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Communication, consensus and order. Who wants to speak first ? *

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Abstract

Parikh and Krasucki [1990] showed that if rational agents communicate the value of a function f according to a protocol upon which they have agreed beforehand, they will eventually reach a consensus about the value of f, provided a *fairness* condition on the protocol and a *convexity* condition on the function f. In this article, we address the issue of how agents agree on a communication protocol in the case where they communicate in order to learn information. We show that if it is common knowledge among a group of agents that some of them disagree about two protocols, then the consensus value of f must be the same according to the two protocols.

JEL Classification: D70, D82.

Keywords : Common knowledge, Consensus, Communication Protocol.

1 Introduction

Alice and Bob are sitting in front of each other, both wearing either a red hat or a white hat. Suppose that the two hats are red. The teacher tells the children that there is at least one red hat, and asks them whether they know the color of their hat. The two children observe that the other's hat is red, but cannot infer the color of their own hat. The only way for them to answer the teacher is to communicate with each other. Suppose that Alice tells Bob

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that she does not know the color of her hat. Bob understands that his own hat is red, for if it had been white, Alice would have known that her hat was red. Now Bob knows the color of his hat. But then if he tells Alice that he knows the color of his hat, Alice will not learn anything, for the message of Bob would have been the same regardless of the color of her hat. Therefore, Alice has no interest to be the first to say whether she knows the color of her hat. This story illustrates the following fact. From the moment that people communicate in order to be better informed, who gets to talk when is important: the communication process is not commutative, for different orders of speech may lead to different outcomes.

It is well known since Geanakoplos and Polemarchakis [1982] that in a group of rational agents, a process of simultaneous communication of posterior beliefs for an event leads to equality of all individual beliefs. Cave [1983] and Bacharach [1985] extended this result to simultaneous communication of decisions, assuming that the decision rule followed by agents satisfies a union consistency property. Yet in most economic situations where agents have to speak together, communication is not simultaneous. It is common sense that each individual speaks one after the other according to a given protocol. Parikh and Krasucki [1990] considered the case where agents of a group communicate with each other, according to a pairwise protocol upon which they have agreed beforehand. They investigated what conditions on the type of messages and on the protocol guarantee that agents eventually reach a consensus, *i.e.* that from some stage on all the communicated values will be the same. They show that if the protocol is *fair*, that is if every participant receives information directly or indirectly from every other participant, and if the function f is *convex*, that is for all pair of disjoint events X, X', there exists $a \in [0, 1]$ such that $f(X \cup X') = af(X) + (1-a)f(X')$, then communication will eventually lead to a consensus about the value of f.

We show that different protocols may lead to different outcomes, in terms of consensus values of f as well as of information learned by the agents during the communication process. This non-commutativity of the order of speech, as well as the fact that agents communicate so as to be better informed, imply that they may have strategic considerations concerning the order of speech. Depending on the state of the world, Alice and Bob may prefer to speak first or second, or may be indifferent. If neither Alice nor Bob wants to speak first, communication can not take place. However, can we conclude that they will not learn anything from each other? The fact that Alice does not want to speak first is informative for Bob. Bob knows that if Alice knew the color of her hat, she wouldn't mind speaking first and saying that she knows the color of her hat. In this paper, we investigate what inferences can be made by rational agents from the common knowledge that some of them disagree about the order of speech.

We show that the following situations are both possible. First, it can be common knowledge in a group of agents that some of them prefer the same order of speech. Second, it can be common knowledge in a group of agents that some of them prefer different orders of speech. However, we show the surprising result that if it is the case, then the consensus value of f must be the same whatever the order of speech. For instance, if it is common knowledge among Alice and Bob that they both want to speak first, then what they will communicate at the end of the day will be the same, whether Alice or Bob speaks first.

The paper is organized as follows. In Section 2 we describe the model and the basic result of Parikh and Krasucki [1990]. Section 3 defines preferences over protocols and develops the result. The proof of the theorem is given in the Appendix.

2 Preliminary notions

Let Ω be the finite set of states of the world, and 2^{Ω} the set of possible events. There are N agents, each agent i being endowed with a partition Π_i of Ω. When the state $\omega \in \Omega$ occurs, agent i just knows that the true state of the world belongs to $\Pi_i(\omega)$, which is the cell of i's partition that contains ω. We say that a partition Π is *finer* than a partition Π' if and only if for all ω, $\Pi(\omega) \subseteq \Pi'(\omega)$ and there exists ω' such that $\Pi(\omega') \subset \Pi'(\omega')$. A partition Π' is coarser than a partition Π if and only if Π is finer than Π'. The partition Π_i represents the ability of agent i to distinguish between the states of the world. The coarser her partition is, the less precise her information is, in the sense that she distinguishes among fewer states of the world. As usual, we say that an agent i endowed with a partition Π_i knows the event E at state ω if and only if $\Pi_i(\omega) \subseteq E$. We define the meet of the partitions $\Pi_1, \Pi_2, \ldots, \Pi_N$ as the finest common coarsening of these partitions, that is the finest partition M such that for all $\omega \in \Omega$ and for all $i = 1, \ldots, N$, $\Pi_i(\omega) \subseteq M(\omega)$.

Common knowledge of an event E at a state ω is the situation that occurs when each agent knows E at ω, each agent knows that each of them knows E at ω, each agent knows that each agent knows that each agent knows... etc. Aumann [1976] showed that, given a set of N agents, the meet M of their N partitions is the partition of common knowledge among these N agents. Hence we say that an event E is common knowledge at state ω iff $M(\omega) \subseteq E$.

Before communicating, agents have to agree on a communication protocol that will be applied throughout the debate. The protocol determines which agents are allowed to speak at each date.

Definition 1 *A protocol α is a pair of functions (s, r) from $\mathbb{N}$ to $2^{\{1,\ldots,N\}}$. If $s(t) = S$ and $r(t) = R$, then we interpret S and R as, respectively, the set of senders and the set of receivers of the communication which takes place at time t.*

We note Γ the set of protocols. Note that the type of protocols we consider are more general than the ones in PK, for we allow for more than one agent to be senders and receivers of the communication at the same time.

Along the debate, agents communicate by sending messages, which we assume to be delivered instantaneously, that is at time t, messages are simultaneously sent by every $i \in s(t)$ and heard by every $j \in r(t)$. We assume that the message sent is the private value of some function f defined from the set of subsets of Ω to $\mathbb{R}$. The private value of f for an agent i at state ω is $f(\Pi_i(\omega))$.

Finally, the set of states of the world Ω, the individual partitions $(\Pi_i)_i$, and the message rule f define an *information model* $I = \langle \Omega, (\Pi_i)_i, f \rangle$.

Two assumptions are made on the protocol and on the function f to guarantee that iterated communication of the value of f leads to a consensus about f. As in PK, we assume that the protocol is *fair*. We adapt PK's definition in our setting, but the meaning remains the same: a protocol is *fair* if and only if every participant in this protocol communicates directly or indirectly with every other participant infinitely many times. This condition is necessary so that nobody is excluded from communication.

Assumption 1 (A1) *The protocol α is fair, that is for all pair of players (i, j), $i \neq j$, there exists an infinite number of finite sequences $t_1, \ldots, t_K$, with $t_k \in \mathbb{N}$ for all $k \in \{1, \ldots, K\}$, such that $i \in s(t_1)$ and $j \in r(t_K)$.*

Assumption 2 (A2) *f is convex, that is for all subsets $E, E' \subseteq \Omega$ such that $E \cap E' = \emptyset$, there exists $\alpha \in]0, 1[$ such that $f(E \cup E') = \alpha f(E) + (1 - \alpha) f(E')$.*

Note that we will have $f(E_1 \cup E_2 \cup \dots \cup E_k) = \sum_{i=1}^{k} \alpha_i f(E_i)$, with $\alpha_i \in]0,1[\ \forall\ i$ and $\sum_{i=1}^{k} \alpha_i = 1$ provided that the E_i are pairwise disjoint events. This condition is obeyed by conditional probabilities and implies union consistency[1] à la Cave [1983].

We now describe how information is aggregated during the debate. At a given date t, the senders $s(t)$ selected by the protocol (s, r) send a message heard by the receivers $r(t)$. Then each individual infers the set of states of the world that are compatible with the messages possibly sent, and updates her partition accordingly. Given an information model $\langle \Omega, (\Pi_i)_i, f \rangle$ and a communication protocol α, we define by induction on t the set $\Pi_i^\alpha(\omega, t)$ of possible states for an agent i at time t, given that the state of the world is ω:

$\Pi_i^\alpha(\omega, 0) = \Pi_i(\omega)$ and for all $t \geq 1$,

$$\Pi_i^\alpha(\omega, t+1) = \Pi_i^\alpha(\omega, t) \cap \{\omega' \in \Omega \mid f(\Pi_j^\alpha(\omega', t)) = f(\Pi_j^\alpha(\omega, t))\ \forall\ j \in s(t)\}\ if\ i \in r(t),$$

$$\Pi_i^\alpha(\omega, t+1) = \Pi_i^\alpha(\omega, t)\ otherwise.$$

The next result states that for all i, for all ω, $f(\Pi_i^\alpha(\omega, t))$ has a limiting value, and that this value does not depend on i. Under assumptions $A1$ and $A2$, participants in the protocol converge to a consensus about the value of f.

Proposition 1 (Parikh and Krasucki (1990)) *Let $\langle \Omega, (\Pi_i)_i, f \rangle$ be an information model, and α a communication protocol. Under assumptions $A1$ and $A2$, there exists a date T such that for all ω, for all i, j, and all $t, t' \geq T$, $f(\Pi_i^\alpha(\omega, t)) = f(\Pi_j^\alpha(\omega, t'))$.*

In the sequel, we will denote $\Pi_i^\alpha(\omega)$ the limiting value of $\Pi_i^\alpha(\omega, t)$, and Π_i^α will be called i's partition of information at consensus. $f(\Pi^\alpha(\omega))$ will denote the limiting value of $f(\Pi_i^\alpha(\omega, t)$, which does not depend on i, and will be called the consensus value of f at state ω, given that the protocol is α.

3 Who wants to speak first?

We know from Parikh and Krasucki [1990] that given any protocol α, under assumptions $A1$ and $A2$, iterated communication of the private value of f eventually leads to a consensus

[1] f is union consistent if for all E, E' such that $E \cap E' = \emptyset$, $f(E) = f(E') \Rightarrow f(E \cup E') = f(E) = f(E')$.

about the value of f. The next proposition states that this value may vary according to the protocol.

Proposition 2 *There exist an information model* $\langle \Omega, (\Pi_i)_i, f \rangle$ *with* f *convex and two fair protocols* α, β *for which there exists* ω *such that* $f(\Pi^\alpha(\omega)) \neq f(\Pi^\beta(\omega))$.

This result can be proved easily for some union consistent functions f. However, to the best of our knowledge, it was not proved for conditional probabilities. As the posterior probabilities of an event are particular union consistent function, it could have been possible that there exist no information model with posterior probabilities such that order matters. We exhibit an example where it does.[2]

Example 1 *Let* $\Omega = \{1, \ldots, 13\}$ *be the set of states of the world. Suppose that Alice and Bob have a uniform prior* P *on* Ω. *They communicate in turn the private value of the function* $f(.) = P(\{2, 3, 4, 8, 12\} \mid .)$, *which is convex, and are endowed with the following partitions of* Ω[3]*:*

$$\Pi_A = \{1,3,7,8\}_{1/2}\{2,6,11,12\}_{1/2}\{4,5,10\}_{1/3}\{9\}_0\{13\}_0$$

$$\Pi_B = \{1,3,5\}_{1/3}\{2\}_1\{4,7,9,10,12,13\}_{1/3}\{6,8\}_{1/2}\{11\}_0$$

If Alice speaks first (protocol α*), individual partitions at consensus are:*

$$\Pi_A^\alpha = \{1,3,7,8\}_{1/2}\{2\}_1\{11\}_0\{6,12\}_{1/2}\{4,10\}_{1/2}\{5\}_0\{9\}_0\{13\}_0$$

$$\Pi_B^\alpha = \{1,3\}_{1/2}\{5\}_0\{2\}_1\{4,10\}_{1/2}\{7,12\}_{1/2}\{9,13\}_0\{6,8\}_{1/2}\{11\}_0$$

If Bob speaks first (protocol β*), individual partitions at consensus are:*

$$\Pi_A^\beta = \{1,3,7\}_{1/3}\{8\}_1\{2\}_1\{6\}_0\{11\}_0\{12\}_1\{4,5,10\}_{1/3}\{9\}_0\{13\}_0$$

$$\Pi_B^\beta = \{1,3,5\}_{1/3}\{2\}_1\{4,7,10,\}_{1/3}\{12\}_1\{9,13\}_0\{6\}_0\{8\}_1\{11\}_0$$

[2] We found it by a numerical search. If somebody has a similar example with less than 13 states of the world, please tell us!

[3] The subscript reflects the posterior belief in each cell.

At state 1, the consensus value of f is $f(\{1,3,7,8\}) = f(\{1,3\}) = 1/2$ if Alice speaks first, whereas it is $f(\{1,3,7\}) = f(\{1,3,5\}) = 1/3$ if Bob speaks first.

We assume that agents communicate in order to be better informed. As a consequence, they prefer protocols that lead them to be better informed at the end of the day. A more precise information is represented by a finer partition. Yet two partitions may not be rankable in the sense of refinement, so we may not be able to say with which partition an agent is better informed. For instance, we cannot say whether an individual is better with $\Pi = \{1\}\{2,3,4\}$ or with $\Pi' = \{1,2,3\}\{4\}$. However, we can say that one is better informed with Π than with Π' at state 1, and better informed with Π' than with Π at state 4.

Definition 2 *We say that an agent is better informed with the partition* Π^α *than with the partition* Π^β *at state* ω *if and only if* $\Pi^\alpha(\omega) \subset \Pi^\beta(\omega)$.

Before communication takes place, the information model $\langle \Omega, (\Pi_i)_i, f \rangle$ is common knowledge among individuals. Therefore, given any protocol α, consensus partitions $(\Pi_i^\alpha)_i$ are also common knowledge. As a consequence, each agent knows *ex interim* which protocol she prefers among any two orders if she's not indifferent.

Definition 3 (Preferences) *Let* $I := \langle \Omega, (\Pi_i)_i, f \rangle$ *be an information model, and* α, β *two protocols. The set of states of the world where agent* i *prefers* α *to* β *is denoted* $B_i^I(\alpha, \beta)$ *and is defined by*

$$B_i^I(\alpha,\beta) = \{\omega \in \Omega \mid \forall \omega' \in \Pi_i(\omega),\ \Pi_i^\alpha(\omega') \subseteq \Pi_i^\beta(\omega') \text{ and } \exists\, \omega'' \in \Pi_i(\omega) \text{ s.t. } \Pi_i^\alpha(\omega'') \subset \Pi_i^\beta(\omega'')\}$$

In **Example 1**, Alice and Bob are both better informed with the protocol α at state 4 and better informed with the protocol β at state 8. Hence at states 4 and 8, they agree on the protocol they prefer. On the contrary, at state 1, Alice and Bob end up strictly better informed when they speak in second. What happens in that case? Suppose that state 1 occurs, and that Alice and Bob stand in front of each other waiting for the other to speak. Alice knows that the state of the world belongs to $\{1,3,7,8\}$. She understands that the state of the world can not be 7 nor 8, for Bob would have spoken first at state 8 and would have been indifferent at state 7. Bob knows that the state of the world belongs to $\{1,3,5\}$. He understands that the state of the world can not be 5, for he knows that Alice prefers to speak first at state 5. Hence knowing that the other does not want to speak first makes Alice and Bob understand that the state of the world is in $\{1,3\}$. From now, they have the same private information at state 1. As they cannot learn information from the communication process, they become indifferent between speaking first or second. This example addresses the question of whether it can be common knowledge among two persons that they disagree about the order of speech. More generally, what inferences can be made by rational agents of a group from the common knowledge that some of them disagree about the order of speech? Our main result states that if it is the case, then the consensus message is the same according to any protocol.

Theorem 1 *Let $I = \langle \Omega, (\Pi_i)_i, f \rangle$ be an information model such that $A1$ and $A2$ are satisfied, and α, β two protocols such that $\alpha \neq \beta$. Consider $a_1, a_2, b_1, b_2 \in \{\alpha, \beta\}$, with $a_1 \neq a_2$ and $b_1 \neq b_2$. Assertions (1), (2) and (3) cannot be true simultaneously.*

(1) $\exists\, i, j$ *such that* $B_i^I(a_1, a_2)$ *and* $B_j^I(b_1, b_2)$ *are common knowledge at* ω.

(2) $\omega \in B_i^I(a_1, a_2) \cap B_j^I(b_1, b_2)$ *and* $a_1 = b_2$.

(3) $f(\Pi^\alpha(\omega)) \neq f(\Pi^\beta(\omega))$.

The meaning of this result is the following.

• If (1) and (2) are true, namely if it is common knowledge at some state ω that Alice and Bob prefer to speak first, then (3) is false, *i.e* the consensus value of f at ω is the same regardless of the person who speaks first.

• If (1) and (3) are true, namely if it is common knowledge at ω that Alice prefers $a_1 \in \{\alpha, \beta\}$ and Bob prefers $b_1 \in \{\alpha, \beta\}$, and if the consensus value of f differs according to whether the protocol is α or β, then (2) is false, *i.e* Alice and Bob prefer the same protocol ($a_1 = b_1$).

• If (2) and (3) are true, namely if Alice and Bob prefer different orders of speech at ω, then (1) is false, *i.e* these preferences are not common knowledge among them at ω.

The result of Theorem 2 is not due to the fact that propositions (1) and (2) or (1) and (3) or (2) and (3) are never true simultaneously.

Proposition 3 *(i) Propositions* (1) *and* (2) *of Theorem 2 can be true simultaneously.*

(ii) Propositions (1) *and* (3) *of Theorem 2 can be true simultaneously.*

(iii) Propositions (2) *and* (3) *of Theorem 2 can be true simultaneously.*

This proposition states that (*i*) it can be common knowledge among them that Alice and Bob prefer different orders of speech, (*ii*) it can be common knowledge among them that Alice and Bob prefer the same order of speech, and (*iii*) it is possible that Alice and Bob prefer different orders of speech which lead to different consensus values of f.

We prove point (*i*) with the following example, which describes a situation where it is common knowledge between Alice and Bob that both of them prefer to speak in second. The fact that both prefer to speak in second in order to be better informed is quite intuitive. When an agent is the second to speak, the first message she hears contains purely private information of the other. When she is the first to speak, the first message she will hear will be a join of the other's private information and her private information, so she may not learn anything. However, we found another example which shows that there exist situations where both agents prefer to speak first.[4]

Example 2 *The set of states of the world be* $\Omega = \{1,2,3,4,5,6,7\}$ *and Alice and Bob are endowed with a uniform prior* P *on* Ω*. They communicate in turn the private value of the function* $f(.) = P(\{1,2,7\} \mid .)$ *and are endowed with the following partitions:*

$$\Pi_A = \{1,2\}_1\{3,4\}_0\{5,6,7\}_{1/3}$$

$$\Pi_B = \{1,7\}_1\{2,3,6\}_{1/3}\{4,5\}_0$$

If Alice speaks first (protocol α*), individual partitions at consensus are:*

$$\begin{cases} \Pi_A^\alpha = \{1,2\}_1\{3,4\}_0\{5,6\}_0\{7\}_1 \\ \Pi_B^\alpha = \{1\}_1\{2\}_1\{3\}_0\{4\}_0\{5\}_0\{6\}_0\{7\}_1 \end{cases}$$

If Bob speaks first (protocol β*), individual partitions at consensus are:*

$$\begin{cases} \Pi_A^\beta = \{1\}_1\{2\}_1\{3\}_0\{4\}_0\{5\}_0\{6\}_0\{7\}_1 \\ \Pi_B^\beta = \{1,7\}_1\{2\}_1\{3,6\}_0\{4,5\}_0 \end{cases}$$

At every state of the world, Alice and Bob both prefer to speak in second: $B_A(\beta,\alpha) = \Omega$ and $B_B(\alpha,\beta) = \Omega$, hence at every state of the world, it is common knowledge among Alice and Bob that Alice prefers the order β and Bob the order α. However, it does not contradict Theorem 2 as for all ω, $f(\Pi^\alpha(\omega)) = f(\Pi^\beta(\omega))$.

[4]Maybe because it is less intuitive, it is also pretty tedious (there are 288 states of the world), so it is available from the authors upon request.

We prove point (ii) with the following example, which shows that it is possible that both agents prefer the same order of speech.

Example 3 *The set of states of the world be* $\Omega = \{1,2,3,4,5,6,7,8,9\}$ *and Alice and Bob are endowed with a uniform prior* P *on* Ω*. They communicate in turn the private value of the function* $f(.) = P(\{1,6,7,9\} \mid .)$ *and are endowed with the following partitions:*

$$\Pi_A = \{1,2,4,5,9\}_{2/5}\{3,6,7,8\}_{1/2}$$

$$\Pi_B = \{1,3,7\}_{1/3}\{2,5,8\}_0\{4,6,9\}_{2/3}$$

If Alice speaks first (protocol α*), individual partitions at consensus are:*

$$\begin{cases} \Pi_A^\alpha = \{1\}_1\{2,5\}_0\{4,9\}_{1/2}\{3,7\}_{1/2}\{6\}_1\{8\}_0 \\ \Pi_B^\alpha = \{1\}_1\{2,5\}_0\{4,9\}_{1/2}\{3,7\}_{1/2}\{6\}_1\{8\}_0 \end{cases}$$

If Bob speaks first (protocol β*), individual partitions at consensus are:*

$$\begin{cases} \Pi_A^\beta = \{1,4,9\}_{2/3}\{2,5\}_0\{3,6,7\}_{2/3}\{8\}_0 \\ \Pi_2^\beta = \{1,3,7\}_{2/3}\{2,5,8\}_0\{4,6,9\}_{2/3} \end{cases}$$

At every state of the world, Alice and Bob prefer that Alice speaks first: $B_A(\alpha,\beta) = B_B(\alpha,\beta) = \Omega$, hence it is common knowledge at any state that both prefer the order α.

Finally, we prove point (iii) with **Example 1** in section 2. The partition of common knowledge is $M = \{1,2,3,4,5,6,7,8,9,10,11,12,13\}$. At state 1, Alice and Bob prefer to speak second, and $f(\Pi^\alpha(1)) = 1/3 \neq f(\Pi^\beta(1) = 1/2$. However, this is not *common knowledge*, for Bob prefers to speak first at states 6 and 8.

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Proof of [Theorem 1]

Consider an information model $I = \langle \Omega, (\Pi_i)_i, f \rangle$, and α, β two protocols such that $\alpha \neq \beta$. Let us show that if points 1) and 2) of theorem 1 are true, then point 3) is false. We show that if there exist two agents i, j and a state ω such that $B_i^I(\alpha, \beta)$ and $B_j^I(\beta, \alpha)$ are common knowledge at ω, then $f(\Pi^\alpha(\omega)) = f(\Pi^\beta(\omega))$. Clearly, the proof still holds if we invert α and β.

Recall that $M(\omega)$ denotes the meet of individual partitions before communication takes place: $M = \bigwedge_{i=1}^n \Pi_i$. We note Π^α the meet of the individual partitions at consensus, given that the protocol is α: $\Pi^\alpha = \bigwedge_{i=1}^n \Pi_i^\alpha$.

If $B_i^I(\alpha, \beta)$ and $B_j^I(\beta, \alpha)$ are common knowledge at ω, then we have

$$M(\omega) \subseteq B_i(\alpha, \beta) \cap B_j(\beta, \alpha)$$

As $\Pi^\alpha(\omega) \subseteq M(\omega)$ and $\Pi^\beta(\omega) \subseteq M(\omega)\ \forall\ \omega$, we have $\Pi^\alpha(\omega) \cap \Pi^\beta(\omega) \subseteq M(\omega)\ \forall\ \omega$. Hence we have

$$\Pi^\alpha(\omega) \cap \Pi^\beta(\omega) \subseteq B_i^I(\alpha, \beta) \cap B_j^I(\beta, \alpha) \tag{1}$$

Consider some $\omega' \in \Pi^\alpha(\omega) \cap \Pi^\beta(\omega)$ (which is not empty as $\omega \in \Pi^\alpha(\omega) \cap \Pi^\beta(\omega)$). By definition of the meet, we have $\Pi_i^\alpha(\omega') \subseteq \Pi^\alpha(\omega')$ and $\Pi_i^\beta(\omega') \subseteq \Pi^\beta(\omega')$. As $\omega' \in \Pi^\alpha(\omega) \cap \Pi^\beta(\omega)$, we have $\Pi^\alpha(\omega') = \Pi^\alpha(\omega)$ and $\Pi^\beta(\omega') = \Pi^\beta(\omega)$. Then we have

$$\Pi_i^\alpha(\omega') \subseteq \Pi^\alpha(\omega) \text{ and } \Pi_i^\beta(\omega') \subseteq \Pi^\beta(\omega) \tag{2}$$

By (1), $\omega' \in B_i^I(\alpha, \beta)$. It implies that $\Pi_i^\alpha(\omega') \subseteq \Pi_i^\beta(\omega')$. Yet $\Pi_i^\beta(\omega') \subseteq \Pi^\beta(\omega)$ by (2). Then we have

$$\Pi_i^\alpha(\omega') \subseteq \Pi^\alpha(\omega) \cap \Pi^\beta(\omega)$$

As this is true for every $\omega' \in \Pi^{\alpha}(\omega) \cap \Pi^{\beta}(\omega)$, we have

$$\Pi^{\alpha}(\omega) \cap \Pi^{\beta}(\omega) = \bigcup_{\omega' \in \Pi^{\alpha}(\omega) \cap \Pi^{\beta}(\omega)} \Pi_i^{\alpha}(\omega')$$

By Proposition 1 of Parikh and Krasucki [1990], $\forall\, i, j,\ f(\Pi_i^{\alpha}(\omega)) = f(\Pi_j^{\alpha}(\omega))$ for all ω. By definition of the meet, it implies that $\forall\, \omega' \in \Pi^{\alpha}(\omega),\ f(\Pi_i^{\alpha}(\omega') = f(\Pi_i^{\alpha}(\omega))$. As f is convex, it is also union consistent, then we have $f(\Pi^{\alpha}(\omega) \cap \Pi^{\beta}(\omega)) = f(\Pi^{\alpha}(\omega))$.

The same reasoning applied to $\Pi_j^{\beta}(\omega)$ boils down to $f(\Pi^{\alpha}(\omega) \cap \Pi^{\beta}(\omega)) = f(\Pi^{\beta}(\omega))$.
Hence $f(\Pi^{\alpha}(\omega)) = f(\Pi^{\alpha}(\omega))$ $\square$