On Intersectoral Asymmetries in Factors Substitutability, «Equilibrium Production Possibility Frontiers» & the Emergence of Indeterminacies

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Abstract†

The existence of asymmetries in factors substitutability between the distinct sectors of a given economy will directly rule the influence that spillover effects have upon its determinacy properties. For leading intersectoral spillover effects, the substitutability of the capital good industry together with a potential relative profit shares reversal — itself conditional to the existence of *asymmetries* between the intrasectoral and intersectoral spillover effects of at least one sector — between the private and the equilibrium level will, *e.g.*, be at the core of the area for local indeterminacies. This proceeds from external dimensions which do not modify the constant returns to scale hypothesis that is retained at the decentralised level of the firm as they directly relate to equilibrium factors costs and outputs prices. The generality of the current approach and the genericity of the associated production set enlighten the role of the irregularities that prevail across the substitutability properties of the various sectors of a given economy but also, in the same vein, of the occurrence of heterogeneities between the intrasectoral and intersectoral spillovers emanating from a given industry, this gap being in turn weighted by the substitutability properties of this industry. It is shown that these multiplicity conclusions directly result from unusual properties of the Equilibrium Production Possibility Frontier that formulate as the occurrence of an equilibrium complementarity between the two outputs.

Keywords: «Equilibrium Production Possibility Frontiers» – Intersectoral Asymmetries in Factors Substitutability and between *Price related* Intrasectoral and Intersectoral Spillovers – Irrelevance of returns to scale for local or global indeterminacies – Equilibrium complementarities between the outputs in a world of heterogeneous goods.

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I – Introduction

The existence of asymmetries in factors substitutability between the distinct sectors of a given economy will be argued to directly rule the influence that intrasectoral or intersectoral spillover effects will have upon its determinacy properties. Somewhat paradoxically, when *both* production technologies tend to exhibit complementarity between the two inputs, and for any equilibrium configuration of intrasectoral and intersectoral spillovers, the economy is characterised by a unique locally determinate steady state. When it happens to be associated with a relative profit shares reversal between the private and the equilibrium levels — the production of the capital industry decreases with the aggregate capital input while the relative rental rate of the capital input increases with the relative price of the capital output, that can only occur when some discrepancies take place between the intrasectoral and intersectoral spillovers that emanate from at least one sector — and for predominant intersectoral spillover effects, arbitrarily low orders for the elasticity of substitution of the capital good industry together with larger substitutability measures in the consumption good industry underlie local indeterminacy. The basic argument proceeds from external spillover dimensions which do not modify the constant returns to scale hypothesis that is retained at the decentralised level of the firm as they directly relate to equilibrium factor costs and outputs prices. More fundamentally and entirely new to the present contribution, the whole argument is established through the introduction of an «equilibrium production possibility frontier» whose features can be characterised at the same level of generality as the canonical formulation of the production possibility frontier, *i.e.*, *aside from any particular parametric formulation*. The generality of the current approach and the genericity of the associated production set enlighten the role of the irregularities that prevail accross the substitutability properties in the productions of the various goods of a given decentralised economy but also of the heterogeneities between the intrasectoral and intersectoral spillovers emanating from a given industry, their influence being in its turn weighted by the associated sectoral elasticity of substitution. Both of these dimensions were out of order in the convex form of the multi-sectoral environment but also happened to having been omitted from the earlier literature on local indeterminacies.

The role of factors substitutability in the emergence of local indeterminacies within competitive economies has recently received a large attention with the contributions of Barinci [1], Cazzavillan [13], Cazzavillan, Pintus & Lloyd-Braga [14], Grandmont, Pintus & de Vilder [19]. Both Barinci [1] and Grandmont, Pintus & de Vilder [14] stress the possibility of locally indeterminate stationary equilibria under complementary technologies. Cazzavillan [13] and Cazzavillan, Pintus & Lloyd-Braga [14] circumvent the undesirable link between complementarity and multipliticity by contemplating an economy exhibiting increasing returns to scale. In particular, Cazzavillan, Pintus & Lloyd-Braga [14] put forth a potentially nonlinear dimension for factors substitutability in indeterminacies: depending upon the degree of increasing returns, both higly complementary and highly substitutable factors could bend the economy towards the multiplicity result. A strong limit of their argument if however inherent to their homogeneous good assumption, that implies a one to one trade-off between consumption and investment: though this admittedly gives v a first convenient benchmark for the link between factors substitutability and indeterminacy, there is no doubt that a more satisfactory argument ought to make an explicit account of the heterogenous determinants of substitutability mechanisms in actually decentralised economies.

In parallel to this, the early literature concerned with the occurrence of local indeterminacies has maintained a constant returns to scale hypothesis at the private level of sectoral production technologies but allowed for increasing returns to scale at the equilibrium level, benchmark studies in this vein being due to Benhabib & Farmer [2, 4], Boldrin & Rustichini [11], Cazzavillan, Pintus & Lloyd-Braga [14] or Venditti [29]. This increasing returns component having been criticised on an empirical data basis — this is lengthly documented by Benhabib & Farmer [3] —, a second research program has, through the contributions of Benhabib & Nishimura [8, 9] or Benhabib, Nishimura & Venditti [10], building upon a two-goods Cobb-Douglas structure and thus a more satisfactory formulation for the production possibility frontier and the trade-off between consumption and investment, retained a decreasing returns to scale hypothesis — and thence profits — at the private level but constant, decreasing or arbitrarily low increasing returns to scale at the equilibrium level. Nishimura & Venditti [22, 23] have parallely developed continuous time and discrete time C.E.S versions of this argument. A shortcoming of all of these approaches springs from, as a direct byproduct of their exclusive focus on parametric formulations, their inability in assessing, being thus in that perspective not as satisfactory as the aggregate arguments of Barinci [1], Cazzavillan [13], Cazzavillan, Pintus & Lloyd-Braga [14] or Grandmont, Pintus & de Vilder [19], the theoretical role of factors substituability in the possibility of multiple equilibria.[1]

Building upon the characterisation of substitutability mechanisms in multi-sectoral economies introduced by Drugeon [15, 18], the current article completes the first unrestricted examination of the potentially asymmetric role of sectoral factors substitutability in the determinacy properties of suboptimal competitive economies. Further, in order to avoid lines of criticisms pertaining to the role of the departure from the constant returns to scale hypothesis — be it at the private or at the equilibrium level — in the occurrence of indeterminacies and in opposition to the previous literature, *the current argument neither departs from constant returns to scale at the equilibrium level nor at the private level: this fundamentally proceeds from the consideration of intrasectoral or intersectoral spillover effects which directly relate to equilibrium levels of factors and output prices*. Finally, the argument is completed for *generic formulations* and a *non-trivial first-order role for asymmetries in substitution mechanisms* is detected in the indeterminacy issue. On methodological grounds, the argument is established through the introduction of an «equilibrium production possibility frontier» — henceforwards E.P.P.F. —, *i.e.*, a P.P.F. characterised *along* the intertemporal competitive equilibrium with

[1] As this will be mentioned in Section III through Remark 4 — a more detailed argument was completed in Drugeon [17] — and in strong contradisctinction with the current contribution, the range of assumptions used by Benhabib & Nishimura [8, 9] or Benhabib, Nishimura & Venditti [10] or Nishimura & Venditti [22, 23] has as a central corollary that the potential for local indeterminacy results, be it for Cobb-Douglas or C.E.S. technologies, rests upon a comparison between private and external factors shares, factors substitutability being then inessential.

externalities — henceforwards C.E.E. — that allows the analysis for attaining a level of generality and theoretical understanding of the mechanics of a suboptimal two-goods world which was absent from earlier studies, noticeably in the regards of the characterisation of the heterogenous components of factors substitutability within multi-sectoral economies. Such a local focus — along a competitive equilibrium with externalities — will be proved to bring anew the possibility of a general understanding of the theoretical problem under examination, *i.e.*, an integrated view between local or global indeterminacy and factors substitutability with a heterogenous goods technological set.

It is first established that the departure from the classical constant returns to scale hypothesis is of no avail for the obtention of local or global indeterminacies as soon as a multi-sectoral environment is explicitly considered — it can be proved that a homogeneous good setting would not have been appropriate in that perspective. As a first illustration, multiple steady states can emerge in any sectoral configuration as soon as both production tecnologies do not tend to exhibit complementarity between the inputs, the examination of a parameterised example further examplifying how multiple steady states may directly result from the existence of intersectoral spillover effects stemming from the consumption good industry. When *both* production technologies tend to exhibit complementarity between the two inputs, and for any equilibrium configuration between intrasectoral and intersectoral spillovers, the economy is characterised by a unique locally determinate steady state. As soon as positive factors substitutability is allowed in at least one sector, the local determinacy properties result from a permanent interplay between the asymmetries in factors substitutability *across* the sectors and the discrepancies between the intrasectoral and the intersectoral spillovers that emanate from a given sector. As an illustration, for dominating intersectoral spillover effects, the substitutability of the capital good industry will be at the core of the area for local indeterminacy. When it happens to be associated with a relative profit shares reversal between the private and the equilibrium levels — the production of the capital industry decreases with the aggregate capital input while the relative rental rate of the capital input increases with the relative price of the capital output, that can only occur for asymmetric intrasectoral and intersectoral spillovers in at least one industry —, arbitrarily low orders for the elasticity of substitution of the capital good industry together with, for small intersectoral spillovers in the production of the consumption good, larger substitutability measures in the consumption good industry underlie local indeterminacy.
A remarkable outcome of the consideration of a suboptimal competitive equilibrium with externalities and of the associated E.P.P.F. results from the features of the latter. An eventual class of wonderings attempts at understanding these multiplicity results in light of its global properties. This proceeds along two distinct roads. First, the scope for an equilibrium convexity the E.P.P.F., in contradiction with the standard concavity of the P.P.F.,though present, does not reveal to be intimately related to the lost of uniqueness, be it from a local or from a global standpoint. Second, a related guess has then to do with the implications of a variation in the competitive equilibrium with externalities relative price of the inputs on the relative demand of these inputs, *i.e.*, the value of the *aggregate* elasticity of substitution between capital and labour in the course of the competitive equilibrium with externalities, an

interesting point in that perspective being that, due to the assumed multiplicative separability assumption between sectoral technologies and the spillovers components of the production set, the canonical sectoral elasticities of substitution still provides an accurate description of substitution mechanisms at the sectoral level. Such a line of argument does however not extend to the aggregate level where it is proved that it is the scope for an E.P.P.F. characterised by an increased role for factors substitutability with respect to its benchmark, best understood as the potential for complementary outputs — this relates to the equilibrium atypical implications of a modification in the relative price of the capital good on the relative levels of the two outputs — that lies at the very core of the scope for local multiplicities.

A new type of constant returns to scale technological set is introduced in Section II and embedded within a competitive setup. The E.P.P.F. and the associated competitive equilibrium with externalities are introduced in Section III. The role of asymmetries in factors substitutabilities between the different industries together with the role of equilibrium complementarities between the outputs are shown in Section IV to directly underlie the scope for multiplicities. Section V finally undertakes a comparison of the current line of argument with the more standard Benhabib & Nishimura's «sectors specific» argument and illustrates the usefulness of the E.P.P.F. for the appraisal of this alternative setting, it also suggests some possible extensions of the current line of argument. The main proofs are gathered in a final appendix.

II – A Multi-Sectoral Competitive Equilibrium with Externalities: The Constant Returns to Scale Hypothesis

II.1 – A Canonical Approach

II.1.1 – A Representative Consumer

Time is discrete. The description of the preferences of the representative consumer amounts to the introduction of an intertemporal utility functional $\mathscr{V}(\cdot)$ defined over a consumption sequence ${}_0\mathscr{C} = \{c_t\}_{t=0}^{\infty}$ that assigns c_t to any $t = 0, 1, 2\ldots$:

$$(1) \qquad \mathscr{V}\left({}_0\mathscr{C}\right) = \sum_{t=0}^{+\infty} \delta^t v(c_t), \delta \in]0, 1],$$

for $v(\cdot)$ an *instantaneous* utility function such that:

Assumption P.1: $v(\cdot)$ is a continuous, concave and increasing map from $\mathbb{R}_+$ into $\mathbb{R}$. Further $v|_{\mathbb{R}_+^*} \in C^2(\mathbb{R}_+^*, \mathbb{R})$ and $\lim_{c\to 0}\partial v(c)/\partial c = +\infty$ and $\lim_{c\to\infty}\partial v(c)/\partial c = 0$.

At each time $t \geq 0$, the representative consumer receives capital and labour income. The capital incomes states as $\omega_t^1 X_{1,t}$, for ω_t^1 the rental payment in units of consumption earned at t by renting one unit of capital at the beginning of time t to the production sector and $X_{1,t}$

capital holdings at date $t = 0, 1, \ldots$ The labour income of the representative household at time t is denoted by $\omega_t^0 X_0$, for ω_t^0 real wage rate measured in units of consumption and $X_{0,t} = X_0$ his invariant labour supply. The household budget constraint then formulates along:

$$(2) \qquad c_t + q_t\big[X_{1,t+1} - (1-\eta)X_{1,t}\big] = \omega_t^1 X_{1,t} + \omega_t^0 X_0,$$

for c_t that denotes consumption at time t, q_t the current price of capital in units of consumption at time t and $\mu \in]0,1]$ the depreciation rate of the capital stock. The household perfectly anticipates the sequence of factor returns $\{\omega_t^0, \omega_t^1\}$ and the sequence of capital goods prices $\{q_t\}$. Given an expected sequence of factors returns, the representative household is to solve the problem of maximising $\mathscr{V}({}_0\mathscr{C})$ subject to (2). The two necessary and sufficient conditions for an optimal solution list as:

$$(3a) \qquad \frac{\partial u}{\partial c}(c_t) - \delta \frac{\partial u}{\partial c}(c_{t+1}) \frac{\omega_{t+1}^1 + (1-\eta)q_{t+1}}{q_t} = 0,$$

$$(3b) \qquad \lim_{t \to +\infty} \delta^t \frac{\partial u}{\partial c}(c_t) q_t X_{1,t} = 0.$$

II.1.2 – A Technological Set

There are two sectors $j = 0, 1$ in the economy At date $t \geq 0$, the first produces a pure consumption good in amount Y_t^0 whereas the second produces a pure capital good in amount Y_t^1. Any of the sectors uses labour and capital as inputs and the outputs of the consumption and investment good sectors respectively satisfy:

$$\begin{aligned}
(4a) \qquad Y_t^0 &\leq \mathscr{F}^0\big(X_{00,t}, X_{10,t}; X_{00,t}^{\mathscr{E}}, X_{10,t}^{\mathscr{E}}, X_{01,t}^{\mathscr{E}}, X_{11,t}^{\mathscr{E}}\big) \\
&= F^0\big(X_{00,t}, X_{10,t}\big) G^{00}\big(X_{00,t}^{\mathscr{E}}, X_{10,t}^{\mathscr{E}}\big) G^{10}\big(X_{01,t}^{\mathscr{E}}, X_{11,t}^{\mathscr{E}}\big), \\
(4b) \qquad Y_t^1 &\leq \mathscr{F}^1\big(X_{01,t}, X_{11,t}; X_{00,t}^{\mathscr{E}}, X_{10,t}^{\mathscr{E}}, X_{01,t}^{\mathscr{E}}, X_{11,t}^{\mathscr{E}}\big) \\
&= F^1\big(X_{01,t}, X_{11,t}\big) G^{01}\big(X_{00,t}^{\mathscr{E}}, X_{10,t}^{\mathscr{E}}\big) G^{11}\big(X_{0j}^{\mathscr{E}}, X_{1j}^{\mathscr{E}}\big),
\end{aligned}$$

where $X_{ij,t}$ denotes the amount of input $i, i = 0, 1$, employed in sector j, $j = 0, 1$ at date $t = 0, 1, \ldots$, for $F^j(\cdot,\cdot)$ that exhibits standard linear homogeneity and concavity properties whereas $G^{jj}(\cdot,\cdot)$ and $G^{jj'}(\cdot,\cdot)$, $j, j' = 0, 1$, $j' \neq j$ respectively denote functions that build from *intrasectoral* and *intersectoral* externalities ; *e.g.*, $G^{00}(\cdot,\cdot)$ and $G^{01}(\cdot,\cdot)$ refer to spillovers stemming from the consumption good industry and respectively taking place in that industry and in the capital good industry.
Both inputs are freely shiftable at any date between the two sectors:

$$(2a) \qquad X_{00,t} + X_{01,t} \leq X_{0,t},$$

$$(2b) \qquad X_{10,t} + X_{11,t} \leq X_{1,t},$$

whereas the value of next period capital stock is subject to:

$$(3) \qquad X_{1,t+1} \leq Y_t^1 + (1-\eta)X_{1,t},$$

for $\eta \in]0,1]$ the depreciation rate of the capital stock.
The properties of the production technologies are restricted to the following list of assumptions:

Assumption T.1: $\forall j \in \{0,1\}$, $F^j(\cdot,\cdot)$ is homogeneous of degree one, concave and continuous over $\mathbb{R}_+ \times \mathbb{R}_+$.

Assumption T.2: $\forall j \in \{0,1\}$, $\forall X_{1j} \in \mathbb{R}_+$, $F^j(0, X_{1j}) = 0$.

Assumption T.3: $\forall X_{01} \in \mathbb{R}_+$, $F^1(X_{01}, 0) = 0$.

Assumption T.4: There exists a level $\bar{X}_1 \in \mathbb{R}_+^*$ such that, for all $X_0 \in \mathbb{R}_+^*$, $X_1 \in \mathbb{R}_+^*$, $F^1(1, X_1/X_0) > X_1/X_0$ for all $X_1/X_0 < \bar{X}_1/X_0$ and $F^1(1, X_1/X_0) < X_1/X_0$ for all $X_1/X_0 > \bar{X}_1/X_0$.

Assumption T.5: $\forall j \in \{0,1\}$, $F^j(\cdot,\cdot)$ is of class C^3 over $\mathbb{R}_+^* \times \mathbb{R}_+^*$.

Assumption T.6: $\forall j \in \{0,1\}$, $\forall (X_{0j}, X_{1j}) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$, $\partial F^j/\partial X_{0j} > 0$, $\partial F^j/\partial X_{1j} > 0$, $\partial^2 F^j/\partial (X_{0j})^2 < 0$, $\partial^2 F^j/\partial (X_{1j})^2 < 0$; $\lim_{X_{0j}\to 0} \partial F^j/\partial X_{0j} = +\infty$, $\lim_{X_{0j}\to\infty} \partial F^j/\partial X_{0j} = 0$, $\lim_{X_{1j}\to 0} \partial F^j/\partial X_{1j} = +\infty$ and $\lim_{X_{1j}\to\infty} \partial F^j/\partial X_{1j} = 0$.

Aside from these standard restrictions, a range of crucial assumptions pertain to the *external dimensions* of the production set:

Assumption E.1: $\forall j, j' \in \{0,1\}$, $j' \neq j$, $G^{jj}(\cdot,\cdot)$ and $G^{j'j}(\cdot,\cdot)$, are homogeneous of degree zero, concave and continuous over $\mathbb{R}_+ \times \mathbb{R}_+$.

Assumption E.2: $\forall j, j' \in \{0,1\}$, $j' \neq j$, $G^{jj}(\cdot,\cdot)$ and $G^{j'j}(\cdot,\cdot)$ are of class C^3 over $\mathbb{R}_+^* \times \mathbb{R}_+^*$.

Assumption E.3: $\forall j, j' \in \{0,1\}$, $j' \neq j$, $\forall (X^{\mathscr{E}}_{0j,t}, X^{\mathscr{E}}_{1j,t}), (X^{\mathscr{E}}_{0j',t}, X^{\mathscr{E}}_{1j',t}) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$, $\partial G^{jj}/\partial X^{\mathscr{E}}_{1j} > 0$, $\partial G^{j'j}/\partial X^{\mathscr{E}}_{1j} > 0$.

A direct implication of Assumption E.1 is that there exists functions — not homogeneous in the general case — $\mathscr{G}^{jj}(\cdot)$ and $\mathscr{G}^{j'j}(\cdot)$ such that $G^{jj}(X^{\mathscr{E}}_{0j,t}, X^{\mathscr{E}}_{1j}) = G^{jj}(1, X^{\mathscr{E}}_{1j}/X^{\mathscr{E}}_{0j}) := \mathscr{G}^{jj}(X^{\mathscr{E}}_{1j}/X^{\mathscr{E}}_{0j})$ and $G^{j'j}(X^{\mathscr{E}}_{0j'}, X^{\mathscr{E}}_{1j'}) = G^{j'j}(1, X^{\mathscr{E}}_{1j'}/X^{\mathscr{E}}_{0j'}) := \mathscr{G}^{j'j}(X^{\mathscr{E}}_{1j'}/X^{\mathscr{E}}_{0j'})$. From Assumptions E.2 and E.3, $\mathscr{G}^{jj}(\cdot)$ and $\mathscr{G}^{j'j}(\cdot)$ are of class C^3 and *increasing as functions of the sectoral capital - labour inputs ratios over* $\mathbb{R}_+^*$. Conceptually, the linear homogeneity of Assumption T.1 implies that factors prices will be homogeneous of degree zero in the arguments of the production technologies.
Firms in sector $j = 0, 1$ take $\{\omega^0_t, \omega^1_t, q_t\}$ as given. They select $\{X_{0j,t}, X_{1j,t}\}$, $j = 0, 1$, in order to maximise their profit functions ζ^j_t, namely:

$$\text{(3a)}\quad \zeta^0_t := \underset{\{X_{00,t}, X_{10,t}\}}{\mathrm{Max}}\; Y^0_t - \omega^0_t X_{00,t} - \omega^1_t X_{10,t}$$
$$\text{s.t. } Y^0_t \leq F^0(X_{00,t}, X_{10,t}) G^{00}(X^{\mathscr{E}}_{00,t}, X^{\mathscr{E}}_{10,t}) G^{10}(X^{\mathscr{E}}_{01,t}, X^{\mathscr{E}}_{11,t}),$$
$$X_{00,t} \geq 0, X_{10,t} \geq 0,$$
$$\text{(3b)}\quad \zeta^1_t := \underset{\{X_{01,t}, X_{11,t}\}}{\mathrm{Max}}\; q_t Y^1_t - \omega^0_t X_{01,t} - \omega^1_t X_{11,t}$$
$$\text{s.t. } Y^1_t \leq F^1(X_{01,t}, X_{11,t}) G^{01}(X^{\mathscr{E}}_{00,t}, X^{\mathscr{E}}_{10,t}) G^{11}(X^{\mathscr{E}}_{01,t}, X^{\mathscr{E}}_{11,t}),$$
$$X_{01,t} \geq 0, X_{11,t} \geq 0.$$

Capital and labour being freely shiftable from one sector to the other, they move so as to equalise their rental rates at each between the two sectors. The constant returns to scale hypothesis at the private level further implying the holding of $\zeta_t^0 = \zeta_t^1 = 0$, the existence of an interior technological equilibrium will summarise to the satisfaction of the following set of equations:

$$
\begin{aligned}
&(4a) \quad \omega_t^0 = \frac{\partial F^0}{\partial X_{00}}(X_{00,t}, X_{10,t}) G^{00}(X_{00,t}^{\mathscr{E}}, X_{10,t}^{\mathscr{E}}) G^{10}(X_{01,t}^{\mathscr{E}}, X_{11,t}^{\mathscr{E}}) \\
&\qquad\quad = q_t \frac{\partial F^1}{\partial X_{01}}(X_{01,t}, X_{11,t}) G^{01}(X_{00,t}^{\mathscr{E}}, X_{10,t}^{\mathscr{E}}) G^{11}(X_{01,t}^{\mathscr{E}}, X_{11,t}^{\mathscr{E}}), \\
&(4b) \quad \omega_t^1 = \frac{\partial F^0}{\partial X_{10}}(X_{00,t}, X_{10,t}) G^{00}(X_{00,t}^{\mathscr{E}}, X_{10,t}^{\mathscr{E}}) G^{10}(X_{01,t}^{\mathscr{E}}, X_{11,t}^{\mathscr{E}}) \\
&\qquad\quad = q_t \frac{\partial F^1}{\partial X_{11}}(X_{01,t}, X_{11,t}) G^{01}(X_{00,t}^{\mathscr{E}}, X_{10,t}^{\mathscr{E}}) G^{11}(X_{01,t}^{\mathscr{E}}, X_{11,t}^{\mathscr{E}}), \\
&(4c) \quad Y_t^0 = F^0(X_{00,t}, X_{10,t}) G^{00}(X_{00,t}^{\mathscr{E}}, X_{10,t}^{\mathscr{E}}) G^{10}(X_{01,t}^{\mathscr{E}}, X_{11,t}^{\mathscr{E}}), \\
&(4d) \quad Y_t^1 = F^1(X_{01,t}, X_{11,t}) G^{01}(X_{00,t}^{\mathscr{E}}, X_{10,t}^{\mathscr{E}}) G^{11}(X_{01,t}^{\mathscr{E}}, X_{11,t}^{\mathscr{E}}), \\
&(4e) \quad X_{0,t} = X_{00,t} + X_{01,t}, \\
&(4f) \quad X_{1,t} = X_{10,t} + X_{11,t}.
\end{aligned}
$$

II.1.3 – A Competitive Equilibrium

Definition 1'. Under Assumptions P.1, T.1-6, E.1-3 and for a given ${}_0\mathfrak{E} := \{\mathscr{E}_t\}_{t=0}^{\infty}$, for $\mathscr{E}_t := \{X_{10,t}^{\mathscr{E}}/X_{00,t}^{\mathscr{E}}, X_{11,t}^{\mathscr{E}}/X_{01,t}^{\mathscr{E}}\}$, an intertemporal competitive equilibrium is a sequence $\{\Phi_t^{\mathfrak{E}}\}_{t=0}^{+\infty}$, $\Phi_t^{\mathfrak{E}} := \{X_{00,t}, X_{01,t}, X_{10,t}, X_{11,t}, \omega_t^0, \omega_t^1, q_t, c_t, X_{0,t}, X_{1,t}; {}_0\mathfrak{E}\} \in \mathbb{R}_+^{10}$, $\{\omega_t^0, \omega_t^1, q_t\} \in \ell^1$, $\{X_{0,t}, X_{1,t}\} \in \ell^\infty$, such that, for any $t \geq 0$:

(i) $(X_{00,t}, X_{10,t}) \in \operatorname{Argmax} \zeta_{0,t}$, $(X_{01,t}, X_{11,t}) \in \operatorname{Argmax} \zeta_{1,t}$, $\forall\, t \geq 0$;

(ii) $c_t = Y_t^0 = F^0(X_{00,t}, X_{10,t}) G^{00}(X_{00,t}^{\mathscr{E}}, X_{10,t}^{\mathscr{E}}) G^{10}(X_{01,t}^{\mathscr{E}}, X_{11,t}^{\mathscr{E}})$;

(iii) $X_{1,t+1} = F^1(X_{01,t}, X_{11,t}) G^{01}(X_{00,t}^{\mathscr{E}}, X_{10,t}^{\mathscr{E}}) G^{11}(X_{01,t}^{\mathscr{E}}, X_{11,t}^{\mathscr{E}}) + (1-\eta) X_{1,t}$;

(iv) $X_{0,t} = X_0$;

(v) $\{c_t, X_{1,t}\}$ maximises $\mathscr{V}({}_0\mathscr{C})$ subject to $c_t + q_t[X_{1,t+1} - (1-\eta)X_{1,t}] = \omega_t^0 X_0 + \omega_t^1 X_{1,t}$, $c_t \geq 0$, $\forall\, t \geq 0$;

(vi) $X_{1j,t}/X_{0j,t} = \Upsilon^j(q_t; X_{10,t}^{\mathscr{E}}/X_{00,t}^{\mathscr{E}}, X_{11,t}^{\mathscr{E}}/X_{01,t}^{\mathscr{E}})$ is defined from (4a–b).

II.2 – A Production Possibility Prontier Approach

II.2.1 – The Benchmark Structure

The subsequent exposition shall then be anchored on a substitutability argument that rests upon an integrated view of outputs, inputs and prices in a two-goods world. This will in turn allows for deriving the expression of the frontier of the production possiblity set, namely the *Production Possibility Frontier*, and will endow the current multi-sectoral appraisal with a simplified argument anchored on the levels of the outputs and the aggregate values of the inputs.

On a formal basis, this will correspond, starting from (4) and for a given $\mathscr{E}_t := \{X^{\mathscr{E}}_{10,t}/X^{\mathscr{E}}_{00,t}, X^{\mathscr{E}}_{11,t}/X^{\mathscr{E}}_{01,t}\}$, to the establishment of a relationship between ω^1_t/ω^0_t and q_t by making use of their articulation with $X_{11,t}/X_{01,t}$ and $X_{10,t}/X_{00,t}$ through (4a) and (4b) and then integrating this link into the equilibrium levels of the outputs (4d) and (4e) once the latters have been amended by the expressions of $\mu_{00,t} = X_{00,t}/X_0$ and $\mu_{01,t} = X_{01,t}/X_0$ that are available from (4c). The equilibrium production of the two goods then deliver an integrated view of outputs, inputs and prices according to

$$
\begin{aligned}
(5a)\quad Y^0_t &= \mu_{00}(X_{1,t}/X_0, q_t) X_0 F^0[1, (X_{10,t}/X_{00,t})(q_t)] G^{00}(1, X^{\mathscr{E}}_{10,t}/X^{\mathscr{E}}_{00,t}) G^{10}(1, X^{\mathscr{E}}_{11,t}/X^{\mathscr{E}}_{01,t}) \\
&= \mathfrak{F}^0(X_0, X_{1,t}, q_t; \mathscr{E}_t), \\
(5b)\quad Y^1_t &= \mu_{01}(X_{1,t}/X_0, q_t) X_0 F^1[1, (X_{11,t}/X_{01,t})(q_t)] G^{01}(1, X^{\mathscr{E}}_{10,t}/X^{\mathscr{E}}_{00,t}) G^{11}(1, X^{\mathscr{E}}_{11,t}/X^{\mathscr{E}}_{01,t}) \\
&= \mathfrak{F}^1(X_0, X_{1,t}, q_t; \mathscr{E}_t),
\end{aligned}
$$

where any of the $\mathfrak{F}^j(\cdot,\cdot,\cdot;\mathscr{E}_t)$ is homogeneous of degree one with respect to X_0 and $X_{1,t}$, $j = 0, 1$. Eliminating the relative price q_t between $\mathfrak{F}^0(\bar{X}_0, X_{1,t}, q_t; \mathscr{E}_t)$ and $\mathfrak{F}^1(\bar{X}_0, X_{1,t}, q_t, \mathscr{E}_t)$, the equation of the frontier of the production possiblity set, namely the one of the *Production Possibility Frontier* that is parameterised by external effects, then describes the *optimal production of the consumption good for a given levels of the investment good and a given pair of inputs* and writes down as

$$
(6)\quad Y^0_t = T(Y^1_t; X_0, X_{1,t}; X^{\mathscr{E}}_{10,t}/X^{\mathscr{E}}_{00,t}, X^{\mathscr{E}}_{11,t}/X^{\mathscr{E}}_{01,t}),
$$

for the function $T(\cdot;\cdot,\cdot;\mathscr{E}_t)$ that features the frontier of the production possibility set. For future reference, building upon Assumption T.2 and from a technical argument available as [15], the homogeneity of degree one of $F^j(\cdot,\cdot)$, $j = 0, 1$, translates as the linear homogeneity of $T(\cdot;\cdot,\cdot;\mathscr{E}_t)$ in Y^1_t, X_0 and $X_{1,t}$. The set of triples $(Y^1_t, X_0, X_{1,t})$ such that the set of feasible allocations is non-void is identical to the hypograph of the function $\mathscr{F}^1(\cdot,\cdot;\mathscr{E}_t)$ as defined by $\underline{\mathrm{Gr}}(\mathscr{F}^1;\mathscr{E}_t) := \{(Y^1, X_0, X_1) \in \mathbb{R}^3_+ \mid Y^1 \leq \mathscr{F}^1(X_0, X_1;\mathscr{E}_t)\}$.
It can be shown that $Y^1/X_0 = \mathscr{F}^1(1, X_1/X_0;\mathscr{E}_t)$ denotes the solution to $T(Y^1; X_0, X_1;\mathscr{E}_t) = 0$, the earlier *private* assumptions T.1-6 restating in the following way in terms of the production possibility frontier:[2]

Assumption T.1': There exists $\bar{X}_1$ such that for any $X_1/X_0 \in \,]0, \bar{X}_1/X_0[$, $T(Y^1; X_0, X_1;\mathscr{E}_t) = 0$ implies $\mathscr{F}^1(1, X_1/X_0;\mathscr{E}_t) > X_1/X_0$ and for any $X_1/X_0 > \bar{X}_1/X_0$, $T(Y^1; X_0, X_1;\mathscr{E}_t)$ implies $\mathscr{F}^1(1, X_1/X_0;\mathscr{E}_t) < X_1/X_0$.

Assumption T.2': Letting $(Y^1; X_0, 0) \in \underline{\mathrm{Gr}}(\mathscr{F}^1;\mathscr{E}_t)$, then $Y^1 = 0$ and , $T(Y^1; X_0, 0;\mathscr{E}_t) = \mathscr{F}^0(X_0, 0;\mathscr{E}_t)$.

Assumption T.3': $T(Y^1; X_0, X_1;\mathscr{E}_t)$ is concave over $\underline{\mathrm{Gr}}(\mathscr{F}^1;\mathscr{E}_t)$, continuous and of class C^2 over $\mathrm{int}[\underline{\mathrm{Gr}}(\mathscr{F}^1;\mathscr{E}_t)]$ with $\partial T/\partial Y^1 < 0$, $\partial T/\partial X_0 > 0$, $\partial T/\partial X_1 > 0$, $\partial^2 T/\partial (Y^1)^2 \leq 0$, $\partial^2 T/\partial (X_0)^2 < 0$, $\partial^2 T/\partial (X_1)^2 < 0$, $\partial^2 T/\partial Y^1 \partial X_0 \lesseqgtr 0$ and $\partial^2 T/\partial Y^1 \partial X_1 \gtreqless 0$ for $X_{11}/X_{01} \gtreqless X_{10}/X_{00}$ and $\partial^2 T/\partial X_0 \partial X_1 > 0$.

[2] A more detailed argument is available in the technical appendix of [15].

The hallmark of this alternative approach of the P.P.F. being however his emphasis on price and substitution mechanisms, it is useful to introduce the respective aggregate shares, *e.g.*, of consumption and profits in national income, as $\pi_{Y^0} := p^0 Y^0 / (p^0 Y^0 + p^1 Y^1)$ and $\pi_{\bar{X}_1} := \omega^1 X_1 / (\omega^0 X_0 + \omega^1 X_1)$ and, at a disaggregated level, the elasticity of substitution between the two inputs and the sectoral share of profits in total production cost accruing to the capital input, *i.e.*, the *share of profits* in sector j, as $\Sigma^j_{X_0 X_1} := (\partial F^j / \partial X_{0j})(\partial F^j / \partial X_{1j}) / F^j (\partial^2 F^j / \partial X_{0j} \partial X_{1j})$ and $\pi^j_{X_1} := \omega^1 X^j_1 / p^j Y^j$, where it is noted that $\pi_{\bar{X}_1} = \pi_{Y^0} \pi^0_{X_1} + \pi_{Y^1} \pi^1_{X_1}$. Reformulating (5) along

$$\frac{Y^0_t}{Y^1_t} = \frac{\mathfrak{F}^0 (1, X_{1,t}/X_0, q_t; X^{\mathscr{E}}_{10,t}/X^{\mathscr{E}}_{00,t}, X^{\mathscr{E}}_{11,t}/X^{\mathscr{E}}_{01,t})}{\mathfrak{F}^1 (1, X_{1,t}/X_0, q_t; X^{\mathscr{E}}_{10,t}/X^{\mathscr{E}}_{00,t}, X^{\mathscr{E}}_{11,t}/X^{\mathscr{E}}_{01,t})},$$

it is proved in Drugeon [15, 18] that, for given level of outputs, the aggregate elasticity of substitution between the two inputs X_0 and X_1 boils down to

$$\Sigma_{\bar{X}_0 \bar{X}_1} = (\pi^0_{X_0} \pi^0_{X_1} \pi_{Y^0} \Sigma^0_{X_0 X_1} + \pi^1_{X_0} \pi^1_{X_1} \pi_{Y^1} \Sigma^1_{X_0 X_1}) / \pi_{\bar{X}_0} \pi_{\bar{X}_1}$$

while, for given amounts of both aggregate inputs, the elasticity of substitution between the two outputs Y^0 and Y^1 is available as $\Sigma_{Y^0 Y^1} = \Sigma_{\bar{X}_0 \bar{X}_1} \pi_{\bar{X}_0} \pi_{\bar{X}_1} / (\pi^1_{X_1} - \pi^0_{X_1})^2 \pi_{Y^0} \pi_{Y^1}$.
The features of the functions $\mathfrak{F}^j (\cdot, \cdot, \cdot; \mathscr{E}_t)$ are easily characterised and provide a first hint to the ones of the P.P.F. From (5), the matrix of the first-order derivatives with respect to the inputs lists as:

$$\frac{1}{\pi^1_{X_1} - \pi^0_{X_1}} \begin{bmatrix} \pi_{Y^0} & 0 \\ 0 & \pi_{Y^1} \end{bmatrix}^{-1} \begin{bmatrix} \pi^1_{X_1} & -\pi^1_{X_0} \\ -\pi^0_{X_1} & \pi^0_{X_0} \end{bmatrix} \begin{bmatrix} \pi_{\bar{X}_0} & 0 \\ 0 & \pi_{\bar{X}_1} \end{bmatrix}$$

while the equilibrium matrix that relates the vector of outputs to the vector of costs states as:

$$-\frac{\Sigma_{\bar{X}_0 \bar{X}_1} \pi_{\bar{X}_0} \pi_{\bar{X}_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \begin{bmatrix} \pi_{Y^0} & 0 \\ 0 & \pi_{Y^1} \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The elasticities of $T(\cdot; \cdot, \cdot; \mathscr{E}_t)$ with respect to its three arguments are then respectively given by π_{Y^1}/π_{Y^0}, π_{X_0}/π_{Y^0} and π_{X_1}/π_{Y^0}, where $q_t = -\partial T(Y^1_t; X_0, X_{1,t}; \mathscr{E}_t)/\partial Y^1$, $\omega^0_t = \partial T(Y^1_t; X_0, X_{1,t}; \mathscr{E}_t)/\partial X_0$ and $\omega^1_t = \partial T(Y^1_t; X_0, X_{1,t}; \mathscr{E}_t)/\partial X_1$. Finally, the components of the negative-definite matrix of *weighted* — by factors and product shares in national income — elasticities —, *e.g.*,

$$\Xi_{Y^1 Y^1} := \frac{\partial^2 T}{\partial (Y^1)^2} \left(T - \frac{\partial T}{\partial Y^1} Y^1 \right) \Big/ \left(\frac{\partial T}{\partial Y^1} \right)^2.$$

associated to the second-order derivatives of $T(Y^1; X_0, X_1; \mathscr{E}_t)$ will repeatedly be refered to. Letting $A' = [\, 1 \quad -\pi^0_{X_1}/(\pi^1_{X_1} - \pi^0_{X_1}) \qquad (1 - \pi^0_{X_1})/(\pi^1_{X_1} - \pi^0_{X_1}) \,]$ and $\mathscr{M}_\Xi$ denote this Hessian Elasticities Matrix of the production possibility frontier, it can be shown to be of rank 1 and available as

$$(7) \qquad \mathscr{M}_\Xi = A \left[-\frac{(\pi^1_{X_1} - \pi^0_{X_1})^2}{\pi_{\bar{X}_0} \pi_{\bar{X}_1} \Sigma_{\bar{X}_0 \bar{X}_1}} \right] A',$$

where it is noted that the intersectoral comparison inherent to the coefficient $\pi^1_{X_1} - \pi^0_{X_1}$ has direct implications on the intrasectoral structures since $(\pi^1_{X_1} - \pi^0_{X_1})\pi_{Y^0}\pi_{Y^1}/\pi_{\bar{X}_0}\pi_{\bar{X}_1} = (\pi^1_{X_1}/\pi_{\bar{X}_1} - \pi^1_{X_0}/\pi_{\bar{X}_0})\pi_{Y^1}$. Two stringent dimensions of the two-sector world described by Assumptions T.1–6 are thus that the matrix $\mathscr{M}_\Xi$ is *of rank* 1 and *symmetrical*.

II.2.2 – A Parameterised Production Possibility Frontier

With respect to its canonical definition associated to a range of assumptions such as T.1-6, the particularity of the current definition of the P.P.F. stems from its parameterisation, under Assumptions E.1-3, by $X^{\mathscr{E}}_{10,t}/X^{\mathscr{E}}_{00,t}$ and $X^{\mathscr{E}}_{11,t}/X^{\mathscr{E}}_{01,t}$. For the *level* definition of the P.P.F., this dependency boils down to a pair of coefficients

$$\pi_{X^{\mathscr{E}}_{1j}/X^{\mathscr{E}}_{0j}} := \left[\partial T/\partial(X^{\mathscr{E}}_{1j}/X^{\mathscr{E}}_{0j})\right](X^{\mathscr{E}}_{1j}/X^{\mathscr{E}}_{0j}) \;/\; \left[T - (\partial T/\partial Y^1)Y^1\right], j = 0, 1.$$

For its first-order derivatives, *i.e.*, for competitive prices, this dependency will be illustrated through a matrix of terms such as

$$\Xi_{X_0(X^{\mathscr{E}}_{1j}/X^{\mathscr{E}}_{0j})} := \frac{\left[\partial^2 T/\partial X_0 \partial(X^{\mathscr{E}}_{1j}/X^{\mathscr{E}}_{0j})\right]\left[(\partial T/\partial X_0)X_0 + (\partial T/\partial X_1)X_1\right]}{(\partial T/\partial X_0)\left[\partial T/\partial(X^{\mathscr{E}}_{1j}/X^{\mathscr{E}}_{0j})\right]}, j = 0, 1.$$

Letting also $\pi^{\mathscr{E}_{jj'}}_{X_1} := (\partial G^{jj'}/\partial X^{\mathscr{E}}_{1j})X^{\mathscr{E}}_{1j}/G^{jj'}$ $j, j' = 0, 1$ and for future reference, the two facets of this parameterisation are gathered in the following statement:

<u>Lemma 1</u> [The Parameterised P.P.F. $T(\cdot;\cdot,\cdot;\mathscr{E}_t)$]. *Under Assumptions* T.1-6, E.1-3*:*

(i) *the first-order level dependency of the P.P.F. with respect to $X^{\mathscr{E}}_{10,t}/X^{\mathscr{E}}_{00,t}$ and $X^{\mathscr{E}}_{11,t}/X^{\mathscr{E}}_{01,t}$ states as the following vector:*

$$\begin{bmatrix}\pi_{X^{\mathscr{E}}_{10}/X^{\mathscr{E}}_{00}} & \pi_{X^{\mathscr{E}}_{11}/X^{\mathscr{E}}_{01}}\end{bmatrix} = \begin{bmatrix}\pi_{Y^0}\pi^{\mathscr{E}_{00}}_{X_1} + \pi_{Y^1}\pi^{\mathscr{E}_{01}}_{X_1} & \pi_{Y^0}\pi^{\mathscr{E}_{10}}_{X_1} + \pi_{Y^1}\pi^{\mathscr{E}_{11}}_{X_1}\end{bmatrix};$$

(ii) *the second-order «price» dependency of the vector $\begin{bmatrix}\omega^0_t & \omega^1_t & q_t\end{bmatrix}'$ with respect to $X^{\mathscr{E}}_{10,t}/X^{\mathscr{E}}_{00,t}$ and $X^{\mathscr{E}}_{11,t}/X^{\mathscr{E}}_{01,t}$ states as the matrix:*

$$\begin{bmatrix}\Xi_{Y^1(X^{\mathscr{E}}_{10}/X^{\mathscr{E}}_{00})} & \Xi_{Y^1(X^{\mathscr{E}}_{11}/X^{\mathscr{E}}_{01})} \\ \Xi_{X_0(X^{\mathscr{E}}_{10}/X^{\mathscr{E}}_{00})} & \Xi_{X_0(X^{\mathscr{E}}_{11}/X^{\mathscr{E}}_{01})} \\ \Xi_{X_0(X^{\mathscr{E}}_{10}/X^{\mathscr{E}}_{00})} & \Xi_{X_0(X^{\mathscr{E}}_{11}/X^{\mathscr{E}}_{01})}\end{bmatrix} = B\begin{bmatrix}\pi_{X^{\mathscr{E}}_{10}/X^{\mathscr{E}}_{00}} & 0 \\ 0 & \pi_{X^{\mathscr{E}}_{11}/X^{\mathscr{E}}_{01}}\end{bmatrix}$$

$$\text{for}\quad B = -\begin{bmatrix} 1 \\ \dfrac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}} \\ -\dfrac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}} \end{bmatrix} \left[-\frac{\left(\pi^1_{X_1}-\pi^0_{X_1}\right)^2}{\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}}\right]$$

$$\times \begin{bmatrix} -\dfrac{\left(\pi^{\mathscr{E}_{01}}_{X_1}-\pi^{\mathscr{E}_{00}}_{X_1}\right)\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}}{\left(\pi^1_{X_1}-\pi^0_{X_1}\right)^2}+\pi^{\mathscr{E}_{01}}_{X_1} \\ -\dfrac{\left(\pi^{\mathscr{E}_{11}}_{X_1}-\pi^{\mathscr{E}_{10}}_{X_1}\right)\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}}{\left(\pi^1_{X_1}-\pi^0_{X_1}\right)^2}+\pi^{\mathscr{E}_{11}}_{X_1} \end{bmatrix}'$$

$$+\begin{bmatrix} 0 & 0 \\ -\dfrac{\pi^0_{X_1}\left(\pi^{\mathscr{E}_{01}}_{X_1}-\pi^{\mathscr{E}_{00}}_{X_1}\right)}{\pi^1_{X_1}-\pi^0_{X_1}}+\pi^{\mathscr{E}_{00}}_{X_1} & -\dfrac{\pi^0_{X_1}\left(\pi^{\mathscr{E}_{11}}_{X_1}-\pi^{\mathscr{E}_{10}}_{X_1}\right)}{\pi^1_{X_1}-\pi^0_{X_1}}+\pi^{\mathscr{E}_{10}}_{X_1} \\ \dfrac{\pi^0_{X_0}\left(\pi^{\mathscr{E}_{01}}_{X_1}-\pi^{\mathscr{E}_{00}}_{X_1}\right)}{\pi^1_{X_1}-\pi^0_{X_1}}+\pi^{\mathscr{E}_{00}}_{X_1} & \dfrac{\pi^0_{X_0}\left(\pi^{\mathscr{E}_{11}}_{X_1}-\pi^{\mathscr{E}_{10}}_{X_1}\right)}{\pi^1_{X_1}-\pi^0_{X_1}}+\pi^{\mathscr{E}_{10}}_{X_1} \end{bmatrix};$$

(iii) *sectoral demands state in terms of the arguments and parameters of the P.P.F. as* $X_{1j,t}/X_{0j,t} = \Psi^j\left(Y^1_t; X_0, X_{1,t}; X^{\mathscr{E}}_{10,t}/X^{\mathscr{E}}_{00,t}, X^{\mathscr{E}}_{11,t}/X^{\mathscr{E}}_{01,t}\right)$, $j=0,1$, *where the features of* $\Psi^j\left(\cdot;\cdot,\cdot;\mathscr{E}_t\right)$ *derive, letting* $\mu_{ij} = X_{ij}/X_i$, $i,j=0,1$, *from:*

$$-\frac{\partial T}{\partial Y^1}\left(Y^1_t; X_0, X_{1,t}; X^{\mathscr{E}}_{10,t}/X^{\mathscr{E}}_{00,t}, X^{\mathscr{E}}_{11,t}/X^{\mathscr{E}}_{01,t}\right) = \frac{\pi^0_{X_1}\left(X_{10,t}/X_{00,t}\right)\pi_{Y^1}\left(Y^1_t/X_0, X_{1,t}/X_0\right)}{\pi^1_{X_1}\left(X_{11,t}/X_{01,t}\right)\pi_{Y^0}\left(Y^1_t/X_0, X_{1,t}/X_0\right)},$$

$$\frac{\pi^0_{X_1}\left(X_{10,t}/X_{00,t}\right)}{\pi^0_{X_0}\left(X_{10,t}/X_{00,t}\right)} = \frac{\pi^1_{X_1}\left(X_{11,t}/X_{01,t}\right)}{\pi^1_{X_0}\left(X_{11,t}/X_{01,t}\right)}\frac{\mu_{01}\mu_{10}}{\mu_{00}\mu_{11}}.$$

Proof : *Vide* Appendix V.1. △

It is worth noticing from (i) how Assumptions E.1-3 have as an immediate implication that the production possibility possibility frontier uncovers a positive dependency with repect to the external acceptations of the sectoral capital-labour ratios, no such clear-cut conclusion being oppositely available for the second-order price dependency described through (ii).

II.2.3 – A Competitive Equilibrium

Definition 1. Under Assumptions P.1, T.1-6, E.1-3 and for a given ${}_0\mathfrak{E} := \{\mathscr{E}_t\}^{\infty}_{t=0}$, an intertemporal competitive equilibrium is a sequence $\{\Phi^{\mathfrak{E}}_t\}^{+\infty}_{t=0}$, $\Phi^{\mathfrak{E}}_t := \{X_{0,t}, X_{1,t}, \omega^0_t, \omega^1_t, q_t, Y^0_t, Y^1_t; {}_0\mathfrak{E}\} \in \mathbb{R}^7_+$, $\{\omega^0_t, \omega^1_t, q_t\} \in \ell^1$, $\{X_{0,t}, X_{1,t}\} \in \ell^\infty$ such that, for any $t \geq 0$:

(i) $(X_{1,t}, Y_{1,t})$ maximises $T\left(Y^1_t; X_0, X_{1,t}; \mathscr{E}_t\right) + q_t Y^1_t - \omega^1_t X_{1,t}$ over the domain of $T(\cdot;\cdot,\cdot;\mathscr{E}_t)$;

(ii) $\omega^0_t X_{0,t} = T\left(Y^1_t; X_0, X_{1,t}; \mathscr{E}_t\right) + q_t Y^1_t - \omega^1_t X_{1,t}$;

(iii) $c_t = Y^0_t = T\left(Y^1_t; X_{0,t}, X_{1,t}; \mathscr{E}_t\right)$;

(iv) $X_{1,t+1} = Y^1_t + (1-\eta)X_{1,t}$;

(v) $X_{0,t} = X_0$;

(vi) $\{c_t, X_{1,t}\}$ maximises $\mathscr{V}({}_0\mathscr{C})$ subject to $c_t + q_t Y_t^1 = \omega_t^0 X_0 + \omega_t^1 X_{1,t}$, $c_t \geq 0$ and $X_{1j,t}/X_{0j,t} = \Psi^j(Y_t^1; X_0, X_{1,t}; X_{10,t}^{\mathscr{E}}/X_{00,t}^{\mathscr{E}}, X_{11,t}^{\mathscr{E}}/X_{01,t}^{\mathscr{E}})$, $j = 0, 1$, $\forall\, t \geq 0$.

Competitive equilibria are then proficiently characterised as solution sequences to a parameterised optimisation problem $\mathrm{P}_\infty({}_0\mathfrak{E})$:

$$(8) \qquad \operatorname*{Max}_{\{X_{1,t}, Y_t^1\}} \sum_{t=0}^{\infty} \delta^t (v \circ T)(Y_t^1; X_0, X_{1,t}; \mathscr{E}_t) \text{ s.t. } (3);\ X_{1,0} \in [0, \overline{X}_1], \delta \in]0,1],$$
$$X_{1j,t}/X_{0j,t} = \Psi^j(Y_t^1; X_0, X_{1,t}; X_{10,t}^{\mathscr{E}}/X_{00,t}^{\mathscr{E}}, X_{11,t}^{\mathscr{E}}/X_{01,t}^{\mathscr{E}}), j = 0, 1.$$

For a given set of external sequences ${}_0\mathfrak{E}$, necessary and sufficient conditions for $\{\lambda_t\}_{t=0}^{\infty}$ to be a sequence of support prices to the optimal trajectory state as:

$$(9a) \qquad \frac{\partial v}{\partial c}[T(Y_t^1; X_0, X_{1,t}; \mathscr{E}_t)] \frac{\partial T}{\partial Y^1}(Y_t^1; X_0, X_{1,t}; \mathscr{E}_t) + \delta \lambda_{t+1} = 0,$$

$$(9b) \qquad \frac{\partial v}{\partial c}[T(Y_t^1; X_0, X_{1,t}; \mathscr{E}_t)] \frac{\partial T}{\partial X_1}(Y_t^1; X_0, X_{1,t}; \mathscr{E}_t) + \delta \lambda_{t+1}(1-\eta) - \lambda_t = 0,$$

$$(9c) \qquad X_{1,t+1} - Y_t^1 - (1-\eta) X_{1,t} = 0,$$

$$(9d) \qquad \lim_{t \to +\infty} \delta^t X_{1,t} \lambda_t = 0.$$

III – The «Equilibrium Production Possibility Frontier» & The associated Competitive Equilibrium with Externalities

This section aims at a characterisation of the benchmark technological set along a competitive equilibrium with externalities

III.1 – The E.P.P.F. & the C.E.E.: A Definition

Assuming that there exists a solution ${}_0\mathfrak{X}^{\mathfrak{E}} := \{X_{1,t}; \mathscr{E}_t\}_{t=0}^{\infty}$ to $\mathrm{P}_\infty({}_0\mathfrak{E})$ and that the associated set of external sequences further satisfies:

$$(10a) \quad q_t = \frac{[\partial F^0(1, X_{10,t}/X_{00,t})/\partial X_{00}]\, G^{00}(1, X_{10,t}/X_{00,t})\, G^{10}(1, X_{11,t}/X_{01,t})}{[\partial F^1(1, X_{11,t}/X_{01,t})/\partial X_{01}]\, G^{01}(1, X_{10,t}/X_{00,t})\, G^{11}(1, X_{11,t}/X_{01,t})},$$
$$= -\frac{\partial T}{\partial Y^1}(Y^1; X_0, X_1; X_{10,t}/X_{00,t}, X_{11,t}/X_{01,t})$$

$$(10b) \quad \frac{\partial F^0(1, X_{10,t}/X_{00,t})/\partial X_{00}}{\partial F^1(1, X_{11,t}/X_{01,t})/\partial X_{01}} = \frac{\partial F^0(1, X_{10,t}/X_{00,t})/\partial X_{10}}{\partial F^1(1, X_{11,t}/X_{01,t})/\partial X_{11}},$$

or, along the notation of Lemma 1(iii) and Definition 1:

$$(10'a) \quad X_{10,t}/X_{00,t} = \Psi^0(Y_t^1; X_0, X_{1,t}; X_{10,t}/X_{00,t}, X_{11,t}/X_{01,t}),$$

$$(10'b) \quad X_{11,t}/X_{01,t} = \Psi^1(Y_t^1; X_0, X_{1,t}; X_{10,t}/X_{00,t}, X_{11,t}/X_{01,t}),$$

an *externality augmented* system of demand functions $X^{\mathscr{E},\star}_{1j,t}/X^{\mathscr{E},\star}_{0j,t} = \Psi^{j,\mathscr{E}}(Y^1_t; X_0, X_{1,t})$ becomes available. It shall more compactly be refered to in vector form throughout the subsequent argument: $\widehat{\mathscr{E}}(Y^1_t, X_0, X_{1,t}) := \{\Psi^{j\mathscr{E}}(Y^1_t; X_0, X_{1,t})\}$, $j = 0, 1$.
From the terminology of Kehoe, Levine & Romer [21], it is the consideration of the extra fixed point-*side* condition (10) that allows for recovering a symmetric C.E.E. through a centralised optimisation problem. Beforehand, it is worth introducing an equilibrium benchmark structure for the production technology:

Definition 2. [The «Equilibrium Production Possibility Frontier»] Under Assumptions T.1-6, E.1-3, P.1, the E.P.P.F. is jointly defined

(i) from a level given by $Y^0_t = T^{\mathscr{E}}(Y^1_t; X_0, X_{1,t}) := T(Y^1_t; X_0, X_{1,t}; \widehat{\mathscr{E}}(Y^1_t, X_0, X_{1,t}))$,

(ii) from first-order derivatives associated to the equilibrium competitive prices as

$$
\begin{aligned}
q_t &= -\left(\frac{\partial T}{\partial Y^1}\right)^{\mathscr{E}}(Y^1_t; X_0, X_{1,t}) := -\frac{\partial T}{\partial Y^1}(Y^1_t; X_0, X_{1,t}; \widehat{\mathscr{E}}(Y^1_t, X_0, X_{1,t}));\\
\omega^0_t &= \left(\frac{\partial T}{\partial X_0}\right)^{\mathscr{E}}(Y^1_t; X_0, X_{1,t}) := \frac{\partial T}{\partial X_0}(Y^1_t; X_0, X_{1,t}; \widehat{\mathscr{E}}(Y^1_t, X_0, X_{1,t}));\\
\omega^1 &= \left(\frac{\partial T}{\partial X_1}\right)^{\mathscr{E}}(Y^1_t; X_0, X_{1,t}) := \frac{\partial T}{\partial X_1}(Y^1_t; X_0, X_{1,t}; \widehat{\mathscr{E}}(Y^1_t, X_0, X_{1,t})).
\end{aligned}
$$

Remark 1: The definition of an E.P.P.F. admittedly introduces a range of unusual formal intricacies: for instance, $\partial T^{\mathscr{E}}/\partial Y^1 \neq (\partial T/\partial Y^1)^{\mathscr{E}}$ while the main features of an E.P.P.F. result from the existence of a fixed-point solution to a infinite dimensional problem over time and thus cannot – in contradiction with a canonical P.P.F. the properties of which are based upon a production set that is invariant accross time –, be analysed on the sole basis of a technological eqauilibrium. Nevertheless, by narrowing the focus to a *local appraisal along the symmetric C.E.E.*, *i.e.*, to *level* and *prices* components respectively available as (i) and (ii), the current approach anew equips the analysis with an E.P.P.F. formulation $T^{\mathscr{E}}(\cdot;\cdot,\cdot)$ that affixes the reasoning upon a tractable and unrestricted aggregate structure, such a generality having been up to now restrained to competitive settings anchored on a standard time-invariant P.P.F. ♢

The definition of a symmetric multi-sectoral competitive equilibria with externalities happens to be significantly simplified.

Definition 3. Under Assumptions P.1, T.1-6, E.1-3, an intertemporal symmetric competitive equilibrium with externalities is a sequence $\{\widehat{\Phi}_t\}_{t=0}^{+\infty}$, $\widehat{\Phi}_t := \{X_{0,t}, X_{1,t}, \omega^0_t, \omega^1_t, q_t, Y^0_t, Y^1_t\} \in \mathbb{R}^7_+$, $\{\omega^0_t, \omega^1_t, q_t\} \in \ell^1$, $\{X_{0,t}, X_{1,t}\} \in \ell^\infty$ such that, for any $t \geq 0$:

(i) $q_t = -[\partial T(Y^1_t; X_0, X_{1,t})/\partial Y^1]^{\mathscr{E}}$, $\omega^0_t = [\partial T(Y^1_t; X_0, X_{1,t})/\partial X_0]^{\mathscr{E}}$, $\omega^1_t = [\partial T(Y^1_t; X_0, X_{1,t})/\partial X_1]^{\mathscr{E}}$;

(ii) $\omega^0_t X_{0,t} = T^{\mathscr{E}}(Y^1_t; X_0, X_{1,t}) + q_t Y^1_t - \omega^1_t X_{1,t}$;

(iii) $c_t = Y^0_t = T^{\mathscr{E}}(Y^1_t; X_{0,t}, X_{1,t})$;

(iv) $X_{1,t+1} = Y^1_t + (1-\eta)X_{1,t}$;

(v) $X_{0,t} = X_0$;

(vi) $\{c_t, X_{1,t}, Y_t^1\}$ maximises $\mathscr{V}({}_0\mathscr{C})$ subject to $c_t + q_t Y_t^1 = \omega_t^0 X_0 + \omega_t^1 X_{1,t}$, $c_t \geq 0$, $\forall\, t \geq 0$.

The intertemporal competitive equilibrium with externalities will hence be represented by the following equilibrium restatement of the necessary and sufficient conditions on $\{\lambda_t\}_{t=0}^{\infty}$:

$$(9'\text{a}) \qquad \frac{\partial v}{\partial c}[T^{\mathscr{E}}(Y_t^1; X_0, X_{1,t})]\left(\frac{\partial T}{\partial Y^1}\right)^{\mathscr{E}}(Y_t^1; X_0, X_{1,t}) + \delta\lambda_{t+1} = 0,$$

$$(9'\text{b}) \qquad \frac{\partial v}{\partial c}[T^{\mathscr{E}}(Y_t^1; X_0, X_{1,t})]\left(\frac{\partial T}{\partial X_1}\right)^{\mathscr{E}}(Y_t^1; X_0, X_{1,t}) + \delta\lambda_{t+1}(1-\eta) - \lambda_t = 0,$$

$$(9'\text{c}) \qquad X_{1,t+1} - Y_t^1 - (1-\eta)X_{1,t} = 0,$$

$$(9'\text{d}) \qquad \lim_{t\to+\infty} \delta^t X_{1,t}\lambda_t = 0.$$

III.2 – The E.P.P.F.: A Characterisation

One of the decisive advantages of letting the appraisal of a multi-sectoral setting rest upon a P.P.F. has to do with the gathering of any equilibrium formal intricacy within the properties of an aggregate function. The aim of this section will then be to complete a related characterisation for an E.P.P.F.

As this is enlightened by the following statement, the features of the *externality augmented* system of demand functions $X_{1j,t}^{\mathscr{E},\star}/X_{0j,t}^{\mathscr{E},\star} = \Psi^{j,\mathscr{E}}(Y_t^1; X_0, X_{1,t})$, that assume almost any of the analytically tractable properties of its standard definition that appear in Lemma 1(ii), are a direct corollary of the competitive equilibrium with externalities relationship between $[\,(X_{10}/X_{00})^{\mathscr{E}} \quad X_{11}/X_{01})^{\mathscr{E}}\,]'$ and the relative price of the capital good q_t:

Lemma 2 [The Equilibrium System of Demands]. *Under Assumptions* T.1-6,E.1-3, P.1, *letting* $\Gamma^{\mathscr{E}} := (\pi_{X_1}^1 - \pi_{X_1}^0) + (\pi_{X_1}^{\mathscr{E}_{01}} - \pi_{X_1}^{\mathscr{E}_{00}})\Sigma_{X_0X_1}^0 + (\pi_{X_1}^{\mathscr{E}_{11}} - \pi_{X_1}^{\mathscr{E}_{10}})\Sigma_{X_0X_1}^1$:

(i) *the vector that relates the equilibrium system of demands to the relative price of the capital good is available as*

$$(\Gamma^{\mathscr{E}})^{-1}\begin{bmatrix}\Sigma_{X_0X_1}^0\\ \Sigma_{X_0X_1}^1\end{bmatrix};$$

(ii) *the matrix that relates the equilibrium system of demands to the aggregate arguments of the P.P.F. states as:*

$$(\Gamma^{\mathscr{E}})^{-1}\begin{bmatrix}\Sigma_{X_0X_1}^0\\ \Sigma_{X_0X_1}^1\end{bmatrix}\left[-\frac{(\pi_{X_1}^1 - \pi_{X_1}^0)^2}{\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}}\right]\begin{bmatrix}1 & -\dfrac{\pi_{X_1}^0}{\pi_{X_1}^1 - \pi_{X_1}^0} & \dfrac{1-\pi_{X_1}^0}{\pi_{X_1}^1 - \pi_{X_1}^0}\end{bmatrix}.$$

Proof : *Vide* Appendix V.2. △

Along Section II.2, a first hint of the features of the E.P.P.F. builds from an integrated view of outputs, stocks and prices, namely:

$$(11\text{a}) \quad Y_t^0 = \mathfrak{F}^0[\bar{X}_0, X_{1,t}, q_t; (X_{10,t}/X_{00,t})^{\mathscr{E}}(q_t), (X_{11,t}/X_{01,t})^{\mathscr{E}}(q_t)] = \mathfrak{F}^{\mathscr{E},0}(\bar{X}_0, \bar{X}_{1,t}, q_t),$$

$$(11\text{b}) \quad Y_t^1 = \mathfrak{F}^1[\bar{X}_0, X_{1,t}, q_t; (X_{10,t}/X_{00,t})^{\mathscr{E}}(q_t), (X_{11,t}/X_{01,t})^{\mathscr{E}}(q_t)] = \mathfrak{F}^{\mathscr{E},1}(\bar{X}_0, \bar{X}_{1,t}, q_t)$$

The first-order features of these equilibrium functions are gathered in the following statement:

LEMMA 3 [THE COMPETITIVE EQUILIBRIUM WITH EXTERNALITIES PRODUCTIONS OF Y_t^0 and Y_t^1].

Under Assumptions P.1, T.1-6, E.1-3*:*

(i) *the equilibrium matrix that relates the vector of outputs to the vector of inputs states as:*

$$\frac{1}{\pi_{X_1}^1-\pi_{X_1}^0}\begin{bmatrix}\pi_{Y^0} & 0\\ 0 & \pi_{Y^1}\end{bmatrix}^{-1}\begin{bmatrix}\pi_{X_1}^1 & -\pi_{X_0}^1\\ -\pi_{X_1}^0 & \pi_{X_0}^0\end{bmatrix}\begin{bmatrix}\pi_{\bar{X}_0} & 0\\ 0 & \pi_{\bar{X}_1}\end{bmatrix};$$

(ii) *the equilibrium matrix that relates the vector of outputs to the relative price of the capital good industry states as:*

$$\frac{\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}}{\left(\pi_{X_1}^1-\pi_{X_1}^0\right)\Gamma^{\mathscr{E}}}\begin{bmatrix}\pi_{Y^0} & 0\\ 0 & \pi_{Y^1}\end{bmatrix}^{-1} \times\begin{bmatrix}-1-\dfrac{\pi_{Y^0}\left(\pi_{X_1}^1-\pi_{X_1}^0\right)}{\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}}\left(\pi_{X_1}^{\mathscr{E}_{00}}\Sigma_{X_0X_1}^0+\pi_{X_1}^{\mathscr{E}_{10}}\Sigma_{X_0X_1}^1\right)\\ 1-\dfrac{\pi_{Y^1}\left(\pi_{X_1}^1-\pi_{X_1}^0\right)}{\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}}\left(\pi_{X_1}^{\mathscr{E}_{01}}\Sigma_{X_0X_1}^0+\pi_{X_1}^{\mathscr{E}_{11}}\Sigma_{X_0X_1}^1\right)\end{bmatrix}.$$

PROOF : *Vide* Appendix V.3. △

Interestingly and as described by Lemma 3(i), the dependencies of the outputs w.r.t. stocks remain unaltered and the details of the Rybczinsky Theorem are unmodified with respect to a convex environment, *e.g.*, the more than unitary coefficient that relies the production of the investment good to the capital stock. In opposition to this and as this appears from Lemma 3(ii), the relation that these outputs assume with respect to the relative price of the capital good is augmented by the existence of spillover effects. It is also noted that while such a dependency solely involved aggregate substitution mechanisms, *i.e.*, the coefficient $\Sigma_{\bar{X}_0\bar{X}_1}$, in a convex structure, the consideration of a *suboptimal environment* introduces an *asymmetry between sectoral substitution mechanisms that cannot any longer be merged into an aggregate measure of substitutability* and hence *opens room for a preeminant role of substitution asymmetries* in the analysis.

Finally introducing the second-order weighted elasticities[3] of the E.P.P.F. along

$$\begin{aligned}\Xi_{Y^1Y^1}^{\mathscr{E}}:=&\frac{\left[\partial\left(\partial T/\partial Y^1\right)^{\mathscr{E}}/\partial Y^1\right]\left[T-\left(\partial T/\partial Y^1\right)^{\mathscr{E}}Y^1\right]}{\left[-\left(\partial T/\partial Y^1\right)^{\mathscr{E}}\right]^2}\\ &=\Xi_{Y^1Y^1}+\Xi_{Y^1(X_{10}^{\mathscr{E}}/X_{00}^{\mathscr{E}})}\Xi_{(\widehat{X}_{10}/\widehat{X}_{00})Y^1}+\Xi_{Y^1(X_{11}^{\mathscr{E}}/X_{01}^{\mathscr{E}})}\Xi_{(\widehat{X}_{11}/\widehat{X}_{01})Y^1}\end{aligned}\tag{12}$$

and looking for a deeper understanding of the sectoral underpinnings of the E.P.P.F. is going to uncover a tractable structure for the Hessian elasticities matrix that will recover such an asymmetry between sectoral substitution mechanisms :

[3] An entirely equivalent way of reaching such a coefficient proceeds from the elimination of q_t between $Y_t^0=\mathfrak{F}^{\mathscr{E},0}\left(\bar{X}_0,\bar{X}_{1,t},q_t\right)$ and $Y_t^1=\mathfrak{F}^{\mathscr{E},1}\left(\bar{X}_0,\bar{X}_{1,t},q_t\right)$.

Proposition 1 [The Hessian Elasticities Matrix of the E.P.P.F.]. *Consider a competitive equilibrium with externalities defined under Assumptions* T.1-6, E.1-3, P.1. *Then, for*

$$Z^{\mathscr{E}} := 1 - \frac{\left(\pi^1_{X_1} - \pi^0_{X_1}\right)\pi_{Y^0}\pi_{Y^1}}{\Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1}}\left(\pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1} + \pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1}\right),$$

assuming $Z^{\mathscr{E}} \neq 0$, the Hessian elasticities matrix of the E.P.P.F. states as:

$$\left[\mathscr{M}_{\Xi^{\mathscr{E}}}\right] = \begin{bmatrix} 1 \\ \dfrac{\pi^0_{X_1} + \pi^{\mathscr{E}_{00}}_{X_1}\Sigma^0_{X_0X_1} + \pi^{\mathscr{E}_{10}}_{X_1}\Sigma^1_{X_0X_1}}{\Gamma^{\mathscr{E}}} \\ -\dfrac{1 - \pi^0_{X_1} - \pi^{\mathscr{E}_{00}}_{X_1}\Sigma^0_{X_0X_1} - \pi^{\mathscr{E}_{10}}_{X_1}\Sigma^1_{X_0X_1}}{\Gamma^{\mathscr{E}}} \end{bmatrix}\left[-\frac{\left(\pi^1_{X_1} - \pi^0_{X_1}\right)\Gamma^{\mathscr{E}}}{\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}Z^{\mathscr{E}}}\right]$$
$$\times \begin{bmatrix} 1 & \dfrac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} & -\dfrac{1 - \pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \end{bmatrix}.$$

Proof : *Vide* Appendix V.4. △

It is noteworthy to underline that, comparing the matrix $\mathscr{M}_{\Xi^{\mathscr{E}}}$ with its canonical formulation $\mathscr{M}_{\Xi}$, it remains of rank 1 and that $\Xi^{\mathscr{E}}_{ij}/\Xi^{\mathscr{E}}_{ik}$ is still equal to $\Xi^{\mathscr{E}}_{jj}/\Xi^{\mathscr{E}}_{jk}$, for $i, j, k = Y^1, X_0, X_1$. The *usual duality* between the outputs Rybczinski effects and the price Stolper-Samuelson effects is however lost with the current E.P.P.F and this implies that $\mathscr{M}^{\mathscr{E}}_{\Xi}$ *is not any longer a symmetrical matrix*. Confirming in that respect the insights of Lemma 3, while $\Xi^{\mathscr{E}}_{ij} \neq \Xi^{\mathscr{E}}_{ji}$, the effects that pertain to the products and the factors inputs, *e.g.*, the ratio $\Xi^{\mathscr{E}}_{Y^1X_1}/\Xi^{\mathscr{E}}_{Y^1Y^1} = -\left(1 - \pi^0_{X_1}\right)/\left(\pi^1_{X_1} - \pi^0_{X_1}\right)$, remain unaltered while the Stolper-Samuelson ones that relate inputs and output prices, *e.g.*,

$$(13)\quad \Xi^{\mathscr{E}}_{X_1Y^1}/\Xi^{\mathscr{E}}_{Y^1Y^1} = -\frac{1 - \pi^0_{X_1} - \pi^{\mathscr{E}_{00}}_{X_1}\Sigma^0_{X_0X_1} - \pi^{\mathscr{E}_{10}}_{X_1}\Sigma^1_{X_0X_1}}{\left(\pi^1_{X_1} - \pi^0_{X_1}\right) + \left(\pi^{\mathscr{E}_{01}}_{X_1} - \pi^{\mathscr{E}_{00}}_{X_1}\right)\Sigma^0_{X_0X_1} + \left(\pi^{\mathscr{E}_{11}}_{X_1} - \pi^{\mathscr{E}_{10}}_{X_1}\right)\Sigma^1_{X_0X_1}},$$

are modified by the consideration of external dimensions. Actually, the newest and most atypical dimension of the matrix $\mathscr{M}^{\mathscr{E}}_{\Xi}$ springs from the *explicit appearance of sectoral elasticities of substitution* in its definition.[4] Otherwise stated, the equilibrium articulation between, *e.g.*, the rental rate of the capital stock and the price of the capital good, is now directly influenced by the elasticity of substitution in the production of the capital good but also by the elasticity of substitution in the production of the consumption good, both effects disappearing when both production technologies tend to exhibit complementarity between factors, *i.e.*, for $\Sigma^j_{X_0X_1} \to 0$, $j = 0, 1$.

Remark 2: The obtention of lists of coefficients such as (12) being however admittedly demanding on a formal basis, an alternative approach for establishing the structure of the

[4]This contradicts with the standards of trade theory where only factors shares were involved but also, as explained through Remark 4, with the most frequently used specifications in the recent characterisation of multi-sectoral competitive equilibria with externalities.

matrix $\mathscr{M}_{\Xi^{\mathscr{E}}}$ proceeds, along the lines of the argument developed by [18] in the convex case, from an analysis that rests upon the equilibrium articulation between the vector of factors costs and the ratio $X_{1,t}/X_0$ for any linearly homogeneous $T^{\mathscr{E}}(\cdot;\cdot,\cdot)$. Restating indeed the price component of the E.P.P.F. according to:

$$\text{(14a)}\quad q_t = -\left(\frac{\partial T}{\partial Y^1}\right)^{\mathscr{E}}\big(Y_t^1/X_0; 1, X_{1,t}/X_0\big);$$

$$\text{(14b)}\quad \omega_t^0 = \left(\frac{\partial T}{\partial X_0}\right)^{\mathscr{E}}\big(Y_t^1/X_0; 1, X_{1,t}/X_0\big);$$

$$\text{(14c)}\quad \omega_t^1 = \left(\frac{\partial T}{\partial X_1}\right)^{\mathscr{E}}\big(Y_t^1/X_0; 1, X_{1,t}/X_0\big).$$

and differentiating this system of equations, the insertion of (14a), namely the expression of Y_t^1/X_0 as a function of X_t^1/X_0 and q_t into (14b) and (14c), allows for expressing ω_t^0 and ω_t^1 as functions of $X_{1,t}/X_0$ and q_t. Noticing however that a simple glance at the symmetric competitive equilibrium with externalities formulation of the decentralised version of the first-order conditions (4a-b), namely:

$$\begin{aligned}
\text{(15a)}\quad \omega_t^0 &= \frac{\partial F^0}{\partial X_{00}}\big(1, X_{10,t}/X_{00,t}\big)G^{00}\big(1, X_{10,t}/X_{00,t}\big)G^{10}\big(1, X_{11,t}/X_{01,t}\big)\\
&= q_t\frac{\partial F^1}{\partial X_{01}}\big(1, X_{11,t}/X_{01,t}\big)G^{01}\big(1, X_{10,t}/X_{00,t}\big)G^{11}\big(1, X_{11,t}/X_{01,t}\big),\\
\text{(15b)}\quad \omega_t^1 &= \frac{\partial F^0}{\partial X_{10}}\big(1, X_{10,t}/X_{00,t}\big)G^{00}\big(1, X_{10,t}/X_{00,t}\big)G^{10}\big(1, X_{11,t}/X_{01,t}\big)\\
&= q_t\frac{\partial F^1}{\partial X_{11}}\big(1, X_{11,t}/X_{01,t}\big)G^{01}\big(1, X_{10,t}/X_{00,t}\big)G^{11}\big(1, X_{11,t}/X_{01,t}\big),
\end{aligned}$$

unambiguously indicates that the dependency of the vector of factors prices with respect to the aggregate capital-labour ratio $X_{1,t}/X_0$ is to cancel down to zero, this in turn implies similar values for any of the components of the following vector computed from the integration of (14a) into (14b) and (14c):

$$\text{(16)}\quad \begin{bmatrix} \left(\Xi^{\mathscr{E}}_{X_0X_1} - \dfrac{\Xi^{\mathscr{E}}_{X_0Y^1}}{\Xi^{\mathscr{E}}_{Y^1Y^1}}\Xi^{\mathscr{E}}_{Y^1X_1}\right)\pi_{\bar{X}_1} \\ \left(\Xi^{\mathscr{E}}_{X_1X_1} - \dfrac{\Xi^{\mathscr{E}}_{X_1Y^1}}{\Xi^{\mathscr{E}}_{Y^1Y^1}}\Xi^{\mathscr{E}}_{Y^1X_1}\right)\pi_{\bar{X}_1} \end{bmatrix}.$$

This hence uncovers the rank one stucture of the Hessian elasticities matrix raised through Proposition 1. The explicit expressions of the coefficients can then be computed by hinging on the linear homogeneity of $T^{\mathscr{E}}(\cdot;\cdot,\cdot)$. $\diamondsuit$

Remark 3: An alternative formulation of the externalities set would stem from a pair of production technologies $F^j(X_{0j}, X_{1j})H^j(X^{\mathscr{E}}_{0,t}, X^{\mathscr{E}}_{1,t})$ for $H^j(\cdot,\cdot)$ that is homogeneous of degree zero. Such formulations are comparable to the ones delimited by Assumptions E.1-3 in the regards that returns to scale are unaffected by the consideration of a suboptimal competitive equilibrium. As an illustration, it can be shown that the equilibrium Hessian elasticities matrix of Proposition 1 then reformulates to:

$$\begin{bmatrix} 1 \\ -\dfrac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}} \\ \dfrac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}} \end{bmatrix} \left[-\frac{\left(\pi^1_{X_1}-\pi^0_{X_1}\right)^2}{\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}}\right] \begin{bmatrix} 1 \\ -\dfrac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}} - \dfrac{\left(\pi^{\mathscr{E}_1}_{X_1}-\pi^{\mathscr{E}_0}_{X_1}\right)\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}}{\left(\pi^1_{X_1}-\pi^0_{X_1}\right)^2} - \pi^{\mathscr{E}_1}_{X_1} \\ \dfrac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}} - \dfrac{\left(\pi^{\mathscr{E}_1}_{X_1}+\pi^{\mathscr{E}_0}_{X_1}\right)\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}}{\left(\pi^1_{X_1}-\pi^0_{X_1}\right)^2} + \pi^{\mathscr{E}_1}_{X_1} \end{bmatrix}'$$

$$+ \begin{bmatrix} 0 \\ -\dfrac{\pi^0_{X_1}\left(\pi^{\mathscr{E}_1}_{X_1}-\pi^{\mathscr{E}_0}_{X_1}\right)}{\pi^1_{X_1}-\pi^0_{X_1}} - \pi^{\mathscr{E}_1}_{X_1} \\ \dfrac{\pi^0_{X_0}\left(\pi^{\mathscr{E}_1}_{X_1}-\pi^{\mathscr{E}_0}_{X_1}\right)}{\pi^1_{X_1}-\pi^0_{X_1}} - \pi^{\mathscr{E}_1}_{X_1} \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \end{bmatrix}.$$

Having a glance at the first element of this summation, this rather complex structure starts from the exact contraposite of the one that is considered in this contribution, *i.e.*, the column price structure is unaffected while the row line components all differ from the one of the convex structure: while $\Xi^{\mathscr{E}}_{ij} \neq \Xi^{\mathscr{E}}_{ji}$, the effects that pertain to the products and the factors inputs, *e.g.*, the ratio $\Xi^{\mathscr{E}}_{Y^1X_1}/\Xi^{\mathscr{E}}_{Y^1Y^1}$, are modified while some, but not all, ratios that relate inputs and output prices, *e.g.*, $\Xi^{\mathscr{E}}_{X_1Y^1}/\Xi^{\mathscr{E}}_{Y^1Y^1}$, are unaffected by the consideration of external dimensions. A great difficulty in pursuing further the characterisation of the associated equilibrium would however spring from the second component of this summation, that entails a loss of symmetry *plus* a rank of two, *i.e.*, $\Xi^{\mathscr{E}}_{jj} - \Xi^{\mathscr{E}}_{jj'}\Xi^{\mathscr{E}}_{j'j}/\Xi^{\mathscr{E}}_{j'j'} \neq 0$, $j, j' = Y^1, X_0, X_1$, $j \neq j'$, for the E.P.P.F. To sum up, though formally related, such a formulation conceptually differs and entails entirely distinct theoretical implications on the production set. $\diamondsuit$

III.3 – The Scope for a Convex E.P.P.F. & Complementary Outputs

A remarkable outcome of the consideration of a suboptimal competitive equilibrium externalities and of the associated E.P.P.F. results from the features of the latter. While the preceding section made clear that some key properties of the Hessian elasticities matrix were left unaffected, it remains to wonder in which regards this E.P.P.F. still describes a traditional trade-off between the consumption and the investment outputs. The rank one E.P.P.F. shall be refered to as being convex if there exists some parameter configuration under which either any of its principal diagonal components happens to become positive or some become positive while others keep on being negative but their sum is of positive sign. The scope for the actual range of such a configuration is delimited through the following statement:

<u>Lemma 4</u> [A Convex E.P.P.F.]. *The E.P.P.F. describes a convex function if one of the two following configurations prevails:*

(i) *$Z^{\mathscr{E}} = 1 - (\pi^{1}_{X_1} - \pi^{0}_{X_1})\pi_{Y^0}\pi_{Y^1}(\pi^{\mathscr{E}_{01}}_{X_1}\Sigma^{0}_{X_0X_1} + \pi^{\mathscr{E}_{11}}_{X_1}\Sigma^{1}_{X_0X_1})/\Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1} < 0$ and the two following inequalities simultaneously hold:*

$$(\pi^{1}_{X_1} - \pi^{0}_{X_1})\big[(\pi^{1}_{X_1} - \pi^{0}_{X_1}) + (\pi^{\mathscr{E}_{01}}_{X_1} - \pi^{\mathscr{E}_{00}}_{X_1})\Sigma^{0}_{X_0X_1} + (\pi^{\mathscr{E}_{11}}_{X_1} - \pi^{\mathscr{E}_{10}}_{X_1})\Sigma^{1}_{X_0X_1}\big] > 0,$$
$$1 - \pi^{0}_{X_1} - \pi^{\mathscr{E}_{00}}_{X_1}\Sigma^{0}_{X_0X_1} - \pi^{\mathscr{E}_{10}}_{X_1}\Sigma^{1}_{X_0X_1} > 0;$$

(ii) *$Z^{\mathscr{E}} = 1 - (\pi^{1}_{X_1} - \pi^{0}_{X_1})\pi_{Y^0}\pi_{Y^1}(\pi^{\mathscr{E}_{01}}_{X_1}\Sigma^{0}_{X_0X_1} + \pi^{\mathscr{E}_{11}}_{X_1}\Sigma^{1}_{X_0X_1})/\Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1} > 0$ and the three following inequalities simultaneously hold:*

$$(\pi^{1}_{X_1} - \pi^{0}_{X_1})\big[(\pi^{1}_{X_1} - \pi^{0}_{X_1}) + (\pi^{\mathscr{E}_{01}}_{X_1} - \pi^{\mathscr{E}_{00}}_{X_1})\Sigma^{0}_{X_0X_1} + (\pi^{\mathscr{E}_{11}}_{X_1} - \pi^{\mathscr{E}_{10}}_{X_1})\Sigma^{1}_{X_0X_1}\big] < 0,$$
$$1 - \pi^{0}_{X_1} - \pi^{\mathscr{E}_{00}}_{X_1}\Sigma^{0}_{X_0X_1} - \pi^{\mathscr{E}_{10}}_{X_1}\Sigma^{1}_{X_0X_1} < 0,$$
$$\pi^{0}_{X_1}(\pi^{0}_{X_1} + \pi^{\mathscr{E}_{00}}_{X_1}\Sigma^{0}_{X_0X_1} + \pi^{\mathscr{E}_{10}}_{X_1}\Sigma^{1}_{X_0X_1}) < (1 - \pi^{0}_{X_1})(1 - \pi^{0}_{X_1} - \pi^{\mathscr{E}_{00}}_{X_1}\Sigma^{0}_{X_0X_1} - \pi^{\mathscr{E}_{10}}_{X_1}\Sigma^{1}_{X_0X_1}).$$

Proof : *Vide* Appendix V.5. △

A related guess has then to do with the implications of a variation in the competitive equilibrium with externalities relative price of the inputs ω^1_t/ω^0_t on the relative demand of these inputs $X_{1,t}/X_0$, *i.e.*, the value of the aggregate elasticity of substitution between capital and labour in the course of the competitive equilibrium with externalities, an interesting point in that perspective being that, due to the multiplicative separability assumption that underlies the representations (4) for the sectoral production technologies, the ensued inputs prices ratio ω^1_t/ω^0_t, when considered along a symmetric competitive equilibrium with externalities, happens not to be modified with respect to its standard representation and hence assumes the same expressions with respect to any of the $X_{1j,t}/X_{0j,t}$, $j = 0, 1$. Otherwise stated, $\Sigma^{j}_{X_0X_1}$ still provides an accurate description substitution mechanisms that occur in sector $j = 0, 1$. As this was however clarified by Lemma 2, the competitive equilibrium with externalities formulations $(X_{1j,t}/X_{0j,t})^{\mathscr{E}}$ are modified as functions of $q_t = -[\partial T(Y^1_t; X_0, X_{1,t})/\partial Y^1]^{\mathscr{E}}$, that is homogeneous of degree zero in Y^1_t, X_0 and $X_{1,t}$. This will in turn entail a modified articulation with, *e.g.*, the ratio $X_{1,t}/X_0$, that in turn lies at the very core of the articulation between ω^1_t/ω^0_t and $X_{1,t}/X_0$ that pictures factors substitutability at the aggregate level. In the same vein, recalling the formal articulation between this coefficient and the one that describes the implications of a modification in the relative price of the capital good q_t on the relative level of the outputs Y^1_t/Y^0_t, it remains to wonder in which regards such a coefficient is modified by the consideration of a competitive equilibrium with externalities.

Lemma 5 [C.E.E. Arbitrages & the Scope for Equilibrium Complementarities]. *The E.P.P.F. is such that:*

(i) *The equilibrium elasticities of substitution between the aggregate inputs $\Sigma^{\mathscr{E}}_{\bar{X}_0\bar{X}_1}$ and the outputs $\Sigma^{\mathscr{E}}_{Y^0Y^1}$ respectively formulate along:*

$$\Sigma^{\mathscr{E}}_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1} = \Sigma^{\mathscr{E}}_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Big[1 - (\pi^{1}_{X_1} - \pi^{0}_{X_1})\pi_{Y^0}\pi_{Y^1}\big[(\pi^{\mathscr{E}_{01}}_{X_1} - \pi^{\mathscr{E}_{00}}_{X_1})\Sigma^{0}_{X_0X_1} + (\pi^{\mathscr{E}_{11}}_{X_1} - \pi^{\mathscr{E}_{10}}_{X_1})\Sigma^{1}_{X_0X_1}\big] \,/\, \Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Big],$$

$$\Sigma^{\mathscr{E}}_{Y^0Y^1}\pi_{Y^0}\pi_{Y^1} = \frac{\Sigma^{\mathscr{E}}_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1}}{(\pi^{1}_{X_1} - \pi^{0}_{X_1})\Gamma^{\mathscr{E}}};$$

(ii) *for* $Z^{\mathscr{E}} = 1 - (\pi^1_{X_1} - \pi^0_{X_1})\pi_{Y^0}\pi_{Y^1}(\pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0 X_1} + \pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0 X_1})/\Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1} < 0$, $\Sigma^{\mathscr{E}}_{\bar{X}_0\bar{X}_1} < 0 < \Sigma_{\bar{X}_0\bar{X}_1}$;

(iii) *for* $Z^{\mathscr{E}} = 1 - (\pi^1_{X_1} - \pi^0_{X_1})\pi_{Y^0}\pi_{Y^1}(\pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0 X_1} + \pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0 X_1})/\Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1} > 0$, $\pi^0_{X_1} > \pi^1_{X_1}$ *and* $(\pi^1_{X_1} - \pi^0_{X_1}) + (\pi^{\mathscr{E}_{01}}_{X_1} - \pi^{\mathscr{E}_{00}}_{X_1})\Sigma^0_{X_0 X_1} + (\pi^{\mathscr{E}_{11}}_{X_1} - \pi^{\mathscr{E}_{10}}_{X_1})\Sigma^1_{X_0 X_1} < 0$, *one obtains* $\Sigma^{\mathscr{E}}_{\bar{X}_0\bar{X}_1} > \Sigma_{\bar{X}_0\bar{X}_1} > 0 > \Sigma^{\mathscr{E}}_{Y^0 Y^1}$.

Proof : *Vide* Appendix V.6. △

The probably most puzzling result of Lemma 5 formulates as the opposite implications of the focus on an E.P.P.F. on the resulting values of $\Sigma^{\mathscr{E}}_{\bar{X}_0\bar{X}_1}$ and $\Sigma^{\mathscr{E}}_{Y^0 Y^1}$ in Lemma 5(iii). The analysis will now aimed at examining the role of the properties listed through Proposition 1 and Lemmas 2-5 in the assessment of the uniqueness issue.

IV – An Asymmetric Role for Substitutability in the Emergence of Indeterminacies

IV.1 – The Global Indeterminacy Issue

Pursuing the analysis of the competitive equilibrium with externalities, under the previous set of assumptions, the system (9) expresses as $[\,X_{1,t+1} \quad \lambda_{t+1}\,]' = \Phi(X_{1,t}, \lambda_t)$ when (9a) has been used for restating Y^1_t in terms of $X_{1,t}$ and λ_{t+1}. Letting $(\pi_{\bar{X}_1})^{\mathscr{E}} := (\partial T/\partial X_1)^{\mathscr{E}}/[T^{\mathscr{E}} - (\partial T/\partial Y^1)^{\mathscr{E}} Y^1]$ and $(\pi_{Y^1})^{\mathscr{E}} := -(\partial T/\partial Y^1)^{\mathscr{E}}/[T^{\mathscr{E}} - (\partial T/\partial Y^1)^{\mathscr{E}} Y^1]$, a benchmark definition is in order:

Definition 3 [Interior Steady State]. Under Assumptions T.1-6, E.1-3, P.1, assume that there exists a symmetric competitive equilibrium with externalities. An interior steady state is then a pair $\{Y^{1\star}, X^\star_1\} \in (\mathbb{R}^*_+)^2$ that solves:

$$Y^{1,\star}/X_0 = \eta X^\star_1/X_0,$$
$$(\pi_{\bar{X}_1})^{\mathscr{E}}(Y^{1\star}/X_0; 1, X^\star_1/X_0)/(\pi_{Y^1})^{\mathscr{E}}(Y^{1\star}/X_0; 1, X^\star_1/X_0) = [1 - \delta(1-\eta)]/\delta\eta.$$

Its existence and uniqueness properties are assessed in the following statement:

<u>Proposition 2</u> [Existence and Uniqueness/Multiplicity of the Steady States]. *Under Assumptions* P.1, T.1-6, E.1-3*:*

(i) *if* $\lim_{X_1/X_0 \to 0}(\pi_{\bar{X}_1})^{\mathscr{E}}/(\pi_{Y^1})^{\mathscr{E}} < [1 - \delta(1-\eta)]/\delta\eta < \lim_{X_1/X_0 \to \infty}(\pi_{\bar{X}_1})^{\mathscr{E}}/(\pi_{Y^1})^{\mathscr{E}}$ *or* $\lim_{X_1/X_0 \to \infty}(\pi_{\bar{X}_1})^{\mathscr{E}}/(\pi_{Y^1})^{\mathscr{E}} < [1 - \delta(1-\eta)]/\delta\eta < \lim_{X_1/X_0 \to 0}(\pi_{\bar{X}_1})^{\mathscr{E}}/(\pi_{Y^1})^{\mathscr{E}}$, *there exists a steady state ;*

(ii) *if* $1 - \pi^1_{X_1} - \pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0 X_1} - \pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0 X_1} < 0$ *(large spillovers in sector* 1 *or high factors substitutability in the production technologies) or* $1 - \pi^1_{X_1} - \pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0 X_1} - \pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0 X_1} > 0$ *(small spillovers in sector* 1 *or complementary factors in the production technologies) uniformly holds over the set of steady states, then there exists at most one unique steady state ;*

(iii) *if* $\lim_{X_1/X_0 \to 0} \left(\pi_{\bar{X}_1}\right)^{\mathscr{E}} / \left(\pi_{Y^1}\right)^{\mathscr{E}} < (>) [1 - \delta(1-\eta)]/\delta\eta$ *and* $\lim_{X_1/X_0 \to \infty} \left(\pi_{\bar{X}_1}\right)^{\mathscr{E}} / \left(\pi_{Y^1}\right)^{\mathscr{E}} < (>) [1 - \delta(1-\eta)]/\delta\eta$ *while there exists a unique solution to* $1 - \pi^1_{X_1} - \pi^{\mathscr{E}_{01}}_{X_1} \Sigma^0_{X_0 X_1} - \pi^{\mathscr{E}_{11}}_{X_1} \Sigma^1_{X_0 X_1} = 0$, *then the curve that depicts the set of steady states is single-peaked (single-caved) and there exists at most two interior steady states.*

Proof : *Vide* Appendix V.7 △

Interestingly, the multiplicity issue, *i.e.*, the sign of $1 - \pi^1_{X_1} - \pi^{\mathscr{E}_{01}}_{X_1} \Sigma^0_{X_0 X_1} - \pi^{\mathscr{E}_{11}}_{X_1} \Sigma^1_{X_0 X_1}$, appears as being unrelated to the prevailing sectoral configuration: while local indeterminacy will reveal in Proposition 3 as being univocally associated to the occurrence of — the one associated to optimal cyclic or chaotic sequences in a convex environment — $\pi^0_{X_1} > \pi^1_{X_1}$, the local uniqueness of the steady state cannot be discarded in the benchmark well-behaved configuration of optimal growth theory for which $\pi^1_{X_1} > \pi^0_{X_1}$.
According to the smoothing properties traditionnally associated with factors substitutability, the uttermost surprising dimension of Proposition 2 however probably results from the paradoxical role of its sectoral values in the global indeterminacy issue. If intersectoral spillovers indeed happen to predominate (are negligible) while intrasectoral ones are negligible (predominate), a *large value for the elasticity of substitution in the production of the consumption (investment) good will favor multiplicity or give rise to a new type of uniqueness result* for which $\lim_{X_1/X_0 \to 0} \left(\pi_{\bar{X}_1}\right)^{\mathscr{E}} / \left(\pi_{Y^1}\right)^{\mathscr{E}} < [1-(1-\eta)]/\delta\eta < \lim_{X_1/X_0 \to \infty} \left(\pi_{\bar{X}_1}\right)^{\mathscr{E}} / \left(\pi_{Y^1}\right)^{\mathscr{E}}$ but not boil down the examination of the uniqueness issue to the benchmark configuration for which $\pi^{\mathscr{E}_{01}}_{X_1} = \pi^{\mathscr{E}_{11}}_{X_1} = 0$. In opposition to this, low values for the elasticity of substitution of *both* sectors will unambiguously restore the uniqueness result.
These conclusions are in some respects close to a recent examination by Cazzavillan, Lloyd-Braga and Pintus [14]. Actually, the coefficient $1 - \pi^1_{X_1} - \pi^{\mathscr{E}_{01}}_{X_1} \Sigma^0_{X_0 X_1} - \pi^{\mathscr{E}_{11}}_{X_1} \Sigma^1_{X_0 X_1}$ is in some regards reminiscent of the expression that they put forth within an aggregate environment with spillovers and increasing returns to scale: they indeed, among others, discuss the sign of $1 - \pi_{\bar{X}_1} - \pi^{\mathscr{E}}_{X_1} \Sigma_{\bar{X}_0 \bar{X}_1}$ where the factor share, here its sectoral definition available as $1 - \pi^1_{X_1}$, is similarly augmented by the *spillover share in production technology* weighted by the aggregate elasticity of substitution between the two inputs. In that perspective, the current line of argument enriches their approach by sectoral concerns: though the present coefficient primarily relates to the capital good industry 1, the existence of intersectoral spillovers effects stemming from the consumption good industry 0, *i.e.*, $\pi^{\mathscr{E}_{01}}_{X_1} > 0$, implies that the substitutability properties of this latter sector will influence, through $\Sigma^0_{X_0 X_1}$, the determination of its sign and thus the case for multiplicity: by itself, letting $\pi^{\mathscr{E}_{11}}_{X_1} \to 0$ is not any longer sufficient to ensure the uniqueness of the steady state. Further, their multiplicity conclusions are associated with an increasing returns to scale assumption on the production technology: in opposition to this, the current line of argument is entirely based upon a standard constant returns to scale assumption, be it on both production technologies and thus on the P.P.F or on the E.P.P.F. Finally, the look for an articulation with the insights of Lemmas 4 and 5 about the potential for atypical global properties of the E.P.P.F. unfortunately appears as being quite disappointing. As a matter of fact, this easily finds its explanation in the preceding comment: the uniqueness issue is fundamentally disconnected from intersectoral comparisons

between factors shares while it is the latters that directly underlie the scopes for both a convex $T^{\mathscr{E}}(\cdot;\cdot,\cdot,\cdot)$ and atypical conclusions on aggregate substitution mechanisms that are gathered in the equilibrium values of $\Sigma^{\mathscr{E}}_{\bar{X}_0\bar{X}_1}$ and $\Sigma^{\mathscr{E}}_{Y^0Y^1}$.
Interestingly, by specialising the theoretical argument to parametric formulations, a clarified picture of the multiplicity issue becomes available:

Corollary 1 [The Multiplicity Issue under a C.E.S. Parameterisation]. *Under Assumptions* P.1, T.1-6, E.1-3, *further let the sectoral elasticities of substitution* $\Sigma^{j}_{X_0X_1}$, $j = 0, 1$, *assume constant values,*

(i) *if any of the functions* $G^{jj'}(\cdot,\cdot)$, $j, j' = 0, 1$, *of Assumptions* E.1-3 *further embraces a unitary elasticity with respect to its argument* $X^{\mathscr{E}}_{1j}/X^{\mathscr{E}}_{0j}$, *then, under the qualifications of Proposition* 2(iii)-(iv)*:*

  a/ *if* $\Sigma^{1}_{X_0X_1} = 1$, *but to a limit case, there generically exists at most one unique steady state ;*

  b/ *if* $\Sigma^{1}_{X_0X_1} > 1$ (< 1), *the set of steady states is a single-peaked (resp. single-caved) curve and there exists two steady states that coincide when* $1 - \pi^{1}_{X_1} - \pi^{\mathscr{E}_{01}}_{X_1}\Sigma^{0}_{X_0X_1} - \pi^{\mathscr{E}_{11}}_{X_1}\Sigma^{1}_{X_0X_1} = 0$ *holds at a steady state ;*

(ii) *if any of the functions* $G^{jj'}(\cdot,\cdot)$ $j, j' = 0, 1$, *of Assumptions* E.1-3 *originating from sector* j *embraces the elasticity of substitution of that sector, i.e.,* $\Sigma^{j}_{X_0X_1}$ *then, further letting* $\Sigma^{1}_{X_0X_1} = 1$ *, under the qualifications of Proposition* 2(iii)*:*

  a/ *if* $\Sigma^{0}_{X_0X_1} = 1$, *but to a limit case, there generically exists at most one unique steady state ;*

  b/ *if* $\Sigma^{0}_{X_0X_1} > 1$ (< 1), *the set of steady states is a single-peaked (resp. single-caved) curve and there exists two steady states that coincide when* $1 - \pi^{1}_{X_1} - \pi^{\mathscr{E}_{01}}_{X_1}\Sigma^{0}_{X_0X_1} - \pi^{\mathscr{E}_{11}}_{X_1} = 0$ *holds at a steady state.*

Proof : *Vide* Appendix V.8. △

The insights of Corollary 1 are twofold. First, the long-run productivity of the capital stock is defined from the marginal productivity of the capital good industry from (i) and along the conclusions of Cazzavillan, Pintus & Lloyd-Braga [14] in an environment with increasing returns to scale, the uniqueness associated with the Cobb-Douglas representation appears as a non-robust configuration that disappears as soon as the fixed elasticity of the capital good industry slightly departs from unitary values. In opposition to this, the insights of (ii) appear as a direct outcome of the existence of intersectoral spillover effects stemming from the consumption good industry ; *even when the elasticity of substitution of the capital good industry is stocked to a unitary value*, it suffices that spillover effects do not proceed from a power function for *non-unitary values for the elasticity of substitution of the consumption good industry to directly underlie the existence of multiple steady states whatever the prevailing sectoral configuration.*

IV.2 – The Local Indeterminacy Issue

Letting the intertemporal elasticity of substitution embrace an infinite value, *i.e.*, for $\Sigma^c := -[\partial v(c)/\partial c]/[\partial^2 v(c)/\partial c^2]c \to \infty$, further assuming that no factor intensity reversal occurs at the steady states positions, *i.e.*, equivalently, $\pi^0_{X_1} \neq \pi^1_{X_1}$ over the set of steady states, it is shown in Appendix V.7 that the Jacobian matrix considered in a neighbourhood of the steady state then assumes a triangular structure, its spectrum being described by:

$$
\begin{aligned}
\text{(11a)}\quad \nu_1 &= \frac{\delta(1-\eta)\Xi^{\mathscr{E}}_{Y^1Y^1} - \eta[1-\delta(1-\eta)]\Xi^{\mathscr{E}}_{Y^1X_1}}{\delta \Xi^{\mathscr{E}}_{Y^1Y^1}} \\
&= \frac{\delta(1-\eta)\left(\pi^1_{X_1} - \pi^0_{X_1}\right) + [1-\delta(1-\eta)]\left(1-\pi^0_{X_1}\right)}{\pi^1_{X_1} - \pi^0_{X_1}} \frac{1}{\delta} \\
&= \frac{\left(\pi^1_{X_1} - \pi^0_{X_1}\right) + [1-\delta(1-\eta)]\left(1-\pi^1_{X_1}\right)}{\pi^1_{X_1} - \pi^0_{X_1}} \frac{1}{\delta}, \\
\text{(11b)}\quad \nu_2 &= \frac{\Xi^{\mathscr{E}}_{Y^1Y^1}}{\delta(1-\eta)\Xi^{\mathscr{E}}_{Y^1Y^1} - [1-\delta(1-\eta)]\Xi^{\mathscr{E}}_{X_1Y^1}} \\
&= \frac{\Gamma^{\mathscr{E}}}{\delta(1-\eta)\Gamma^{\mathscr{E}} + [1-\delta(1-\eta)]\left(1 - \pi^0_{X_1} - \pi^{\mathscr{E}_{00}}_{X_1}\Sigma^0_{X_0X_1} - \pi^{\mathscr{E}_{10}}_{X_1}\Sigma^1_{X_0X_1}\right)} \\
&= \frac{\Gamma^{\mathscr{E}}}{\Gamma^{\mathscr{E}} + [1-\delta(1-\eta)]\left(1 - \pi^1_{X_1} - \pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1} - \pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1}\right)}.
\end{aligned}
$$

In contradiction with an optimal growth environment and the associated P.P.F., the consideration of an E.P.P.F. — that differs from the standard one as a result of the consideration of an externalities augmented production set — *reintroduces an explicit first-order role* for substitution mechanisms in the stability issue. It is worthwhile noticing that this happens in spite of the retainment of $\Sigma^c \to \infty$ on the preferences side: this latter assumption however still implies that $\Sigma_{\bar{X}_0\bar{X}_1}$ is erased from the expression of the eigenvalues while it appeared in $\mathscr{M}_{\Xi}$. More precisely, the influence of the external effects which take place in a given sector in the determination of the *equilibrium relative sectoral profit shares index* $\Gamma^{\mathscr{E}}$ will be proportionate to the elasticity of substitution between the inputs of the sector from which they originate. As an illustration, if $\Sigma^0_{X_0X_1}$ was made arbitrarily large, the sign of $\Gamma^{\mathscr{E}}$ would be directly ruled by the one of $\pi^{\mathscr{E}_{01}}_{X_1} - \pi^{\mathscr{E}_{00}}_{X_1}$ independently of the *private* sign of $\pi^1_{X_1} - \pi^0_{X_1}$ or of the spillovers that stem from sector 1, *i.e.*, $\pi^{\mathscr{E}_{11}}_{X_1} - \pi^{\mathscr{E}_{10}}_{X_1}$.

At that stage, it is also worth remarking that, while the first *stocks* eigenvalue is let unmodified by the consideration of a competitive equilibrium with externalities in place of a convex environment, the expression of the second *prices* one undergoes a twofold modification. First, and as this was already mentionned in the discussion of the multiplicity issue, though the coefficient $1 - \pi^1_{X_1} - \pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1} - \pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1}$ corresponds to a formulation that is largely reminiscent of an expression put forth in the aggregate environment of Cazzavillan, Lloyd-Braga and Pintus [14], the existence of intersectoral spillovers effects stemming from the consumption good industry 0 underlies a fundamentally distinct interpretation. In parallel to this, the second ingredient that underlies the construction of ν_2 is specific to the current two-goods structure and states as $\Gamma^{\mathscr{E}} := \left(\pi^1_{X_1} - \pi^0_{X_1}\right) + \left(\pi^{\mathscr{E}_{01}}_{X_1} - \pi^{\mathscr{E}_{00}}_{X_1}\right)\Sigma^0_{X_0X_1} + \left(\pi^{\mathscr{E}_{11}}_{X_1} - \pi^{\mathscr{E}_{10}}_{X_1}\right)\Sigma^1_{X_0X_1}$. The current *intrasectoral vs intersectoral spillovers and sectoral factors substitutability augmented relative profit shares coefficient* is hence the first to stress in an explicit manner the need for asymmetries between intrasectoral and intersectoral spillovers stemming from a given industry in the stability of a given long-run equilibrium — *if the equilibrium intrasectoral and intersectoral spillovers stemming from a given sector happen to be identical*, then $\Gamma^{\mathscr{E}}$ recovers its canonical expression and no role is any longer allowed for sectoral factors substitutability — but also to emphasise the central role of heterogeneity in sectoral factors substitutability in that perspective. The area for local indeterminacies is more carefully circumscribed through the following statement:

<u>Proposition 3</u> [The Benchmark Saddlepoint Property & the Areas for Local Indeterminacy]. *Under assumptions* P.1, T.1-6, E.1-3, *consider an interior steady state position along Definition* 3 . *Then:*

(i) *if* $\pi^1_{X_1} > \pi^0_{X_1}$, *it cannot be locally indeterminate and is characterised by a unique convergent trajectory if and only if one of the two following conditions is satisfied:*

a/ $\left(1 - \pi^1_{X_1} - \pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1} - \pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1}\right)\Gamma^{\mathscr{E}} > 0$;

b/ *if* $\left(1 - \pi^1_{X_1} - \pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1} - \pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1}\right)\Gamma^{\mathscr{E}} < 0$ *when the following inequality further holds:*

$$\left|\Gamma^{\mathscr{E}}\right| < [1 - \delta(1-\eta)]\left|\left(1 - \pi^1_{X_1} - \pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1} - \pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1}\right)\right|/2.$$

(ii) *if* $\pi^0_{X_1} > \pi^1_{X_1}$, *further let* $\left|\pi^1_{X_1} - \pi^0_{X_1}\right|(1 + 1/\delta) > [1/\delta - (1-\eta)]\left(1 - \pi^1_{X_1}\right)$*:*

a/ *if* $\left(1 - \pi^1_{X_1} - \pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1} - \pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1}\right)\Gamma^{\mathscr{E}} > 0$, *the steady state is locally indeterminate;*

b/ *if* $\left(1 - \pi^1_{X_1} - \pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1} - \pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1}\right)\Gamma^{\mathscr{E}} < 0$, *the steady state is locally indeterminate if and only if the following inequality further holds:*

$$\left|\Gamma^{\mathscr{E}}\right| < [1 - \delta(1-\eta)]\left|\left(1 - \pi^1_{X_1} - \pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1} - \pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1}\right)\right|/2.$$

Proof : *Vide* Appendix V.9. △

Proposition 3 first clarifies the conditions under which a given steady state exhibits the saddlepoint property: unsurprisingly, under the holding of the standard assumption according to which the investment industry uses relatively more capital units than the consumption one, namely $\pi^1_{X_1} > \pi^0_{X_1}$, the conditions for the obtention of a locally unique steady state are more stringent. While these are trivially satisfied in an optimal accumulation since the $\left(1 - \pi^1_{X_1}\right)\left(\pi^1_{X_1} - \pi^0_{X_1}\right) > 0$ is embedded in Proposition 3(i)a/, the formal condition delivered by this one allows for the simultaneous holding of $1 - \pi^1_{X_1} - \pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1} - \pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1} < 0$ and $\Gamma^{\mathscr{E}} < 0$, that both entail strongly atypical dimensionsz of the competitive equilibrium with externalities. Paradoxically enough, the slightly more conventional configuration described through Proposition 3(i)a/ reveals as being much more difficult with local uniqueness. From a broader perspective, a first implication of the joint contemplation of Propositions 2 and 3 lies in the erasure of any indeterminacy area when both production technologies tend to complementarity, *i.e.*, $\Sigma^j_{X_0X_1} \to 0$, $j = 0, 1$: there indeed then exists at most one unique locally

determinate steady state. Proposition 3(i)-(ii) also makes clear that a necessary condition for local indeterminacy states as a private configuration where the share of profits in the consumption good industry is greater than its counterpart in the capital good industry. Otherwise stated and from Proposition 2, though an equilibrium configuration such that $\pi^1_{X_1} > \pi^0_{X_1}$ can be characterised by an alternance of locally unstable and saddlepoint steady states, no local indeterminacy is any longer conceivable.
From Proposition 3(ii), it reveals that a *relative sectoral profits shares reversal* between the private and the equilibrium level, *i.e.*, the simulaneous occurrence of $\pi^1_{X_1} - \pi^0_{X_1} < 0$ and $\Gamma^{\mathscr{E}} := (\pi^1_{X_1} - \pi^0_{X_1}) + (\pi^{\mathscr{E}_{01}}_{X_1} - \pi^{\mathscr{E}_{00}}_{X_1})\Sigma^0_{X_0X_1} + (\pi^{\mathscr{E}_{11}}_{X_1} - \pi^{\mathscr{E}_{10}}_{X_1})\Sigma^1_{X_0X_1} > 0$ — equivalently and from Lemma 3(i), *the equilibrium production of the capital good industry becomes a decreasing function of the relative price of the capital good* —, plays a role in the emergence of local indeterminacies. While it may then seem at first sight somewhat difficult to infer a clear-cut articulation between sectoral factors substitutability, only part of the ingredients of Corollary 2 are of true interest: the outstanding ones uncover a very specific construction of insufficient generality to deserve a careful analysis. To perceive this, consider the continuous time counterpart of the current environment with an unchanged set of of assumptions on the technology : letting $\delta \in \mathbb{R}_+$ and $\rho \in \mathbb{R}_+$ respectively denote the depreciation rates of the capital stock and the rate of time preference of the representative consumer, the local properties of dynamical equilibria around a steady state are then described, for $\Sigma^c \to \infty$ and from Proposition 1, by the following eigenvalues:

$$
\begin{aligned}
\nu_1 &= -\left[\frac{\Xi^{\mathscr{E}}_{Y^1X_1}}{\Xi^{\mathscr{E}}_{Y^1Y^1}} + \frac{\delta}{\delta+\rho}\right](\delta+\rho) \\
&= \left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} - \frac{\delta}{\delta+\rho}\right](\delta+\rho), \\
\nu_2 &= \left[\frac{\Xi^{\mathscr{E}}_{X_1Y^1}}{\Xi^{\mathscr{E}}_{Y^1Y^1}} + 1\right](\delta+\rho) \\
&= \left[-\frac{1-\pi^1_{X_1} - \pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1} - \pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1}}{(\pi^1_{X_1} - \pi^0_{X_1}) + (\pi^{\mathscr{E}_{01}}_{X_1} - \pi^{\mathscr{E}_{00}}_{X_1})\Sigma^0_{X_0X_1} + (\pi^{\mathscr{E}_{11}}_{X_1} - \pi^{\mathscr{E}_{10}}_{X_1})\Sigma^1_{X_0X_1}}\right](\delta+\rho).
\end{aligned}
$$

From this, it is straightforward that, in accordance with Proposition 3(i), the simple holding of $\pi^0_{X_1} > \pi^1_{X_1}$ would ensure that $\nu_1 < 0$ and thus provide a necessary condition for local indeterminacy ; a sufficient condition would then be univocally associated with $(1 - \pi^1_{X_1} - \pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1} - \pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1})\Gamma^{\mathscr{E}} > 0$, that corresponds to Proposition 3(ii)a/: otherwise stated, such a conjunction is at the core of the indeterminacy conclusions based upon the technological set described by Assumptions T.1-6, E.1-3. In opposition to this, Proposition 3(ii)b/ is specific to the discrete time formulation: indeterminacy statements then happen to be directly related to the depreciation rate of the capital stock. For the current purpose,[5] those solutions seem of too poor a generality to complete a generic understanding of the role of asymmetries in factors substitutability and thus deserve a more advanced characterisation. The subsequent

[5]The appraisal of this limit configuration is available upon request.

statement then gives a more accurate picture of the substitutability underpinnings of local indeterminacy in the benchmark configuration of Proposition 3(ii)a/.

Proposition 4. [Asymmetric Factors Substitutability, Intra vs Inter–Sectoral Spillovers & Local Indeterminacies] *Under assumptions* P.1, T.1-6, E.1-3, *consider an interior steady state such that* $\pi^1_{X_1} - \pi^0_{X_1} < 0$, $|\pi^1_{X_1} - \pi^0_{X_1}|(1+1/\delta) > [1/\delta - (1-\eta)](1-\pi^1_{X_1})$ *and further assume that*

(i) *intersectoral spillovers are predominant and intrasectoral spillover effects are negligible; the steady state will become indeterminate if the equilibrium values of* $\Sigma^0_{X_0X_1}$ *and* $\Sigma^1_{X_0X_1}$ *respectively satisfy:*

a/ *when* $\Gamma^{\mathscr{E}} > 0$, $\Sigma^1_{X_0X_1} < [(\pi^1_{X_1} - \pi^0_{X_1}) + \pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1}]/\pi^{\mathscr{E}_{10}}_{X_1}$ *and* $(\pi^0_{X_1} - \pi^1_{X_1})/\pi^{\mathscr{E}_{01}}_{X_1} < \Sigma^0_{X_0X_1} < (1-\pi^1_{X_1})/\pi^{\mathscr{E}_{01}}_{X_1}$;

b/ *when* $\Gamma^{\mathscr{E}} < 0$, $\Sigma^1_{X_0X_1} > [(\pi^1_{X_1} - \pi^0_{X_1}) + \pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1}]/\pi^{\mathscr{E}_{10}}_{X_1}$ *and* $\Sigma^0_{X_0X_1} > (1-\pi^1_{X_1})/\pi^{\mathscr{E}_{01}}_{X_1}$;

(ii) *intrasectoral spillovers are predominant and intersectoral spillover effects are negligible; the steady state will become indeterminate if the equilibrium values of* $\Sigma^0_{X_0X_1}$ *and* $\Sigma^1_{X_0X_1}$ *respectively satisfy:*

a/ *when* $\Gamma^{\mathscr{E}} > 0$, $\Sigma^0_{X_0X_1} < [(\pi^1_{X_1} - \pi^0_{X_1}) + \pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1}]/\pi^{\mathscr{E}_{00}}_{X_1}$ *and* $(\pi^0_{X_1} - \pi^1_{X_1})/\pi^{\mathscr{E}_{11}}_{X_1} < \Sigma^1_{X_0X_1} < (1-\pi^1_{X_1})/\pi^{\mathscr{E}_{11}}_{X_1}$;

b/ *when* $\Gamma^{\mathscr{E}} < 0$, $\Sigma^0_{X_0X_1} > [(\pi^1_{X_1} - \pi^0_{X_1}) + \pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1}]/\pi^{\mathscr{E}_{00}}_{X_1}$ *and* $\Sigma^1_{X_0X_1} > (1-\pi^1_{X_1})/\pi^{\mathscr{E}_{11}}_{X_1}$.

Proof : *Vide* Appendix V.9. △

The main insights of Proposition 4 may then be listed as follows. First and from Proposition 4(i), it is clear that for dominating intersectoral spillover effects, the substitutability properties of the capital good industry will be at the core of the area for local indeterminacy. When it happens to be associated with a relative profit shares reversal between the private and the equilibrium level, *i.e.*, $\Gamma^{\mathscr{E}} > 0$ — *the relative cost of the capital input becomes an increasing function of the relative price of the capital good while the production of the capital output is a decreasing function of the capital stock*, and the equilibrium production of the capital good decreases as a function of its relative price, arbitrarily low orders for the elasticity of substitution of the capital good industry together with relative low orders for the elasticity of substitution of the consumption good — the smaller the intersectoral spillovers stemming from sector 1 and taking place in that sector, the larger the substitutability measures that are compatible with local indeterminacy : more than unitary values for the elasticity of substitution being, *e.g.*, allowed for $\pi^{\mathscr{E}_{10}}_{X_1} > 1 - \pi^1_{X_1}$. Oppositely and still from Proposition 4(i), large values for the elastiticity of substitution of the consumption good industry and arbitrarily large ones for the elasticity of substitution of the capital good one will favour the emergence of local indeterminacies when the latter obtains under the dominating influence of intersectoral spillovers but the production of the capital good remains an increasing function of its price. Under a configuration closer to the earlier concerns of the literature and from Proposition 4(ii), *i.e.*, under dominating intrasectoral spillover effects, it is now the substitutability properties of the consumption industry which will be central to the potential for local indeterminacy: while a relative sectoral profit shares reversal will still be associated with low orders for that value, a configuration where an increasing relation is maintained between the production of the capital good and its relative price will allow, as soon as intrasectoral spillovers in that sector are relatively small, for the emergence of local indeterminacy based upon high orders for the elasticity of substitution in the consumption good industry.

Remark 4: The sole earlier available argument to provide a detailed account of indeterminacy stemming from *intersectoral spillover effects* and which was straightly anchored on parametric forms is due to Nishimura & Venditti [20, 21] who focus on formulations $F^0(X_{00}, X_{10}; X_{10}^{\mathscr{E}}, X_{11}^{\mathscr{E}}) = (X_{00})^{\alpha_{00}}(X_{10})^{\alpha_{10}}[\theta X_{10}^{\mathscr{E}} + (1-\theta)X_{11}^{\mathscr{E}}]^a$, $\theta \in [0,1]$, $\alpha_{00}+\alpha_{10}+a = 1$ and $F^1(X_{01}, X_{11}) = (X_{01})^{\alpha_{01}}(X_{11})^{\alpha_{11}}$, $\alpha_{01} + \alpha_{11} = 1$. The specificity of such a range of assumptions with respect to the present one arises from the case of *aggregate capital stock* externalities $\theta = 1/2$ which cannot be embedded into the current framework. The current argument departs from theirs from a constant returns to scale assumption at the private level and unrestricted formulations for the production technologies, but also the *price-related* nature of spillover effects. In that latter respect, a formulation whose properties would more closely mimic theirs is the specification $F^j(X_{0j}, X_{1j})H^j(X_0^{\mathscr{E}}, X_1^{\mathscr{E}})$ sketched through Remark 1. ♢

Remark 5: The formulation (1) for the production technologies is somewhat specific in postulating a multiplicative separability property between the intrasectoral and the intersectoral external blocks in the production technologies. Though it could be modified to a formulation $F^j(X_{0j}, X_{1j})G^j(X_{1j}^{\mathscr{E}}/X_{0j}^{\mathscr{E}}, X_{1j'}^{\mathscr{E}}/X_{0j'}^{\mathscr{E}})$ where $G^j(\cdot,\cdot)$ would, *e.g.*, assumed a C.E.S. shape, this would let essentially unaffected the current argument. A more stimulating extension would build from a specification $F^j(X_{0j}, X_{1j}; X_{1j}^{\mathscr{E}}/X_{0j}^{\mathscr{E}}, X_{1j'}^{\mathscr{E}}/X_{0j'}^{\mathscr{E}})$ when $\Sigma_{X_0X_1}^j \neq 1$: at the equilibrium, substitution mechanisms between the private inputs would be directly affected by spillovers and in which regards this could relax the conditions for indeterminacies is unclear at that stage. ♢

An eventual class of wonderings pertains to the articulation of these multiplicity results with the potentially unusual features of the E.P.P.F. described through Lemmas 4 and 5. As for the scope for a convex E.P.P.F., Proposition 3(ii)a/ is unambiguous since it clearly assesses that the occurrence of $(1 - \pi_{X_1}^1 - \pi_{X_1}^{\mathscr{E}_{01}}\Sigma_{X_0X_1}^0 - \pi_{X_1}^{\mathscr{E}_{11}}\Sigma_{X_0X_1}^1)\Gamma^{\mathscr{E}} > 0$, that underlies the details of Proposition 4, cannot be reconciled with the satisfaction of Lemma 4(ii) when $\pi_{X_1}^0 > \pi_{X_1}^1$.[6] In opposition to this, the consideration of Lemma 5 and of the associated modified definition of substitutability mechanisms that underlie the introduction of an E.P.P.F. uncovers an interesting articulation:

<u>Corollary 2</u> [An-E.P.P.F. Understanding of the Multiplicity Results]. *Under Assumptions* P.1, T.1-6, E.1-3, *consider an interior steady state, then it is locally indeterminate if* $\pi_{X_1}^1 > \pi_{X_1}^0$, $|\pi_{X_1}^1 - \pi_{X_1}^0|(1 + 1/\delta) > [1/\delta - (1-\eta)](1 - \pi_{X_1}^1)$, $\Sigma_{Y^0Y^1}^{\mathscr{E}} < 0$ *and*

[6] Though the consideration of Proposition 3(ii)b/ would deliver more conclusive results in this perspective, it has been argued above as being non-generic and thus not well-suited for the generality which has been sought throughout this contribution.

$1 - \pi^1_{X_1} - \pi^{\mathscr{E}_{01}}_{X_1} \Sigma^0_{X_0 X_1} - \pi^{\mathscr{E}_{11}}_{X_1} \Sigma^1_{X_0 X_1} < 0.$

Proof : Follows from the consideration of Proposition 3(ii)a/ and Lemma 5(iii). △

Otherwise stated, it is the potential for complementary outputs and the increased role factors substitutability that derive from $\Sigma^{\mathscr{E}}_{Y^0 Y^1} < 0$ and $\Sigma^{\mathscr{E}}_{\bar{X}_0 \bar{X}_1} > \Sigma_{\bar{X}_0 \bar{X}_1}$ that lies at the very core of the scope for local multiplicities.

Sections II, III and IV having been strongly concerned with methodological issues and the interest of an E.P.P.F. approach, the subsequent section will finally be aimed at examplifying its potential interest for the appraisal of suboptimal heterogeneous goods environments under alternative ranges of assumptions on technology and preferences.

V – A Comparison with Related Literature & Some Extensions

V.1 – Appraising an Alternative Conception of Spillover Effects through an E.P.P.F. Approach: the «Sector-Specific» Assumption

A generalised — unrestricted — formulation of the class of externalities augmented Cobb-Douglas production technologies introduced by Benhabib & Nishimura [10] and then generalised to a n goods argument, an unbounded growth environment and to discrete time formulations by respectively Benhabib & Nishimura [11], Benhabib, Meng & Nishimura [7] and Benhabib, Nishimura & Venditti [12], Nishimura & Venditti [29, 30] and surveyed by Nishimura & Venditti [31, 32], arises from consumption and investment good sectors which would respectively satisfy:

(12a) $Y^0_t \leq F^0\big(X_{00,t}, X_{10,t}; X^{\mathcal{E}}_{00,t}, X^{\mathcal{E}}_{10,t}\big),$

(12b) $Y^1_t \leq F^1\big(X_{01,t}, X_{11,t}; X^{\mathcal{E}}_{01,t}, X^{\mathcal{E}}_{11,t}\big),$

where $F^0(\cdot,\cdot; X^{\mathcal{E}}_{00,t}, X^{\mathcal{E}}_{10,t})$ and $F^1(\cdot,\cdot; X^{\mathcal{E}}_{01,t}, X^{\mathcal{E}}_{11,t})$ are supposed to fit Assumptions T.1-6 but the fact that they are now both assumed to be such that their associated scale elasticities satisfy:

$$\mathscr{S}^j = \big[(\partial F^j/\partial X_{0j})X_{0j} + (\partial F^j/\partial X_{1j})X_{1j}\big]/F^j < 1.$$

The consideration of a competitive equilibrium with externalities will however modify this decreasing returns to scale property that will turn into a constant returns one at the equilibrium level, namely and for $\mathscr{S}^{j,\mathscr{E}}$ that features, along the E.P.P.F. range of ideas, the scale elasticity along a competitive equilibrium with externalities:

$$\begin{aligned}\mathscr{S}^{j,\mathscr{E}} = \big[(\partial F^j/\partial X_{0j})X_{0j} + (\partial F^j/\partial X_{1j})X_{1j} + (\partial F^j/\partial X^{\mathscr{E}}_{0j})X^{\mathscr{E}}_{0j} \\ + (\partial F^j/\partial X^{\mathscr{E}}_{1j})X^{\mathscr{E}}_{1j}\big]/F^j = 1.\end{aligned}$$

Otherwise stated, it has systematically been assumed by these authors that $\pi^j_{X_0} + \pi^j_{X_1} < 1$ while $\pi^j_{X_0} + \pi^j_{X_1} + \pi^{\mathscr{E}}_{X_{0j}} + \pi^{\mathscr{E}}_{X_{1j}} = 1$, decreasing returns assumptions being retained at the private level whereas constant returns to scale were assumed to prevail at the level of the competitive equilibrium with externalities. As this already appears in the expression above, external effects in the production set also univocally stem from the specific sector where they appear and this proceeds in the following way:

Assumption E.4: $\forall (X_{0j}^{\mathcal{E}}, X_{1j}^{\mathcal{E}}) \in \mathbb{R}_{+}^{*} \times \mathbb{R}_{+}^{*}$, $\partial F^j / \partial X_{0j}^{\mathcal{E}\,j} > 0$, $\partial F^j / \partial X_{1j}^{\mathcal{E}\,j} > 0$, $j = 0, 1$.

A major difficulty emerges in the perspective of reconducing the gradual formulation of the E.P.P.F. which was completed in sections II, III and through the introduction of $T(\cdot;\cdot,\cdot;\mathcal{E}_t)$, the matrix $\mathscr{M}_{\Xi}$, Lemmas 1-2 and Proposition 1: under Assumption E.4 and for unitary *equilibrium* scale elasticities, $T(\cdot;\cdot,\cdot;\mathcal{E}_t)$ exhibits decreasing

returns to scale. Such a configuration implies, among other complicated features, that the elasticities Hessian matrix $\mathscr{M}_{\Xi}$ is of rank three, a one-to-one articulation between costs and prices and the entailed Stolper-Samuelson theorem being no longer available.

In order to remedy to those difficulties, an alternative — less satisfactory on a theoretical basis — approach will proceed from replacing the integrated view of the P.P.F. and the E.P.P.F. by a sole focus on the properties of the E.P.P.F. Letting again $\Sigma_{X_0 X_1}^{j}$ feature the sectoral elasticity of substitution between X_{0j} and X_{1j} in sector j and, building on their formally related formulations, $\Sigma_{X_{0j} X_{1j}^{\mathscr{E}}}^{\mathscr{E}} := (\partial F^j / \partial X_{0j})(\partial F^j / \partial X_{1j}^{\mathscr{E}}) \;/\; (\partial^2 F^j / \partial X_{0j} \partial X_{1j}^{\mathscr{E}}) F^j$ and $\Sigma_{X_{1j} X_{0j}^{\mathscr{E}}}^{\mathscr{E}} := (\partial F^j / \partial X_{1j})(\partial F^j / \partial X_{0j}^{\mathscr{E}}) \;/\; (\partial^2 F^j / \partial X_{1j} \partial X_{0j}^{\mathscr{E}}) F^j$ denote related *substitution-like*[7] external coefficients, the following symmetry assumption will be essential to the possibility of reaching clear-cut conclusions on the second-order features of the E.P.P.F. with sector-specific externalities:

Assumption E.5: Production technologies are C.E.S.: $\Sigma_{X_0 X_1}^{j} = \Sigma_{X_{0j} X_{1j}^{\mathscr{E}}}^{\mathscr{E}} = \Sigma_{X_{1j} X_{0j}^{\mathscr{E}}}^{\mathscr{E}}$, $j = 0, 1$.

Considering a competitive equilibrium with externalities defined under Assumptions P.1, E.4-5 and C.E.S. production technologies, it can be shown that, for

$$
\begin{aligned}
Z^{\mathscr{E}} &:= \Sigma_{\bar{X}_0 \bar{X}_1} \pi_{\bar{X}_0} \pi_{\bar{X}_1} \\
&\qquad + \left[\left(\pi_{X_0}^{0} \pi_{Y^0} / \pi_{\bar{X}_0} \right) \pi_{X_1}^{\mathscr{E}_{00}} + \left(\pi_{X_0}^{0} \pi_{\bar{X}_1} / \pi_{X_1}^{0} \pi_{\bar{X}_0} \right) \left(\pi_{X_1}^{0} \pi_{Y^0} / \pi_{\bar{X}_1} \right) \pi_{X_0}^{\mathscr{E}_{00}} \right] \Sigma_{X_0 X_1}^{0} \\
&\qquad + \left[\left(\pi_{X_0}^{1} \pi_{Y^1} / \pi_{\bar{X}_0} \right) \pi_{X_1}^{\mathscr{E}_{11}} + \left(\pi_{X_0}^{0} \pi_{\bar{X}_1} / \pi_{X_1}^{0} \pi_{\bar{X}_0} \right) \left(\pi_{X_1}^{1} \pi_{Y^1} / \pi_{\bar{X}_1} \right) \pi_{X_0}^{\mathscr{E}_{11}} \right] \Sigma_{X_0 X_1}^{1}, \\
\Gamma^{\mathscr{E}} &:= \left(\pi_{X_1}^{1} + \pi_{X_1}^{\mathscr{E}_{11}} \right) \left(\pi_{X_0}^{0} + \pi_{X_0}^{\mathscr{E}_{00}} \right) - \left(\pi_{X_1}^{0} + \pi_{X_1}^{\mathscr{E}_{00}} \right) \left(\pi_{X_0}^{1} + \pi_{X_0}^{\mathscr{E}_{11}} \right) \\
&= \left(\pi_{X_1}^{1} + \pi_{X_1}^{\mathscr{E}_{11}} \right) - \left(\pi_{X_1}^{0} + \pi_{X_1}^{\mathscr{E}_{00}} \right),
\end{aligned}
$$

[7]These expressions do not anymore satisfy the *symmetry dimension* associated with the canonical definition of the elasticities of substitution.

the matrix of the weighted second-order elasticities of the E.P.P.F. in turn states as:

$$[\mathcal{M}_{\Xi^{\mathscr{E}}}] = \begin{bmatrix} 1 \\ -\dfrac{\pi^0_{X_1} + \pi^{\mathscr{E}_{00}}_{X_1}}{\Gamma^{\mathscr{E}}} \\ \dfrac{1 - \pi^0_{X_1} - \pi^{\mathscr{E}_{00}}_{X_1}}{\Gamma^{\mathscr{E}}} \end{bmatrix} \left[-\frac{\left(\pi^1_{X_1}\pi^0_{X_0} - \pi^1_{X_0}\pi^0_{X_1}\right)\Gamma^{\mathscr{E}}}{\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}Z^{\mathscr{E}}} \right]$$
$$\times \begin{bmatrix} 1 & -\dfrac{\pi^0_{X_1}}{\pi^1_{X_1}\pi^0_{X_0} - \pi^1_{X_0}\pi^0_{X_1}} & \dfrac{1 - \pi^0_{X_1}}{\pi^1_{X_1}\pi^0_{X_0} - \pi^1_{X_0}\pi^0_{X_1}} \end{bmatrix}.$$

Facing with the scope for a convex E.P.P.F., it derives that $\Xi^{\mathscr{E}}_{Y^1Y^1} > 0$ for $\pi^1_{X_1}\pi^0_{X_0} - \pi^1_{X_0}\pi^0_{X_1} < 0$ and $\left(\pi^0_{X_0} + \pi^{\mathscr{E}_{00}}_{X_0}\right)\left(\pi^1_{X_1} + \pi^{\mathscr{E}_{11}}_{X_1}\right) - \left(\pi^0_{X_1} + \pi^{\mathscr{E}_{00}}_{X_1}\right)\left(\pi^1_{X_0} + \pi^{\mathscr{E}_{11}}_{X_0}\right) > 0$ since $Z^{\mathscr{E}} > 0$ but that $\Xi^{\mathscr{E}}_{X_0X_0} < 0$ and $\Xi^{\mathscr{E}}_{X_1X_1} < 0$ are to continue to hold since $1 - \pi^0_{X_1} - \pi^{\mathscr{E}_{00}}_{X_1} = \pi^0_{X_0} + \pi^{\mathscr{E}_{00}}_{X_0} > 0$. A convex E.P.P.F. would then obtain for $\Xi^{\mathscr{E}}_{Y^1Y^1} + \Xi^{\mathscr{E}}_{X_0X_0} + \Xi^{\mathscr{E}}_{X_1X_1} > 0$ but the underlying conditions are more stringent than for the current argument. Also and along the range of ideas of Section III, the competitive equilibrium with externalities trade-offs between the inputs and the outputs are now described by

$$\begin{aligned}\Sigma^{\mathscr{E}}_{\bar{X}_0\bar{X}_1} &= \Sigma_{\bar{X}_0\bar{X}_1} \\ &\quad + \left[\left(\pi^0_{X_0}\pi_{Y^0}/\pi_{\bar{X}_0}\right)\pi^{\mathscr{E}_{00}}_{X_1} + \left(\pi^0_{X_0}\pi_{\bar{X}_1}/\pi^0_{X_1}\pi_{\bar{X}_0}\right)\left(\pi^0_{X_1}\pi_{Y^0}/\pi_{\bar{X}_1}\right)\pi^{\mathscr{E}_{00}}_{X_0}\right]\Sigma^0_{X_0X_1} \\ &\quad + \left[\left(\pi^1_{X_0}\pi_{Y^1}/\pi_{\bar{X}_0}\right)\pi^{\mathscr{E}_{11}}_{X_1} + \left(\pi^0_{X_0}\pi_{\bar{X}_1}/\pi^0_{X_1}\pi_{\bar{X}_0}\right)\left(\pi^1_{X_1}\pi_{Y^1}/\pi_{\bar{X}_1}\right)\pi^{\mathscr{E}_{11}}_{X_0}\right]\Sigma^1_{X_0X_1}, \\ \Sigma^{\mathscr{E}}_{Y^0Y^1} &= \frac{\Sigma^{\mathscr{E}}_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1}}{\left(\pi_{X_1} - \pi^0_{X_1}\right)\left[\left(\pi^1_{X_1} + \pi^{\mathscr{E}_{11}}_{X_1}\right) - \left(\pi^0_{X_1} + \pi^{\mathscr{E}_{00}}_{X_1}\right)\right]},\end{aligned}$$

a configuration with $\Sigma^{\mathscr{E}}_{Y^0Y^1} < 0$ and $\Sigma^{\mathscr{E}}_{\bar{X}_0\bar{X}_1} > \Sigma_{\bar{X}_0\bar{X}_1}$ being again available for $\pi^1_{X_1}\pi^0_{X_0} - \pi^1_{X_0}\pi^0_{X_1} < 0$ and $\left(\pi^0_{X_0} + \pi^{\mathscr{E}_{00}}_{X_0}\right)\left(\pi^1_{X_1} + \pi^{\mathscr{E}_{11}}_{X_1}\right) - \left(\pi^0_{X_1} + \pi^{\mathscr{E}_{00}}_{X_1}\right)\left(\pi^1_{X_0} + \pi^{\mathscr{E}_{11}}_{X_0}\right) > 0$.
Facing then with the area for multiplicities and from a straightforward adaptation of the line of reasoning developed for the proof of Proposition 1, it reveals that the treatment of the existence/uniqueness issue is significantly simplified with respect to the one of the present contribution since, for $\lim_{X_1/X_0 \to 0}\left(\pi_{\bar{X}_1}\right)^{\mathscr{E}}/\left(\pi_{Y^1}\right)^{\mathscr{E}} > 1/\delta - (1-\eta) > \lim_{X_1/X_0 \to \infty}\left(\pi_{\bar{X}_1}\right)^{\mathscr{E}}/\left(\pi_{Y^1}\right)^{\mathscr{E}}$, there exists a steady state that is unique.

Following the approach of Proposition 3 and adaptating the expressions of the system (11) to the contents of Proposition 5, it is readily proved that when such an E.P.P.F. is considered in the neighbourhood of a steady state and $\Sigma^c \to \infty$ is further assumed, the explicit formulation of the eigenvalues is available as:

$$\text{(13a)} \quad \nu_1 = \frac{\delta(1-\eta)\left(\pi^1_{X_1}\pi^0_{X_0} - \pi^0_{X_1}\pi^1_{X_0}\right) + [1 - \delta(1-\eta)]\pi^0_{X_0}}{\pi^1_{X_1}\pi^0_{X_0} - \pi^0_{X_1}\pi^1_{X_0}}\frac{1}{\delta},$$

$$\text{(13b)} \quad \nu_2 = \frac{\left(\pi^1_{X_1} + \pi^{\mathscr{E}_{11}}_{X_1}\right) - \left(\pi^0_{X_1} + \pi^{\mathscr{E}_{00}}_{X_1}\right)}{\delta(1-\eta)\left[\left(\pi^1_{X_1} + \pi^{\mathscr{E}_{11}}_{X_1}\right) - \left(\pi^0_{X_1} + \pi^{\mathscr{E}_{00}}_{X_1}\right)\right] + [1 - \delta(1-\eta)]\left(1 - \pi^0_{X_1} - \pi^{\mathscr{E}_{00}}_{X_1}\right)}.$$

The area for local indeterminacies and the role of factors substitutability in that perspective are then delimited by first noticing that, for $\pi^1_{X_1}/\pi^1_{X_0} > \pi^0_{X_1}/\pi^0_{X_0}$, the steady state cannot be

locally indeterminate. In opposition to this and for $\pi^0_{X_1}/\pi^0_{X_0} > \pi^1_{X_1}/\pi^1_{X_0}$, further let $\eta = 1$, a sufficient condition for local indeterminacy boils down to the holding of $\pi^1_{X_1}/\pi^1_{X_0} - \pi^0_{X_1}/\pi^0_{X_0} < 0$ and $(\pi^1_{X_1} + \pi^{\mathscr{E}_{11}}_{X_1})/(\pi^1_{X_0} + \pi^{\mathscr{E}_{11}}_{X_0}) - (\pi^0_{X_1} + \pi^{\mathscr{E}_{00}}_{X_1})/(\pi^0_{X_0} + \pi^{\mathscr{E}_{00}}_{X_0}) > 0$.
Some limitations to the actual revelance of such a result however emerge:

– First, the above insights directly result from the retainment of $\mathscr{S}^j < 1$ and $\mathscr{S}^{j,\mathscr{E}} = 1$, $j = 0, 1$ that, when combined with Assumption E.8, implies decreasing returns to scale at the private level. Such microfoundations may not be entirely acceptable as the departure from constant returns to scale at the private level entails the existence of profits which are not explicitly taken into account in the analysis. Notwithstanding such general concerns about the actual micro-foundations of such a conjunction on returns to scale, it may be observed that the approach clearly rests upon a departure from a standard constant returns to scale hypothesis at the private level — and thus an *atypical* configuration — in order to raise irregular conclusions at the equilibrium level, that limits, at least a first sight, their actual interest.

– As soon as any plausible *intersectoral* dimension for external effects is considered together with an Assumption such as E.4, be it with or without E.5, any possibility of a clear-cut argument resting upon factors intensities in the spirit of Propositions 1–3 definitively becomes out of order.

– Perhaps more fundamentally , *any first-order role is erased for sectoral substitutability in the determinacy issue*, *i.e.*, and as established in [15, 18], *exactly the same unappealing features as a convex bi-sectoral optimal growth environment*. But this is an artificial and direct outcome of Assumption E.5 — this boils down to assume, along the recent contributions of Nishimura & Venditti [25, 26] and their surveys [27, 28], C.E.S. parametric formulations for the two sectoral production technologies, $\Sigma^j_{X_0X_1}$, $j = 0, 1$ being thus a constant — that would not generalise to an arbitrary theoretical production technology under the «sectors-specific» assumption E.4.

V.2 – Allowing for Non-Linear Utilities

V.2.1 – A Stationary Environment

While the preceding statements enlighten, through Propositions 2, 3 and 4, the role of the heterogeneous measures of factors substitutability, namely $\Sigma^0_{X_0X_1}$ and $\Sigma^1_{X_0X_1}$, in the determinacy properties of a competitive economy, they make no explicit account of the role of aggregate factors substitutability, *i.e.*, $\Sigma_{\bar{X}_0\bar{X}_1}$. As previously mentioned, this is a direct byproduct of the retainment of $\Sigma^c \to \infty$ on the intertemporal preferences of the agent. An undesirable implication of such a limitation results from the direct articulation between this elasticity of substitution between the aggregate values of the inputs and the elasticity of substitution between the two outputs $\Sigma_{Y^0Y^1}$: otherwise stated, by letting $\Sigma^c \to \infty$, one cannot directly appraise the role of finite values for the elasticity of substitution between the two outputs and thus of a nonlinear P.P.F. in the emergence of local indeterminacies. The formal difficulty in relaxing such an assumption on preferences essentially results from the expression of the equilibrium first-order derivatives that are concerned with the *level* component (i) of the E.P.P.F. in Definition 2 — letting $\Sigma^c \to \infty$ oppositely allowed to specialise the analysis on the *prices* component (ii) of the E.P.P.F. in Definition 2. Considering indeed Definition 2(i) and introducing, along $\Xi^{\mathscr{E}}_{Y^1Y^1}$ through (12) and from Lemma 1(i), *e.g.*,

$$\begin{aligned}\pi^{\mathscr{E}}_{\bar{X}_0} &:= \frac{\partial T^{\mathscr{E}}/\partial X_0)X_0}{T^{\mathscr{E}} - \left(\partial T/\partial Y^1\right)^{\mathscr{E}} Y^1} \\ &= \pi_{X_0} + \pi_{X^{\mathscr{E}}_{10}/X^{\mathscr{E}}_{00}} \Xi_{(\widehat{X}_{10}/\widehat{X}_{00})X_0} + \pi_{X^{\mathscr{E}}_{11}/X^{\mathscr{E}}_{01}} \Xi_{(\widehat{X}_{11}/\widehat{X}_{01})X_0},\end{aligned}$$

it is immediatly derived, along the constructive approach to Proposition 1 and letting $V^{\mathscr{E}} := \left(\pi_{Y^0}\pi^{\mathscr{E}_{00}}_{X_1} + \pi_{Y^1}\pi^{\mathscr{E}_{01}}_{X_1}\right)\Sigma^0_{X_0X_1} + \left(\pi_{Y^0}\pi^{\mathscr{E}_{10}}_{X_1} + \pi_{Y^1}\pi^{\mathscr{E}_{11}}_{X_1}\right)\Sigma^1_{X_0X_1}$, that:

$$\begin{bmatrix} \pi^{\mathscr{E}}_{Y^1} \\ \pi^{\mathscr{E}}_{X_0} \\ \pi^{\mathscr{E}}_{X_1} \end{bmatrix}' = \begin{bmatrix} \pi_{Y^1} - \dfrac{V^{\mathscr{E}}\left(\pi^1_{X_1} - \pi^0_{X_1}\right)}{Z^{\mathscr{E}}\Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1}} \\ \pi_{\bar{X}_0} - \dfrac{V^{\mathscr{E}}\pi^0_{X_1}}{Z^{\mathscr{E}}\Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1}} \\ \pi_{\bar{X}_1} + \dfrac{V^{\mathscr{E}}\left(1 - \pi^0_{X_1}\right)}{Z^{\mathscr{E}}\Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1}} \end{bmatrix}'.$$

For $\mathscr{W}(\nu) = (\nu - \nu_1)(\nu - \nu_2) = 0$ the characteristic polynomial of the Jacobian Matrix, necessary and sufficient conditions for local indeterminacy list as the joint holding of $\mathscr{W}(0) < 1$, $\mathscr{W}(-1) > 0$, $\mathscr{W}(1) > 0$. The first of these conditions is of special interest — it is also the conceptually simplest one since is merely invoves one coefficient of the characteristic polynomial,*i.e.*, the product of the eigenvalues — as it corresponds to the breaking of one of the building blocks of optimal growth theory, *i.e.*, the pair roots structure of the Jacobian matrix. While its canonical value usually states as $1/\delta$, it is readily shown[8] that the condition $\mathscr{W}(0) < 1$ here writes down as:

$$\begin{aligned}&\Bigg(\eta\left\{\frac{1/\Sigma^c}{\pi_{Y^0}}\left[\pi_{\bar{X}_1} - \frac{V^{\mathscr{E}}\left(1 - \pi^0_{X_1}\right)}{Z^{\mathscr{E}}\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}}\right] + \frac{\Gamma^{\mathscr{E}}\left(1 - \pi^0_{X_1}\right)\pi_{\bar{X}_1}}{Z^{\mathscr{E}}\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}}\right\} \\ &+ (1-\eta)\left\{\frac{1/\Sigma^c}{\pi_{Y^0}}\left[\pi_{Y^1} - \frac{V^{\mathscr{E}}\left(\pi^1_{X_1} - \pi^0_{X_1}\right)}{Z^{\mathscr{E}}\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}}\right] + \frac{\Gamma^{\mathscr{E}}\left(\pi^1_{X_1} - \pi^0_{X_1}\right)\pi_{\bar{X}_1}}{Z^{\mathscr{E}}\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}}\right\}\Bigg)^{-1} \\ &\times\Bigg(\delta\eta\frac{\pi_{\bar{X}_1}}{\pi_{Y^1}}\left\{\frac{1/\Sigma^c}{\pi_{Y^0}}\left[\pi_{Y^1} - \frac{V^{\mathscr{E}}\left(\pi^1_{X_1} - \pi^0_{X_1}\right)}{Z^{\mathscr{E}}\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}}\right] + \frac{W^{\mathscr{E}}\left(\pi^1_{X_1} - \pi^0_{X_1}\right)\pi_{Y^1}}{Z^{\mathscr{E}}\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}}\right\} \\ &+ \delta(1-\eta)\left\{\frac{1/\Sigma^c}{\pi_{Y^0}}\left[\pi_{Y^1} - \frac{V^{\mathscr{E}}\left(\pi^1_{X_1} - \pi^0_{X_1}\right)}{Z^{\mathscr{E}}\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}}\right] + \frac{\Gamma^{\mathscr{E}}\left(\pi^1_{X_1} - \pi^0_{X_1}\right)\pi_{\bar{X}_1}}{Z^{\mathscr{E}}\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}}\right\}\Bigg) < 1,\end{aligned}$$

for $W^{\mathscr{E}} := 1 - \pi^0_{X_1} - \pi^{\mathscr{E}_{00}}_{X_1}\Sigma^0_{X_0X_1} - \pi^{\mathscr{E}_{10}}_{X_1}\Sigma^1_{X_0X_1}$. Otherwise stated, even for one of three coefficients, an integrated view of local indeterminacy that makes explicit account of intertemporal substitutability in consumption, aggregate measures of outputs substitutability, asymmetric sectoral understandings of factors substitutability and intra versus intersectoral spillovers

[8]The details are available upon request.

sounds unreachable. A much more appropriate environment in that perspective is ought to build from a two-period overlapping generations setting of the kind considered by Cazzavillan [13] in an appraisal of an aggregate increasing returns technological set. The simplicity of the basic understanding of intertemporal preferences inherent to these structures should permit to circumvent the current difficulties associated with the appraisal of $\Sigma^c < \infty$. ♢

V.2.2 – Unbounded Growth

As stressed through the above section, a limit to the insights of Proposition 3 and Corollary 2 stems from the assumption $\Sigma^c \to \infty$. An alternative and simpler way of handling finite values for the intertemporal elasticity of substitution would stem from the characterisation of unbounded equilibrium growth solutions within a two capital goods environment, one of which would be a mixed good, say 1, that can be consumed or accumulated, while the other, say 2, is a pure accumulation one. The Production Possibility Frontier would then formulate at date $t \geq 0$ as: $Y_t^0 = T(Y_t^1, Y_t^2; X_{1,t}, X_{2,t}; \mathscr{E}_t)$ Along the approach of Proposition 1, it is readily shown that the Hessian elasticities matrix states as

$$[\mathscr{M}_{\Xi^{\mathscr{E}\mathscr{G}}}] = \begin{bmatrix} 0 \\ 1 \\ -\dfrac{\pi^1_{X_1} + \pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_1X_2} - \pi^{\mathscr{E}_{21}}_{X_1}\Sigma^2_{X_1X_2}}{(\pi^1_{X_1} - \pi^2_{X_1}) + (\pi^{\mathscr{E}_{11}}_{X_1} - \pi^{\mathscr{E}_{12}}_{X_1})\Sigma^1_{X_1X_2} + (\pi^{\mathscr{E}_{21}}_{X_2} - \pi^{\mathscr{E}_{22}}_{X_1})\Sigma^2_{X_1X_2}} \\ \dfrac{1 - \pi^1_{X_1} - \pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_1X_2} + \pi^{\mathscr{E}_{21}}_{X_1}\Sigma^2_{X_1X_2}}{(\pi^1_{X_1} - \pi^2_{X_1}) + (\pi^{\mathscr{E}_{11}}_{X_1} - \pi^{\mathscr{E}_{12}}_{X_1})\Sigma^1_{X_1X_2} + (\pi^{\mathscr{E}_{21}}_{X_2} - \pi^{\mathscr{E}_{22}}_{X_1})\Sigma^2_{X_1X_2}} \end{bmatrix}$$

$$\times \left[-\frac{(\pi^1_{X_1} - \pi^2_{X_1})\Gamma^{\mathscr{E}\mathscr{G}}}{\pi_{\bar{X}_1}\pi_{\bar{X}_2}\Sigma_{\bar{X}_1\bar{X}_2}Z^{\mathscr{E}\mathscr{G}}}\right] \begin{bmatrix} 0 & 1 & -\dfrac{\pi^1_{X_1}}{\pi^1_{X_1} - \pi^2_{X_1}} & \dfrac{1 - \pi^1_{X_1}}{\pi^1_{X_1} - \pi^2_{X_1}} \end{bmatrix},$$

for $Z^{\mathscr{E}\mathscr{G}}$ a coefficient formally related to $Z^{\mathscr{E}}$ in Proposition 1. As this is proved in Drugeon [17], a block diagonal structure is recovered for the Jacobian Matrix and the local behaviour around an equilibrium growth ray $\kappa \in \mathbb{R}^*_+$ will be governed by a spectrum made of $\nu_1 = 1$, $\nu_2 = 1/\delta\kappa^{1-1/\Sigma^c}$ and

$$\begin{aligned} \nu_3 = {} & \frac{1}{\delta\kappa^{1-1/\Sigma^c}} + \left[\frac{1 - \pi^2_{X_1}}{\pi^1_{X_1} - \pi^2_{X_1}} - 1\right]\frac{[\kappa^{1/\Sigma^c} - \delta(1 - \eta_1)]}{\delta\kappa} \\ & + \left[\frac{\pi^1_{X_1}}{\pi^1_{X_1} - \pi^2_{X_1}} - 1\right]\frac{[\kappa^{1/\Sigma^c} - \delta(1 - \eta_2)]}{\delta\kappa}, \\ \nu_4 = {} & \Bigg\{1 + \left[1 - \frac{\delta(1 - \eta_2)}{\kappa^{1/\Sigma^c}}\right]\left[\frac{\pi^1_{X_1} + \pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_1X_2} + \pi^{\mathscr{E}_{21}}_{X_1}\Sigma^2_{X_1X_2}}{\Gamma^{\mathscr{E}\mathscr{G}}} - 1\right] \\ & + \left[1 - \frac{\delta(1 - \eta_1)}{\kappa^{1/\Sigma^c}}\right]\left[\frac{1 - \pi^2_{X_1} - \pi^{\mathscr{E}_{12}}_{X_1}\Sigma^1_{X_1X_2} - \pi^{\mathscr{E}_{22}}_{X_1}\Sigma^2_{X_1X_2}}{\Gamma^{\mathscr{E}\mathscr{G}}} - 1\right]\Bigg\}^{-1}, \end{aligned}$$

where $1/\delta\kappa^{1-1/\Sigma^c} > 1$, $\kappa^{1/\Sigma^c} > \delta(1 - \eta_1)$ and $\kappa^{1/\Sigma^c} > \delta(1 - \eta_2)$. A detailed characterisation, noticeably the need for $\pi^2_{X_1} > \pi^1_{X_1}$ in order for indeterminate growth rays to emerge, is available in Drugeon [17], noticeably in the non-trivial role of the heterogenous depreciation rates of the capital goods in that perspective. ♢

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VII - Proofs

VII.1 – Proof of Lemma 1.

(i) Two relations between the technological parameters shall extensively be used in the subsequent argument:

$$\frac{\pi^0_{X_1}/(1-\pi^0_{X_1})}{\pi^1_{X_1}/(1-\pi^1_{X_1})}=\frac{\mu_{01}/(1-\mu_{01})}{\mu_{11}/(1-\mu_{11})},$$
$$\frac{1-\pi_{Y^1}}{\pi_{Y^1}}=\frac{\pi^1_{X_1}}{\pi^0_{X_1}}\frac{1-\mu_{11}}{\mu_{11}}=\frac{1-\pi^1_{X_1}}{1-\pi^0_{X_1}}\frac{1-\mu_{01}}{\mu_{01}}.$$

For $\epsilon := [1 \quad 1]'$, letting Ω^i, $\mathscr{Q}$, $\mathscr{X}_{ij}$ and $\mathscr{X}^{\mathscr{E}}_{1j}-\mathscr{X}^{\mathscr{E}}_{0j}$, $i,j=0,1$, respectively denote the differentiated expressions of the rental rate of input i, the relative price of the capital good, the amount of input i used in sector j and of the ratio $X^{\mathscr{E}}_{1j}/X^{\mathscr{E}}_{0j}$, the system of sectoral optimality conditions implies :

$$\begin{bmatrix}\Omega^0\\ \Omega^1\end{bmatrix}=\begin{bmatrix}\dfrac{\pi^0_{X_1}}{\Sigma^0_{X_0X_1}}\\ -\dfrac{\pi^0_{X_0}}{\Sigma^0_{X_0X_1}}\end{bmatrix}(\mathscr{X}_{10}-\mathscr{X}_{00})+\epsilon\pi^{\mathscr{E}_{00}}_{X_1}(\mathscr{X}^{\mathscr{E}}_{10}-\mathscr{X}^{\mathscr{E}}_{00})+\epsilon\pi^{\mathscr{E}_{10}}_{X_1}(\mathscr{X}^{\mathscr{E}}_{11}-\mathscr{X}^{\mathscr{E}}_{01})$$

$$\begin{bmatrix}\Omega^0\\ \Omega^1\end{bmatrix}-\epsilon\mathscr{Q}=\begin{bmatrix}\dfrac{\pi^1_{X_1}}{\Sigma^1_{X_0X_1}}\\ -\dfrac{\pi^1_{X_0}}{\Sigma^1_{X_0X_1}}\end{bmatrix}(\mathscr{X}_{11}-\mathscr{X}_{01})+\epsilon\pi^{\mathscr{E}_{01}}_{X_1}(\mathscr{X}^{\mathscr{E}}_{10}-\mathscr{X}^{\mathscr{E}}_{00})+\epsilon\pi^{\mathscr{E}_{11}}_{X_1}(\mathscr{X}^{\mathscr{E}}_{11}-\mathscr{X}^{\mathscr{E}}_{01})$$

In parallel to this, letting $\mathscr{Y}^j$, $\mathscr{X}_i$ and $\mathscr{M}_{0j}$, $i,j=0,1$, respectively denote the differentiated expressions of the production of good j, the available amount of input j and the share of input 0 used for the production of good j, equilibrium levels of production give:

$$\begin{bmatrix}\mathscr{Y}^0\\ \mathscr{Y}^1\end{bmatrix}=\begin{bmatrix}\mathscr{M}_{00}\\ \mathscr{M}_{01}\end{bmatrix}+\epsilon\mathscr{X}_0+\begin{bmatrix}\pi^0_{X_1} & 0\\ 0 & \pi^1_{X_1}\end{bmatrix}\begin{bmatrix}\mathscr{X}_{10}-\mathscr{X}_{00}\\ \mathscr{X}_{11}-\mathscr{X}_{01}\end{bmatrix}+\begin{bmatrix}\pi^{\mathscr{E}_{00}}_{X_1} & \pi^{\mathscr{E}_{10}}_{X_1}\\ \pi^{\mathscr{E}_{01}}_{X_1} & \pi^{\mathscr{E}_{11}}_{X_1}\end{bmatrix}\begin{bmatrix}\mathscr{X}^{\mathscr{E}}_{10}-\mathscr{X}^{\mathscr{E}}_{00}\\ \mathscr{X}^{\mathscr{E}}_{11}-\mathscr{X}^{\mathscr{E}}_{01}\end{bmatrix},$$

for the differentiated expressions of factors shares that derive from the full employment equation $(1-\mu_{01})(X_{10}/X_{00})+\mu_{01}(X_{11}/X_{01})=X_1/X_0$ as:

$$\begin{bmatrix}\mathscr{M}_{00}\\ \mathscr{M}_{01}\end{bmatrix}\left[\mu_{11}-\frac{\mu_{10}\mu_{01}}{\mu_{00}}\right]=\begin{bmatrix}\dfrac{\mu_{10}\mu_{01}}{\mu_{00}} & \dfrac{\mu_{11}\mu_{01}}{\mu_{00}}\\ -\mu_{10} & -\mu_{11}\end{bmatrix}\begin{bmatrix}\mathscr{X}_{10}-\mathscr{X}_{00}\\ \mathscr{X}_{11}-\mathscr{X}_{01}\end{bmatrix}+\begin{bmatrix}-\dfrac{\mu_{01}}{\mu_{00}}\\ 1\end{bmatrix}[\mathscr{X}_1-\mathscr{X}_0].$$

Hinging upon the relation $\mu_{ij}/\pi_{Y^j}=\pi^j_{X_i}/\pi_{\bar{X}_i}$, $i,j=0,1$, it is obtained that this reformulates to:

$$\begin{bmatrix}\mathscr{M}_{00}\\ \mathscr{M}_{01}\end{bmatrix}=\left[\frac{\pi^1_{X_0}\pi_{Y^1}}{\pi_{\bar{X}_1}}\left(\frac{\pi^1_{X_1}}{\pi^1_{X_0}}-\frac{\pi^0_{X_1}}{\pi^0_{X_0}}\right)\right]^{-1}\left\{\begin{bmatrix}-\dfrac{\pi^1_{X_0}\pi_{Y^1}}{\pi^0_{X_0}\pi_{Y^0}}\\ 1\end{bmatrix}[\mathscr{X}_1-\mathscr{X}_0]\right.$$
$$\left.+\begin{bmatrix}\dfrac{\pi^0_{X_1}\pi^1_{X_0}}{\pi_{\bar{X}_1}\pi^0_{X_0}} & \dfrac{\pi^1_{X_1}\pi^1_{X_0}\pi_{Y^1}\pi_{Y^1}}{\pi_{\bar{X}_1}\pi^0_{X_0}\pi_{Y^0}}\\ -\dfrac{\pi^0_{X_1}\pi_{Y^0}}{\pi_{\bar{X}_1}} & -\dfrac{\pi^1_{X_1}\pi_{Y^1}}{\pi_{\bar{X}_1}}\end{bmatrix}\begin{bmatrix}\mathscr{X}_{10}-\mathscr{X}_{00}\\ \mathscr{X}_{11}-\mathscr{X}_{01}\end{bmatrix}\right\}.$$

...A.1

It is derived that:

$$\begin{bmatrix} \epsilon\big(\Omega^0-\Omega^1\big) \\ \mathscr{Q} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\Sigma^0_{X_0X_1}} & 0 \\ 0 & \dfrac{1}{\Sigma^1_{X_0X_1}} \\ \dfrac{\pi^0_{X_1}}{\Sigma^0_{X_0X_1}} & -\dfrac{\pi^1_{X_1}}{\Sigma^1_{X_0X_1}} \end{bmatrix} \begin{bmatrix} \mathscr{X}_{10}-\mathscr{X}_{00} \\ \mathscr{X}_{11}-\mathscr{X}_{01} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -\big(\pi^{\mathscr{E}_{01}}_{X_1}-\pi^{\mathscr{E}_{00}}_{X_1}\big) & -\big(\pi^{\mathscr{E}_{11}}_{X_1}-\pi^{\mathscr{E}_{10}}_{X_1}\big) \end{bmatrix} \begin{bmatrix} \mathscr{X}^{\mathscr{E}}_{10}-\mathscr{X}^{\mathscr{E}}_{00} \\ \mathscr{X}^{\mathscr{E}}_{11}-\mathscr{X}^{\mathscr{E}}_{01} \end{bmatrix},$$

whence an articulated view between relative costs and prices that is parameterised by the vector of spillover effects:

$$\Omega^1-\Omega^0 = \Big[\mathscr{Q}+\big(\pi^{\mathscr{E}_{01}}_{X_1}-\pi^{\mathscr{E}_{00}}_{X_1}\big)\big(\mathscr{X}^{\mathscr{E}}_{10}-\mathscr{X}^{\mathscr{E}}_{00}\big) + \big(\pi^{\mathscr{E}_{11}}_{X_1}-\pi^{\mathscr{E}_{10}}_{X_1}\big)\big(\mathscr{X}^{\mathscr{E}}_{11}-\mathscr{X}^{\mathscr{E}}_{01}\big)\Big] \Big/ \big(\pi^1_{X_1}-\pi^0_{X_1}\big).$$

Integrating this relation into the differentiated expressions of Y^0 and Y^1, the expressions of outputs as functions of stocks and prices derive, noticing that:

$$\big(\pi^1_{X_1}-\pi^0_{X_1}\big)\pi_{Y^0}\pi_{Y^1}/\pi_{\bar{X}_0}\pi_{\bar{X}_1} = \big(\pi^1_{X_1}/\pi_{\bar{X}_1}-\pi^1_{X_0}/\pi_{\bar{X}_0}\big)\pi_{Y^1},$$

as

$$\begin{bmatrix} \mathscr{Y}^0 \\ \mathscr{Y}^1 \end{bmatrix} = \begin{bmatrix} \dfrac{\pi^1_{X_1}\pi_{\bar{X}_0}/\pi_{Y^0}}{\pi^1_{X_1}-\pi^0_{X_1}} & -\dfrac{\pi^1_{X_0}\pi_{\bar{X}_1}/\pi_{Y^0}}{\pi^1_{X_1}-\pi^0_{X_1}} \\ -\dfrac{\pi^0_{X_1}\pi_{\bar{X}_0}/\pi_{Y^1}}{\pi^1_{X_1}-\pi^0_{X_1}} & \dfrac{\pi^0_{X_0}\pi_{\bar{X}_1}/\pi_{Y^1}}{\pi^1_{X_1}-\pi^0_{X_1}} \end{bmatrix} \begin{bmatrix} \mathscr{X}_0 \\ \mathscr{X}_1 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \mathscr{Q} + \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} \mathscr{X}^{\mathscr{E}}_{10}-\mathscr{X}^{\mathscr{E}}_{00} \\ \mathscr{X}^{\mathscr{E}}_{11}-\mathscr{X}^{\mathscr{E}}_{01} \end{bmatrix},$$

for the components of the vector that relates outputs to prices that are given, integrating the expression of the elasticity of substitution between the aggregate values of the inputs, by:

$$\begin{aligned} B_1 &= -\left\{\left[\pi^0_{X_1}+\frac{\pi^0_{X_1}\pi^1_{X_0}}{\pi^1_{X_1}-\pi^0_{X_1}}\right]\Sigma^0_{X_0X_1}+\frac{\pi^1_{X_1}\pi^1_{X_0}}{\pi^1_{X_1}-\pi^0_{X_1}}\frac{\pi_{Y^1}}{\pi_{Y^0}}\Sigma^1_{X_0X_1}\right\} \Big/ \big(\pi^1_{X_1}-\pi^0_{X_1}\big) \\ &= -\frac{\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}}{\big(\pi^1_{X_1}-\pi^0_{X_1}\big)^2\pi_{Y^0}}; \\ B_2 &= -\left\{-\frac{\pi^0_{X_0}\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\frac{\pi_{Y^0}}{\pi_{Y^1}}\Sigma^0_{X_0X_1}+\left[\pi^1_{X_1}-\frac{\pi^0_{X_0}\pi^1_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\right]\Sigma^1_{X_0X_1}\right\} \Big/ \big(\pi^1_{X_1}-\pi^0_{X_1}\big) \\ &= \frac{\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}}{\big(\pi^1_{X_1}-\pi^0_{X_1}\big)^2\pi_{Y^1}}; \end{aligned}$$

while the matrix that relates the outputs to the various components of the externalities set details as:

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} 1/\pi_{Y^0} \\ \\ -1/\pi_{Y^1} \end{bmatrix} \frac{\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \begin{bmatrix} \pi_{X_1}^{\mathscr{E}_{01}} - \pi_{X_1}^{\mathscr{E}_{00}} & \pi_{X_1}^{\mathscr{E}_{11}} - \pi_{X_1}^{\mathscr{E}_{10}} \end{bmatrix} + \begin{bmatrix} \pi_{X_1}^{\mathscr{E}_{00}} & \pi_{X_1}^{\mathscr{E}_{10}} \\ \\ \pi_{X_1}^{\mathscr{E}_{01}} & \pi_{X_1}^{\mathscr{E}_{11}} \end{bmatrix}.$$

Rearranging, it is finally obtained that:

$$\begin{aligned} &\left(\pi^1_{X_1} - \pi^0_{X_1}\right) \begin{bmatrix} \pi_{Y^0} & 0 \\ 0 & \pi_{Y^1} \end{bmatrix} \begin{bmatrix} \mathscr{Y}^0 \\ \mathscr{Y}^1 \end{bmatrix} - \begin{bmatrix} \pi^1_{X_1} & -\pi^1_{X_0} \\ -\pi^0_{X_1} & \pi^0_{X_0} \end{bmatrix} \begin{bmatrix} \pi_{\bar{X}_0} & 0 \\ 0 & \pi_{\bar{X}_1} \end{bmatrix} \begin{bmatrix} \mathscr{X}_0 \\ \mathscr{X}_1 \end{bmatrix} \\ &= \frac{\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -\left(\pi_{X_1}^{\mathscr{E}_{01}} - \pi_{X_1}^{\mathscr{E}_{00}}\right) & -\left(\pi_{X_1}^{\mathscr{E}_{11}} - \pi_{X_1}^{\mathscr{E}_{10}}\right) \end{bmatrix} \begin{bmatrix} \mathscr{Q} \\ \mathscr{X}_{10}^{\mathscr{E}} - \mathscr{X}_{00}^{\mathscr{E}} \\ \mathscr{X}_{11}^{\mathscr{E}} - \mathscr{X}_{01}^{\mathscr{E}} \end{bmatrix} \\ &\quad + \left(\pi^1_{X_1} - \pi^0_{X_1}\right) \begin{bmatrix} \pi_{Y^0} & 0 \\ 0 & \pi_{Y^1} \end{bmatrix} \begin{bmatrix} \pi_{X_1}^{\mathscr{E}_{00}} & \pi_{X_1}^{\mathscr{E}_{10}} \\ \pi_{X_1}^{\mathscr{E}_{01}} & \pi_{X_1}^{\mathscr{E}_{11}} \end{bmatrix} \begin{bmatrix} \mathscr{X}_{10}^{\mathscr{E}} - \mathscr{X}_{00}^{\mathscr{E}} \\ \mathscr{X}_{11}^{\mathscr{E}} - \mathscr{X}_{01}^{\mathscr{E}} \end{bmatrix}. \end{aligned}$$

Summing over these two equations, it emerges that:

$$\begin{bmatrix} \pi_{Y^0} & \pi_{Y^1} \end{bmatrix} \begin{bmatrix} \mathscr{Y}^0 \\ \mathscr{Y}^1 \end{bmatrix} = \begin{bmatrix} \pi_{\bar{X}_0} & \pi_{\bar{X}_1} & \pi_{Y^0}\pi_{X_1}^{\mathscr{E}_{00}} + \pi_{Y^1}\pi_{X_1}^{\mathscr{E}_{01}} & \pi_{Y^0}\pi_{X_1}^{\mathscr{E}_{10}} + \pi_{Y^1}\pi_{X_1}^{\mathscr{E}_{11}} \end{bmatrix} \times \begin{bmatrix} \mathscr{X}_0 \\ \mathscr{X}_1 \\ \mathscr{X}_{10}^{\mathscr{E}} - \mathscr{X}_{00}^{\mathscr{E}} \\ \mathscr{X}_{11}^{\mathscr{E}} - \mathscr{X}_{01}^{\mathscr{E}} \end{bmatrix}.$$

The statement follows.

(ii) From a simple rearrangement of the expression of the production of the investment good, it is obtained that:

$$\mathscr{Q} = \left[\frac{\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\right]^{-1} \left\{ \begin{bmatrix} \pi_{Y^1}\left(\pi^1_{X_1} - \pi^0_{X_1}\right) & -\pi^0_{X_1}\pi_{\bar{X}_0} & \pi^0_{X_0}\pi_{\bar{X}_1} \end{bmatrix} \begin{bmatrix} \mathscr{Y}^1 \\ \mathscr{X}_0 \\ \mathscr{X}_1 \end{bmatrix} + \begin{bmatrix} \dfrac{\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}\left(\pi_{X_1}^{\mathscr{E}_{01}} - \pi_{X_1}^{\mathscr{E}_{00}}\right)}{\pi^1_{X_1} - \pi^0_{X_1}} - \pi_{X_1}^{\mathscr{E}_{01}}\pi_{Y^1} \\ \dfrac{\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}\left(\pi_{X_1}^{\mathscr{E}_{01}} - \pi_{X_1}^{\mathscr{E}_{00}}\right)}{\pi^1_{X_1} - \pi^0_{X_1}} - \pi_{X_1}^{\mathscr{E}_{11}}\pi_{Y^1} \end{bmatrix}' \begin{bmatrix} \mathscr{X}_{10}^{\mathscr{E}} - \mathscr{X}_{00}^{\mathscr{E}} \\ \mathscr{X}_{11}^{\mathscr{E}} - \mathscr{X}_{01}^{\mathscr{E}} \end{bmatrix} \right\}.$$

Noticing that the system of first-order conditions parallely implies

$$\begin{aligned} \begin{bmatrix} \Omega^0 \\ \Omega^1 \end{bmatrix} &= \begin{bmatrix} \dfrac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \\ -\dfrac{\pi^0_{X_0}}{\pi^1_{X_1} - \pi^0_{X_1}} \end{bmatrix} \mathscr{Q} \\ &+ \begin{bmatrix} -\dfrac{\pi^0_{X_1}\left(\pi_{X_1}^{\mathscr{E}_{01}} - \pi_{X_1}^{\mathscr{E}_{00}}\right)}{\pi^1_{X_1} - \pi^0_{X_1}} + \pi_{X_1}^{\mathscr{E}_{00}} & -\dfrac{\pi^0_{X_1}\left(\pi_{X_1}^{\mathscr{E}_{11}} - \pi_{X_1}^{\mathscr{E}_{10}}\right)}{\pi^1_{X_1} - \pi^0_{X_1}} + \pi_{X_1}^{\mathscr{E}_{10}} \\ \dfrac{\pi^0_{X_0}\left(\pi_{X_1}^{\mathscr{E}_{01}} - \pi_{X_1}^{\mathscr{E}_{00}}\right)}{\pi^1_{X_1} - \pi^0_{X_1}} + \pi_{X_1}^{\mathscr{E}_{00}} & \dfrac{\pi^0_{X_0}\left(\pi_{X_1}^{\mathscr{E}_{11}} - \pi_{X_1}^{\mathscr{E}_{10}}\right)}{\pi^1_{X_1} - \pi^0_{X_1}} + \pi_{X_1}^{\mathscr{E}_{10}} \end{bmatrix} \begin{bmatrix} \mathscr{X}_{10}^{\mathscr{E}} - \mathscr{X}_{00}^{\mathscr{E}} \\ \mathscr{X}_{11}^{\mathscr{E}} - \mathscr{X}_{01}^{\mathscr{E}} \end{bmatrix} \end{aligned}$$

…A.3

Rearranging, the expressions of the statement become available. △

(iii) The differentiation of the equations of the statement — that follow from the assumption of freely shiftable inputs between the two sectors — uncovers the features of $\Psi^j(\cdot), j = 0, 1$ as respectively:

$$\left\{\pi_{X_1}^{\mathscr{E}_{01}} + \pi_{X_1}^{\mathscr{E}_{11}} \frac{\Sigma^1_{X_0X_1}}{\Sigma^0_{X_0X_1}} - \frac{\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}}{(\pi^1_{X_1} - \pi^0_{X_1})\Sigma^0_{X_0X_1}}\right\}(\mathscr{X}_{10} - \mathscr{X}_{00}) = \left(\frac{\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\right)\frac{\mathscr{Q}}{\pi^1_{X_1} - \pi^0_{X_1}}$$
$$+ \begin{bmatrix} \dfrac{\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}(\pi_{X_1}^{\mathscr{E}_{01}} - \pi_{X_1}^{\mathscr{E}_{00}})}{\pi^1_{X_1} - \pi^0_{X_1}} - \pi_{X_1}^{\mathscr{E}_{01}}\pi_{Y^1} \\ \dfrac{\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}(\pi_{X_1}^{\mathscr{E}_{01}} - \pi_{X_1}^{\mathscr{E}_{00}})}{\pi^1_{X_1} - \pi^0_{X_1}} - \pi_{X_1}^{\mathscr{E}_{11}}\pi_{Y^1} \end{bmatrix}' \begin{bmatrix} \mathscr{X}^{\mathscr{E}}_{10} - \mathscr{X}^{\mathscr{E}}_{00} \\ \mathscr{X}^{\mathscr{E}}_{11} - \mathscr{X}^{\mathscr{E}}_{01} \end{bmatrix},$$

and $(\mathscr{X}_{11} - \mathscr{X}_{01}) = (\Sigma^1_{X_0X_1}/\Sigma^0_{X_0X_1})(\mathscr{X}_{10} - \mathscr{X}_{00})$, for the expression of $\mathscr{Q}$ that was available from (ii). △

VII.2 – Proof of Lemma 2.

Integrating the definition of the equilibrium, the fixed-point to be solved by the system of demands expresses as:

$$-\frac{\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\left\{\frac{1}{\Sigma^0_{X_0X_1}} + \frac{\pi_{X_1}^{\mathscr{E}_{01}} - \pi_{X_1}^{\mathscr{E}_{00}}}{\pi^1_{X_1} - \pi^0_{X_1}} + \frac{(\pi_{X_1}^{\mathscr{E}_{11}} - \pi_{X_1}^{\mathscr{E}_{10}})\Sigma^1_{X_0X_1}}{(\pi^1_{X_1} - \pi^0_{X_1})\Sigma^0_{X_0X_1}}\right\}(\mathscr{X}_{10} - \mathscr{X}_{00}) = \frac{\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \times \frac{\mathscr{Q}}{\pi^1_{X_1} - \pi^0_{X_1}}.$$

Simplifying:

$$\mathscr{X}_{10} - \mathscr{X}_{00} = -\frac{\Sigma^0_{X_0X_1}\mathscr{Q}}{(\pi^1_{X_1} - \pi^0_{X_1}) + (\pi_{X_1}^{\mathscr{E}_{01}} - \pi_{X_1}^{\mathscr{E}_{00}})\Sigma^0_{X_0X_1} + (\pi_{X_1}^{\mathscr{E}_{11}} - \pi_{X_1}^{\mathscr{E}_{10}})\Sigma^1_{X_0X_1}}$$

and integrating that $(\mathscr{X}_{11} - \mathscr{X}_{01}) = (\Sigma^1_{X_0X_1}/\Sigma^0_{X_0X_1})(\mathscr{X}_{10} - \mathscr{X}_{00})$, the statement follows. △

VII.3 – Proof of Lemma 3.

Integrating Lemma 2, it is immediate that the matrix that related the outputs to the various components of the externalities set in Lemma 1 reformulates as a vector that relates these outputs to the relative price of the capital good. For $\Gamma^{\mathscr{E}} := (\pi^1_{X_1} - \pi^0_{X_1}) + (\pi_{X_1}^{\mathscr{E}_{01}} - \pi_{X_1}^{\mathscr{E}_{00}})\Sigma^0_{X_0X_1} + (\pi_{X_1}^{\mathscr{E}_{11}} - \pi_{X_1}^{\mathscr{E}_{10}})\Sigma^1_{X_0X_1}$, the first of its components details as:

$$-\frac{\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}}{(\pi^1_{X_1} - \pi^0_{X_1})\pi_{Y^0}}\left[1 - \frac{(\pi_{X_1}^{\mathscr{E}_{01}} - \pi_{X_1}^{\mathscr{E}_{00}})\Sigma^0_{X_0X_1} + (\pi_{X_1}^{\mathscr{E}_{11}} - \pi_{X_1}^{\mathscr{E}_{10}})\Sigma^1_{X_0X_1}}{\Gamma^{\mathscr{E}}}\right] + \pi_{X_1}^{\mathscr{E}_{00}}\Sigma^0_{X_0X_1} + \pi_{X_1}^{\mathscr{E}_{10}}\Sigma^1_{X_0X_1}$$
$$= -\frac{\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}}{(\pi^1_{X_1} - \pi^0_{X_1})\Gamma^{\mathscr{E}}\pi_{Y^0}}\left[1 + \frac{(\pi_{X_1}^{\mathscr{E}_{00}}\Sigma^0_{X_0X_1} + \pi_{X_1}^{\mathscr{E}_{10}}\Sigma^1_{X_0X_1})(\pi^1_{X_1} - \pi^0_{X_1})\pi_{Y^0}}{\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}}\right].$$

...A.4...

Completing the same approach, the coefficient that relates the equilibrium production of the capital good to its relative price is similarly available as:

$$\frac{\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}}{(\pi^1_{X_1}-\pi^0_{X_1})\Gamma^{\mathscr{E}}\pi_{Y^1}}\left[1-\frac{(\pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1}+\pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1})(\pi^1_{X_1}-\pi^0_{X_1})\pi_{Y^1}}{\pi_{\bar{X}_0}\pi_{\bar{X}_1}\Sigma_{\bar{X}_0\bar{X}_1}}\right].$$

Merging and noticing that the matrix that relative equilibrium outputs to stocks is let unaffected by the consideration of a competitive equilibrium with externalities, the statement follows. △

VII.4 – Proof of Proposition 1.

Solving between the differentiated expressions of Y^0 and Y^1, it derives that:

$$\begin{aligned}
\Xi^{\mathscr{E}}_{Y^1Y^1} &= -\frac{\pi_{Y^1}}{\pi_{\bar{X}_0}\pi_{\bar{X}_1}}\frac{\Gamma^{\mathscr{E}}}{Z^{\mathscr{E}}}(\pi^1_{X_1}-\pi^0_{X_1}),\\
\Xi^{\mathscr{E}}_{Y^1X_0} &= -\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\Xi^{\mathscr{E}}_{Y^1Y^1}\\
\Xi^{\mathscr{E}}_{Y^1X_1} &= -\frac{\pi^0_{X_0}}{\pi^1_{X_1}-\pi^0_{X_1}}\Xi^{\mathscr{E}}_{Y^1Y^1}.
\end{aligned}$$

In parallel to this, noticing that

$$\begin{bmatrix}\Omega^0\\ \Omega^1\end{bmatrix} = (\Gamma^{\mathscr{E}})^{-1}\begin{bmatrix}-(\pi^0_{X_1}+\pi^{\mathscr{E}_{00}}_{X_1}\Sigma^0_{X_0X_1}+\pi^{\mathscr{E}_{10}}_{X_1}\Sigma^1_{X_0X_1})\\ 1-\pi^0_{X_1}-\pi^{\mathscr{E}_{00}}_{X_1}\Sigma^0_{X_0X_1}-\pi^{\mathscr{E}_{10}}_{X_1}\Sigma^1_{X_0X_1}\end{bmatrix}\mathscr{Q}$$

and rearranging, the expression of the matrix $\mathscr{M}_\Xi$ becomes available. △

VII.5 – Proof of Lemma 4.

Firstly restating the principal diagonal components of the Hessian Elasticies Matrix:

$$\begin{aligned}
\Xi^{\mathscr{E}}_{Y^1Y^1} &= -\frac{\pi_{Y^1}}{\pi_{\bar{X}_0}\pi_{\bar{X}_1}}\frac{\Gamma^{\mathscr{E}}}{Z^{\mathscr{E}}}(\pi^1_{X_1}-\pi^0_{X_1}),\\
\Xi^{\mathscr{E}}_{X_0X_0} &= -\frac{\pi^0_{X_1}(\pi^0_{X_1}+\pi^{\mathscr{E}_{00}}_{X_1}\Sigma^0_{X_0X_1}+\pi^{\mathscr{E}_{10}}_{X_1}\Sigma^1_{X_0X_1})}{\Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1}\left[1-(\pi^1_{X_1}-\pi^0_{X_1})\pi_{Y^1}(\pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1}+\pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1})/\Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1}\right]}\\
\Xi^{\mathscr{E}}_{X_1X_1} &= -\frac{\pi^0_{X_0}(1-\pi^0_{X_1}-\pi^{\mathscr{E}_{00}}_{X_1}\Sigma^0_{X_0X_1}-\pi^{\mathscr{E}_{10}}_{X_1}\Sigma^1_{X_0X_1})}{\Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1}\left[1-(\pi^1_{X_1}-\pi^0_{X_1})\pi_{Y^1}(\pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1}+\pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1})/\Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1}\right]}.
\end{aligned}$$

The scope for $\Xi^{\mathscr{E}}_{Y^1Y^1} > 0$ hence happens to be univocally associated with the possible occurrence of $\Gamma^{\mathscr{E}}Z^{\mathscr{E}} < 0$. Such a conjunction in its turn emerges under two distinct configurations:

$$\begin{aligned}
&\text{a/}\quad \pi^1_{X_1} > \pi^0_{X_1},\quad 1-\frac{(\pi^1_{X_1}-\pi^0_{X_1})\pi_{Y^1}}{\Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1}}(\pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1}+\pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1}) < 0, \Gamma^{\mathscr{E}} > 0,\\
&\text{b/}\quad Z^{\mathscr{E}} > 0,\quad (\pi^1_{X_1}-\pi^0_{X_1})\left[(\pi^1_{X_1}-\pi^0_{X_1})+(\pi^{\mathscr{E}_{01}}_{X_1}-\pi^{\mathscr{E}_{00}}_{X_1})\Sigma^0_{X_0X_1}\right.\\
&\qquad\qquad\qquad\left.+(\pi^{\mathscr{E}_{11}}_{X_1}-\pi^{\mathscr{E}_{10}}_{X_1})\Sigma^1_{X_0X_1}\right] < 0.
\end{aligned}$$

…A.5

In parallel to this, the occurrence of $\Xi^{\mathscr{E}}_{X_0X_0} > 0$ is obtained for

$$\pi^1_{X_1} > \pi^0_{X_1}, \quad 1 - \frac{(\pi^1_{X_1} - \pi^0_{X_1})\pi_{Y^1}}{\Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1}}(\pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1} + \pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1}) < 0.$$

Finally, the scope for $\Xi^{\mathscr{E}}_{X_1X_1} > 0$ emerges under the occurrence of one of the two following configurations:

$$\text{a/} \quad \pi^1_{X_1} > \pi^0_{X_1}, \quad 1 - \frac{(\pi^1_{X_1} - \pi^0_{X_1})\pi_{Y^1}}{\Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1}}(\pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1} + \pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1}) < 0,$$
$$1 - \pi^0_{X_1} - \pi^{\mathscr{E}_{00}}_{X_1}\Sigma^0_{X_0X_1} - \pi^{\mathscr{E}_{10}}_{X_1}\Sigma^1_{X_0X_1} > 0,$$
$$\text{b/} \quad 1 - \pi^0_{X_1} - \pi^{\mathscr{E}_{00}}_{X_1}\Sigma^0_{X_0X_1} - \pi^{\mathscr{E}_{10}}_{X_1}\Sigma^1_{X_0X_1} < 0,$$
$$1 - \frac{(\pi^1_{X_1} - \pi^0_{X_1})\pi_{Y^1}}{\Sigma_{\bar{X}_0\bar{X}_1}\pi_{\bar{X}_0}\pi_{\bar{X}_1}}(\pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1} + \pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1}) > 0.$$

1 – To sum up and for $Z^{\mathscr{E}} < 0$, the simultaneous occurrences of $\Xi^{\mathscr{E}}_{Y^1Y^1} > 0$, $\Xi^{\mathscr{E}}_{X_0X_0} > 0$ and $\Xi^{\mathscr{E}}_{X_1X_1} > 0$ would be guaranteed when the following inequalities are further satisfied:

$$\text{(i)} \quad (\pi^1_{X_1} - \pi^0_{X_1})\left[(\pi^1_{X_1} - \pi^0_{X_1}) + (\pi^{\mathscr{E}_{01}}_{X_1} - \pi^{\mathscr{E}_{00}}_{X_1})\Sigma^0_{X_0X_1} + (\pi^{\mathscr{E}_{11}}_{X_1} - \pi^{\mathscr{E}_{10}}_{X_1})\Sigma^1_{X_0X_1}\right] > 0,$$
$$\text{(ii)} \quad 1 - \pi^0_{X_1} - \pi^{\mathscr{E}_{00}}_{X_1}\Sigma^0_{X_0X_1} - \pi^{\mathscr{E}_{10}}_{X_1}\Sigma^1_{X_0X_1} > 0.$$

2 – In opposition to this and for $Z^{\mathscr{E}} > 0$, $\Xi^{\mathscr{E}}_{X_0X_0} < 0$, so that a convex E.P.P.F. can only be obtained under the simultaneous occurrences of $\Xi^{\mathscr{E}}_{Y^1Y^1} > 0$, $\Xi^{\mathscr{E}}_{X_1X_1} > 0$ and $\Xi^{\mathscr{E}}_{Y^1Y^1} + \Xi^{\mathscr{E}}_{X_0X_0} + \Xi^{\mathscr{E}}_{X_1X_1} > 0$, namely

$$\text{(i)} \quad (\pi^1_{X_1} - \pi^0_{X_1})\left[(\pi^1_{X_1} - \pi^0_{X_1}) + (\pi^{\mathscr{E}_{01}}_{X_1} - \pi^{\mathscr{E}_{00}}_{X_1})\Sigma^0_{X_0X_1} + (\pi^{\mathscr{E}_{11}}_{X_1} - \pi^{\mathscr{E}_{10}}_{X_1})\Sigma^1_{X_0X_1}\right] < 0,$$
$$\text{(ii)} \quad 1 - \pi^0_{X_1} - \pi^{\mathscr{E}_{00}}_{X_1}\Sigma^0_{X_0X_1} - \pi^{\mathscr{E}_{10}}_{X_1}\Sigma^1_{X_0X_1} < 0$$

for the two first relations while a sufficient condition for the holding of the third is available as the satisfaction of $\Xi^{\mathscr{E}}_{X_0X_0} + \Xi^{\mathscr{E}}_{X_1X_1} > 0$ that in turn formulates, under $Z^{\mathscr{E}} > 0$, as:

$$\text{(iii)} \quad \pi^0_{X_1}(\pi^0_{X_1} + \pi^{\mathscr{E}_{00}}_{X_1}\Sigma^0_{X_0X_1} + \pi^{\mathscr{E}_{10}}_{X_1}\Sigma^1_{X_0X_1}) < (1 - \pi^0_{X_1})(1 - \pi^0_{X_1} - \pi^{\mathscr{E}_{00}}_{X_1}\Sigma^0_{X_0X_1} - \pi^{\mathscr{E}_{10}}_{X_1}\Sigma^1_{X_0X_1}).$$

Items 1 and 2 provide a set of sufficient conditions under which the E.P.P.F. as defined through Definition 2 from $T^{\mathscr{E}}(\cdot;\cdot,\cdot)$, $[\partial T(\cdot;\cdot,\cdot)/\partial Y^1]^{\mathscr{E}}$, $[\partial T(\cdot;\cdot,\cdot)/\partial X_0]^{\mathscr{E}}$ and $[\partial T(\cdot;\cdot,\cdot)/\partial X_1]^{\mathscr{E}}$ becomes a convex equilibrium function. △

VII.6 – Proof of Lemma 5.

The derivation of $\Sigma^{\mathscr{E}}_{\bar{X}_0\bar{X}_1}$ and $\Sigma^{\mathscr{E}}_{Y^0Y^1}$ proceeds from the consideration of:

$$\frac{Y^0_t}{Y^1_t} = \frac{\mathfrak{F}^{\mathscr{E},0}}{\mathfrak{F}^{\mathscr{E},1}}(1, X_{1,t}/X_0, q_t),$$

…A.6…

that, in differentiated terms, states as:

$$\begin{aligned}\mathscr{Y}^0-\mathscr{Y}^1=&\frac{(\pi^1_{X_1}/\pi_{Y^0}+\pi^0_{X_1}/\pi_{Y^1})}{\pi^1_{X_1}-\pi^0_{X_1}}\pi_{\bar X_0}\mathscr{X}_0-\frac{(\pi^1_{X_0}/\pi_{Y^0}+\pi^0_{X_0}/\pi_{Y^1})}{\pi^1_{X_1}-\pi^0_{X_1}}\pi_{\bar X_1}\mathscr{X}_1\\&+\frac{\pi_{\bar X_0}\pi_{\bar X_1}\Sigma_{\bar X_0\bar X_1}}{(\pi^1_{X_1}-\pi^0_{X_1})\Gamma^{\mathscr{E}}}\Big[-(1/\pi_{Y^0}+1/\pi_{Y^1})\\&\qquad+\frac{(\pi^1_{X_1}-\pi^0_{X_1})\big[(\pi^{\mathscr{E}_{01}}_{X_1}-\pi^{\mathscr{E}_{00}}_{X_1})\Sigma^0_{X_0X_1}+(\pi^{\mathscr{E}_{11}}_{X_1}-\pi^{\mathscr{E}_{10}}_{X_1})\Sigma^1_{X_0X_1}\big]}{\Sigma_{\bar X_0\bar X_1}\pi_{\bar X_0}\pi_{\bar X_1}}\Big]\end{aligned}$$

and from the subsequent consideration of the equilibrium coefficient that relates q_t to the ratio ω^1_t/ω^0_t, namely $(\pi^1_{X_1}-\pi^0_{X_1})+(\pi^{\mathscr{E}_{01}}_{X_1}-\pi^{\mathscr{E}_{00}}_{X_1})\Sigma^0_{X_0X_1}+(\pi^{\mathscr{E}_{11}}_{X_1}-\pi^{\mathscr{E}_{10}}_{X_1})\Sigma^1_{X_0X_1}$, that eventually gives

$$\begin{aligned}\Sigma^{\mathscr{E}}_{\bar X_0\bar X_1}\pi_{\bar X_0}\pi_{\bar X_1}&=(\pi^0_{X_0}\pi^0_{X_1}\pi_{Y^0}\Sigma^0_{X_0X_1}+\pi^1_{X_0}\pi^1_{X_1}\pi_{Y^1}\Sigma^1_{X_0X_1})\\&\quad\times\Big[1-\frac{(\pi^1_{X_1}-\pi^0_{X_1})\big[(\pi^{\mathscr{E}_{01}}_{X_1}-\pi^{\mathscr{E}_{00}}_{X_1})\Sigma^0_{X_0X_1}+(\pi^{\mathscr{E}_{11}}_{X_1}-\pi^{\mathscr{E}_{10}}_{X_1})\Sigma^1_{X_0X_1}\big]}{(\pi_{Y^0}\pi_{Y^1})^{-1}\Sigma_{\bar X_0\bar X_1}\pi_{\bar X_0}\pi_{\bar X_1}}\Big],\\\Sigma^{\mathscr{E}}_{Y^0Y^1}\pi_{Y^0}\pi_{Y^1}&=\frac{\Sigma^{\mathscr{E}}_{\bar X_0\bar X_1}\pi_{\bar X_0}\pi_{\bar X_1}}{(\pi^1_{X_1}-\pi^0_{X_1})\Gamma^{\mathscr{E}}}.\end{aligned}$$

An interesting property of the two above expressions states as follows. While it can be shown that, in a standard convex two-goods world, $\Sigma_{Y^0Y^1}>\Sigma_{\bar X_0\bar X_1}$, this inequality can be reversed in the course of a competitive equilibrium with externalities. Already refering to a configuration that shall reveal as being at the core of the subsequent multiplicity argument, one may indeed notice that, for $\pi^0_{X_1}>\pi^1_{X_1}$ and $(\pi^1_{X_1}-\pi^0_{X_1})+(\pi^{\mathscr{E}_{01}}_{X_1}-\pi^{\mathscr{E}_{00}}_{X_1})\Sigma^0_{X_0X_1}+(\pi^{\mathscr{E}_{11}}_{X_1}-\pi^{\mathscr{E}_{10}}_{X_1})\Sigma^1_{X_0X_1}<0$, one obtains $\Sigma^{\mathscr{E}}_{\bar X_0\bar X_1}>\Sigma_{\bar X_0\bar X_1}>0>\Sigma^{\mathscr{E}}_{Y^0Y^1}$, namely a configuration where the substituability between the two aggregate values of the inputs is increased with respect to its standard formulation while the two outputs end up as being complements in place of substitutes in the benchmark convex case. △

VII.7 – Proof of Proposition 2.

(i) This directly follows from the definition of the steady state.
(ii)-(iii) The steady state expression of the rate of return on the capital stock reformulates as:

$$\frac{(\pi_{\bar X_1})^{\mathscr{E}}(\eta X_1/X_0;1,X_1/X_0)}{(\pi_{Y^1})^{\mathscr{E}}(\eta X_1/X_0;1,X_1/X_0)}=\big[1-\delta(1-\eta)\big]/\delta\eta.$$

Letting $\Upsilon(\cdot)$ denote the L.H.S. of this equation and $\xi_{\Upsilon/(X_1/X_0)}:=\Upsilon'(X_1/X_0)\times(X_1/X_0)/\Upsilon(X_1/X_0)$, it derives that:

$$\begin{aligned}\xi_{\Upsilon/(X_1/X_0)}&=(\Xi^{\mathscr{E}}_{X_1Y^1}+\Xi^{\mathscr{E}}_{Y^1Y^1})\pi_{Y^1}+(\Xi^{\mathscr{E}}_{X_1X_1}+\Xi^{\mathscr{E}}_{Y^1X_1})\pi_{\bar X_1}\\&=\Big[1-\frac{1-\pi^0_{X_1}-\pi^{\mathscr{E}_{00}}_{X_1}\Sigma^0_{X_0X_1}-\pi^{\mathscr{E}_{10}}_{X_1}\Sigma^1_{X_0X_1}}{(\pi^1_{X_1}-\pi^0_{X_1})+(\pi^{\mathscr{E}_{01}}_{X_1}-\pi^{\mathscr{E}_{00}}_{X_1})\Sigma^0_{X_0X_1}+(\pi^{\mathscr{E}_{11}}_{X_1}-\pi^{\mathscr{E}_{10}}_{X_1})\Sigma^1_{X_0X_1}}\Big]\\&\qquad\qquad\times(\Xi^{\mathscr{E}}_{Y^1X_1}\pi_{\bar X_1}+\Xi^{\mathscr{E}}_{Y^1Y^1}\pi_{Y^1})\\&=-(1-\pi^1_{X_1}-\pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1}-\pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1})\frac{\pi_{Y^1}}{\pi_{\bar X_0}\pi_{\bar X_1}\Sigma_{\bar X_0\bar X_1}Z^{\mathscr{E}}}.\end{aligned}$$

…A.7

The statement follows. △

VII.8 – Proof of Corollary 1.

(i) In the C.E.S. case and for Cobb-Douglas functions for the spillover effects, *i.e.*, for :

$$\begin{aligned}
F^0(X_{00}, X_{10}) &= \left[(1-\alpha_{X_{10}})(X_{00})^{1-1/\Sigma^0_{X_0X_1}} + \alpha_{X_{10}}(X_{10})^{1-1/\Sigma^0_{X_0X_1}}\right]^{1/(1-1/\Sigma^0_{X_0X_1})},\\
F^1(X_{01}, X_{11}) &= \left[(1-\alpha_{X_{11}})(X_{01})^{1-1/\Sigma^1_{X_0X_1}} + \alpha_{X_{11}}(X_{11})^{1-1/\Sigma^1_{X_0X_1}}\right]^{1/(1-1/\Sigma^1_{X_0X_1})},\\
G^{00}(X^{\mathscr{E}}_{00}, X^{\mathscr{E}}_{10}) &= (X^{\mathscr{E}}_{10}/X^{\mathscr{E}}_{00})^{\alpha^{\mathscr{E}_{00}}_{X_1}}, G^{10}(X^{\mathscr{E}}_{01}, X^{\mathscr{E}}_{11}) = (X^{\mathscr{E}}_{11}/X^{\mathscr{E}}_{01})^{\alpha^{\mathscr{E}_{10}}_{X_1}},\\
G^{01}(X^{\mathscr{E}}_{00}, X^{\mathscr{E}}_{10}) &= (X^{\mathscr{E}}_{10}/X^{\mathscr{E}}_{00})^{\alpha^{\mathscr{E}_{01}}_{X_1}}, G^{11}(X^{\mathscr{E}}_{01}, X^{\mathscr{E}}_{11}) = (X^{\mathscr{E}}_{11}/X^{\mathscr{E}}_{01})^{\alpha^{\mathscr{E}_{11}}_{X_1}},
\end{aligned}$$

the definition of the steady state boils down to:

$$\frac{\pi^1_{X_1}}{\mu_{11}} = [1-\delta(1-\eta)]/\delta\eta,$$

for

$$\begin{aligned}
\pi^1_{X_1} &= \frac{\alpha_{X_{11}}(X_{11}/X_{01})^{1-1/\Sigma^1_{X_0X_1}}}{(1-\alpha_{X_{11}}) + \alpha_{X_{11}}(X_{11}/X_{01})^{1-1/\Sigma^1_{X_0X_1}}},\\
\eta &= \mu_{11}(X_{11}/X_{01})^{-1}\left[(1-\alpha_{X_{11}}) + \alpha_{X_{11}}(X_{11}/X_{01})^{1-1/\Sigma^1_{X_0X_1}}\right]^{1/\left(1-1/\Sigma^1_{X_0X_1}\right)}\\
&\qquad\qquad \times (X_{10}/X_{00})^{\alpha^{\mathscr{E}_{01}}_{X_1}}(X_{11}/X_{01})^{\alpha^{\mathscr{E}_{11}}_{X_1}},\\
0 &= \frac{\alpha_{X_{11}}(X_{11}/X_{01})^{-1/\Sigma^1_{X_0X_1}}}{1-\alpha_{X_{11}}} - \frac{\alpha_{X_{10}}(X_{10}/X_{00})^{-1/\Sigma^0_{X_0X_1}}}{1-\alpha_{X_{10}}}.
\end{aligned}$$

Integrating the third equation, the second above equation implies the local holding of

$$\mathscr{M}_{11} = \left[\pi^1_{X_0} - \alpha^{\mathscr{E}_{01}}_{X_1}(\Sigma^0_{X_0X_1}/\Sigma^1_{X_0X_1}) - \alpha^{\mathscr{E}_{11}}_{X_1}\right](\mathscr{X}_{11} - \mathscr{X}_{01}).$$

The L.H.S. of the definition of the steady state thus emerges as a function $\Psi(X_{11}/X_{01})$ whose slope can be computed to:

$$\begin{aligned}
\xi_{\Psi/(X_{11}/X_{01})} &= (1-1/\Sigma^1_{X_0X_1})\,\pi^1_{X_0} - \left[\pi^1_{X_0} - \alpha^{\mathscr{E}_{01}}_{X_1}(\Sigma^0_{X_0X_1}/\Sigma^1_{X_0X_1}) - \alpha^{\mathscr{E}_{11}}_{X_1}\right]\\
&= -\Sigma^1_{X_0X_1}\left[\pi^1_{X_0}(X_{11}/X_{01}) - \alpha^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1} - \alpha^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1}\right].
\end{aligned}$$

For $\Sigma^1_{X_0X_1} \neq 1$, its second-order properties emerge as:

$$\xi_{[\Psi/(X_{11}/X_{01})]/(X_{11}/X_{01})} = -\left(1-1/\Sigma^1_{X_0X_1}\right)\pi^1_{X_1}.$$

Considering then a value of $X^\star_{11}/X^\star_{01}$ that solves $\xi_{\Psi/(X_{11}/X_{01})} = 0$ and thus describes the case for multiplicity. From the expression of $\xi_{\Psi/(X_{11}/X_{01})}$, it is necessarily unique in the C.E.S. case and its expression is given by

$$X^\star_{11}/X^\star_{01} = \left[\frac{\alpha_{X_{11}}}{\alpha^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1} + \alpha^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1}}\right]^{-\left(1-1/\Sigma^1_{X_0X_1}\right)}.$$

…A.8…

From the expression of $\xi_{[\Psi/(X_{11}/X_{01})]/(X_{11}/X_{01})}$, it respectively corresponds to a minimum and a maximum of the function $\Psi(\cdot)$ for $\Sigma^1_{X_0X_1} < 1$ and $\Sigma^1_{X_0X_1} > 1$. The details of the statement follow.

(ii) In the Cobb-Douglas case for the capital good industry but when a C.E.S. is retained for the spillover effects stemming from the consumption good industry, *i.e.*,

$$\begin{aligned} G^{00}(X^{\mathscr{E}}_{00}, X^{\mathscr{E}}_{10}) &= \left[(1-\alpha^{\mathscr{E}_{00}}_{X_1}) + \alpha^{\mathscr{E}_{00}}_{X_1}(X^{\mathscr{E}}_{10}/X^{\mathscr{E}}_{00})^{1-1/\Sigma^0_{X_0X_1}}\right]^{1/(1-1/\Sigma^0_{X_0X_1})}, \\ G^{01}(X^{\mathscr{E}}_{00}, X^{\mathscr{E}}_{10}) &= \left[(1-\alpha^{\mathscr{E}_{01}}_{X_1}) + \alpha^{\mathscr{E}_{01}}_{X_1}(X^{\mathscr{E}}_{10}/X^{\mathscr{E}}_{00})^{1-1/\Sigma^0_{X_0X_1}}\right]^{1/(1-1/\Sigma^0_{X_0X_1})}, \end{aligned}$$

the first and third ingredients of the definition of the steady state respectively simplify to $\pi^1_{X_1} = \alpha_{X_{11}}$ and $\alpha_{X_{11}}(X_{11}/X_{01})^{-1}/(1-\alpha_{X_{11}}) = \alpha_{X_{10}}(X_{10}/X_{00})^{-1/\Sigma^0_{X_0X_1}}/(1-\alpha_{X_{10}})$ while the second is modified to

$$\begin{aligned} \eta = \mu_{11}(X_{11}/X_{01})^{\alpha_{X_{11}}}\left[(1-\alpha^{\mathscr{E}_{01}}_{X_1}) + \alpha^{\mathscr{E}_{01}}_{X_1}(X_{10}/X_{00})^{-1/(1-1/\Sigma^0_{X_0X_1})}\right]^{-(1-1/\Sigma^0_{X_0X_1})} \\ \times (X_{11}/X_{10})^{\alpha^{\mathscr{E}_{11}}_{X_1}}, \end{aligned}$$

that implies

$$\mathscr{M}_{11} = \left[1-\alpha_{X_{11}} - \pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1} - \alpha^{\mathscr{E}_{11}}_{X_1}\right](\mathscr{X}_{11} - \mathscr{X}_{01}).$$

The L.H.S. of the definition of the steady state again emerge as a function $\Psi(X_{11}/X_{01})$ whose slope and second-order properties can be computed to:

$$\begin{aligned} \xi_{\Psi/(X_{11}/X_{01})} &= -\left[1-\alpha_{X_{11}} - \pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1} - \alpha^{\mathscr{E}_{11}}_{X_1}\right], \\ \xi_{[\Psi/(X_{11}/X_{01})]/(X_{11}/X_{01})} &= -\left(1-1/\Sigma^0_{X_0X_1}\right)\pi^{\mathscr{E}_{01}}_{X_1}. \end{aligned}$$

The argument then follows from the same line of reasoning as the one developed in (i). $\triangle$

VII.9 – Proof of Propositions 3 and 4.

Under the earlier list of assumptions on preferences and the technology and from an application of the Implicit Function Theorem in a neighbourhood of the steady state position, for $\Sigma^c := -1/\xi_{cc} \to \infty$, the elasticities of the demand of Y^1_t as a function of $X_{1,t}$ and λ_{t+1} can be computed from (8a) as: $\mathscr{Y}^1_t = \left[\Delta_{Y^1X_1} \quad \Delta_{Y^1\lambda'}\right]\left[\mathscr{X}_{1,t} \quad \Lambda_{t+1}\right]'$, for $\mathscr{Y}^1_t := (Y^1_t - Y^{1\star})/Y^{1\star}$, $\mathscr{X}_{1,t} := (X_{1,t} - X^\star_1)/X^\star_1$, $\Lambda_{t+1} := (\lambda_{t+1} - \lambda^\star)/\lambda^\star$ and:

$$\begin{aligned} \Delta_{Y^1X_1} &= -\left[\Xi^{\mathscr{E}}_{Y^1Y^1}\pi_{\bar{X}_1}\right]^{-1}\left[\Xi^{\mathscr{E}}_{Y^1X_1}\pi_{Y^1}\right], \\ \Delta_{Y^1\lambda'} &= -\left[\Xi^{\mathscr{E}}_{Y^1Y^1}\pi_{Y^1}\right]^{-1}. \end{aligned}$$

The linearised formulation of the ensued dynamical system defined from (8b-c) in a neighbourhood of a steady state $\{X^\star_1, \lambda^\star\}$ is then given by:

$$\begin{bmatrix} \mathscr{X}_{1,t+1} \\ \Lambda_{t+1} \end{bmatrix} = \begin{bmatrix} (\Delta_{Y^1X_1} + \Delta_{Y^1\lambda'}\Delta_{\lambda'X_1})\eta + (1-\eta) & (\Delta_{Y^1\lambda'}\Delta_{\lambda'\lambda})\eta \\ \Delta_{\lambda'X_1} & \Delta_{\lambda'\lambda} \end{bmatrix}\begin{bmatrix} \mathscr{X}_{1,t} \\ \Lambda_t \end{bmatrix}, \qquad \ldots \text{A.9}$$

the components of the price equation being computed as:

$$\Delta_{\lambda' X_1} = -\frac{\Xi^{\mathscr{E}}_{X_1 X_1}\pi_{\bar{X}_1} - \Xi^{\mathscr{E}}_{X_1 Y^1}\pi_{Y^1}\Delta_{Y^1 X_1}}{[1-\delta(1-\eta)]^{-1}[\delta(1-\eta)] + \Xi^{\mathscr{E}}_{X_1 Y^1}\pi_{Y^1}\Delta_{Y^1\lambda'}},$$
$$\Delta_{\lambda'\lambda} = \frac{[1-\delta(1-\eta)]^{-1}}{[1-\delta(1-\eta)]^{-1}[\delta(1-\eta)] + \Xi^{\mathscr{E}}_{X_1 Y^1}\pi_{Y^1}\Delta_{Y^1\lambda'}}.$$

Noticing that, taking advantage of Proposition 1, $\Delta_{\lambda' X_1} = 0$, a diagonal structure becomes available for the Jacobian Matrix. The eigenvalues are real and available in the main text.
(i) First, and as this was to be expected from Proposition 2, the component of the Jacobian matrix that pertains to the stock equation, *i.e.*, ν_1, remains unaffected by the consideration of a suboptimal equilibrium with external effects in place of an optimum. Hence, although the pair structure of the Jacobian Matrix is unambiguously lost — that ν_1 is an eigenvalue does not anymore imply that $1/\delta\nu_1$ is also an eigenvalue — an equilibrium configuration with a greater share of profits within the investment good sector, *i.e.*, for which $\pi^1_{X_1} > \pi^0_{X_1}$ prevails, is associated with $\nu_1 > 1$ since this latter condition restates as

$$[1/\delta - (1-\eta)]\frac{1-\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} > 1-(1-\eta).$$

It thus erases any area for local indeterminacies.
a/ Consider, *e.g.*, the scope for the saddlepoint property when $\Gamma^{\mathscr{E}}\big(1-\pi^1_{X_1} - \pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1} - \pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1}\big) > 0$. As this implies the holding of $\nu_2 > 0$, the saddlepoint property will then be available for $\nu_2 < 1$. For $\Gamma^{\mathscr{E}} > 0$, this requires the satisfaction of $1-\pi^1_{X_1} - \pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1} - \pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1} > 0$ that holds by assumption. Similarly and for $\Gamma^{\mathscr{E}} < 0$, comparison with respect to 1 suggests the need for $1-\pi^1_{X_1} - \pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1} - \pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1} > 0$, that holds by assumption.
b/ In opposition to this and for $\Gamma^{\mathscr{E}}\big(1-\pi^1_{X_1} - \pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1} - \pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1}\big) < 0$, the requesites for the holding of the saddlepoint property are slightly more stringent. For $\nu_2 < 0$ and $\Gamma^{\mathscr{E}} > 0$, they indeed boil down to the satisfaction of

$$\Gamma^{\mathscr{E}} < 2\Gamma^{\mathscr{E}} < -[1-\delta(1-\eta)]\big(1-\pi^1_{X_1} - \pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1} - \pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1}\big)$$

while a related line of reasoning for $\Gamma^{\mathscr{E}} < 0$ allows for completing the statement.
(ii) This follows from the expressions of ν_1 and ν_2: the first condition ensures that $\nu_1 \in]-1, 1[$. The steady state will then be indeterminate for $\nu_2 \in]-1, 1[$, *i.e.*

$$-1 < \frac{\Gamma^{\mathscr{E}}}{\Gamma^{\mathscr{E}} + [1-\delta(1-\eta)]\big(1-\pi^1_{X_1} - \pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1} - \pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1}\big)} < 1,$$

that restates as:

$$-2 < -\frac{[1-\delta(1-\eta)]\big(1-\pi^1_{X_1} - \pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1} - \pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1}\big)}{\Gamma^{\mathscr{E}} + [1-\delta(1-\eta)]\big(1-\pi^1_{X_1} - \pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1} - \pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1}\big)} < 0.$$

…A.10…

But

$$\left|\frac{[1-\delta(1-\eta)]\big(1-\pi^1_{X_1}-\pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1}-\pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1}\big)}{\Gamma^{\mathscr{E}}+[1-\delta(1-\eta)]\big(1-\pi^1_{X_1}-\pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1}-\pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1}\big)}\right|<1$$

for $(1-\pi^1_{X_1}-\pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1}-\pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1})\Gamma^{\mathscr{E}}>0$, that establishes a/.
For $(1-\pi^1_{X_1}-\pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1}-\pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1})\Gamma^{\mathscr{E}}<0$, a first necessary condition for the previous restatement of the indeterminacy condition lies in the obtention of the same sign for the numerator and the denominator, that can only obtain for

$$\big|\Gamma^{\mathscr{E}}\big|<[1-\delta(1-\eta)]\big|\big(1-\pi^1_{X_1}-\pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1}-\pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1}\big)\big|.$$

First consider the case $1-\pi^1_{X_1}-\pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1}-\pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1}<0$ and where $\Gamma^{\mathscr{E}}>0$ satisfies the above condition. The previous indeterminacy condition then reformulates to:

$$\begin{aligned}-2\big\{\Gamma^{\mathscr{E}}+[1-\delta(1-\eta)]\big(1-\pi^1_{X_1}-\pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1}-\pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1}\big)\big\}\\ >-[1-\delta(1-\eta)]\big(1-\pi^1_{X_1}-\pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1}-\pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1}\big)\end{aligned},$$

that rearranges as

$$\Gamma^{\mathscr{E}}<-[1-\delta(1-\eta)]\big|\big(1-\pi^1_{X_1}-\pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1}-\pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1}\big)/2,$$

that is more stringent than the condition $\Gamma^{\mathscr{E}}<-[1-\delta(1-\eta)]\big|\big(1-\pi^1_{X_1}-\pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1}-\pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1}\big)$ previously raised when $\Gamma^{\mathscr{E}}>0$.
Then consider the case $1-\pi^1_{X_1}-\pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1}-\pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1}>0$ and where $\Gamma^{\mathscr{E}}<0$ satisfies the previous condition. Completing the same line of reasoning gives

$$-\Gamma^{\mathscr{E}}<[1-\delta(1-\eta)]\big|\big(1-\pi^1_{X_1}-\pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1}-\pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1}\big)/2.$$

that is in turn more stringent than the condition $-\Gamma^{\mathscr{E}}<[1-\delta(1-\eta)]\big|\big(1-\pi^1_{X_1}-\pi^{\mathscr{E}_{01}}_{X_1}\Sigma^0_{X_0X_1}-\pi^{\mathscr{E}_{11}}_{X_1}\Sigma^1_{X_0X_1}\big)$ previously raised when $\Gamma^{\mathscr{E}}>0$.
Merging the two cases establishes b/. The details of Proposition 4 derive from straightforward specialisations of the above lines of reasoning. △