

An extension of Reny's theorem without quasiconcavity.

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Abstract

In a recent but well known paper, Reny has proved the existence of Nash equilibria for compact and quasiconcave games, with possibly discontinuous payoff functions. In this paper, we prove that the quasiconcavity assumption in Reny's theorem can be weakened: roughly, we introduce a measure allowing to localize the lack of quasiconcavity; this allows to refine the analysis of equilibrium existence.

Keywords: Nash equilibrium, existence, discontinuous games, non quasiconcave.

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1 Introduction

The purpose of this paper is to relax the quasiconcavity assumption in the standard Nash equilibrium existence results. Several papers have weakened the continuity assumption of payoff functions (see, for example, Dasgupta and Maskin (1986), Reny (1999) or Topkis (1979)), with various applications, for example to Hotelling's model of price competition or to patent races. Yet, only a few papers have tried to weaken the quasiconcavity assumption, although many games in the economic literature have non quasiconcave payoff functions. Such papers could be classified in several categories, observing the method used to circumvent the non quasiconcavity:

- a first possible method is to relax directly the convexity assumption of the best reply correspondences (see, for example, Friedman and Nishimura (1981) or McClendon (2005)). Unfortunately, the properties assumed on the best reply correspondences are generally not derived from hypotheses on the payoff functions. Thus, such technique may be difficult to use in practice;

- a second method is to use the convexification of preferences when the number of players becomes sufficiently large (see, for example, Starr (1969)). A drawback of this approach is that it depends on the number of players;

- a third approach is to enlarge the definition of a pure Nash equilibrium, for instance by considering mixed-strategy equilibria, or generalized equilibria (see Kostreva (1989));

- last, another answer to the nonconvexity issue is to look at particular classes of games, as supermodular games (see, for example, Topkis (1979)), for which the standard topological fixed point theorems can be avoided, using lattice-theoretical techniques.

In this paper, we propose a new approach to obtain the existence of a (standard) Nash equilibrium in pure strategies, without assumptions on the best reply correspondences or on the number of agents, and we allow non quasiconcavity of payoff functions.

First, for every player i , we introduce a mapping $\rho_i : X \rightarrow \mathbb{R}$ (where X is the product of the pure strategy sets of the agents) which measures the non quasiconcavity of the payoff function of player i , and which is easy to compute for many games.

Then, we exhibit a condition, using the measures ρ_i , which provides the existence of a Nash equilibrium in pure strategies. Moreover, in order to cover the case of discontinuous games, our approach generalizes the main result of Reny (1999). More precisely, our main condition says that for every

non equilibrium strategy profile x^* and every payoff vector u^* resulting from strategies approaching x^* , some player i has a strategy yielding a payoff strictly above $u_i^* + \rho_i(x^*)$ even if the others deviate slightly from x^* . Since for quasiconcave games we obtain $\rho_i(x^*) = 0$ for every player, in this case the last condition is exactly the better-reply security assumption of Reny.

On a technical level, the proof of our main Nash equilibrium existence result is in the spirit of Reny's proof: in particular, it rests on Reny's idea of approximating the discontinuous payoff functions by a sequence of continuous payoff functions. But it differs in the fact that it uses the standard Kakutani's fixed point theorem.

The motivation for the weakening of the quasiconcavity assumption, aside from the fact that many games exhibit non quasiconcavity, could be also a better understanding of the existence or non existence issue of equilibria: up to now, most attention in the literature has been concentrated on the continuity problem, and one of the aim of this paper is to offer a new tool to refine the analysis of equilibrium existence in game theory, in particular to be able to localize the non quasiconcavity issues.

The remainder of this paper is organized as follows: in Section 2, we describe the non quasiconcavity measure and its main properties. In Section 3, we use the idea introduced in Section 2 to define our class of games, strongly better-reply secure games, which strictly contains the class of quasiconcave and better reply-secure games. Then, our main pure strategy equilibrium existence result is stated and proved. In Section 4, the previous results are extended to quasisymmetric games, for which the non quasiconcavity measure can be restricted along the diagonal of payoffs. This permits to extend some standard equilibrium existence results for quasisymmetric games, as Reny's one (1999) or Baye et al's one (1993), to a nonconvex framework.

2 Measure of lack of quasiconcavity

In this section, we define a measure ρ_f of lack of quasiconcavity for every real-valued mapping f defined on a nonempty convex subset of a topological space. The idea we introduce will be used in the next section to measure the lack of quasiconcavity of payoff functions. Roughly, we want to overcome the dichotomy "to be quasiconcave or not to be quasiconcave", by defining a local index of non quasiconcavity.

For every $n \in \mathbb{N}^*$, let Δ^{n-1} be the simplex of \mathbb{R}^n , defined by

$$\Delta^{n-1} = \{(t_1, \dots, t_n) \in \mathbb{R}_+^n, \sum_{i=1}^n t_i = 1\}.$$

Let X be a topological vector space. For every $n \in \mathbb{N}^*$, $t \in \Delta^{n-1}$ and $(x_1, \dots, x_n) \in X^n$, we denote $t \cdot x = \sum_{i=1}^n t_i x_i$. Let Y be a convex subset of X , and consider a mapping $f : Y \rightarrow \mathbb{R}$.

Recall that f is said to be quasiconcave if the following condition is true:

$$\forall n \in \mathbb{N}^*, \forall (t, y) \in \Delta^{n-1} \times Y^n, f(t \cdot y) \geq \min\{f(y_1), \dots, f(y_n)\}.$$

Now, we propose to measure how much the previous inequality can be false at $x \in Y$. For this purpose, we introduce the mapping $\pi_f(x)$ defined as follows:

$$\pi_f(x) = -f(x) + \sup_{n \in \mathbb{N}^*, (t, y) \in \Delta^{n-1} \times Y^n, t \cdot y = x} \min\{f(y_1), \dots, f(y_n)\}$$

Our final measure of lack of quasiconcavity of f is the upper semi-continuous regularization of the previous mapping:

$$\forall x \in Y, \rho_f(x) = \limsup_{x' \rightarrow x} \pi_f(x').$$

Definition 2.1 *The mapping ρ_f defined above is called the measure of lack of quasiconcavity of f*

Remark 2.1 *If f is not bounded, $\rho_f(x)$ may be equal to $+\infty$ for some $x \in Y$.*

Figure 1 gives an example of non quasiconcave mapping and of its measure of lack of quasiconcavity. Now, the following proposition summarizes all the important properties satisfied by ρ_f .

Proposition 2.2 *i) $\forall x \in Y, \rho_f(x) \geq 0$.*

ii) ρ_f is upper semi-continuous.

iii) f is quasiconcave if and only if for every $x \in Y, \rho_f(x) = 0$.

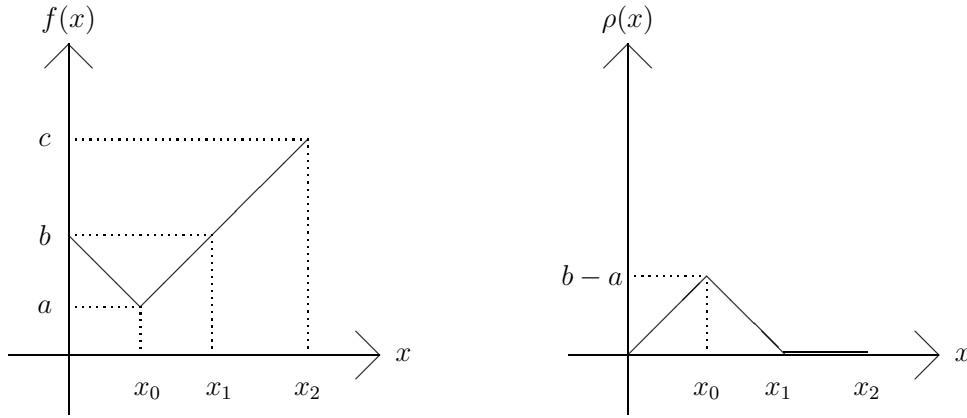


Figure 1: graph of a non-quasiconcave mapping f and of its measure of lack of quasiconcavity.

iv) Let $\varepsilon > 0$ and let I be a closed range of \mathbb{R} . If $f : I \rightarrow \mathbb{R}$ is ε -non-decreasing or ε -non-increasing¹ then for every $x \in I$, $\rho_f(x) \leq \varepsilon$.

v) Let $\varepsilon > 0$. If $f : Y \rightarrow \mathbb{R}$ is ε -quasiconcave² then for every $x \in Y$, $\rho_f(x) \leq \varepsilon$.

vi) Let $\varepsilon > 0$. If $f : Y \rightarrow \mathbb{R}$ is ε -concave³ then for every $x \in Y$, $\rho_f(x) \leq \varepsilon$.

vii) If $\|f\|_\infty = \sup_{x \in Y} |f(x)|$ is finite, then for every $x \in Y$, $\rho_f(x) \leq 2 \cdot \|f\|_\infty$.

viii) If f is continuous, if X is a N -dimensional vector space and if Y is compact, then one has:

$$\forall x \in Y, \rho_f(x) = -f(x) + \max_{t \in \Delta^N, y \in Y^{N+1}, t \cdot y = x} \min\{f(y_1), \dots, f(y_{N+1})\}.$$

ix) If f is continuous and Y is a non-empty compact and convex subset

¹A mapping $f : I \rightarrow \mathbb{R}$ is ε -non-decreasing (resp. ε -non-increasing) if for every $(x, y) \in I^2$, $x \leq y$ implies $f(x) \leq f(y) + \varepsilon$ (resp. for every $(x, y) \in I^2$, $x \leq y$ implies $f(x) \geq f(y) + \varepsilon$).

²A mapping $f : Y \rightarrow \mathbb{R}$ is ε -quasiconcave if for every $n \in \mathbb{N}^*$, $x \in Y^n$ and $t \in \Delta^{n-1}$, one has $f(t \cdot x) \geq \min\{f(x_1), \dots, f(x_n)\} + \varepsilon$.

³A mapping $f : Y \rightarrow \mathbb{R}$ is ε -concave if for every $n \in \mathbb{N}^*$, $x \in Y^n$ and $t \in \Delta^{n-1}$, one has $f(t \cdot x) \geq \sum_{k=1}^n t_k \cdot f(x_k) + \varepsilon$

of \mathbb{R} , then

$$\forall x \in Y, \rho_f(x) = -f(x) + \min\{\max_{y \leq x} f(y), \max_{z \geq x} f(z)\}.$$

x) Let \tilde{f} be the quasiconcave hull of f (see [5], p.33), defined by

$$\tilde{f} = \inf\{h : Y \rightarrow \mathbb{R}, f \leq h, h \text{ quasiconcave}\}.$$

Then, $\pi_f(x) = \tilde{f}(x) - f(x)$.

Remark 2.2 *The last property x) provides another possible definition of ρ_f : it is the upper semi-continuous regularization of the distance between f and its quasiconcave hull. For a practical purpose, it is the definition we shall often use.*

3 The class of strongly better-reply secure games

The aim of this section is to define a class of non quasiconcave games for which a Nash equilibrium exists. First, in the following subsection, we extend the definition of the measure of lack of quasiconcavity to payoff functions.

3.1 Definition of a game and measure of lack of quasiconcavity of payoff functions

Consider a game with N players. The pure strategy set of each player i , denoted by X_i , is a non-empty, compact and convex subset of a topological vector space. Each agent i has a bounded payoff function

$$u_i : X = \prod_{i=1}^N X_i \rightarrow \mathbb{R}.$$

A game G is a couple $G = ((X_i)_{i=1}^N, (u_i)_{i=1}^N)$. Throughout this paper, a game G satisfying the above assumptions will be called a compact game.

For every $x \in X$ and every $i \in \{1, \dots, N\}$, we denote

$$x_{-i} = (x_j)_{j \neq i},$$

and

$$X_{-i} = \prod_{j \neq i} X_j.$$

We say that the game G is quasiconcave if for every player i and every strategy $x_{-i} \in X_{-i}$, the mapping $u_i(\cdot, x_{-i})$, defined on X_i , is quasiconcave.

Recall that $x^* = (x_1^*, \dots, x_N^*) \in X$ is a Nash equilibrium if for every player i , one has

$$\forall x_i \in X_i, u_i(x_i^*, x_{-i}^*) \geq u_i(x_i, x_{-i}^*).$$

For instance, it is well known that for every compact and quasiconcave game, if the payoff functions are continuous then there exists a Nash equilibrium.

To weaken the standard quasiconcavity assumption, we introduce the measure of lack of quasiconcavity of payoff functions as follows, using the previous section: in the following definition, for every player i and every $x = (x_i, x_{-i}) \in X$, $u_i(\cdot, x_{-i})$ denotes the mapping defined from X_i to \mathbb{R} by $u_i(\cdot, x_{-i})(x_i) = u_i(x)$ for every $x_i \in X_i$.

Definition 3.1 *For every $i = 1, \dots, N$ and every $x \in X$, we define the measure $\rho_i : X \rightarrow \mathbb{R}$ of lack of quasiconcavity of player i 's payoff function as follows:*

$$\rho_i(x) = \limsup_{x' \rightarrow x} \pi_{u_i(\cdot, x'_{-i})}(x'_i),$$

where the definition of π is given in Section 2.

Thus, from Statement x) of Proposition 2.2, the measure $\rho_i : X \rightarrow \mathbb{R}$ of lack of quasiconcavity of player i 's payoff function at $x = (x_i, x_{-i})$ is the upper semi-continuous regularization (with respect to the strategy profile x) at x of the distance between the quasiconcave hull \tilde{u}_i of u_i (with respect to the action of player i) and u_i . Clearly, for every $x \in X$, $\rho_i(x) \geq 0$, and a compact game G is quasiconcave if and only if one has $\rho_i = 0$ for every player i . Besides, by definition, ρ_i is upper semi-continuous.

3.2 The class of better-reply secure games

Before defining our class of games, we recall the definition of better-reply secure games. This notion was introduced by Reny (1999), who has proved that every quasiconcave, compact and better-reply secure game has a Nash equilibrium, thus extending most of the previous Nash equilibrium existence results.

Let $u = (u_1, \dots, u_N) \in \mathbb{R}^N$ and let $\Gamma = \overline{\{(x, u(x)), x \in X\}}$.

Definition 3.2 *Player i can secure a payoff strictly above $u_i \in \mathbb{R}$ at $x = (x_i, x_{-i}) \in X$ if there exists $x'_i \in X_i$ and $V_{x_{-i}}$, an open neighborhood of x_{-i} , such that*

$$\forall x'_{-i} \in V_{x_{-i}}, u_i(x'_i, x'_{-i}) > u_i.$$

Notice that Player i can secure a payoff strictly above u_i at $x = (x_i, x_{-i}) \in X$ if and only if

$$\sup_{x'_i \in X_i} \liminf_{x'_{-i} \rightarrow x_{-i}} u_i(x'_i, x'_{-i}) > u_i.$$

Definition 3.3 *A game G is better-reply secure if for every $(x^*, u^*) \in \Gamma$ such that x^* is not a Nash equilibrium, some player i can secure a payoff strictly above u_i^* .*

3.3 The class of strongly better-reply secure games

In this subsection, we define our class of games, call strongly better-reply secure games:

Definition 3.4 *A game G is said to be strongly better-reply secure if for every $(x^*, u^*) \in \Gamma$ such that x^* is not a Nash equilibrium, some player i can secure a payoff strictly above $u_i^* + \rho_i(x^*)$.*

Remark 3.1 *Clearly, our definition strengthens Reny's Definition: every strongly better-reply secure game is better-reply secure. But the class of compact and strongly better-reply secure games strictly generalizes the class of compact, quasiconcave and better-reply secure games, as stated in the following proposition.*

Proposition 3.5 *If a game G is quasiconcave, then it is strongly better-reply secure if and only if it is better reply secure. Moreover, there exists some compact games which are strongly better-reply secure and which are not quasiconcave.*

Proof. The first assertion is clear, since one has $\rho_i = 0$ for every quasiconcave game and every player i . To prove the second assertion, see Example 1 and Example 2 of Section 3 or Example 3 of Section 4, where are defined compact games which are strongly better-reply secure and which are not quasiconcave.

3.4 Existence of Nash equilibria in compact and strongly better-reply secure games

The purpose of this subsection is to prove our main equilibrium existence result:

Theorem 3.2 *If G is a compact and strongly better-reply secure game, then it admits a pure strategy Nash equilibrium.*

The proof is in the spirit of Reny's proof ([8]). Consider G a compact and strongly better-reply secure game. First begin with the following lemma, which is a simple translation of the definition of strong better-reply security:

Lemma 3.3 *A game is strongly better-reply secure if and only if for every $(x^*, u^*) \in \Gamma = \overline{\{(x, u(x)), x \in X\}}$ such that x^* is not a Nash equilibrium, there exists a player i such that*

$$\sup_{x_i \in X_i} \liminf_{x_{-i} \rightarrow x_{-i}^*} u_i(x_i, x_{-i}) > u_i^* + \rho_i(x^*).$$

Following Reny, we denote $\underline{u}_i(x_i, x_{-i})$ for $\liminf_{x_{-i} \rightarrow x_{-i}^*} u_i(x_i, x_{-i}')$.

The following lemma will be needed in the proof of Theorem 3.2. Roughly, it will permit to approximate in a nice way the strategy spaces by finite strategy space.

Lemma 3.4 *If G has no Nash equilibrium, then there exists a finite set $\prod_{i=1}^N X'_i \subset X$ such that for every $(x^*, u^*) \in \Gamma$, there exists $i \in \{1, \dots, N\}$ such that*

$$\sup_{x_i \in X'_i} \underline{u}_i(x_i, x_{-i}^*) > u_i^* + \rho_i(x^*).$$

Proof of Lemma 3.4 Suppose there is no Nash equilibrium. Since the game is strongly better-reply secure, for every $(x^*, u^*) \in \Gamma$ there exists some player i and some strategy $a_i \in X_i$ such that one has

$$\rho_i(x^*) + u_i^* < \underline{u}_i(a_i, x_{-i}^*). \quad (1)$$

Since $\underline{u}_i(x_i, x_{-i})$ is lower semi-continuous with respect to the second variable x_{-i} for every $x_i \in X_i$ (from its definition), and since ρ is upper semi-continuous with respect to x , there exists an open neighborhood $V_{x^*, u^*}(a_i)$ of (x^*, u^*) in Γ such that one has

$$\forall (x, u) \in V_{x^*, u^*}(a_i), \quad \rho_i(x) + u_i < \underline{u}_i(a_i, x_{-i}). \quad (2)$$

Now, recall that Γ is a compact set (because G is a compact game). Hence, there exists a finite covering \mathcal{O} of Γ by some open neighborhoods $V_{x^*(j), u^*(j)}(a(j))$ (where $(x^*(j), u^*(j)) \in \Gamma$ and $a(j) \in \bigcup_{i=1}^N X_i$ for every $j \in I$, I being a finite subset of \mathbb{N}^*). Then, for every player i one can define

$$X'_i = \{a(j), j \in I\} \cap X_i$$

if $\{a(j), j \in I\} \cap X_i$ is non-empty, and X'_i be any element of X_i otherwise. The sets X'_i clearly fulfill the conditions of Lemma 3.4.

Now, we begin the proof of Theorem 3.2. We make a proof by contradiction: suppose there is no Nash equilibrium. In the following, for every subset A of a vector space, $\text{co}A$ denotes the convex hull of A .

First, let $X' = \prod_{i=1}^N X'_i$, where X'_i is defined by Lemma 3.4. Notice that for every $i = 1, \dots, N$ and for every $x_i \in X_i$, the restriction of $\underline{u}_i(x_i, \cdot)$ to the compact metric space $\text{co}X'_{-i}$ is lower semi-continuous. Thus, using an approximation result (see Lemma 3.5 in [8]), we know that for every $i = 1, \dots, N$ and for every $x_i \in X_i$, there exists a sequence of real-valued function $u_i^n(x_i, \cdot)$, continuous on $\text{co}X'_{-i}$, such that :

$$\forall x_{-i} \in \text{co}X'_{-i}, u_i^n(x_i, x_{-i}) \leq \underline{u}_i(x_i, x_{-i}) \quad (3)$$

and such that for every sequence x_{-i}^n converging to x_{-i} in $\text{co}X'_{-i}$ one has

$$\liminf_{n \rightarrow +\infty} u_i^n(x_i, x_{-i}^n) \geq \underline{u}_i(x_i, x_{-i}). \quad (4)$$

For every integer n , consider the correspondence Φ^n from $\text{co}X'$ to $\text{co}X'$, defined for every $x \in \text{co}X'$ by

$$\Phi^n(x) = \text{co}\{x' \in X', \forall a \in X', \forall i = 1, \dots, N, u_i^n(x'_i, x_{-i}) \geq u_i^n(a_i, x_{-i})\}. \quad (5)$$

Let us check that the correspondence Φ^n satisfies the standard properties of Kakutani's theorem:

- 1) It has convex values (from its definition).

2) It has non-empty values: indeed, for every $i = 1, \dots, N$ and every $x \in X$, and since the set X' is finite, there exists $\bar{x}_i \in X'_i$ such that

$$u_i^n(\bar{x}_i, x_{-i}) = \text{Arg max}_{a_i \in X'_i} u_i^n(a_i, x_{-i})$$

and one has $\bar{x} = (\bar{x}_1, \dots, \bar{x}_N) \in \Phi^n(x)$.

3) It has a closed graph, which is an easy consequence of the continuity of u_i^n with respect to the second variable, and the finiteness of X' .

Thus, from Kakutani's Theorem, for every integer n there exists $x^n \in \text{co}X'$ which is a fixed point of Φ^n . It means that there exists an integer K and $x^n(1), \dots, x^n(K)$ in X' such that for every $k = 1, \dots, K$, one has

$$\forall a \in X', \forall i = 1, \dots, N, u_i^n(x_i^n(k), x_{-i}^n) \geq u_i^n(a_i, x_{-i}^n) \quad (6)$$

and such that

$$x^n \in \text{co}\{x^n(1), \dots, x^n(K)\} \quad (7)$$

From Equations 3, Equations 6 and from $\underline{u}_i \leq u_i$, we obtain

$$\forall a \in X', \forall i = 1, \dots, N, \forall k = 1, \dots, K, u_i(x_i^n(k), x_{-i}^n) \geq u_i^n(a_i, x_{-i}^n). \quad (8)$$

It implies that for every a in X' and every $i = 1, \dots, N$, one has

$$\sup_{k \in \mathbb{N}^*, (t, y) \in \Delta^{k-1} \times X_i^k, t \cdot y = x_i^n} \min\{u_i(y_1, x_{-i}^n), \dots, u_i(y_k, x_{-i}^n)\} \geq u_i^n(a_i, x_{-i}^n) \quad (9)$$

or equivalently, subtracting $u_i(x^n)$ to the equation above and using the definition of π ,

$$\pi_{u_i(\cdot, x_{-i}^n)}(x_i^n) \geq u_i^n(a_i, x_{-i}^n) - u_i(x^n). \quad (10)$$

Finally, recalling that $\rho_i(x^n) \geq \pi_{u_i(\cdot, x_{-i}^n)}(x_i^n)$, we obtain

$$\forall a \in X', \forall i = 1, \dots, N, \rho_i(x^n) + u_i(x_i^n, x_{-i}^n) \geq u_i^n(a_i, x_{-i}^n) \quad (11)$$

Without any loss of generality, we can suppose (extracting a subsequence if necessary) that $(x^n, u(x^n))$, which is a sequence of the compact set Γ ,

converges to $(x^*, u^*) \in \Gamma$. Taking the lower limit in Equation 11, from Equation 4, and since ρ is u.s.c., we obtain

$$\forall a \in X', \forall i = 1, \dots, N, \rho_i(x^*) + u_i^* \geq \underline{u}_i(a_i, x_{-i}^*). \quad (12)$$

But this is a contradiction with the choice of X' and with Lemma 3.4: thus the assumption "there is no Nash equilibrium" is absurd.

If G is quasiconcave, we obtain as a corollary:

Corollary 3.5 (*Reny (1999)*)

If G is a compact, quasiconcave and better-reply secure game, then it admits a pure strategy Nash equilibrium.

Two examples illustrate Theorem 3.2:

Example 1. Consider the following game G : there are two players $i = 1, 2$; the strategy sets of each player are $X_1 = [0, V_1]$ and $X_2 = [0, V_2]$, where $V_1 > 0$ and $V_2 > 0$; the payoff functions are defined as follows, where $-i$ denotes 2 if $i = 1$ and 1 if $i = 2$:

$$u_i(x_i, x_{-i}) = -x_i \text{ if } x_i < x_{-i}$$

$$u_i(x_i, x_{-i}) = V_i - x_i \text{ if } x_i \geq x_{-i}.$$

Clearly, G is not quasiconcave (see figure 2) but is compact. To compute the measure of lack of quasiconcavity ρ_i , from Statement x) of Proposition 2.2, we only have to find $\tilde{u}_i(\cdot, x_{-i})$, the envelop of $u_i(\cdot, x_{-i})$ ($x_{-i} \in X_{-i}$ being fixed). Then, $\rho_i(x_i, x_{-i})$ is the upper semi-continuous regularization (with respect to (x_i, x_{-i})) of $\tilde{u}_i(x) - u_i(\cdot, x_{-i})$. See figure 2 and figure 3 for a representation of $u_i(\cdot, x_{-i})$, $\tilde{u}_i(\cdot, x_{-i})$ and $\rho_i(\cdot, x_{-i})$.

Now, to prove that G is strongly better-reply secure, let $(x_1^*, x_2^*, u_1^*, u_2^*) \in \Gamma = \overline{\{(x, u_1(x), u_2(x)), x \in X_1 \times X_2\}}$ such that (x_1^*, x_2^*) is not an equilibrium. Thus, $x_1^* \neq x_2^*$, because for every $a \in [0, \min\{V_1, V_2\}]$, (a, a) is a Nash equilibrium of G . Without any loss of generality, one can suppose that $x_1^* < x_2^*$. Consequently, $\rho_2(x^*) = 0$ and $x_1^* < V_2$. Let $\varepsilon > 0$ such that $x_2^* - \varepsilon > x_1^*$. By playing $x_2 = x_2^* - \varepsilon$, player 2 obtains $V_2 - x_2^* + \varepsilon$. Since $u_2^* + \rho_2(x^*) = V_2 - x_2^*$,

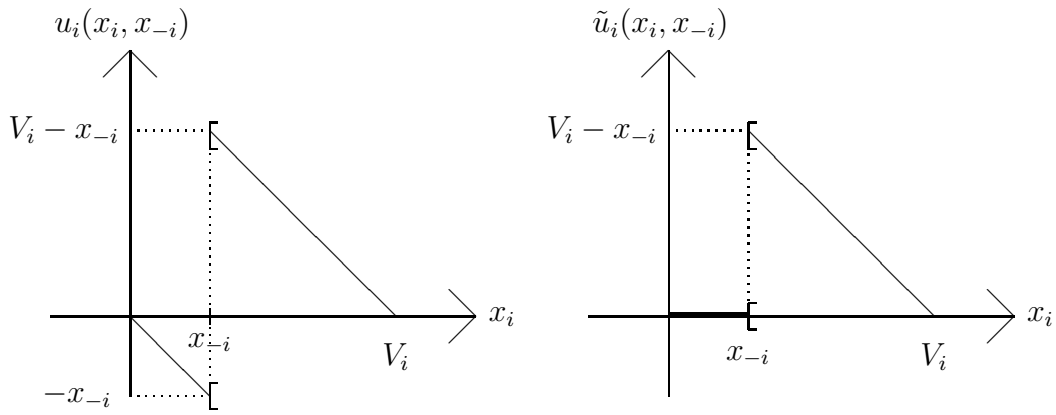


Figure 2: Graph of $u_i(\cdot, x_{-i})$ and $\tilde{u}_i(\cdot, x_{-i})$ in Example 1.

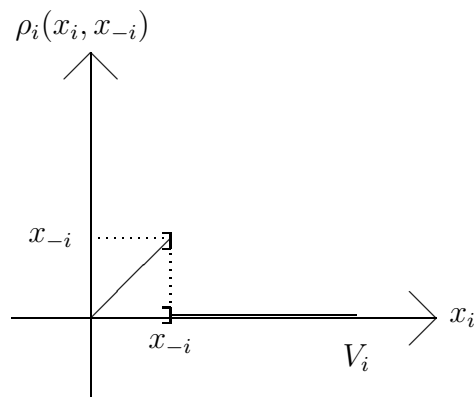


Figure 3: Graph of $\rho_i(\cdot, x_{-i})$ in Example 1.

it proves that player 2 can secure a payoff strictly above $u_2^* + \rho_2(x^*)$ by playing x_2 (because the payoff of player 2 moves continuously when the strategy $x_1^* \neq x_2$ of player 1 is slightly modified).

In this other example, we provide a continuous and compact game which is not quasiconcave, but which is strongly better-reply secure:

Example 2. Consider the following location game G : there are two players $i = 1, 2$; the strategy sets of each player are $X_1 = X_2 = [0, 1]$; the payoff functions are defined as follows, where $-i$ denotes 2 if $i = 1$ and 1 if $i = 2$:

$$u_1(x, y) = - |x - y|$$

$$u_2(x, y) = \left(\frac{1}{2} - x\right) \cdot |x - y|$$

In this game, player 1 would like to choose the same location as player 2, whereas the behaviour of player 2 depends on the location of player 1: he would like to be far from player 1 if $x < \frac{1}{2}$, would like to be close to player 1 if $x > \frac{1}{2}$, and does not care for $x = \frac{1}{2}$.

G is not quasiconcave, because $u_2(x, \cdot)$ is not quasiconcave for $x < \frac{1}{2}$ (see figure 4). More precisely, since $u_1(\cdot, y)$ is quasiconcave for every $y \in X_2$, one has $\rho_1 = 0$, and we now compute ρ_2 to measure the lack of quasiconcavity of this game. Figure 3 represents the graph of $u_2(x, \cdot)$ and of its quasiconcave envelop $\tilde{u}_2(x, \cdot)$ for $x < \frac{1}{2}$. From Statement x) of Proposition 2.2, and since the payoff functions are continuous, one has $\rho_2(x, y) = \tilde{u}_2(x, y) - u_2(x, y)$. Moreover, for $x \leq \frac{1}{2}$, $u_2(x, \cdot)$ is quasiconcave, thus $\rho_2(x, \cdot) = 0$ in this case.

Now, to prove that G is strongly better-reply secure, let $(x^*, y^*, u_1^*, u_2^*) \in \Gamma = \overline{\{(x, y, u_1(x, y), u_2(x, y)), (x, y) \in X_1 \times X_2\}}$ such that (x^*, y^*) is not an equilibrium. First notice that if $x^* \neq y^*$, then player 1, whose payoff function is continuous and quasiconcave with respect to x , can strictly secure a payoff of $u_1^* + \rho_1(x^*) = u_1(x^*) = - |x^* - y^*|$, by playing y^* . Thus, now suppose that $x^* = y^*$. Since (a, a) is an equilibrium for every $a \in [\frac{1}{2}, 1]$, one has $x^* < \frac{1}{2}$. This implies $\rho_2(x^*, x^*) = (\frac{1}{2} - x^*)x^*$. Consequently, player 2 can strictly secure $u_2^* + \rho_2(x^*, x^*) = (\frac{1}{2} - x^*)x^*$ by playing $2x^* + \varepsilon \in]0, 1[$ for $\varepsilon > 0$ small enough: indeed, it gives him a payoff of $(\frac{1}{2} - x^*)(x^* + \varepsilon)$. Thus, G is strongly better-reply secure.

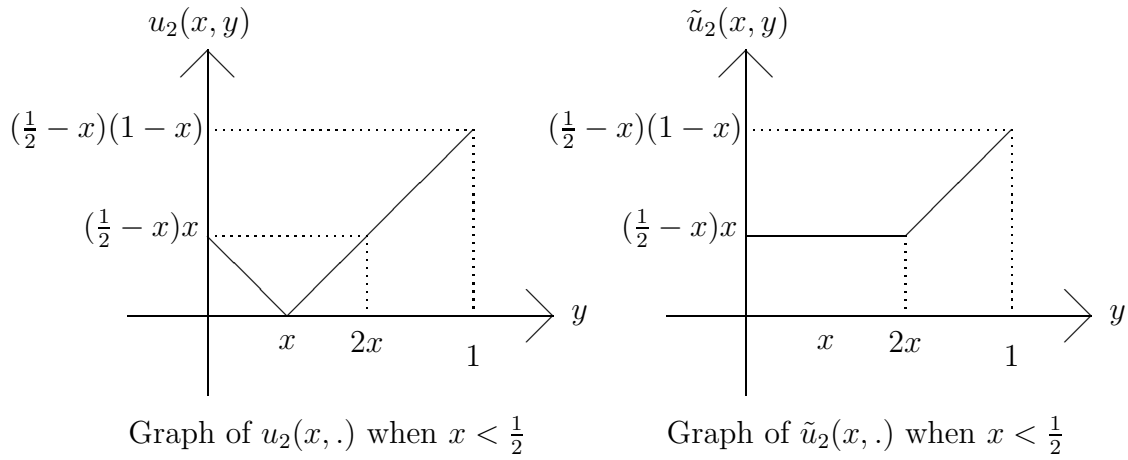


Figure 4: Graph of $u_2(x, \cdot)$ and of $\tilde{u}_2(x, \cdot)$ when $x < \frac{1}{2}$.

4 Symmetric equilibria

In this section, we improve the results of the previous section, by considering the more restricted class of quasisymmetric games.

Recall that a game $G = ((X_i)_{i=1}^N, (u_i)_{i=1}^N)$ is quasisymmetric if $X_1 = X_2 = \dots = X_N$ and if $u_1(x, y, y, \dots, y) = u_2(y, x, y, y, \dots, y) = \dots = u_N(y, y, \dots, y, x)$ for every $x \in X_1$ and for every $y \in X_1$. For $N = 2$, a quasisymmetric game is called a symmetric game. In the following, we let $X = X_1$, and the quasisymmetric game will be denoted $G = (X, (u_i)_{i=1}^N)$. In such games, one can define the diagonal payoff function $v : X \rightarrow \mathbb{R}$ by $v(x) = u_1(x, \dots, x)$ for every $x \in X$.

First of all, we define a measure of non quasiconcavity for quasisymmetric games:

Definition 4.1 *Let $G = (X, (u_i)_{i=1}^N)$ be a quasisymmetric game. For every $x \in X$, we define the measure $\rho : X \rightarrow \mathbb{R}$ of lack of quasiconcavity of G at x as follows:*

$$\rho(x) = \limsup_{x' \rightarrow x} (-v(x') + \sup_{n \in \mathbb{N}^*, (t, y) \in \Delta^{n-1} \times X^n, t \cdot y = x'} \min\{u_1(y_1, x', \dots, x'), \dots, u_n(y_n, x', \dots, x')\})$$

It is worthwhile to note that in the above definition, one can replace player 1 by any player without changing the value of ρ . Moreover, from Statement x) of Proposition 2.2, one can relate the previous measure to the

notion of quasiconcave envelop as follows: for every $x' \in X$, define the quasiconcave envelop of $u_1(\cdot, x', \dots, x')$ with respect to the first variable, denoted $\tilde{u}_1(\cdot, x', x', \dots, x')$. Then, one has

$$\rho(x) = \limsup_{x' \rightarrow x} (\tilde{u}_1(x', x', x', \dots, x') - u_1(x', \dots, x')) \quad (13)$$

Now, recall that G is said to be diagonally quasiconcave (see Reny (1999)) if X is convex and if for each $x' \in X$ and y_1, \dots, y_n in X such that $x' \in \text{co}\{y_1, \dots, y_n\}$, one has

$$-v(x') + \min\{u_1(y_1, x', \dots, x'), \dots, u_1(y_n, x', \dots, x')\} \leq 0.$$

Thus, if X is convex, then G is diagonally quasiconcave if and only if $\rho(x) = 0$ for every $x \in X$.

Following Reny (1999), we say that player i secures a payoff of $\alpha \in \mathbb{R}$ along the diagonal at $(x, x, \dots, x) \in X^N$ if there exists $\bar{x} \in X$ such that $u_i(x', x', \dots, \bar{x}, x', \dots, x') \geq \alpha$ for all x' in some neighborhood of $x \in X$. Remark that if G is quasisymmetric, then player i secures a payoff of $\alpha \in \mathbb{R}$ along the diagonal at $(x, x, \dots, x) \in X^N$ if and only if player j secures a payoff of $\alpha \in \mathbb{R}$ along the diagonal at $(x, x, \dots, x) \in X^N$, for every $j = 1, \dots, N$.

We now adapt Definition 3.4 to quasisymmetric games:

Definition 4.2 *A quasisymmetric game $G = (X, (u_i)_{i=1}^N)$ is strongly diagonally better-reply secure if whenever $(x^*, v^*) \in X \times \mathbb{R}$ is in the closure of the graph of its diagonal payoff function and (x^*, \dots, x^*) is not an equilibrium, some player i can secure a payoff strictly above $v^* + \rho(x^*)$ along the diagonal at (x^*, \dots, x^*) .*

If G is diagonally quasiconcave, we have seen that $\rho = 0$: in this case, the previous definition is exactly the definition of diagonally better-reply secure games, introduced by Reny. Remark also that since G is quasisymmetric, in the definition above, "some player i " can be replaced by "every player i " without altering this definition.

The following theorem is an extension of Theorem 3.2 to quasisymmetric games. The proof, which is similar to the Proof of Theorem 3.2, is given in the appendix.

Theorem 4.1 *If $G = (X, (u_i)_{i=1}^N)$ is quasisymmetric, compact and strongly diagonally better-reply secure, then it admits a symmetric pure Nash equilibrium.*

An immediate corollary is Reny's result:

Corollary 4.2 *If $G = ((X_i)_{i=1}^N, (u_i)_{i=1}^N)$ is quasisymmetric, compact, diagonally quasiconcave and diagonally better-reply secure, then it admits a symmetric pure Nash equilibrium.*

We now give an example of quasisymmetric, compact, strongly diagonally better-reply secure game which is not diagonally quasiconcave:

Example 3 In their paper, Baye et al. (1993) introduce the following game G : two duopolists with zero costs set prices $(p_1, p_2) \in [0, T] \times [0, T]$, where $T > 0$. For $i = 1, 2$, the payoff functions are

$$\bar{u}_i(p_i, p_{-i}) = p_i \quad \text{if } p_i \leq p_{-i},$$

and

$$\bar{u}_i(p_i, p_{-i}) = p_i - c \quad \text{otherwise,}$$

where $0 < c < T$, and where $-i$ denotes 1 if $i = 2$ and 2 if $i = 1$. The interpretation is that each firm pays brand-loyal consumers a penalty of c if the other firm beats its price. It is easy to prove that this game is symmetric, compact, diagonally quasiconcave and diagonally better-reply secure. Thus, one could apply Corollary 4.2, or also the main result of [1], in order to obtain the existence of a pure Nash equilibrium.

Now, let $\varepsilon \in]0, \min\{c, T - c\}[$, and consider the following modification G_ε of the previous game: suppose that the penalty of firm i is reinforced for $p_{-i} \leq T - c$: in this case, firm i pays brand-loyal consumers a penalty of c if $p_{-i} < p_i + \varepsilon$, and nothing otherwise. Thus, firm i may have to pay the penalty even if p_{-i} is larger than p_i

Consequently, one can write the modified payoff function:

$$u_i(p_i, p_{-i}) = p_i \quad \text{if } p_i \leq p_{-i} - \varepsilon(\mathbf{1}_{p_{-i} \leq T - c}(p_{-i})),$$

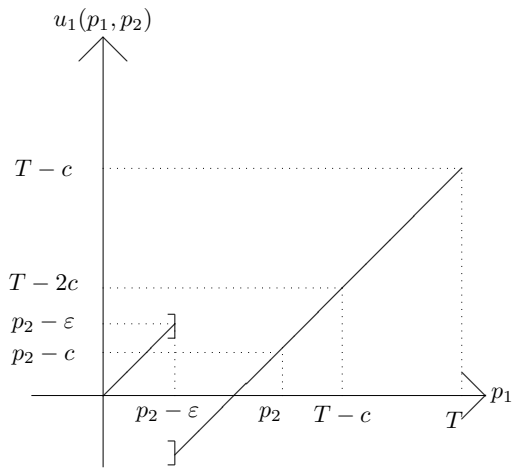
and

$$u_i(p_i, p_{-i}) = p_i - c \quad \text{otherwise,}$$

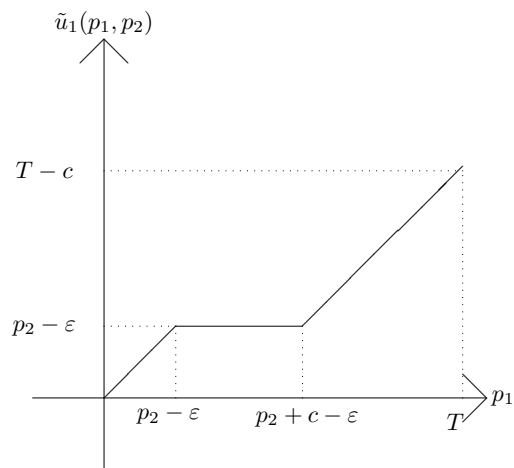
where $\mathbf{1}_{p_{-i} \leq T - c}(p_{-i}) = 1$ if $p_{-i} \leq T - c$ and $\mathbf{1}_{p_{-i} \leq T - c}(p_{-i}) = 0$ otherwise.

Remark that for $\varepsilon = 0$, one has $G_\varepsilon = G$. Besides, clearly, G_ε is symmetric and compact. We now prove that it is strongly diagonally better-reply secure.

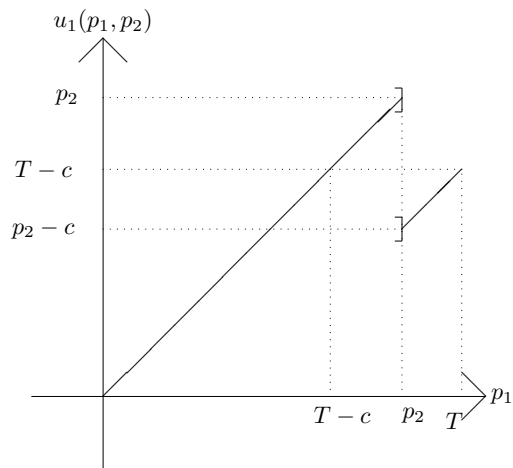
First, we compute ρ , the measure of non quasi-concavity of G_ε . For this purpose, recall that for every $p_2 \in [0, T]$, $\tilde{u}_1(\cdot, p_2)$ denotes the quasiconcave



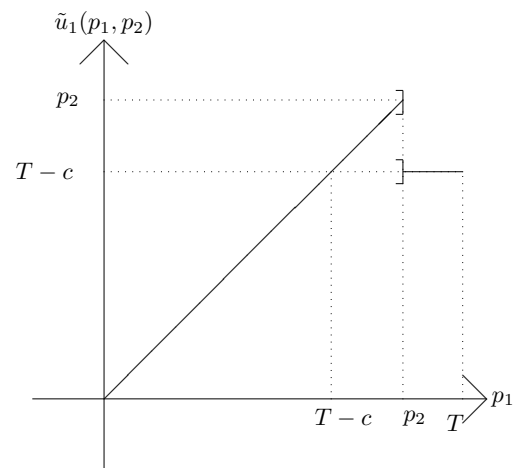
Graph of $u_1(\cdot, p_2)$ when $\varepsilon \leq p_2 \leq T - c$.



Graph of the quasi-concave envelope of $u_1(\cdot, p_2)$ when $\varepsilon \leq p_2 \leq T - c$



Graph of $u_1(\cdot, p_2)$ when $p_2 > T - c$.



Graph of of the quasi-concave envelope of $u_1(\cdot, p_2)$ when $p_2 > T - c$.

Figure 5: graph of $u_1(\cdot, p_2)$ and $\tilde{u}_1(\cdot, p_2)$ in Example 3

envelop of the mapping $u_1(\cdot, p_2)$, defined in Section 2. Now, consider the three following cases:

i) Suppose $p_2 < \varepsilon$ (which implies $p_2 < T - c$). In this case, one has $u_1(p_1, p_2) = p_1 - c$ for every $p_1 \in [0, T]$. Thus, the mapping $u_1(\cdot, p_2)$ is quasiconcave, and $\tilde{u}_1(p_1, p_2) = p_1 - c$ for every $p_1 \in [0, T]$.

ii) Suppose $p_2 \in [\varepsilon, T - c]$. One has $u_1(p_1, p_2) = p_1$ if $p_1 \leq p_2 - \varepsilon$ and $u_1(p_1, p_2) = p_1 - c$ if $p_1 > p_2 - \varepsilon$. Thus (see figure 5), one has $\tilde{u}_1(p_1, p_2) = p_1$ if $p_1 \leq p_2 - \varepsilon$, $\tilde{u}_1(p_1, p_2) = p_2 - \varepsilon$ if $p_1 \in [p_2 - \varepsilon, p_2 + c - \varepsilon]$ and $\tilde{u}_1(p_1, p_2) = p_1 - c$ if $p_1 > p_2 + c - \varepsilon$.

iii) Last, suppose $p_2 > T - c$. One has $u_1(p_1, p_2) = p_1$ if $p_1 \leq p_2$, and $u_1(p_1, p_2) = p_2 - c$ if $p_1 > p_2$. Thus (see figure 5), one has $\tilde{u}_1(p_1, p_2) = p$ if $p_1 \leq p_2$, and $\tilde{u}_1(p_1, p_2) = T - c$ if $p_1 > p_2$.

From the three previous cases, one has $\tilde{u}_1(p, p) - u_1(p, p) = 0$ for every $p < \varepsilon$, $\tilde{u}_1(p, p) - u_1(p, p) = c - \varepsilon$ for every $p \in [\varepsilon, T - c]$ and $\tilde{u}_1(p, p) - u_1(p, p) = 0$ for every $p > T - c$. Thus, from Equation 13, one obtains $\rho(p) = 0$ for every $p < \varepsilon$, $\rho(p) = c - \varepsilon$ for every $p \in [\varepsilon, T - c]$ and $\rho(p) = 0$ for every $p > T - c$.

Eventually, to prove that G_ε is strongly diagonally better-reply secure, consider (p^*, v^*) in the closure of the graph of its diagonal payoff function, such that (p^*, p^*) is not an equilibrium. Thus $p^* \leq T - c$, because for every $p > T - c$, (p, p) is an equilibrium (see figure 5). Now, if $p^* < \varepsilon$ then consumer 1 can strictly secure $v^* + \rho(p^*) = v^* = p^* - c$ by playing strictly above p^* . On the other hand, if $p^* \in [\varepsilon, T - c]$, then consumer 1 can strictly secure $v^* + \rho(p^*) = p^* - c + c - \varepsilon = p^* - \varepsilon$ by playing $p^* + c$.

5 Appendix

Proof of Proposition 2.2. To prove i), notice that for every $x \in Y$, $\pi_f(x) \geq 0$, since one can take $y = (x, x, \dots, x)$ in the supremum of the definition of π_f . For ii), just recall that $\rho_f(x)$ is the upper limit of π_f at x . For Statement iii), just notice that f is quasiconcave if and only if for every $x \in Y$, $\pi_f(x) = 0$. To prove iv), let $x \in Y$, $n \in \mathbb{N}^*$, $(t, y) \in \Delta^{n-1} \times Y^n$ such that $t \cdot y = x$. Let $y_i = \min\{y_1, \dots, y_n\}$ for some $i \in \{1, \dots, n\}$. From $t \cdot y = x$, one has $y_i \leq x$. Thus, if f is ε -non-increasing, one has $f(y_i) \leq f(x) + \varepsilon$. Consequently, $\min\{f(y_1), \dots, f(y_n)\} \leq f(y_i) \leq f(x) + \varepsilon$, which implies $\pi_f(x) \leq \varepsilon$. Since this last inequality is true for every $x \in Y$, one has $\rho_f(x) \leq \varepsilon$ for every $x \in Y$. The case where f is ε -non-decreasing is similar. The proof of v) is similar to the proof of iv), and vi) is a consequence of v), because ε -quasiconcavity implies

ε -concavity. Statement vii) is an immediate consequence of the definition of π_f . Now we prove Statement viii). Let $x \in Y$. First, notice that from Caratheodory's theorem, every convex hull of $\{y_1, \dots, y_n\}$, a finite subset of $Y \subset X$, can be written as the convex hull of $N + 1$ elements of $\{y_1, \dots, y_n\}$. So, the integer n in the definition of π_f can be replaced by $N + 1$. Second, define for every $y = (y_1, \dots, y_{N+1}) \in Y^{N+1}$ and every $x \in Y$

$$\phi(y) = \min\{f(y_1), \dots, f(y_{N+1})\}$$

and

$$\Phi(x) = \{(t, y) \in \Delta^N \times Y^{N+1}, t \cdot y = x\}.$$

Clearly, ϕ is a continuous mapping and Φ is a continuous correspondence (i.e. lower-semi-continuous and upper-semi-continuous) with non-empty values. Thus, from Berge's maximum theorem, $\sup_{(t,y) \in \Phi(x)} \phi(y)$ is a continuous mapping with respect to x . This proves that π_f is continuous, so that $\rho_f = \pi_f$, which proves Statement viii). To prove Statement ix), first notice that from Statement viii), one has to prove that

$$\min\{\max_{y \leq x} f(y), \max_{z \geq x} f(z)\} = \max_{t \in [0,1], (y,z) \in Y^2, ty+(1-t)z=x} \min\{f(y), f(z)\}$$

Clearly, one has

$$\min\{\max_{y \leq x} f(y), \max_{z \geq x} f(z)\} \geq \max_{t \in [0,1], (y,z) \in Y^2, ty+(1-t)z=x} \min\{f(y), f(z)\}.$$

To prove the converse inequality, we notice that there exists some $y_0 \in Y$ with $y_0 \leq x$ (for example) such that $\min\{\max_{y \leq x} f(y), \max_{z \geq x} f(z)\} = f(y_0)$, where $\max_{y \geq x} f(y) = f(z_0) \geq f(y_0)$ for some $z_0 \geq x$ and $z_0 \in Y$. But there exists $t \in [0, 1]$ such that $x = ty_0 + (1 - t)z_0$. Thus,

$$\max_{t \in [0,1], (y,z) \in Y^2, ty+(1-t)z=x} \min\{f(y), f(z)\} \geq \min\{f(y_0), f(z_0)\},$$

the last term being equal to $f(y_0) = \min\{\max_{y \leq x} f(y), \max_{z \geq x} f(z)\}$, which ends the proof of Statement ix).

To finish, one has to prove Statement x), which is equivalent to prove that the mapping $g : Y \rightarrow \mathbb{R}$ defined by

$$\forall y \in Y, g(y) = \sup_{n \in \mathbb{N}^*, (t,y) \in \Delta^{n-1} \times Y^n, t \cdot y = x} \min\{f(y_1), \dots, f(y_n)\}$$

is equal to \tilde{f} . We let the reader check that g is the smallest quasiconcave mapping above f , which ends the proof of Statement x).

Proof of Theorem 4.1

For every $(x, y) \in X \times X$, let

$$\underline{u}(x, y) = \liminf_{y' \rightarrow y} u_1(x, y', y', \dots, y').$$

For every $x \in X$, it is clearly a lower semicontinuous mapping with respect to y . Besides, player 1 can secure a payoff strictly above $v^* + \rho(x^*)$ along the diagonal at (x^*, \dots, x^*) if and only $\sup_{x \in X} \underline{u}(x, x^*) > v^* + \rho(x^*)$. Now, define

$$\Gamma = \overline{\{(x, v(x)), x \in X\}}.$$

Suppose G is quasisymmetric, compact, diagonally better-reply secure and suppose that there is no Nash equilibrium. Following Lemma 3.4, there exists a finite set $X' \subset X$ such that for every $(x^*, v^*) \in \Gamma$, one has

$$\sup_{x \in X'} \underline{u}(x, x^*) > v^* + \rho(x^*).$$

Then, from Lemma 3.5 in [8], for every $x \in X$, there exists a sequence of real-valued function $u^n(x, \cdot)$, continuous on $\text{co}X'$, such that :

$$\forall x' \in \text{co}X', u^n(x, x') \leq \underline{u}(x, x') \tag{14}$$

and such that for every sequence x'^n converging to x' in $\text{co}X'$ one has

$$\liminf_{n \rightarrow +\infty} u^n(x, x'^n) \geq \underline{u}(x, x') \tag{15}$$

For every integer n , consider the correspondence Φ^n from $\text{co}X'$ to $\text{co}X'$, defined for every $x \in \text{co}X'$ by

$$\Phi^n(x) = \text{co}\{x' \in X', \forall a \in X', u^n(x', x) \geq u^n(a, x)\} \tag{16}$$

The correspondence Φ^n satisfies the standard properties of Kakutani's theorem.

Thus, from Kakutani's Theorem, for every integer n there exists $x^n \in \text{co}X'$ which is a fixed point of Φ^n . It means that there exists an integer K (we can suppose it does not depend on n because X' is finite) and $x'^n(1), \dots, x'^n(K)$ in X' such that for every $k = 1, \dots, K$, one has

$$\forall a \in X', \forall k = 1, \dots, K, u^n(x^n(k), x^n) \geq u^n(a, x^n) \quad (17)$$

and such that

$$x^n \in co\{x^n(1), \dots, x^n(K)\} \quad (18)$$

From Equations 14, Equations 17 and from $\underline{u} \leq u$, we obtain

$$\forall a \in X', \forall k = 1, \dots, K, u(x^n(k), x^n) \geq u^n(a, x^n) \quad (19)$$

So, using the definition of ρ , we have:

$$\forall a \in X', \forall i = 1, \dots, N, \rho(x^n) + v(x^n) \geq u^n(a, x^n) \quad (20)$$

Without any loss of generality, we can suppose (extracting a subsequence if necessary) that $(x^n, v(x^n))$, which is a sequence of the compact set Γ , converges to $(x^*, v^*) \in \Gamma$. Taking the lower limit in Equation (20), from Equation 15, and since ρ is u.s.c., we obtain

$$\forall a \in X', \rho(x^*) + v^* \geq \underline{u}(a, x^*). \quad (21)$$

But this is a contradiction with the definition of X' . Thus the assumption that there is no Nash equilibrium is absurd.

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