subset \mathcal{A}$ be regular with respect to $\succsim$. Say that $\succsim$ is* smooth *on an event E with respect to $\mathcal{B}$ if for all $f, g, h, \ell \in \mathcal{B}$ such that $f_E g \sim h_E \ell$ and all $\alpha \in (0, 1)$, $\alpha f_E g + (1 - \alpha) h_E \ell \sim f_E g$. Furthermore, say that $\succsim$ is smooth with respect to $\mathcal{B}$ if it is smooth on any event.*

Smoothness is defined by saying that the preferences are uncertainty neutral, as defined in Gilboa and Schmeidler (1989), with respect to binary acts whose components are in a regular set $\mathcal{B}$. It is thus a weak form of uncertainty neutrality and we are not aware of a decision model in which preferences can be uncertainty neutral with respect to such binary acts and at the same time exhibit uncertainty aversion (or a taste for uncertainty) for some other acts in $\mathcal{A}$. SEU preferences are obviously smooth with respect to $\mathcal{A}$, while c-linear biseparable preferences are smooth with respect to $\mathcal{A}^c$ only on events for which $\rho(E) = 1 - \rho(E^c)$. For instance, MMEU preferences are not smooth with respect to $\mathcal{A}^c$ on the events for which $\bar{p}(E) \neq \underline{p}(E)$. Our last result in this section shows that, as hinted above, k^E is an indicator of whether preferences are smooth on E.

Proposition 2 *Under the representation of Proposition 1, $\succsim$ is smooth on event E with respect to $\mathcal{B}$ if and only if $k^E = 0$.*

Before turning to the main result of the paper, let us summarize the decision theoretic foundations on which it is built. We considered a very wide class of preferences in the Anscombe-Aumann setting, that includes c-linear biseparable preferences and therefore MMEU and CEU preferences (and state-dependent versions thereof). We defined a set of acts on which such preferences admit a pseudo-additive representation and characterized what it means for such a representation to be smooth on an event. In the following section, we argue that smoothness is the crucial property that delimits the frontier between the possibility and impossibility of linear aggregation.

3 Aggregation

In this Section, we first state our main theorem and an important corollary for c-linear biseparable preferences. We then present informally the main argument of the proof, pointing out how it applies were one to abandon the Anscombe-Aumann setting. We end with a discussion of the tension that the theorem uncovers and argue that dropping the sure-thing principle on binary acts for the society's preferences might be a reasonable way to restore the possibility of aggregation.

3.1 Main Theorem

We adopt the weak form of the Pareto axiom.

Axiom 2 (Pareto) *For all $f, g \in \mathcal{A}$, $[\forall i \in N', f \succ_i g \Rightarrow f \succ_0 g]$.*

For the aggregation problem to be interesting, one needs to impose some diversity among the preferences that one seeks to aggregate. The next definition provides one such condition (see Mongin (1998)).

Definition 4 *The n binary relations $\{\succsim_i\}_{i \in N'}$ satisfy the* Independent Prospects Property *on a set $\mathcal{B} \subset \mathcal{A}$ if for all $i \in N'$, there exist $h_i^\star, h_{\star i} \in \mathcal{B}$ such that:*

$$h_i^\star \succ_i h_{\star i} \text{ and } h_i^\star \sim_j h_{\star i} \, \forall j \in N' \setminus \{i\}.$$

Theorem 1 *Let $\{\succsim_i\}_{i \in N}$ be binary relations on $\mathcal{A}$ and $\{\mathcal{B}_i\}_{i \in N}$ be non-empty subsets of $\mathcal{A}$. Assume that*

1. *for all $i \in N$, $\succsim_i$ satisfies Axiom 1;*
2. *for all $i \in N$, $\mathcal{B}_i$ is regular with respect to $\succsim_i$;*
3. *$\{\succsim_i\}_{i \in N'}$ satisfy the Independent Prospects Property on $\cap_{i \in N} \mathcal{B}_i$.*

Then, Axiom 2 holds if and only if,

(i) there exists a $\mathcal{B}_i$-affine representation V_i of $\succsim_i$ for all $i \in N$, unique weights $(\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n_+ \setminus \{0\}$, $\mu \in \mathbb{R}$ such that

$$\forall f \in \mathcal{A}, V_0(f) = \sum_{i \in N'} \lambda_i V_i(f) + \mu$$

(ii) $\forall i, j \in N'$, $i \neq j$, $\lambda_i \times \lambda_j \neq 0 \Leftrightarrow \forall E \in \Sigma, k_i^E = k_j^E = 0.$

In words, under the assumptions of the theorem, either society's preferences are a linear aggregation of *smooth* individuals' preferences or there is a dictator. It cannot be the case that society's preferences are the result of the aggregation of an uncertainty averse agent with any other type (uncertainty averse, loving or neutral) of agent. A consequence of this is that if society's preferences are uncertainty averse (of the limited kind corresponding to the fact that it is not smooth on some event with respect to $\mathcal{B}_0$), then it must be dictatorial. Remark that the theorem is in a sense stronger than Harsanyi's since smoothness of the preferences is a consequence and not an assumption of the theorem.[7]

When specified for the general class of c-linear biseparable preferences, the following corollary is readily deduced from the theorem.

Corollary 1 *Assume $\succsim_i$ is c-linear biseparable (and not smooth) for all $i \in N$ and that the Independent Prospect Property holds on $\mathcal{A}^c$, then Axiom 2 holds if and only if there exists $j \in N'$ such that $\succsim_0 = \succsim_j$.*

This is a direct consequence of the fact that $\mathcal{A}^c$ is regular for c-linear biseparable preferences and, as we established in the previous section, that these preferences are not smooth with respect to that set. Two important particular cases covered by this corollary are when agents and society have MMEU preferences and when they have CEU preferences. Hence, for instance, *it is not possible to aggregate MMEU preferences into an MMEU social preferences*, irrespective of the fact that the sets of priors are identical among agents. Whereas in an expected utility setting it is possible to aggregate agents with the same beliefs, this does not generalize to non-expected utility settings.

3.2 Main arguments in the proof and how they generalize

The proof is divided into two distinct parts. The first one is a direct application of Proposition 2 in De Meyer and Mongin (1995). It states that, given the underlying convex structure introduced via the Anscombe-Aumann setting, the Pareto axiom implies that V_0 is a weighted sum of the V_is. Hence, aggregation has to be linear. The second part can itself be divided in two.

[7]This fact has already been noted in Blackorby, Donaldson, and Mongin (2004) study of the aggregation of rank dependent expected utility agents. As they put it " the EU-like conditions are to be found here in the conclusion, whereas Harsanyi put them in the assumption; apparently, he did not realize the logical power of his own framework."

First, the Independent Prospect Property on $\cap_{i \in N} \mathcal{B}_i$ states that for any i, there exist h^*, h_* in $\cap_{i \in N} \mathcal{B}_i$ such that $h^* \succ_i h_*$ and $h^* \sim_j h_*$, $\forall j \in N' \setminus \{i\}$. Using these acts for any i, one can establish that for any agent i that has a non zero weight λ_i, $k_i^E = k_0^E$ for any event E. Thus, all agents that are taken into account in V_0 must have the same attitude towards uncertainty.

Second, we prove that $k_0^E = 0$ as soon as there are two agents with non zero weights. Assume for simplicity that only agent 1 and 2 have non zero weight. The argument relies on the fact that, using the Independent Prospect Property and mixing acts, one can find two acts $f, g \in \cap_{i \in N} \mathcal{B}_i$ such that $f \succ_1 g$ and $f \prec_2 g$, while $f \sim_0 g$. The smoothness of the preferences can then be established by computing $V_0(f_E g) + V_0(g_E f) - V_0(f) - V_0(g)$ in two different ways. The first one is direct and establishes that this quantity is zero since $f \sim_0 g$. The second one is to compute it decomposing V_0 as the sum of $\lambda_1 V_1$ and $\lambda_2 V_2$. Using the fact that $k_1^E = k_2^E = k_0^E$, this last part establishes that $k_0^E = 0$. In this last argument, the fact that we have a mixture operation available is used only to build the acts f and g.

Now, if one were to relax the Anscombe-Aumann structure and move to a setting in which there is no well defined notion of mixture, Proposition 2 in De Meyer and Mongin (1995) cannot be used. As a consequence, there might be other forms of aggregation than simply taking a weighted sum of the V_is. But, if there exist two acts f and g that have the property described above, then our argument applies and shows that if an aggregation of the V_is is possible, then it has to be non-linear (non utilitarian). These two acts can be constructed from the h^*, h_* identified above (which exist when the Independent Prospect Property holds) whenever some joint continuity among the preferences of the agents is satisfied.

The conclusion we'd like to draw from this is that whenever one can identify a set of acts (which does not necessarily have a mixture space structure) on which preferences have a pseudo-additive representation, and if there exists such a set that is common to all agents and the society, then, under the Independent Prospect Property, linear aggregation of preferences that are non smooth is not possible (under mild joint regularity conditions on agents' preferences). Thus, for invariant bi-separable preferences, Rank Dependent Utility preferences, CEU and MMEU preferences in a Savage setting, the philosophy of the result applies: linear aggregation is possible only if preferences are smooth. If non smooth, then either there is a dictator or there might be a non-linear aggregation.

To illustrate this discussion, take the MMEU model and assume that agents all have their set of priors equal to the simplex, so that $V_i(f) = \min_s u_i(f(s))$. In the Anscombe-Aumann version (the one originally axiomatized by Gilboa and Schmeidler (1989)), $f(s)$ is a lottery, and we showed that aggregation is not possible. Now, in the Savage version of the MMEU model in which $f(s)$ is an arbitrary consequence, *linear* aggregation is not possible but taking $V_0(f) = \min_i V_i(f)$ is a perfectly legitimate preference for the society, that respects the Pareto axiom and is of the MMEU type since one can define u_0 by $u_0(f(s)) = \min_i u_i(f(s))$ and $V_0(f)$ is then simply equal to $\min_s u_0(f(s))$.

3.3 On the possibility of aggregating non-smooth preferences.

As shown by Mongin (1998) and Chambers and Hayashi (2003) a way to circumvent the impossibility of aggregating SEU agents when they have different beliefs is to enrich the possible domain for society's preferences. Specifically, they allowed for state-dependence in society's preferences (while remaining in the SEU class). Since state-dependent preferences are already included in our class of preferences, this particular way of enlarging the domain is not relevant. However, Theorem 1 makes it clear that several options are available to restore the "possibility of possibility".

One option is to relax condition 3 in the theorem, so that the Independent Prospect Property does not hold on $\cap_{i\in N}\mathcal{B}_i$. An uninteresting way would be to assume that tastes are not diverse (for instance, aggregation is trivially possible if all agents are identical). A less trivial way to relax that condition is to say that it does not have any bite because $\cap_{i\in N}\mathcal{B}_i$ is empty, and more specifically because the regular set for society's preferences does not intersect $\cap_{i\in N'}\mathcal{B}_i$. However, in general, regular sets are not easy to characterize, so we cannot provide a general characterization of when is it that $\cap_{i\in N}\mathcal{B}_i$ is empty. To illustrate, take the particular case in which agents have c-bilinear preferences (and hence $\mathcal{A}^c$ is a regular set) that are non smooth. Our theorem then says that aggregation is impossible if $\mathcal{A}^c$ is regular with respect to $\succsim_0$ (and the Independent Prospect Property holds on $\mathcal{A}^c$), but leaves open the fact that some pseudo-additive preferences whose regular set of acts does not intersect $\mathcal{A}^c$ might aggregate agents' preferences.

The second option is to relax condition 2 in Theorem 1, and to assume that there is no set of acts which is regular with respect to society's preferences. This in turn can be due either to the fact that there are no acts f such that for all $g, h \in \mathcal{A}$, $\alpha \in (0, 1]$, $g \succsim h \Leftrightarrow \alpha g + (1-\alpha)f \succsim \alpha h + (1-\alpha)f$, or that there is no set of acts $\mathcal{B}$ such that for all acts f, g, h, h' in $\mathcal{B}$ and events E, $f_E h \succ g_E h \Rightarrow f_E h' \succsim g_E h'$, that is, the binary sure-thing principle is violated. We explore this latter avenue in more detail now and show why it is the tension between this principle and the Pareto principle that is at the heart of the impossibility result and why smoothness allows one to bypass the problem.

As we noticed in Section 2 when we introduced the binary sure-thing principle (Definition 1), it is a consequence of monotonicity in the MMEU model.[8] Thus a violation of this axiom (by society) would mean that monotonicity is not satisfied by $\succsim_0$. There are actually compelling reasons as to why society might want to violate monotonicity. Consider the following society, made of two agents 1 and 2. There are two states of nature s_1 and s_2. The following matrix gives the (certain) utility associated in each state to acts f, g, h, and ℓ for both agents.

[8] Monotonicity requires that if $f(s) \succsim g(s)$ for all s, then $f \succsim g$.

	s_1	s_2
f	$(1,1)$	$(1,1)$
g	$(10,0)$	$(0,10)$
h	$(0,10)$	$(0,10)$
ℓ	$(10,0)$	$(10,0)$

Assume agents are MMEU with the simplex as a set of priors. Then both agents would prefer f to g. The Pareto condition then yields that society should prefer f to g. Now, to make the argument as simple as possible, imagine that it were possible to aggregate agents' preferences and assume that $V_0 = \frac{1}{2}V_1 + \frac{1}{2}V_2$. Then, according to these preferences the constant act $g(s_1)$ is preferred by society to $f(s_1)$ and similarly $g(s_2)$ is preferred to $f(s_2)$. Thus, monotonicity yields that g is preferred to f by society, hence violating the conclusion obtained from the Pareto condition. As expected, $\succsim_0$ does not satisfy the sure-thing principle on binary acts: $V_0(f_{s_1}\ell) = \frac{1}{2}V_1(f_{s_1}\ell) + \frac{1}{2}V_2(f_{s_1}\ell) = \frac{1}{2}\times 1 + \frac{1}{2}\times 0 = \frac{1}{2}$ and similarly $V_0(h_{s_1}\ell) = 0$. Hence, $f_{s_1}\ell \succ_0 h_{s_1}\ell$. On the other hand, $V_0(h_{s_1}h) = 5$ and $V_0(f_{s_1}h) = \frac{1}{2}$, and therefore $h_{s_1}h \succ_0 f_{s_1}h$. We hence get that $f_{s_1}\ell \succ_0 h_{s_1}\ell$ together with $h_{s_1}h \succ_0 f_{s_1}h$, which is a violation of the sure-thing principle for binary acts.

We now generalize the lesson from this example. Consider a society in which some set $\mathcal{B}$ is regular for all agents' preferences (assumed to satisfy Axiom 1). Preferences hence admit $\mathcal{B}$-affine representations V_i. Assume furthermore that the $\{\succeq_i\}_{i\in N'}$ satisfy the Independent Prospect Property on $\mathcal{B}$. When is it that the preference relation represented by a weighted sum of the V_i, say $V_0 = \sum_i \lambda_i V_i$, is acceptable for the society?

Observe first that V_0 is $\mathcal{B}-$affine. Therefore, $\mathcal{B}$ satisfies condition 2 of Definition 1 with respect to $\succeq_0$. Furthermore, $\mathcal{B}$ is a mixture set, since it is a regular set with respect to the $\succeq_i$. Finally, $\succeq_0$ (represented by V_0) obviously satisfies Axiom 2. Hence, the impossibility result should come from a conflict between Axiom 2 and the requirement that social's preferences satisfy the sure-thing principle for binary acts on $\mathcal{B}$, when agents' preferences are not smooth. Furthermore, this conflict should disappear whenever preferences are smooth.

First, note that $\succsim_0$ can be viewed as a preference relation over the agents' utility levels, i.e., by a preference relation $\hat{\succeq}_0$ on $K = \{(V_1(f), \cdots, V_n(f))\,|\,f \in \mathcal{A}\}$. Then, Axiom 2 can be read as a monotonicity of $\hat{\succeq}_0$ on K. This monotonicity is to be understood as a component-wise monotonicity, where each component is an agent's utility level.

On the other hand, the requirement that $\succeq_0$ satisfies the sure-thing principle for binary acts on $\mathcal{B}$ can be interpreted as an event-wise monotonicity condition on the preference relation $\succeq_0$ (restricted to binary acts) on $\mathcal{B}$. To see this, define a binary relation $\trianglerighteq_0^E$ on $\mathcal{B}$ by $f \trianglerighteq_0^E g$ if and only if $f_E h \succeq_0 g_E h$ for all $h \in \mathcal{B}$, and by $\triangleright_0^E$ its asymmetric part. The sure-thing principle on binary acts implies that $\trianglerighteq_0^E$ is a weak order and represents conditional preferences on E. By definition, it satisfies the independence axiom on $\mathcal{B}$. Therefore, $\trianglerighteq_i^E$ can be represented by a vNM function. Hence, by Harsanyi's Theorem, $\trianglerighteq_0^E$ can be represented by a linear combination of the vNM representations of the $\trianglerighteq_i^E$. This implies in particular that $\succsim_0$ is event-wise monotonic in the following sense: for any $f, g, h, \ell \in \mathcal{B}$, if $f \trianglerighteq_0^E g$ and $h \trianglerighteq_0^{E^c} \ell$, then $f_E h \succeq_0 g_E \ell$.

Now, consider what happens if all agents have smooth preferences. In this case, we can choose for $\succeq_i$ a representation of the following form on binary acts: $V_i(f_E g) = V_i^E(f) + V_i^{E^c}(g)$, where V_i^E and $V_i^{E^c}$ are vNM representations of $\trianglerighteq_i^E$ and $\trianglerighteq_i^{E^c}$, respectively. Therefore, $V_0(f_E g) = \sum_i \lambda_i V_i(f_E g) = (\sum_i \lambda_i V_i^E(f)) + (\sum_i \lambda_i V_i^{E^c}(g))$. On the other hand, applying Harsanyi's Theorem, V_0^E is a linear combination of the V_i^E (and similarly with E^c instead of E). Therefore, $V_0^E(f) = \sum_i \lambda_i^E V_i^E(f)$ and $V_0^{E^c}(g) = \sum_i \lambda_i^{E^c} V_i^{E^c}(g)$, which in turn implies (since obviously $\succeq_0$ has to be smooth, and $\trianglerighteq_0^E$ should coincide with $\trianglerighteq_0^{E^c}$) that $V_0(f_E g) = (\sum_i \lambda_i^E V_i^E(f)) + (\sum_i \lambda_i^{E^c} V_i^{E^c}(g))$. Identifying the two expressions we got for V_0, and using the Independent Prospect Property, we finally get that $\lambda_i = \lambda_i^E = \lambda_i^{E^c}$. Hence, it does not matter if one first chooses to aggregate the agents' preferences event by event, and then build the representation of $\succeq_0$, or if one directly aggregates agents' preferences on binary acts. Therefore, in this case, the two monotonicity requirements do not conflict. The crucial point, of course, is that one gets the same linear structure among events at the individual level, and among agents. The possibility result is a direct consequence of the well-known fact that two steps linear aggregations are equivalent whatever the order in which they are done.

Now, assume that some individual has non-smooth preferences. Then, one still get that V_0 is a linear aggregation of the V_i. In order to apply the same trick as above, one should aggregate the $\bar{V}_i^E$ and the $\underline{V}_i^E$ separately, and then build the V_0 functional. But this can be done only for binary acts $f_E g$ such that all individuals agree that, e.g., $f \succeq_i g$. By the Independent Prospect Property, there is some pair of acts for which such an agreement does not exist, which prevents the possibility of building V_0^E in this way. Hence, aggregation is not possible, unless one is willing to relax the sure-thing principle on binary acts.

4 Appendix

4.1 Appendix A

Proposition 1.

Condition 1 follows from a usual vNM kind of proof and is omitted here.

Condition 2 & 3 (for sake of simplicity we prove the two conditions at the same time)

For any event E and acts $f, g \in \mathcal{B}$, say that $f \trianglerighteq_E g$ if for all act $h \in \mathcal{B}$, $f_E h \succsim g_E h$. This relation is well-defined since $\succsim$ satisfies the sure thing principle for binary acts. We note $\triangleright_E$ and $\approx_E$ respectively for the strict preference part and the indifference part. It can be checked that by definition of $\mathcal{B}$, $\trianglerighteq_E$ satisfies the vNM axioms.

Suppose there exists $f^*, f_* \in \mathcal{B}$ such that $V(f^*_E f_*) + V(f_{*E} f^*) \neq V(f^*) + V(f_*)$. As a first step, we show that either $\trianglerighteq_E = \trianglerighteq_{E^c}$ or $\trianglerighteq_{E^c}$ is a reverse order of $\trianglerighteq_E$, in the sense that $f \trianglerighteq_{E^c} g$ if and only if $f \trianglerighteq_E g$, for all $f, g \in \mathcal{B}$. In step 2, we complete the proof of conditions 2 and 3.

Step 1 Suppose that $f^* \succsim f_*$. Then, we necessarily must be in one of the following case:

- $f^* \rhd_E f_*$ and $f^* \rhd_{E^c} f_*$,
- $f^* \rhd_E f_*$ and $f_* \unrhd_{E^c} f^*$,
- $f_* \unrhd_E f^*$ and $f^* \rhd_{E^c} f_*$
- $f_* \unrhd_E f^*$ and $f_* \unrhd_{E^c} f^*$.

Note that this last case is not possible. Indeed, $f_* \unrhd_E f^*$ implies that $f_* \succsim f^*_E f_*$ and $f_{*E} f^* \succsim f^*$ while $f_* \unrhd_{E^c} f^*$ implies that $f_* \succsim f_{*E} f^*$ and $f^*_E f_* \succsim f^*$. Thus $f_* \succsim f^*_E f_*, f_{*E} f^* \succsim f^*$ while by assumption $f^* \succsim f_*$ and therefore $f^*_E f_* \sim f_{*E} f^* \sim f_* \sim f^*$ and thus $V(f^*_E f_*) + V(f_{*E} f^*) = V(f^*) + V(f_*)$ which leads to a contradiction.

Therefore, we have essentially two cases to consider : (a) $f^* \rhd_E f_*$ and $f^* \rhd_{E^c} f_*$, and (b) $f^* \rhd_E f_*$ and $f_* \unrhd_{E^c} f^*$ (the third case being the symmetric of case (b)).

<u>Case (a)</u>: $f^* \rhd_E f_*$ and $f^* \rhd_{E^c} f_*$.

Let us prove that $\unrhd_E = \unrhd_{E^c}$. Assume to the contrary that there exist $f, g \in \mathcal{B}$ such that $f \rhd_E g$ while $g \unrhd_{E^c} f$. W.l.o.g, we can take these acts such that $f^* \rhd_E f \rhd_E g \rhd_E f_*$ and $f^* \unrhd_{E^c} g \unrhd_{E^c} f \unrhd_{E^c} f_*$. Indeed, we can always exhibit two acts satisfying our conditions by mixing f, g with either f^*or f_*. Then there exist a, $a^c, b, b^c \in (0,1)$ such that $1 \geq a > b \geq 0$ and $1 \geq b^c \geq a^c \geq 0$ and

$$\begin{aligned} f &\approx_E && af^* + (1-a) f_* \\ f &\approx_{E^c} && a^c f^* + (1-a^c) f_* \\ g &\approx_E && bf^* + (1-b) f_* \\ g &\approx_{E^c} && b^c f^* + (1-b^c) f_*. \end{aligned}$$

Let suppose $a > a^c$. By definition of $\mathcal{B}$, $f \sim (af^* + (1-a)f_*)_E \, (a^c f^* + (1-a^c) f_*)$. Hence,

$$\begin{aligned} V(f) &= V\left((af^* + (1-a)f_*)_E \, (a^c f^* + (1-a^c) f_*)\right) \\ &= V\left(\frac{a-a^c}{1-a^c} f^*_E \, (a^c f^* + (1-a^c) f_*) + \frac{1-a}{1-a^c} \, (a^c f^* + (1-a^c f_*)\right) \\ &= \frac{a-a^c}{1-a^c} V\left(f^*_E \, (a^c f^* + (1-a^c) f_*)\right) + \frac{1-a}{1-a^c} V\left(a^c f^* + (1-a^c) f_*\right) \\ &= \frac{a-a^c}{1-a^c} \left(a^c V\left(f^*\right) + (1-a^c) V\left(f^*_E f_*\right)\right) + \frac{1-a}{1-a^c} \left(a^c V\left(f^*\right) + (1-a^c) V\left(f_*\right)\right) \\ &= a^c V\left(f^*\right) + \left(a-a^c\right) V\left(f^*_E f_*\right) + (1-a) V\left(f_*\right) \end{aligned}$$

Since $f \in \mathcal{B}$,

$$
\begin{aligned}
V\left(\tfrac{1}{1+a-a^c} f\right. & \left. + \tfrac{a-a^c}{1+a-a^c} f_{*E} f^*\right) \\
& = \tfrac{1}{1+a-a^c} V(f) + \tfrac{a-a^c}{1+a-a^c} V\left(f_{*E} f^*\right) \\
& = \tfrac{1}{1+a-a^c}\left(a^c V\left(f^*\right) + \left(a-a^c\right) V\left(f_E^* f_*\right) + (1-a) V\left(f_*\right)\right) + \tfrac{a-a^c}{1+a-a^c} V\left(f_{*E} f^*\right)
\end{aligned}
$$

But we also have by definition of $\mathcal{B}$,

$$
\begin{aligned}
& V\left(\tfrac{1}{1+a-a^c} f + \tfrac{a-a^c}{1+a-a^c} f_{*E} f^*\right) \\
& = V\left(\left(\tfrac{1}{1+a-a^c} f + \tfrac{a-a^c}{1+a-a^c} f_*\right)_E \left(\tfrac{1}{1+a-a^c} f + \tfrac{a-a^c}{1+a-a^c} f^*\right)\right) \\
& = V\left(\left(\tfrac{1}{1+a-a^c}\left(a f^* + (1-a) f_*\right) + \tfrac{a-a^c}{1+a-a^c} f_*\right)_E \left(\tfrac{1}{1+a-a^c}\left(a^c f^* + (1-a^c) f_*\right) + \tfrac{a-a^c}{1+a-a^c} f^*\right)\right) \\
& = V\left(\left(\tfrac{a}{1+a-a^c} f^* + \tfrac{1-a^c}{1+a-a^c} f_*\right)_E \left(\tfrac{a}{1+a-a^c} f^* + \tfrac{1-a^c}{1+a-a^c} f_*\right)\right) \\
& = \tfrac{a}{1+a-a^c} V\left(f^*\right) + \tfrac{1-a^c}{1+a-a^c} V\left(f_*\right).
\end{aligned}
$$

Therefore,

$$
\left(a^c V\left(f^*\right) + \left(a-a^c\right) V\left(f_E^* f_*\right) + (1-a) V\left(f_*\right)\right) + \left(a-a^c\right) V\left(f_{*E} f^*\right) = a V\left(f^*\right) + \left(1-a^c\right) V\left(f_*\right)
$$

which is equivalent to

$$
\left(a-a^c\right)\left(V\left(f_E^* f_*\right) + V\left(f_{*E} f^*\right)\right) = \left(a-a^c\right)\left(V\left(f^*\right) + V\left(f_*\right)\right)
$$

This contradicts the fact that $V(f_E^* f_*) + V\left(f_{*E} f^*\right) \neq V(f^*) + V(f_*)$ and $a > a^c$. In the case where $a \leq a^c$, then either $a < a^c$ or $a = a^c$ but $b < b^c$ and the proof can be easily adapted in both cases.

Hence, $\trianglerighteq_E = \trianglerighteq_{E^c}$.

Case (b) : $f^* \vartriangleright_E f_*$ and $f_* \trianglerighteq_{E^c} f^*$.

In this case, we show that $\trianglerighteq_{E^c}$ is a reverse order of $\trianglerighteq_E$, that is, for all $f, g \in \mathcal{B}$, $f \trianglerighteq_E g$ if and only if $g \trianglerighteq_{E^c} f$.

Observe first that it has to be the case that $f_* \vartriangleright_{E^c} f^*$. Indeed, if $f_* \approx_{E^c} f^*$, then by definition of $\mathcal{B}$, $f^* \sim f_E^* f_*$ and $f_* \sim f_{*E} f^*$ and thus $V(f_E^* f_*) + V(f_{*E} f^*) = V(f^*) + V(f_*)$.

Suppose $\trianglerighteq_{E^c}$ is not a reverse order of $\trianglerighteq_E$, that is, there exist $f, g \in \mathcal{B}$, such that $f \vartriangleright_E g$ while $f \trianglerighteq_{E^c} g$. As in case (a), we can assume w.l.o.g that $f^* \vartriangleright_E f \vartriangleright_E g \vartriangleright_E f_*$ and $f_* \trianglerighteq_{E^c} f \trianglerighteq_{E^c} g$

$\trianglerighteq_{E^c} f^*$. Then, there exist a, $a^c, b, b^c \in (0,1)$ with $a > b$ and $a^c \leq b^c$ such that

$$\begin{aligned} f &\approx_E af^* + (1-a)f_* \\ f &\approx_{E^c} a^c f^* + (1-a^c)f_* \\ g &\approx_E bf^* + (1-b)f_* \\ g &\approx_{E^c} b^c f^* + (1-b^c)f_* \end{aligned}$$

Either $a > a^c$ or $a < a^c$ or $a = a^c$ but $b < b^c$. In case $a > a^c$, we can replicate the proof made before to show that

$$(a-a^c)\left(V\left(f^*_E f_*\right) + V\left(f_{*E} f^*\right)\right) = (a-a^c)\left(V\left(f^*\right) + V\left(f_*\right)\right)$$

We can adapt the proof to the other cases to show a similar contradiction.

Step 2 As a second step, we prove conditions 2 and 3 when there exists $f, g \in \mathcal{B}$ such that $V(f_E g) + V\left(g_E f\right) \neq V(f) + V(g)$.

Case (a) Suppose $\trianglerighteq_E = \trianglerighteq_{E^c}$. Given that $\succsim$ is not degenerate on $\mathcal{B}$, there exist $f^*, f_* \in \mathcal{B}$ such that $f^* \succ f_*$.

Thus, define for any f

$$\begin{aligned} \overline{V}_E(f) &= \frac{V(f^*_E f_*) - V(f_*)}{V(f^*) - V(f_*)} V(f) \\ \underline{V}_E(f) &= \frac{V(f^*) - V(f_{*E} f^*)}{V(f^*) - V(f_*)} V(f) \\ \overline{V}_{E^c}(f) &= \frac{V(f_{*E} f^*) - V(f_*)}{V(f^*) - V(f_*)} V(f) \\ \underline{V}_{E^c}(f) &= \frac{V(f^*) - V(f^*_E f_*)}{V(f^*) - V(f_*)} V(f) \end{aligned}$$

Let us prove that for all $f, g \in \mathcal{B}$,

$$\begin{aligned} V(f_E g) &= \overline{V}_E(f) + \underline{V}_{E^c}(g) \text{ if } f \succsim g \\ &= \underline{V}_E(f) + \overline{V}_{E^c}(g) \text{ if } f \precsim g \end{aligned}$$

Consider $f, g \in \mathcal{B}$ such that $f \succsim g$ and consider the case where $V(f^*) \geq V(f) \geq V(g) \geq V(f_*)$. We have that

$$f \approx_E \frac{V(f) - V(f_*)}{V(f^*) - V(f_*)} f^* + \left(1 - \frac{V(f) - V(f_*)}{V(f^*) - V(f_*)}\right) f_*$$

and

$$g \approx_{E^c} \frac{V(g) - V(f_*)}{V(f^*) - V(f_*)} f^* + \left(1 - \frac{V(g) - V(f_*)}{V(f^*) - V(f_*)}\right) f_*$$

By definition of $\mathcal{B}$, $f_E g \sim (af^* + (1-a)f_*)_E\,(bf^* + (1-b)f_*)$ where $a = \frac{V(f)-V(f_*)}{V(f^*)-V(f_*)}$ and $b = \frac{V(g)-V(f_*))}{V(f^*)-V(f_*)}$. Thus

$$\begin{aligned}
V(f_E g) &= V\left((af^* + (1-a)f_*)_E\,(bf^* + (1-b)f_*)\right) \\
&= bV\left(f^*\right) + (a-b)\,V\left(f^*_E f_*\right) + (1-a)V\left(f_*\right) \\
&= \frac{(V(g)-V(f_*))\,V\left(f^*\right) + (V(f)-V(g))\,V\left(f^*_E f_*\right) + (V\left(f^*\right)-V(f))V\left(f_*\right)}{V(f^*)-V(f_*)} \\
&= \frac{\left(V\left(f^*_E f_*\right) - V\left(f_*\right)\right)V(f) + \left(V\left(f^*\right) - V\left(f^*_E f_*\right)\right)V(g)}{V(f^*)-V(f_*)} \\
&= \overline{V}_E(f) + \underline{V}_{E^c}(g)
\end{aligned}$$

In the case where $V(f^*) \geq V(g) \geq V(f) \geq V(f_*)$, a similar computation shows that $V(f_E g) = \underline{V}_E(f) + \overline{V}_{E^c}(g)$.

In the other cases, the proof can be easily adapted to show that

$$\begin{aligned}
V(f_E g) &= \overline{V}_E(f) + \underline{V}_{E^c}(g) \text{ if } f \succsim g \\
&= \underline{V}_E(f) + \overline{V}_{E^c}(g) \text{ if } f \precsim g
\end{aligned}$$

Define $k_E = \frac{V(f^*_E f_*)+V(f_{*E} f^*)-V(f^*)-V(f_*)}{V(f^*)-V(f_*)}$.

If $f \succsim g$,

$$V(f_E g) + V(g_E f) - V(f) - V(g) =$$

$$= \overline{V}_E(f) + \underline{V}_{E^c}(g) + \underline{V}_E(g) + \overline{V}_{E^c}(f) - \overline{V}_E(f) - \underline{V}_{E^c}(f) - \underline{V}_E(g) - \overline{V}_{E^c}(g)$$

$$= \overline{V}_{E^c}(f) - \underline{V}_{E^c}(f) + \underline{V}_{E^c}(g) - \overline{V}_{E^c}(g)$$

$$= \left(\tfrac{V(f_{*E} f^*)-V(f_*)}{V(f^*)-V(f_*)} - \tfrac{V(f^*)-V(f^*_E f_*)}{V(f^*)-V(f_*)}\right)V(f) - \left(\tfrac{V(f_{*E} f^*)-V(f_*)}{V(f^*)-V(f_*)} - \tfrac{V(f^*)-V(f^*_E f_*)}{V(f^*)-V(f_*)}\right)V(g)$$

$$= k_E\left(V(f) - V(g)\right)$$

If $f \precsim g$,

$$V(f_E g) + V(g_E f) - V(f) - V(g) =$$

$$= \underline{V}_E(f) + \overline{V}_{E^c}(g) + \overline{V}_E(g) + \underline{V}_{E^c}(f) - \underline{V}_E(f) - \overline{V}_{E^c}(f) - \overline{V}_E(g) - \underline{V}_{E^c}(g)$$

$$= \underline{V}_{E^c}(f) - \overline{V}_{E^c}(f) + \overline{V}_{E^c}(g) - \underline{V}_{E^c}(g)$$

$$= k_E\left(V(g) - V(f)\right)$$

Case (b): Suppose $\trianglerighteq_{E^c}$ is a reverse order of $\trianglerighteq_E$.

Let $f^*, f_* \in \mathcal{B}$ be such that $V(f^*_E f_*) + V\left(f_{*E} f^*\right) \neq V(f^*) + V(f_*)$. Without loss of generality, suppose that $f^* \succsim f_*$, $f^* \rhd_E f_*$ and $f_* \rhd_{E^c} f^*$.

Consider $\overline{V}_E$, $\underline{V}_E$ the vNM utility functions representing $\trianglerighteq_E$ and $\overline{V}_{E^c}$, $\underline{V}_{E^c}$ the vNM utility functions representing $\trianglerighteq_{E^c}$ such that

- $\overline{V}_E(f^*) = \overline{V}_{E^c}(f^*) = V(f^*)$,
- $\underline{V}_E(f^*) = \underline{V}_{E^c}(f^*) = 0$,
- $\overline{V}_E(f_*) = V(f^*) + V(f_*) - V(f^*_E f_*)$,
- $\underline{V}_E(f_*) = V(f_{*E} f^*) - V(f^*)$,
- $\overline{V}_{E^c}(f_*) = V(f^*) + V(f_*) - V(f_{*E} f^*)$,
- $\underline{V}_{E^c}(f_*) = V(f^*_E f_*) - V(f^*)$.

Note that it is possible to choose this normalization for these vNM utility functions since $f^* \vartriangleright_E f_*$ and $f_* \vartriangleright_{E^c} f^*$ and thus

$$V(f^*_E f_*) > V(f^*), V(f_*) > V(f_{*E} f^*)$$

which implies that $\overline{V}_E(f^*) > \overline{V}_E(f_*)$, $\underline{V}_E(f^*) > \underline{V}_E(f_*)$, $\overline{V}_{E^c}(f_*) > \overline{V}_{E^c}(f^*)$ and $\underline{V}_{E^c}(f_*) > \underline{V}_{E^c}(f^*)$.

Let us prove that for all $f, g \in \mathcal{B}$,

$$\begin{aligned} V(f_E g) &= \overline{V}_E(f) + \underline{V}_{E^c}(g) \text{ if } f \succsim g \\ &= \underline{V}_E(f) + \overline{V}_{E^c}(g) \text{ if } f \precsim g \end{aligned}$$

Let $f, g \in \mathcal{B}$ such that $f \succsim g$. Consider a first case where $f^* \trianglerighteq_E f \trianglerighteq_E f_*$ and $f^* \trianglerighteq_E g \trianglerighteq_E f_*$. Then there exist a, $b \in (0,1)$ such that

$$\begin{aligned} f &\approx_E af^* + (1-a)f_* \\ g &\approx_E bf^* + (1-b)f_* \end{aligned}$$

Since $\trianglerighteq_{E^c}$ is a reverse order of $\trianglerighteq_E$, we also have that

$$\begin{aligned} f &\approx_{E^c} af^* + (1-a)f_* \\ g &\approx_{E^c} bf^* + (1-b)f_* \end{aligned}$$

Then, by definition of $\mathcal{B}$, $f \sim af^* + (1-a)f_*$ and $g \sim bf^* + (1-b)f_*$. Since $f \succsim g$ and $f^* \succsim f_*$, we get that $a \geq b$. Thus,

$$\begin{aligned} V(f_E g) &= V\left((af^* + (1-a)f_*)_E (bf^* + (1-b)f_*)\right) \\ &= bV(f^*) + (a-b)V(f^*_E f_*) + (1-a)V(f_*) \\ &= aV(f^*) + (1-a)(V(f^*) + V(f_*) - V(f^*_E f_*)) + 0.b + (1-b)(V(f^*_E f_*) - V(f^*)) \\ &= a\overline{V}_E(f^*) + (1-a)\overline{V}_E(f_*) + b\underline{V}_{E^c}(f^*) + (1-b)\underline{V}_{E^c}(f_*) \\ &= \overline{V}_E(af^* + (1-a)f_*) + \underline{V}_{E^c}(bf^* + (1-b)f_*) \\ &= \overline{V}_E(f) + \underline{V}_{E^c}(g) \end{aligned}$$

Consider a second case where $f \trianglerighteq_E f^*$ and $f_* \trianglerighteq_E g$. Then, there exist a, $b \in (0,1)$ such that

$$\begin{aligned} f^* &\approx_E af + (1-a)g \\ f_* &\approx_E bf + (1-b)g \end{aligned}$$

and

$$\begin{aligned} f^* &\approx_{E^c} af + (1-a)g \\ f_* &\approx_{E^c} bf + (1-b)g \end{aligned}$$

and $f^* \sim af + (1-a)g$ and $f_* \sim bf + (1-b)g$. Thus $a > b$ and

$$\begin{aligned} V(f^*_E f_*) &= V\left((af + (1-a)g)_E (bf + (1-b)g)\right) \\ &= bV(f) + (a-b)V(f_E g) + (1-a)V(g) \end{aligned}$$

Thus

$$V(f_E g) = \frac{V(f^*_E f_*) - bV(f) - (1-a)V(g)}{a-b}$$

We also have

$$\begin{aligned} \overline{V}_E(f) &= \frac{(1-b)\overline{V}_E(f^*) - (1-a)\overline{V}_E(f_*)}{a-b} \\ \underline{V}_{E^c}(g) &= \frac{b\underline{V}_{E^c}(f^*) - a\underline{V}_{E^c}(f_*)}{b-a} \end{aligned}$$

and thus

$$\begin{aligned} \overline{V}_E(f) + \underline{V}_{E^c}(g) &= \frac{(1-b)\overline{V}_E(f^*) - (1-a)\overline{V}_E(f_*) - b\underline{V}_{E^c}(f^*) + a\underline{V}_{E^c}(f_*)}{a-b} \\ &= \frac{(1-b)\overline{V}_E(f^*) - (1-a)\overline{V}_E(f_*) - b\underline{V}_{E^c}(f^*) + a\underline{V}_{E^c}(f_*)}{a-b} \\ &= \frac{(1-b)V(f^*) - (1-a)\left(V(f^*) + V(f_*) - V(f^*_E f_*)\right) + a\left(V(f^*_E f_*) - V(f^*)\right)}{a-b} \\ &= \frac{V(f^*_E f_*) - bV(f^*) - (1-a)V(f_*)}{a-b} \\ &= \frac{V(f^*_E f_*) - b\left(aV(f) + (1-a)V(g)\right) - (1-a)\left(bV(f) + (1-b)V(g)\right)}{a-b} \\ &= \frac{V(f^*_E f_*) - bV(f) - (1-a)V(g)}{a-b} \end{aligned}$$

which proves that $V(f_E g) = \overline{V}_E(f) + \underline{V}_{E^c}(g)$.

The proof can be adapted in the cases where $f \trianglerighteq_E f^*$ and $g \trianglerighteq_E f^*$ (or $f^* \trianglerighteq_E g \trianglerighteq_E f_*$) or $f^* \trianglerighteq_E g \trianglerighteq_E f_*$ and $f_* \trianglerighteq_E g$ or $f_* \trianglerighteq_E f$ and $f_* \trianglerighteq_E g$.

Let suppose now that $f^* \trianglerighteq_E f \trianglerighteq_E f_*$ and $g \triangleright_E f^*$. Then, there exist $a, b \in (0,1)$ such that

$$\begin{aligned} f &\approx_E af^* + (1-a)f_* \\ f^* &\approx_E bg + (1-b)f_* \end{aligned}$$

Then we also have

$$\begin{aligned} f &\approx_{E^c} af^* + (1-a)f_* \\ f^* &\approx_{E^c} bg + (1-b)f_* \end{aligned}$$

and thus

$$\begin{aligned} f &\sim af^* + (1-a)f_* \\ f^* &\sim bg + (1-b)f_* \end{aligned}$$

which yields a contradiction to the fact that $f \succsim g$.

We can prove that a similar contradiction occur if we suppose $f_* \trianglerighteq_E f$ and $g \triangleright_E f_*$.

Since $\overline{V}_E$, $\underline{V}_E$ are vNM representation of $\trianglerighteq_E$, $\overline{V}_{E^c}$, $\underline{V}_{E^c}$ are vNM representation of $\trianglerighteq_{E^c}$ and since they are two reverse orders, the uniqueness conditions imply that

- $\underline{V}_E = \frac{V(f^*)-V(f_{*E}f^*)}{V\left(f^*_E f_*\right)-V(f_*)}\left(\overline{V}_E - V\left(f^*\right)\right)$
- $\overline{V}_{E^c} = \frac{V(f_*)-V(f_{*E}f^*)}{V(f_*)-V\left(f^*_E f_*\right)}\left(\overline{V}_E - V\left(f^*\right)\right) + V\left(f^*\right)$
- $\underline{V}_{E^c} = \frac{V\left(f^*_E f_*\right)-V(f^*)}{V(f_*)-V\left(f^*_E f_*\right)}\left(\overline{V}_E - V\left(f^*\right)\right)$

Note that for all $f \in \mathcal{B}$, $V(f) = \frac{V(f^*)-V(f_*)}{V\left(f^*_E f_*\right)-V(f_*)}\overline{V}_E(f) + \frac{V\left(f^*_E f_*\right)-V(f^*)}{V\left(f^*_E f_*\right)-V(f_*)}V\left(f^*\right)$. Let's now check that the representation of condition 3 obtains.

If $f \succsim g$,

$$\begin{aligned} V(f_E g) + V(g_E f) - V(f) - V(g) &= \overline{V}_{E^c}(f) - \underline{V}_{E^c}(f) + \underline{V}_{E^c}(g) - \overline{V}_{E^c}(g) \\ &= \left(\frac{V(f_*)-V(f_{*E}f^*)}{V(f_*)-V\left(f^*_E f_*\right)} - \frac{V\left(f^*_E f_*\right)-V(f^*)}{V(f_*)-V\left(f^*_E f_*\right)}\right)\left(\overline{V}_E(f) - \overline{V}_E(g)\right) \\ &= \frac{V(f^*)-V(f_*)}{V\left(f^*_E f_*\right)-V(f_*)}\left(\overline{V}_E(f) - \overline{V}_E(g)\right) \\ &= V(f) - V(g) \end{aligned}$$

If $f \precsim g$,

$$\begin{aligned} V(f_E g) + V(g_E f) - V(f) - V(g) &= \underline{V}_{E^c}(f) - \overline{V}_{E^c}(f) + \overline{V}_{E^c}(g) - \underline{V}_{E^c}(g) \\ &= V(g) - V(f) \end{aligned}$$

Step 3 As a third step, we consider the case where for all $f, g \in \mathcal{B}$, $V(f_E g) + V(g_E f) - V(f) - V(g) = 0$.

If for all $f, g \in \mathcal{B}$, $f_E g \sim f$, then for $V_E = V$ and $V_{E^c} = 0$, we have that $V(f_E g) = V_E(f) + V_{E^c}(g)$ which proves that conditions 2 and 3 hold.

Suppose now that there exist $f^*, f_* \in \mathcal{B}$ such that $f^*_E f_* \nsim f^*$. Since $V(f^*_E f_*) + V(f_{*E} f^*) = V(f^*) + V(f_*)$, w.l.o.g we can restrict our attention to two cases: (a) $V(f^*) > V(f^*_E f_*), V(f_{*E} f^*) > V(f_*)$ and (b) $V(f^*_E f_*) > V(f^*) > V(f_*) > V(f_{*E} f^*)$.

In either case, consider V_E and V_{E^c} the vNM utility functions representing $\trianglerighteq_E$ and $\trianglerighteq_{E^c}$ such that $V_E(f^*) = V(f^*)$, $V_{E^c}(f^*) = 0$, $V_E(f_*) = V(f_{*E} f^*)$, $V_{E^c}(f_*) = V(f^*_E f_*) - V(f^*)$. Note that it is possible to choose this normalization for these vNM utility functions. Indeed, in case (a), we have $f^* \triangleright_E f_*$ and $f^* \triangleright_{E^c} f_*$ and the normalization proposed is such that $V_E(f^*) > V_E(f_*)$ and $V_{E^c}(f^*) > V_{E^c}(f_*)$, while in case (b), we have $f^* \triangleright_E f_*$ and $f_* \triangleright_{E^c} f^*$ and the normalization proposed is such that $V_E(f^*) > V_E(f_*)$ and $V_{E^c}(f^*) < V_{E^c}(f_*)$.

Let $f, g \in \mathcal{B}$ and consider a first case where $f^* \trianglerighteq_E f \trianglerighteq_E f_*$ and g is between f^* and f_* according to $\trianglerighteq_{E^c}$. Then there exists $a, b^c \in (0,1)$ such that

$$\begin{aligned} f &\approx_E a f^* + (1-a) f_* \\ g &\approx_{E^c} b^c f^* + (1-b^c) f_* \end{aligned}$$

If $a \geq b^c$, then by definition of $\mathcal{B}$ and since $V(f^*_E f_*) + V(f_{*E} f^*) = V(f^*) + V(f_*)$,

$$\begin{aligned} V(f_E g) &= V\left((af^* + (1-a)f_*)_E (b^c f^* + (1-b^c) f_*)\right) \\ &= b^c V(f^*) + (a - b^c) V(f^*_E f_*) + (1-a) V(f_*) \\ &= aV(f^*) + (1-a)(V(f^*) + V(f_*) - V(f^*_E f_*)) + 0.b^c + (1-b^c)(V(f^*_E f_*) - V(f^*)) \\ &= aV(f^*) + (1-a)V(f_{*E} f^*) + 0.b^c + (1-b^c)(V(f^*_E f_*) - V(f^*)) \\ &= aV_E(f^*) + (1-a)V_E(f_*) + b^c V_{E^c}(f^*) + (1-b^c) V_{E^c}(f_*) \\ &= V_E(af^* + (1-a)f_*) + V_{E^c}(b^c f^* + (1-b^c) f_*) \\ &= V_E(f) + V_{E^c}(g) \end{aligned}$$

If $b^c \geq a$, then by definition of $\mathcal{B}$ and since $V(f^*_E f_*) + V(f_{*E} f^*) = V(f^*) + V(f_*)$,

$$\begin{aligned} V(f_E g) &= V\left((af^* + (1-a)f_*)_E (b^c f^* + (1-b^c) f_*)\right) \\ &= aV(f^*) + (b^c - a) V(f_{*E} f^*) + (1-b^c) V(f_*) \\ &= aV(f^*) + (1-a)V(f_{*E} f^*) + 0.b^c + (1-b^c)(V(f_*) - V(f_{*E} f^*)) \\ &= aV(f^*) + (1-a)V(f_{*E} f^*) + 0.b^c + (1-b^c)(V(f^*_E f_*) - V(f^*)) \\ &= aV_E(f^*) + (1-a)V_E(f_*) + b^c V_{E^c}(f^*) + (1-b^c) V_{E^c}(f_*) \\ &= V_E(f) + V_{E^c}(g) \end{aligned}$$

Consider now a second case, where $f \trianglerighteq_E f^*$ and g is between f^* and f_* according to $\trianglerighteq_{E^c}$. Then there exist $a, b^c \in (0,1)$ such that

$$\begin{aligned} f^* &\approx_E a f + (1-a) f_* \\ g &\approx_{E^c} b^c f^* + (1-b^c) f_* \end{aligned}$$

Therefore by definition of $\mathcal{B}$,

$V(f_E^* g) = V\left((af + (1-a)f_*)_E\, g\right) \Leftrightarrow V(f_E^*\left(b^c f^* + (1-b^c)f_*\right)) = aV(f_E g) + (1-a)V(f_{*E} g)$

$\Leftrightarrow b^c V(f^*) + (1-b^c)V\left(f_E^* f_*\right) = aV(f_E g) + (1-a)\left(b^c V(f_{*E} f^*) + (1-b^c)V\left(f_*\right)\right)$

$\Leftrightarrow V(f_E g) = \frac{b^c V(f^*) + (1-b^c)V\left(f_E^* f_*\right) - (1-a)(b^c V(f_{*E} f^*) + (1-b^c)V(f_*))}{a}$

Using the fact that $V\left(f_*\right) = V(f_{*E} f^*) + V\left(f_E^* f_*\right) - V(f^*)$, we get that

$$V(f_E g) = \frac{(1-a+ab^c)V\left(f^*\right) - (1-a)V(f_{*E} f^*) + a(1-b^c)V\left(f_E^* f_*\right)}{a}$$

We also have that

$$\begin{aligned} V_E\left(f\right) &= \frac{V_E\left(f^*\right) - (1-a)V_E\left(f_*\right)}{a} \\ V_{E^c}\left(g\right) &= b^c V_{E^c}\left(f^*\right) + (1-b^c)V_{E^c}\left(f_*\right) \end{aligned}$$

Thus

$$\begin{aligned} V_E\left(f\right) + V_{E^c}\left(g\right) &= \frac{V_E\left(f^*\right) - (1-a)V_E\left(f_*\right) + a\left(b^c V_{E^c}\left(f^*\right) + (1-b^c)V_{E^c}\left(f_*\right)\right)}{a} \\ &= \frac{V\left(f^*\right) - (1-a)V(f_{*E} f^*) + a(1-b^c)\left(V\left(f_E^* f_*\right) - V\left(f^*\right)\right)}{a} \\ &= \frac{(1-a+ab^c)V\left(f^*\right) - (1-a)V(f_{*E} f^*) + a(1-b^c)V\left(f_E^* f_*\right)}{a} \end{aligned}$$

and therefore $V(f_E g) = V_E\left(f\right) + V_{E^c}\left(g\right)$.

In the other cases the proof can be adapted to show that $V(f_E g) = V_E\left(f\right) + V_{E^c}\left(g\right)$.

Finally, remark that condition 3 is satisfied with $k^E = 0$. ■

Proposition 2.

Suppose $\succsim$ is smooth on E with respect to $\mathcal{B}$. Let us prove that for all $f, g \in \mathcal{B}$, $V(f_E g) + V\left(g_E f\right) = V(f) + V(g)$ and thus that $k^E = 0$.

Let $f, g \in \mathcal{B}$. Suppose first that $f \sim g$.

If $f \trianglerighteq_E g$ and $f \trianglerighteq_{E^c} g$, then $f \succsim f_E g$, $g_E f \succsim g$ and thus $f \sim f_E g \sim g_E f \sim g$. Therefore, $V(f_E g) + V\left(g_E f\right) = V(f) + V(g)$.

If $f \trianglerighteq_E g$ and $f \vartriangleleft_{E^c} g$, then $f_E g \succsim f \sim g \succsim g_E f$. If $f \sim f_E g \sim g_E f \sim g$ then $V(f_E g) + V\left(g_E f\right) = V(f) + V(g)$. However, w.l.o.g let us suppose that $f_E g \succ f$. Since $\succsim$ is not degenerate on $\mathcal{B}$, there exists $h \in \mathcal{B}$ such that $h \nsim f$. Suppose $h \succ f$ and w.l.o.g, suppose that $f_E g \succ h \succ f \sim g \succsim g_E f$. Then

$$\frac{1}{2}f + \frac{1}{2}h \sim af_E g + (1-a)f \sim bg_E f + (1-b)h$$

where $a = \frac{1}{2}\frac{V(h)-V(f)}{V(f_E g)-V(f)}$ and $b = \frac{1}{2}\frac{V(h)-V(f)}{V(h)-V(g_E f)}$. Since

$$\frac{1}{2}f + \frac{1}{2}h \sim f_E\left(ag + (1-a)f\right) \sim \left(bg + (1-b)h\right)_E\left(bf + (1-b)h\right)$$

and $\succsim$ is smooth on E, then

$$\begin{aligned}\left(\tfrac{b}{a+b}f+\tfrac{a}{a+b}\left(bg+(1-b)h\right)\right)_E\left(\tfrac{b}{a+b}\left(ag+(1-a)f\right)+\tfrac{a}{a+b}\left(bf+(1-b)h\right)\right) &\sim f_E\left(ag+(1-a)f\right)\\ &\sim \tfrac{1}{2}f+\tfrac{1}{2}h\end{aligned}$$

Note that

$$\begin{aligned}&\left(\tfrac{b}{a+b}f+\tfrac{a}{a+b}\left(bg+(1-b)h\right)\right)_E\left(\tfrac{b}{a+b}\left(ag+(1-a)f\right)+\tfrac{a}{a+b}\left(bf+(1-b)h\right)\right)\\ &\quad\sim \tfrac{(1+a)b}{a+b}\left(\tfrac{1}{1+a}f+\tfrac{a}{1+a}g\right)+\tfrac{a(1-b)}{a+b}h\\ &\quad\sim \tfrac{(1+a)b}{a+b}f+\tfrac{a(1-b)}{a+b}h\end{aligned}$$

Thus we have that

$$\frac{(1+a)b}{a+b}V(f)+\frac{a(1-b)}{a+b}V(h)=\frac{1}{2}V(f)+\frac{1}{2}V(h)$$

which is equivalent to

$$\begin{aligned}\tfrac{1}{4}\left(\tfrac{2V(f_Eg)+V(h)-3V(f)}{V(f_Eg)-V(f)}\right)\left(\tfrac{V(h)-V(f)}{V(h)-V(g_Ef)}\right)V(f)+\tfrac{1}{4}\left(\tfrac{V(h)-V(f)}{V(f_Eg)-V(f)}\right)\left(\tfrac{V(h)+V(f)-2V(g_Ef)}{V(h)-V(g_Ef)}\right)V(h)\\ =\tfrac{1}{4}\left(\tfrac{V(h)-V(f)}{V(f_Eg)-V(f)}+\tfrac{V(h)-V(f)}{V(h)-V(g_Ef)}\right)\left(V(f)+V(h)\right)\end{aligned}$$

equivalent to

$$\begin{aligned}\left(2V(f_Eg)+V(h)-3V(f)\right)V(f)+\left(V(h)+V(f)-2V(g_Ef)\right)V(h)\\ =\left(V(h)-V(g_Ef)+V(f_Eg)-V(f)\right)\left(V(f)+V(h)\right)\end{aligned}$$

and finally to

$$\left(2V(f)-V(f_Eg)-V(g_Ef)\right)\left(V(h)-V(f)\right)=0$$

Since $V(h)>V(f)$, we must have $V(f_Eg)+V\left(g_Ef\right)=2V(f)=V(f)+V(g)$.

The proof is similar for the other cases ($f\succ h$ or $f\lhd_E g$ and $f\unrhd_{E^c} g$).

Suppose now that $f\succ g$ and consider a first case where $f\unrhd_E g$ and $f\unrhd_{E^c} g$ and thus $f\succsim f_Eg$, $g_Ef\succsim g$. First note that if $f\sim f_Eg$, then $g_Ef\sim g$ and thus $V(f_Eg)+V\left(g_Ef\right)=V(f)+V(g)$.

If $f\succ f_Eg\succsim g_Ef$, then $f_Eg\sim(af+(1-a)g)_E f$ where $a=\frac{V(f_Eg)-V(g_Ef)}{V(f)-V(g_Ef)}$. Since $\succsim$ is smooth on E,

$$\left(\frac{1-a}{2-a}f+(1-\frac{1-a}{2-a})\left(af+(1-a)g\right)\right)_E\left(\frac{1-a}{2-a}g+(1-\frac{1-a}{2-a})f\right)\sim f_Eg$$

Note that

$$\left(\frac{1-a}{2-a}f+(1-\frac{1-a}{2-a})\left(af+(1-a)g\right)\right)_E\left(\frac{1-a}{2-a}g+(1-\frac{1-a}{2-a})f\right)=\frac{1}{2-a}f+\frac{1-a}{2-a}g$$

We also have $f_Eg\sim bf+(1-b)g$ where $b=\frac{V(f_Eg)-V(g)}{V(f)-V(g)}$. Since $f\succ g$, $b=\frac{1}{2-a}$; this is equivalent to

$$
\begin{aligned}
& 2-\frac{V(f_E g)-V(g_E f)}{V(f)-V(g_E f)}=\frac{V(f)-V(g)}{V(f_E g)-V(g)} \\
\Leftrightarrow \quad & (2V(f)-V(g_E f)-V(f_E g))(V(f_E g)-V(g))=(V(f)-V(g_E f))\,(V(f)-V(g)) \\
\Leftrightarrow \quad & -V(f)V(g)+2V(f)V(f_E g)-V(g_E f)V(f_E g)+V(g_E f)V(f)-V(f_E g)V(f_E g)+ \\
& +V(f_E g)V(g)-V(f)V(f)=0 \\
\Leftrightarrow \quad & (V(f)-V(f_E g))(-V(f)-V(g)+V(g_E f)+V(f_E g))=0
\end{aligned}
$$

Since $f \succ f_E g$, therefore $V(f_E g)+V\,(g_E f)=V(f)+V(g)$. The proof is similar in the case where $f \succ g_E f \succsim f_E g$.

Conversely, suppose that $k^E=0$. Consider the utility functions $\overline{V}_E$, $\underline{V}_E$, $\overline{V}_{E^c}$ and $\underline{V}_{E^c}$. As shown in the proof of Proposition 1 these functions are linear with respect to mixture on $\mathcal{B}$. Note that $k^E=0$ implies that for all $f,g\in\mathcal{B}$, $\overline{V}_E(f)+\underline{V}_{E^c}(g)=\underline{V}_E(f)+\overline{V}_{E^c}(g)$.

Let consider $f,g,h,\ell\in\mathcal{B}$ such that $f_E g\sim h_E\ell$ and $\alpha\in(0,1)$.

$$
\begin{aligned}
V((\alpha f+(1-\alpha)h)_E\,(\alpha g+(1-\alpha)\ell)) &= \overline{V}_E(\alpha f+(1-\alpha)h)+\underline{V}_{E^c}(\alpha g+(1-\alpha)\ell) \\
&= \alpha\left(\overline{V}_E(f)+\underline{V}_{E^c}(g)\right)+(1-\alpha)\left(\overline{V}_E(h)+\underline{V}_{E^c}(\ell)\right) \\
&= \alpha V(f_E g)+(1-\alpha)V(h_E\ell)
\end{aligned}
$$

and thus $(\alpha f+(1-\alpha)h)_E\,(\alpha g+(1-\alpha)\ell)\sim f_E g$. ■

4.2 Appendix B

In this Appendix, we provide the proof of our main result. We decompose the proof into 4 lemmas. Although not always explicitly stated in the lemma, all the assumptions of Theorem 1 are made throughout this Appendix. The following Lemma is adapted from Weymark (1993, Lemma 1):

Lemma 1 *Let $(V_i)_{i\in N}$ be a collection of $\mathcal{B}_i$-affine representation of $\succsim_i$ for all $i\in N$ and assume conditions 1, 2, 3 of Theorem 1 are satisfied. Then, $(V_1,\cdots,V_n)$ are affinely independent on $\cap_{i\in N}\mathcal{B}_i$.*

Proof. Suppose on the contrary that $(V_1,\cdots,V_n)$ are affinely dependent on $\cap_{i\in N}\mathcal{B}_i$, that is, there exists $(\lambda_1,\cdots,\lambda_n)\in\mathbb{R}^n$ and $\mu\in\mathbb{R}$ such that $\sum_{i=1}^n\lambda_iV_i(f)+\mu=0$ for all $f\in\cap_{i\in N}\mathcal{B}_i$ with at least one $\lambda_j\neq 0$. Without loss of generality, assume that $\lambda_1=-1$. We then have:

$$
V_1(f)=\sum_{i\neq 1}\lambda_iV_i(f)+\mu,\ \forall f\in\cap_{i\in N}\mathcal{B}_i. \tag{1}
$$

Let f and g in $\cap_{i\in N}\mathcal{B}_i$ be such that $f\sim_i g$ for all $i\neq 1$ and $f\succ_1 g$ (such acts exist, since $\{\succsim_i\}_{i\in N'}$ satisfy the independent prospects property on $\cap_i\mathcal{B}_i$). But equation (1) implies that $V_1(f)=V_1(g)$, a contradiction. ■

Lemma 2 *There exist* $\bar{f}, \underline{f} \in \cap_{i\in N}\mathcal{B}_i$ *such that* $\bar{f} \succ_i \underline{f}$ *for all* $i \in N'$.

Proof. For all $i \in N'$, let $\bar{f}_i, \underline{f}_i \in \cap_{i\in N}\mathcal{B}_i$ be such that $\bar{f}_i \succ_i \underline{f}_i$ and $\bar{f}_i \sim_j \underline{f}_i$ for all $j \neq i$ (such acts exist since $\{\succsim_i\}_{i\in N'}$ satisfy the independent prospects property). Consider $\alpha_j \in]0,1[$ for $j = 2, .., n$ and define recursively $\bar{f}^j, \underline{f}^j$ by

- $\bar{f}^2 = \alpha_2 \bar{f}_1 + (1-\alpha_2)\bar{f}_2$, $\underline{f}^2 = \alpha_2 \underline{f}_1 + (1-\alpha_2)\underline{f}_2$
- for $j = 3, .., n$, $\bar{f}^j = \alpha_j \bar{f}^{j-1} + (1-\alpha_j)\bar{f}_j$, $\underline{f}^j = \alpha_j \underline{f}^{j-1} + (1-\alpha_j)\underline{f}_j$.

Since $\cap_{i\in N}\mathcal{B}_i$ is a mixture space, $\bar{f}^n, \underline{f}^n \in \cap_{i\in N}\mathcal{B}_i$ and it we can check that $\bar{f}^n \succ_i \underline{f}^n$ for all $i \in N'$. ■

Lemma 3 *Let* $(V_i)_{i\in N}$ *be a collection of* $\mathcal{B}_i$*-affine representation of* $\succsim_i$ *for all* $i \in N$ *and assume conditions 1, 2, 3 of Theorem 1 are satisfied. There exist unique weights* $(\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n_+ \setminus \{0\}$, $\mu \in \mathbb{R}$, *such that*

$$\forall f \in \mathcal{A}, V_0(f) = \sum_{i\in N'} \lambda_i V_i(f) + \mu.$$

Proof. Define $F : \mathcal{A} \to \mathbb{R}^{n+1}$ by $F(f) = (V_0(f), V_1(f), \cdots, V_n(f))$ and let $K_f = co\{f, \cap_{i\in N}\mathcal{B}_i\}$ for all $f \in \mathcal{A}$. Clearly, for all $f \in \mathcal{A}$, K_f is a convex set, $\cap_{i\in N}\mathcal{B}_i \subseteq K_f$, and $\bigcup_{f\in\mathcal{A}} K_f = \mathcal{A}$.

We first prove that $F(K_f)$ is convex for all $f \in \mathcal{A}$. Let f be fixed, and consider $g_1, g_2 \in K_f$, with $g_1 \neq g_2$. Let $\gamma = tF(g_1) + (1-t)F(g_2)$, with $t \in (0,1)$. By definition, there exist $\alpha_1, \alpha_2 \in [0,1]$, and $h_1, h_2 \in \cap_{i\in N'}\mathcal{B}_i$ such that $g_1 = \alpha_1 f + (1-\alpha_1)h_1$ and $g_2 = \alpha_2 f + (1-\alpha_2)h_2$. Let $g_3 = tg_1 + (1-t)g_2$. Let $h_3 = \frac{t(1-\alpha_1)}{t(1-\alpha_1)+(1-t)(1-\alpha_2)}h_1 + \frac{(1-t)(1-\alpha_2)}{t(1-\alpha_1)+(1-t)(1-\alpha_2)}h_2$[9]. It is easy to see that $g_3 = [t\alpha_1 + (1-t)\alpha_2]f + [1-(t\alpha_1 + (1-t)\alpha_2)]h_3$. Note that $\cap_{i\in N}\mathcal{B}_i$ is a mixture set and thus $h_3 \in K_f$.

We hence have, by affinity of the V_i

$$\begin{aligned}
V_i(g_3) &= [t\alpha_1 + (1-t)\alpha_2]V_i(f) + [1-(t\alpha_1 + (1-t)\alpha_2)]V_i(h_3) \\
&= [t\alpha_1 + (1-t)\alpha_2]V_i(f) + \\
&\quad [1-(t\alpha_1 + (1-t)\alpha_2)]\left[\frac{t(1-\alpha_1)}{t(1-\alpha_1)+(1-t)(1-\alpha_2)}V_i(h_1)\right. \\
&\quad \left.+\frac{(1-t)(1-\alpha_2)}{t(1-\alpha_1)+(1-t)(1-\alpha_2)}V_i(h_2)\right] \\
&= t[\alpha_1 V_i(f) + (1-\alpha_1)V_i(h_1)] + (1-t)[\alpha_2 V_i(f) + (1-\alpha_2)V_i(h_2)] \\
&= tV_i(\alpha_1 f + (1-\alpha_1)h_1) + (1-t)V_i(\alpha_2 f + (1-\alpha_2)h_2) \\
&= tV_i(g_1) + (1-t)V_i(g_2).
\end{aligned}$$

Hence $F(g_3) = \gamma$, which proves that $F(K_f)$ is convex.

[9] Since $g_1 \neq g_2$, $\alpha_1 \neq \alpha_2$, and therefore $t(1-\alpha_1) + (1-t)(1-\alpha_2) \neq 0$.

By Proposition 2 in De Meyer and Mongin (1995), the convexity of $F(K_f)$, axiom 2 and the existence of two acts f, g such that $f \succ_i g$ for all $i \in N'$ imply that there exist non-negative numbers $\lambda_1(f), \cdots, \lambda_n(f)$, not all equal to zero, and a real number $\mu(f)$ such that, for all $g \in K_f$,

$$V_0(g) = \sum_{i=1}^{n} \lambda_i(f) V_i(g) + \mu(f).$$

Now, consider f_1 and f_2 in $\mathcal{A}$. Since $\cap_{i \in N} \mathcal{B}_i \subseteq K_{f_1} \cap K_{f_2}$, for all act $h \in \cap_{i \in N'} \mathcal{B}_i$, we have:

$$\begin{cases} V_0(h) = \sum_{i=1}^{n} \lambda_i(f_1) V_i(h) + \mu(f_1) \\ V_0(h) = \sum_{i=1}^{n} \lambda_i(f_2) V_i(h) + \mu(f_2). \end{cases}$$

This implies that for all $h \in \cap_{i \in N'} \mathcal{B}_i$, $\sum_{i=1}^{n} [\lambda_i(f_1) - \lambda_i(f_2)] u_i(h) + [\mu(f_1) - \mu(f_2)] = 0$. Since by lemma 1, the V_i are affinely independent on $\cap_{i \in N'} \mathcal{B}_i$, $\lambda_i(f_1) = \lambda_i(f_2)$ $i \in N'$ and $\mu(f_1) = \mu(f_2)$. Therefore, there exist n non-negative numbers, not all equal to zero, $(\lambda_1, \cdots, \lambda_n)$ and a number μ, such that for all $f \in \mathcal{A}$,

$$V_0(f) = \sum_{i=1}^{n} \lambda_i V_i(f) + \mu.$$

Finally, it remains to show that the weights $(\lambda_1, \cdots, \lambda_n)$ and μ are unique. Since the $\{\succsim_i\}_{i \in N'}$ satisfy the independent prospects property, there exist for all $i \in N'$ h_i^*, h_{i*} in $\cap_{i \in N'} \bar{\mathcal{A}}_i$ such that

$$\begin{cases} h_i^* \succ_i h_{i*} \\ h_i^* \sim_j h_{i*}, \forall j \in N' \setminus \{i\}. \end{cases}$$

We have $V_0(h_i^*) - V_0(h_{i*}) = \lambda_i (V_i(h_i^*) - V_i(h_{i*}))$ and thus λ_i is unique. This is true for all $i \in N'$. But since $(\lambda_1, \cdots, \lambda_n)$ are unique, so is μ. ■

Lemma 4 *Let $(V_i)_{i \in N}$ be a collection of $\mathcal{B}_i$-affine representation of $\succsim_i$ for all $i \in N$ and assume conditions 1, 2, 3 of Theorem 1 are satisfied. Let the weights $(\lambda_1, \cdots, \lambda_n) \in \mathbb{R}_+^n \setminus \{0\}$, $\mu \in \mathbb{R}$, be such that*

$$\forall f \in \mathcal{A}, V_0(f) = \sum_{i \in N'} \lambda_i V_i(f) + \mu.$$

If there exist $i, j \in N'$ such that $\lambda_i, \lambda_j > 0$, then these two agents have smooth preferences.

Proof. First, remark that for any $i \in N'$ such that $\lambda_i > 0$, for any event E, $k_i^E = k_0^E$. Indeed, since the $\{\succsim_i\}_{i \in N'}$ satisfy the independent prospects property, there exists h^*, h_* in $\cap_{i \in N'} \mathcal{B}_i$ such that

$$\begin{cases} h^* \succ_i h_* \\ h^* \sim_j h_*, \forall j \in N' \setminus \{i\}. \end{cases}$$

We have that

$$\begin{aligned} V_0(h^*_E h_*) + V_0(h_{*E} h^*) - (V_0(h^*) + V_0(h_*)) &= k_0^E (V_0(h^*) - V_0(h_*)) \\ &= k_0^E \lambda_i (V_i(h^*) - V_i(h_*)) \end{aligned}$$

but also

$$\begin{aligned} V_0(h^*_E h_*) + V_0(h_{*E} h^*) - (V_0(h^*) + V_0(h_*)) &= \lambda_i (V_i(h^*_E h_*) + V_i(h_{*E} h^*) - (V_i(h^*) + V_i(h_*))) \\ &= k_i^E \lambda_i (V_i(h^*) - V_i(h_*)) \end{aligned}$$

and thus $k_0^E = k_i^E$.

Suppose now that there exist $i, j \in N'$ such that $\lambda_i, \lambda_j > 0$. Consider h_i^*, h_{i*}, h_j^*, h_{j*} in $\cap_{i \in N'} \mathcal{B}_i$ such that

$$\begin{cases} h_i^* \succ_i h_{i*} \\ h_i^* \sim_h h_{i*}, \forall h \in N' \setminus \{i\} \end{cases}$$

and

$$\begin{cases} h_j^* \succ_j h_{j*} \\ h_j^* \sim_h h_{j*}, \forall h \in N' \setminus \{j\} \end{cases} .$$

Note that for $\alpha = \frac{V_0(h_j^*) - V_0(h_{j*})}{V_0(h_i^*) - V_0(h_{i*}) + V_0(h_j^*) - V_0(h_{j*})} \in [0, 1]$, we have $V_0\left(\alpha h_i^* + (1-\alpha) h_{j*}\right) = V_0\left(\alpha h_{i*} + (1-\alpha) h_j^*\right)$. We also have that $V_i\left(\alpha h_i^* + (1-\alpha) h_{j*}\right) > V_i\left(\alpha h_{i*} + (1-\alpha) h_j^*\right)$ and $V_j\left(\alpha h_i^* + (1-\alpha) h_{j*}\right) < V_j\left(\alpha h_{i*} + (1-\alpha) h_j^*\right)$.

Thus, for an event E,

$$\begin{aligned} V_0 &\left(\left(\alpha h_i^* + (1-\alpha) h_{j*}\right)_E \left(\alpha h_{i*} + (1-\alpha) h_j^*\right)\right) + V_0\left(\left(\alpha h_{i*} + (1-\alpha) h_j^*\right)_E \left(\alpha h_i^* + (1-\alpha) h_{j*}\right)\right) \\ &- \left(V_0(\alpha h_i^* + (1-\alpha) h_{j*}) + V_0(\alpha h_{i*} + (1-\alpha) h_j^*)\right) = 0 \end{aligned}$$

but it must also be the case that

$$\begin{aligned} V_0 &\left(\left(\alpha h_i^* + (1-\alpha) h_{j*}\right)_E \left(\alpha h_{i*} + (1-\alpha) h_j^*\right)\right) + V_0\left(\left(\alpha h_{i*} + (1-\alpha) h_j^*\right)_E \left(\alpha h_i^* + (1-\alpha) h_{j*}\right)\right) \\ &\quad - \left(V_0(\alpha h_i^* + (1-\alpha) h_{j*}) + V_0(\alpha h_{i*} + (1-\alpha) h_j^*)\right) \\ &= \lambda_i k_i^E \left[V_i\left(\alpha h_i^* + (1-\alpha) h_{j*}\right) - V_i\left(\alpha h_{i*} + (1-\alpha) h_j^*\right)\right] \\ &\quad + \lambda_j k_j^E \left[V_j\left(\alpha h_{i*} + (1-\alpha) h_j^*\right) - V_j\left(\alpha h_i^* + (1-\alpha) h_{j*}\right)\right] \\ &= k_0^E \left[\lambda_i \left[V_i(\alpha h_i^* + (1-\alpha) h_{j*}) - V_i(\alpha h_{i*} + (1-\alpha) h_j^*)\right]\right. \\ &\quad \left. + \lambda_j \left[V_j(\alpha h_{i*} + (1-\alpha) h_j^*) - V_j(\alpha h_i^* + (1-\alpha) h_{j*})\right]\right] . \end{aligned}$$

Since

$$\begin{aligned} &[\lambda_i \left[V_i\left(\alpha h_i^* + (1-\alpha) h_{j*}\right) - V_i\left(\alpha h_{i*} + (1-\alpha) h_j^*\right)\right] \\ &\qquad + \lambda_j \left[V_j\left(\alpha h_{i*} + (1-\alpha) h_j^*\right) - V_j\left(\alpha h_i^* + (1-\alpha) h_{j*}\right)\right]] > 0 \end{aligned}$$

we must have $k_0^E = k_i^E = k_j^E = 0$. ■

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