

PRODUCTION EQUILIBRIA

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ABSTRACT. This paper studies production economies in a commodity space that is an ordered locally convex space. We establish a general theorem on the existence of equilibrium without requiring that the commodity space or its dual be a vector lattice. Such commodity spaces arise in models of portfolio trading where the absence of some option usually means the absence of a vector lattice structure. The conditions on preferences and production sets are at least as general as those imposed in the literature dealing with vector lattice commodity spaces. The main assumption on the order structure is that the Riesz–Kantorovich functionals satisfy a uniform properness condition that can be formulated in terms of a duality property that is readily checked. This condition is satisfied in a vector lattice commodity space but there are many examples of other commodity spaces that satisfy the condition, which are not vector lattices, have no order unit, and do not have either the decomposition property or its approximate versions.

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1. INTRODUCTION

This paper is part of a series of papers studying Walrasian equilibria in ordered vector spaces that need not be vector lattices. The other papers in this series are Aliprantis, Florenzano, and Tourky (2004b, 2005); Aliprantis, Florenzano, Martins-da-Rocha, and Tourky (2004a). The paper extends to production economies results obtained in Aliprantis, Florenzano, and Tourky (2004b) for exchange economies.

In our papers Aliprantis, Florenzano, and Tourky (2004b, 2005) we established necessary and sufficient conditions for the existences of equilibrium in exchange economies with an ordered topological vector commodity space and preferences that satisfy the properness assumption. We were motivated by models in finance with commodity spaces that are not vector lattices. In models of

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portfolio trading we have a pair of function spaces L and X , where L is the portfolio space and X is the space of contingent claims, together with a positive linear operator $A: L \rightarrow X$ that pulls back the order intervals of $A(L)$ to closed and bounded subsets of L . The portfolio dominance ordering, a notion introduced in Aliprantis, Brown, and Werner (2000), is the only ordering on the portfolio space which is relevant for purposes of economic analysis of arbitrage free security prices. However, if the portfolio space is reordered by the portfolio dominance ordering, then it is seldom a vector lattice and the existence of equilibrium cannot be deduced from theorems that require lattice properties on the commodity space. Our papers Aliprantis, Florenzano, and Tourky (2004b, 2005) solve this problem for exchange economies in the sense that they provide necessary and sufficient conditions on the order structure of the commodity (portfolio) space, and therefore in terms of the assets structure $A: L \rightarrow X$. In particular, if our conditions do not hold, then one can construct an economy satisfying the usual assumptions on preferences that has no linear price equilibrium. When our conditions hold the usual assumptions guarantee the existence of equilibrium with linear prices. If there is no linear price equilibrium, then one must invoke the theory of value with personalized prices developed in Aliprantis, Tourky, and Yannelis (2001).

In this paper, we study the existence of general equilibrium for production economies with an ordered locally convex commodity space. Our assumptions are quite general. Preferences are not ordered, we do not assume any monotonicity of preferences or free disposal in production. We also assume general properness assumptions on preference and production sets. The modern literature establishing the existence of equilibrium with general assumptions on preferences and production sets is divided into two groups. The first considers the setting of Mas-Colell (1986a), where the commodity space is a locally solid topological vector lattice. The main papers in this group include Aliprantis, Brown, and Burkinshaw (1987b); Zame (1987). The second group follows Mas-Colell and Richard (1991) and assumes that the commodity space is a locally convex vector lattice whose topological dual is a vector sublattice of the order dual. That is, they drop the local solidness assumption of the earlier literature. The main contributions in this setting include Richard (1989); Tourky (1999); Florenzano and Marakulin (2001). In both literatures the lattice structures of the commodity-price duality are used in non-trivial ways.

The novelty of this paper is to replace the lattice properties of the commodity-price duality by a condition on the order structure of the commodity space. As with our the other papers in the series, the present paper uses the duality property of the Riesz–Kantorovich functionals whose study was initiated in Aliprantis, Tourky, and Yannelis (2000, 2001); Aliprantis and Tourky (2002) and was further motivated by the three assets portfolio trading example of Aliprantis, Monteiro, and Tourky (2004c).

The paper is organized as follows. In section 2, we recall the notion of the Riesz–Kantorovich functional associated with a finite list of (linear) functionals and the concept of sup-convolution in optimization. In section 3, we define the model and set out our assumptions. Section 4 is devoted to the decentralization of Edgeworth equilibria with linear prices and thus establishes the existence of equilibrium.

2. PRELIMINARIES

For details regarding Riesz spaces that are not explained below we refer the reader to Aliprantis and Burkinshaw (1985, 2003). This paper will utilize the notion of the Riesz–Kantorovich functional studied in Aliprantis, Tourky, and Yannelis (2000); Aliprantis and Tourky (2002) and used extensively in Aliprantis, Tourky, and Yannelis (2001); Aliprantis, Florenzano, and Tourky (2004b,

2005). We shall briefly introduce this formula here and refer the reader to Aliprantis and Tourky (2002) for a more complete discussion.

Let L be an ordered vector space. Recall that the **decomposition property** states that if $x, y_1, y_2 \in L_+$ satisfy $0 \leq x \leq y_1 + y_2$, then there exist vectors x_1 and x_2 such that $0 \leq x_1 \leq y_1$, $0 \leq x_2 \leq y_2$ and $x = x_1 + x_2$. The **order dual** L^\sim is the vector space consisting of all linear functionals on L which map order intervals of L to order bounded subsets of \mathbb{R} , ordered by the relation $f \geq g$ whenever $f(x) \geq g(x)$ for all $x \in L_+$.

We start with the following classical result from the theory of partially ordered vector spaces due to F. Riesz and L. V. Kantorovich.

Theorem 2.1 (Riesz–Kantorovich). *If L is an ordered vector space with a generating cone and the decomposition property, then the order dual L^\sim is a Riesz space and for each $f, g \in L^\sim$ and $x \in L_+$ its lattice operations are given by:*

$$\begin{aligned} [f \vee g](x) &= \sup\{f(y) + g(z) : y, z \in L_+ \text{ and } y + z = x\}, \text{ and} \\ [f \wedge g](x) &= \inf\{f(y) + g(z) : y, z \in L_+ \text{ and } y + z = x\} \end{aligned}$$

In particular, if L has the decomposition property, then for any finite collection of linear functionals $f_1, f_2, \dots, f_m \in L^\sim$ their supremum in L^\sim at each $x \in L_+$ is given by

$$\left[\bigvee_{i=1}^m f_i \right](x) = \sup \left\{ \sum_{i=1}^m f_i(x_i) : x_i \in L_+ \text{ for each } i \text{ and } \sum_{i=1}^m x_i = x \right\}. \quad (\star)$$

For any positive integer m and $x \in L_+$ define

$$\mathcal{A}_x^m = \left\{ (x_1, \dots, x_m) \in L_+^m : \sum_{i=1}^m x_i = x \right\}.$$

The formula (\star) that gives the supremum of the order bounded linear functionals f_1, \dots, f_m is called the Riesz–Kantorovich formula of these functionals. The useful observation here is that if each f_i is an arbitrary function from L_+ to $(-\infty, \infty]$, then the right-hand side of (\star) still defines an extended real number for each $x \in L_+$. That is, the formula appearing in (\star) defines a function from L_+ to $(-\infty, \infty]$ called the **Riesz–Kantorovich functional** of the m -tuple of functions $f = (f_1, \dots, f_m)$ and denoted \mathcal{R}_f . In other words, $\mathcal{R}_f : L_+ \rightarrow (-\infty, \infty]$ is defined for each $x \in L_+$ by

$$\mathcal{R}_f(x) := \sup \left\{ \sum_{i=1}^m f_i(x_i) : (x_1, \dots, x_m) \in \mathcal{A}_x^m \right\}.$$

If each f_i is a function that carries order intervals to bounded from above subsets of \mathbb{R} , then the Riesz–Kantorovich functional is real-valued. Moreover, if each f_i is super-additive and positively homogeneous, then the Riesz–Kantorovich functional \mathcal{R}_f is likewise super-additive and positively homogeneous; in particular, it is a concave function.

Let us now associate with each $f_i : L_+ \rightarrow (-\infty, \infty)$ the function $\widehat{f}_i : L \rightarrow [-\infty, \infty)$ defined by

$$\widehat{f}_i(x) := \begin{cases} f_i(x) & \text{if } x \in L_+, \\ -\infty & \text{otherwise.} \end{cases}$$

If each f_i is finite-valued, it is easy to see that for $x \in L_+$, $\mathcal{R}_f(x)$ is given by the value at x of the **sup-convolution**¹ of functions \widehat{f}_i defined by

$$[\nabla_{i=1}^m \widehat{f}_i](x) := \sup \left\{ \sum_{i=1}^m \widehat{f}_i(x_i) : \sum_{i=1}^m x_i = x \right\}.$$

Definition 2.2. We will say that \mathcal{R}_f is **exact** at x with respect to a vector $(x_1, \dots, x_m) \in \mathcal{A}_x^m$ if

$$\mathcal{R}_f(x) = [\nabla_{i=1}^m \widehat{f}_i](x) = \sum_{i=1}^m \widehat{f}_i(x_i) = \sum_{i=1}^m f_i(x_i).$$

Let $\langle X, X' \rangle$ be a dual system and let $f \in \overline{\mathbb{R}}^X$. Recall that a vector $y' \in X'$ is called a **supergradient** of f at x if $f(x)$ is finite and $f(y) - f(x) \leq \langle y - x, y' \rangle$ for all $y \in X$. The (possibly empty) set of all supergradients of f at x is called the **superdifferential** at x of the function f and denoted $\partial f(x)$.

The following standard result, see for example (Laurent, 1972, Proposition 6.6.4), will be used in our work.

Theorem 2.3 (Moreau, 1967). *Assume that $\langle X, X' \rangle$ is an arbitrary dual system. For each $i = 1, \dots, m$, let $g_i: X \rightarrow [-\infty, \infty]$ be a non identically equal to $-\infty$ function. If the sup-convolution $\nabla_{i=1}^m g_i$ is exact at x with respect to some $(x_1, \dots, x_m) \in X^m$ that satisfies $x = \sum_{i=1}^m x_i$, then*

$$\partial[\nabla_{i=1}^m g_i](x) = \bigcap_{i=1}^m \partial g_i(x_i).$$

3. THE ECONOMIC MODEL

In what follows, if L is an ordered linear space, then for the sake of notational convenience, L^+ or L_+ will denote the positive cone of L . The **commodity space** of our model is an ordered vector space L equipped with a Hausdorff locally convex topology τ such that:

- (A1) The positive cone L_+ of L is generating (i.e., $L = L_+ - L_+$) and τ -closed.
- (A2) The order intervals of L are τ -bounded.

The *topological dual* of (L, τ) (i.e., the vector space of all τ -continuous linear functionals on L) will be denoted L' . The *algebraic dual* of L (i.e., the vector space of all linear functionals on L) is denoted L^* . The *order dual* of L (i.e., the vector space of all order-bounded linear functionals on L) is denoted L^\sim . Since every order interval of L is τ -bounded, it follows that $L' \subseteq L^\sim \subseteq L^*$.

¹ Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ denote the set of extended reals. Using the convention $+\infty + (-\infty) = -\infty + (+\infty) = -\infty$, the addition in \mathbb{R} can be extended to $\overline{\mathbb{R}}$. Then for $f \in \overline{\mathbb{R}}^L$ and $g \in \overline{\mathbb{R}}^L$, the formula $[f \nabla g](x) = \sup \{ f(y) + g(z) : y, z \in L \text{ and } y + z = x \}$ defines an extended real-valued function $f \nabla g$ called the *sup-convolution* of f and g . The expression $f \nabla g$ is also called by Rockafellar and Wets (1998) the *hypo-addition* of functions f and g , because if $\text{hypo } f$ denotes the hypograph of f , one has $\text{hypo}(f \nabla g) = \text{hypo } f + \text{hypo } g$, as long as the supremum defining $[f \nabla g](x)$ is attained when finite.

More generally, for any finite collection of functions $g_1, g_2, \dots, g_m \in \overline{\mathbb{R}}^L$, the formula $\nabla_{i=1}^m g_i(x) = \sup \{ \sum_{i=1}^m g_i(x_i) : \sum_{i=1}^m x_i = x \}$ defines the *sup-convolution* of functions g_i . The sup-convolution $\nabla_{i=1}^m g_i$ is said to be *exact* at $x = \sum_{i=1}^m x_i$ if $\nabla_{i=1}^m g_i(x) = \sum_{i=1}^m g_i(x_i)$.

Given the commodity space L , a private ownership production economy is an ordered tuple

$$\mathcal{E} = \left((X_i, P_i, \omega_i)_{i \in I}, (Y_j)_{j \in J}, (\theta_{ij})_{i \in I, j \in J} \right)$$

where $I = \{1, \dots, m\}$ is a finite set of $m (\geq 2)$ consumers, $J = \{1, \dots, n\}$ is a finite set of $n (\geq 1)$ producers. Each consumer i is characterized by a non-empty **consumption set** $X_i \subseteq L$, an **initial endowment** $\omega_i \in X_i$ and an irreflexive **preference correspondence** $P_i: X_i \rightarrow X_i$, i.e., $x_i \notin P_i(x_i)$ for each $x_i \in X_i$. Each producer j is characterized by a non-empty **production set** $Y_j \subseteq L$. For every producer and each consumer, the firm shares $0 \leq \theta_{ij} \leq 1$ classically represent a contractual claim of consumer i to the profit of producer j and $\sum_{i \in I} \theta_{ij} = 1$ for each $j \in J$. In a core and Edgeworth equilibrium approach, the relative shares θ_{ij} reflect consumers' stockholdings that represent proprietorships of production possibilities and $\theta_{ij}Y_j$ is the portion of the j producer's technology set at i 's disposal.

Let $\omega = \sum_{i \in I} \omega_i$ be the **total endowment**, and let \mathcal{A}_ω be the set of all **allocations** of \mathcal{E} , that is,

$$\mathcal{A}_\omega = \left\{ (x, y) = ((x_i)_{i \in I}, (y_j)_{j \in J}) \in \prod_{i \in I} X_i \times \prod_{j \in J} Y_j : \sum_{i \in I} x_i = \omega + \sum_{j \in J} y_j \right\}.$$

The set X_ω , the projection of \mathcal{A}_ω on $\prod_{i \in I} X_i$,² is the set of all **consumption allocations**.

We recall the following standard notions of equilibria for an economy \mathcal{E} .

Definition 3.1. A 3-tuple $(\bar{x}, \bar{y}, \bar{p})$ consisting of an allocation (\bar{x}, \bar{y}) and a non-zero linear functional \bar{p} is said to be:

- (1) a **quasi-equilibrium**, if
 - (a) for every $i \in I$ we have $\bar{p}(\bar{x}_i) = \bar{p}(\omega_i) + \sum_{j \in J} \theta_{ij} \bar{p}(\bar{y}_j)$ and $x_i \in P_i(\bar{x}_i)$ implies $\bar{p}(x_i) \geq \bar{p}(\bar{x}_i)$, and
 - (b) for every $j \in J$ and every $y_j \in Y_j$ we have $\bar{p}(y_j) \leq \bar{p}(\bar{y}_j)$;
- (2) an **equilibrium**, if it is a quasi-equilibrium and if $x_i \in P_i(\bar{x}_i)$ implies $\bar{p}(x_i) > \bar{p}(\bar{x}_i)$.

We emphasize that the notion of quasi-equilibrium in the infinite dimensional setting is vacuous without an additional restriction.

Definition 3.2. A quasi-equilibrium $(\bar{x}, \bar{y}, \bar{p})$ is said to be **non-trivial** if for some $i \in I$ we have

$$\inf\{\bar{p}(z_i) : z_i \in X_i\} < \bar{p}(\bar{x}_i).$$

In this paper, we will be interested only in non-trivial quasi-equilibria. If $(\bar{x}, \bar{y}, \bar{p})$ is some trivial quasi-equilibrium, then for every feasible allocation (x, y) , the pair $((x, y), \bar{p})$ is also a quasi-equilibrium. On the other hand, if the quasi-equilibrium $(\bar{x}, \bar{y}, \bar{p})$ is non-trivial, then it is well-known that, under some continuity condition on preferences or concavity for utility functions and some weak irreducibility assumption on the economy, the quasi-equilibrium $(\bar{x}, \bar{y}, \bar{p})$ is actually an equilibrium.

The following notions of optimality are also standard.

Definition 3.3. A consumption allocation $\bar{x} \in X_\omega$ is said to be:

- (1) **weakly Pareto optimal**, if there is no consumption allocation $x \in X_\omega$ satisfying $x_i \in P_i(\bar{x}_i)$ for each $i \in I$,

²That is, $X_\omega = \{x = (x_i)_{i \in I} \in \prod_{i \in I} X_i : \sum_{i \in I} x_i = \omega + \sum_{j \in J} y_j \text{ for some } (y_j)_{j \in J} \in \prod_{j \in J} Y_j\}$.

- (2) a **core allocation**, if it cannot be blocked by any coalition in the sense that there is no coalition $S \subseteq I$ and some $x \in \prod_{i \in S} X_i$ such that:
- (a) $\sum_{i \in S} x_i \in \sum_{i \in S} \omega_i + \sum_{i \in S} \sum_{j \in J} \theta_{ij} Y_j$, and
 - (b) $x_i \in P_i(\bar{x}_i)$ for all $i \in S$,
- (3) an **Edgeworth equilibrium**, if it belongs to the core of every r -fold replica of \mathcal{E} ,³
- (4) a **fuzzy core allocation**, if there exist no $\tau = (\tau_i)_{i \in I} \in [0, 1]^I \setminus \{0\}$ and $x \in \prod_{i \in I} X_i$ such that:
- (a) $\sum_{i \in I} \tau_i x_i \in \sum_{i \in I} \tau_i \omega_i + \sum_{i \in I} \tau_i \sum_{j \in J} \theta_{ij} Y_j$, and
 - (b) $x_i \in P_i(\bar{x}_i)$ for all $i \in I$ with $\tau_i > 0$.

From now on, we impose on \mathcal{E} the following conditions.

Standard assumptions and ω -properness: That is,

- (1) For each consumer i , the consumption set X_i is convex, and $\omega = \sum_{i \in I} \omega_i > 0$.
- (2) For each i and every weakly Pareto optimal consumption allocation $x = (x_i)_{i \in I}$, $x_i \in \text{cl} P_i(x_i)$, $P_i(x_i)$ is open in X_i for some linear topology on L (or is induced by a concave utility function) and is ω -**proper** at x_i in the following sense adapted from Tourky (1998): There exist in L a convex set $\hat{P}_i(x_i)$ and a convex set $Z_i(x_i)$ such that
 - (a) the vector $x_i + \omega$ is a τ -interior point of $\hat{P}_i(x_i)$,
 - (b) $\hat{P}_i(x_i) \cap Z_i(x_i) = P_i(x_i)$,
 - (c) $x_i, 0, \omega_i \in Z_i(x_i)$, $Z_i(x_i) + L_+ \subseteq Z_i(x_i)$, and
 - (d) for every $u > 0$ and every couple $\{z_i, z'_i\}$ of elements of $Z_i(x_i)$, if $-u \leq z_i$ and $-u \leq z'_i$, then there exists $z \in Z_i(x_i)$ such that $-u \leq z \leq z_i$ and $-u \leq z \leq z'_i$.
- (3) For each producer j , the production set Y_j is convex and $0 \in Y_j$.
- (4) For each j and every $y_j \in Y_j$ associated with a weakly Pareto optimal consumption allocation, Y_j is ω -**proper at y_j** in the following sense adapted from Tourky (1999): There exist in L a convex set $\hat{Y}_j(y_j)$ and a convex set $Z_j(y_j)$ such that
 - (a) the vector $y_j - \omega$ is a τ -interior point of $\hat{Y}_j(y_j)$,
 - (b) $\hat{Y}_j(y_j) \cap Z_j(y_j) = Y_j$, and
 - (c) $0 \in Z_j(y_j)$ and $Z_j(y_j) - L_+ \subseteq Z_j(y_j)$,
 - (d) for every $u > 0$ and every couple $\{z_j, z'_j\}$ of elements of $Z_j(y_j)$, if $z_j \leq u$ and $z'_j \leq u$, then there exists $z \in Z_j(y_j)$ such that $z_j \leq z \leq u$ and $z'_j \leq z \leq u$.

An economy satisfying these assumptions will be referred to as an ω -**proper economy**.

Compactness: For some Hausdorff linear topology σ on L , the set X_ω of all consumption allocations is σ^m -compact and preferences have σ -open (in X_i) lower sections $P_i^{-1}(x_i) = \{x'_i \in X_i: x_i \in P_i(x'_i)\}$.

It is worth noticing that in the consumption case if L is a Riesz space, then our assumptions on the sets $Z_i(x_i)$ are satisfied by the assumption of Tourky (1998) where each $Z_i(x_i)$ is a convex lattice containing $0, \omega_i, x_i$ and satisfying $Z_i(x_i) + L_+ \subseteq Z_i(x_i)$. Our assumptions however do not imply that L_+ is a lattice cone. For the production case, the sets $Z_j(y_j)$ correspond to the *pretechnology sets* defined by Mas-Colell (1986b) and Richard (1989).

The following lemma follows from the previous assumptions and Florenzano (1990).

³The ideas in this definition go back to Debreu and Scarf (1963). An important reference is also Aubin (1979). Edgeworth equilibria were introduced and studied in Aliprantis, Brown, and Burkinshaw (1987a).

Lemma 3.4. *In our economy, Edgeworth equilibria exist and belong to the fuzzy core.*

As seen in Allouch and Florenzano (2004), one can prove that if preference correspondences are derived from quasiconcave utility functions defined on general consumption sets, our compactness condition can be replaced by a weaker compactness assumption made in the utility space on the “utility set”.⁴

4. DECENTRALIZING EDGEWORTH EQUILIBRIA

For the discussion of this section, we fix an Edgeworth equilibrium consumption allocation $(\bar{x}_i)_{i \in I}$, thus a fuzzy core consumption allocation of \mathcal{E} , and $(\bar{y}_j)_{j \in J}$, the associated production allocation. Such an allocation $((\bar{x}_i)_{i \in I}, (\bar{y}_j)_{j \in J})$ exists by Lemma 3.4. The proof of its possible decentralization by prices in L' will adapt ideas of (Florenzano, 2003, Chapter 5.3.4) where is proved the decentralization of Edgeworth equilibria of an ω -proper production economy defined on a vector lattice commodity space. For technical details not covered in this paper, we refer to that chapter.

Since L_+ is by **(A1)** generating, we can choose $u > 0$ in L such that the order interval $[-u, u]$ contains $\bar{x}_i, \omega_i, \bar{y}_j$ for all $i \in I$ and $j \in J$. Now consider the ordered vector subspace $L_u = \bigcup_{\lambda > 0} \lambda[-u, u]$ equipped with the order topology. This space is Archimedean (a property inherited from L of which the positive cone is τ -closed) and has u as an order unit.⁵ Its order topology (i.e., the finest locally convex topology on L_u for which every order interval is bounded) is normable [see (Schaefer, 1971, Chapter V 6.2)]. More precisely, the gauge $\|\cdot\|_u$ of $[-u, u]$, defined for each $z \in L_u$ by

$$\|z\|_u = \inf\{\lambda > 0: -\lambda u \leq z \leq \lambda u\},$$

is a norm on L_u that generates the order topology whose closed unit ball is precisely the order interval $[-u, u]$ and u is an interior point of $L_u^+ = L_u \cap L_+$. Moreover, it follows from **(A2)** (namely from the fact that $[-u, u]$ is τ -bounded) that on L_u the order topology is finer than the topology induced by τ . In addition, it is not difficult to see that the cone L_u^+ is $\|\cdot\|_u$ -closed in L_u .⁶

Let \mathcal{E}_u be the economy \mathcal{E} restricted to L_u in an obvious way. For each consumer i , $X_i^u = X_i \cap L_u$, and for each $x_i \in X_i^u$, $P_i^u(x_i) = P_i(x_i) \cap L_u$; for each producer j , $Y_j^u = Y_j \cap L_u$. It is easily seen that the consumption allocation $(\bar{x}_i)_{i \in I}$ is an Edgeworth equilibrium and belongs to the fuzzy core of \mathcal{E}_u . It follows from the ω -properness of \mathcal{E} that for each i and any $0 < \alpha \leq 1$, the vector $\bar{x}_i + \alpha\omega$ belongs to the norm-interior of $P_i^u(\bar{x}_i)$. But then from the standard decentralization result in presence of an interiority property that there exists a nonzero $\bar{p}_u \in (L_u, \|\cdot\|_u)'$ such that $(\bar{x}, \bar{y}, \bar{p}_u)$ is a quasi-equilibrium of the restricted economy \mathcal{E}_u and $\bar{p}_u(\omega) > 0$.

Before going further, let us recall an extension lemma due to Podczeck (1996) whose a proof can be found in (Florenzano, 2003, Lemma 5.3.1, p. 134).

Lemma 4.1 (Podczeck). *Let (L, τ) be an ordered topological vector space, let K be a vector subspace of L (endowed with the induced order), let A be a convex subset of K such that $A + K_+ \subseteq A$, let*

⁴That is, the set of feasible and individually rational utility vectors.

⁵That is, the order interval $[-u, u]$ is radial at the origin.

⁶Indeed, if $\{x_n\} \subseteq L_+$ satisfies $-\frac{1}{n}u \leq x - x_n \leq \frac{1}{n}u$ for all n , then from $x = (x - x_n) + x_n \geq x - x_n \geq -\frac{1}{n}u$ and the Archimedean property we easily obtain $x \geq 0$.

V be a convex τ -open subset of L such that $V \cap A \neq \emptyset$, and let $a \in A \cap \text{cl}V$. If p is a linear functional on K satisfying

$$p \cdot a \leq p \cdot x, \text{ for all } x \in V \cap A,$$

then there exists some $\pi \in L'$ such that $\pi|_K \leq p$, and

$$p \cdot (a - x) = \pi \cdot (a - x) \text{ for each } x \in A \text{ with } x \leq a.$$

The following obvious consequence of the preceding lemma was stated in Florenzano and Marakulin (2001).

Corollary 4.2. *Let (L, τ) be an ordered topological vector space, let K be a vector subspace of L (endowed with the induced order), let A be a convex subset of K such that $A - K_+ \subseteq A$, let V be a convex τ -open subset of L such that $V \cap A \neq \emptyset$, and let $a \in A \cap \text{cl}V$. If p is a linear functional on K satisfying*

$$p \cdot a \geq p \cdot x, \text{ for all } x \in V \cap A,$$

then there exists some $\pi \in L'$ such that $\pi|_K \leq p$, and

$$p \cdot (a - y) = \pi \cdot (a - y) \text{ for each } y \geq a, y \in A.$$

Our next step associates with the quasi-equilibrium price \bar{p}_u a finite list of τ -continuous linear functionals $\pi^u = ((\pi_i^u)_{i \in I}, (\pi_j^u)_{j \in J})$ defined on the whole commodity space. These continuous linear functionals can be thought of as personalized supporting prices for each consumer and each producer at the corresponding component of the allocation (\bar{x}, \bar{y}) .

Proposition 4.1. *In our economy, there exist $(\pi_i^u)_{i \in I}, (\pi_j^u)_{j \in J}$ in L' such that:*

- (1) *For each consumer i we have*
 - (a) $\pi_i^u \leq \bar{p}_u$ on L_u ,
 - (b) $\pi_i^u \cdot \hat{P}_i(\bar{x}_i) \geq \pi_i^u \cdot \bar{x}_i$, and
 - (c) *if $z_i \in Z_i(\bar{x}_i) \cap L_u$ satisfies $z_i \leq \bar{x}_i$, then*

$$\bar{p}_u \cdot (\bar{x}_i - z_i) = \pi_i^u \cdot (\bar{x}_i - z_i) = \mathcal{R}_{\pi^u}(\bar{x}_i - z_i).$$

- (2) *For each producer j we have*
 - (a) $\pi_j^u \leq \bar{p}_u$ on L_u ,
 - (b) $\pi_j^u \cdot \hat{Y}_j(\bar{y}_j) \leq \pi_j^u \cdot \bar{y}_j$, and
 - (c) *if $z_j \in Z_j(\bar{y}_j) \cap L_u$ satisfies $z_j \geq \bar{y}_j$, then*

$$\bar{p}_u \cdot (z_j - \bar{y}_j) = \pi_j^u \cdot (z_j - \bar{y}_j) = \mathcal{R}_{\pi^u}(z_j - \bar{y}_j).$$

- (3) *Moreover,*

- (a) $\mathcal{R}_{\pi^u} \leq \bar{p}_u$ on L_u , and
- (b) *if $v \leq \omega$ satisfies $v = \sum_{i \in I} v_i - \sum_{j \in J} w_j \in \left[\sum_{i \in I} (Z_i(\bar{x}_i) \cap L_u) - \sum_{j \in J} (Z_j(\bar{y}_j) \cap L_u) \right]$, then*

$$\bar{p}_u(\omega - v) = \mathcal{R}_{\pi^u}(\omega - v).$$

Proof. Using Lemma 4.1 and its corollary with $K = L_u$ and for each i , $a = \bar{x}_i$, $A = Z_i(\bar{x}_i) \cap L_u$, $V = \text{int } \hat{P}_i(\bar{x}_i)$, for each j , $a = \bar{y}_j$, $A = Z_j(\bar{y}_j) \cap L_u$, $V = \text{int } \hat{Y}_j(\bar{y}_j)$, the existence of $(\pi_i^u)_{i \in I}, (\pi_j^u)_{j \in J}$ in L' satisfying (1), (2) and (3)(a) is a straightforward consequence of the fact that $(\bar{x}, \bar{y}, \bar{p}_u)$ is

a quasi-equilibrium of the restricted economy \mathcal{E}_u and of the ω -properness assumptions at each component of the allocation (\bar{x}, \bar{y}) .

To prove (3)(b), assume that $v \leq \omega$, $v = \sum_{i \in I} v_i - \sum_{j \in J} w_j$ with each $v_i \in Z_i(\bar{x}_i) \cap L_u$ and each $w_j \in Z_j(\bar{y}_j) \cap L_u$. Let $z_i \in Z_i(\bar{x}_i) \cap L_u$ be such that $z_i \leq \{v_i, \bar{x}_i\}$ and $z_j \in Z_j(\bar{y}_j) \cap L_u$ be such that $z_j \geq \{w_j, \bar{y}_j\}$. Such z_i and z_j exist in view of the definition of L_u and our assumptions on sets $Z_i(\bar{x}_i)$ and $Z_j(\bar{y}_j)$. On one hand, using (1)(c), (2)(c) and the superadditivity of \mathcal{R}_{π^u} , it follows from (1)(a) and (2)(a) that

$$\begin{aligned} (\bar{p}_u - \mathcal{R}_{\pi^u})(\omega - v) &= (\bar{p}_u - \mathcal{R}_{\pi^u}) \left(\sum_{i \in I} \bar{x}_i - \sum_{j \in J} \bar{y}_j - \sum_{i \in I} v_i + \sum_{j \in J} w_j \right) \\ &\leq (\bar{p}_u - \mathcal{R}_{\pi^u}) \left(\sum_{i \in I} (\bar{x}_i - z_i) + \sum_{j \in J} (z_j - \bar{y}_j) \right) = 0. \end{aligned}$$

On the other hand, it follows also from (1)(a) and (2)(a) that $(\bar{p}_u - \mathcal{R}_{\pi^u})(\omega - v) \geq 0$, which implies $(\bar{p}_u - \mathcal{R}_{\pi^u})(\omega - v) = 0$ ■

The following corollary can be seen as an analogue for the production economy \mathcal{E}_u of Proposition 5.1 in Aliprantis et al. (2004b) which extends at several instances statement (1) of Theorem 7.5 in Aliprantis et al. (2001).

Corollary 4.3. *We have in addition:*

- (1) $x_i \in P_i^u(\bar{x}_i) \implies \mathcal{R}_{\pi^u}(x_i - z_i) \geq \mathcal{R}_{\pi^u}(\bar{x}_i - z_i)$ for every $z_i \in Z_i(\bar{x}_i) \cap L_u$, $z_i \leq \{x_i, \bar{x}_i\}$.
- (2) $y_j \in Y_j^u \implies \mathcal{R}_{\pi^u}(z_j - y_j) \geq \mathcal{R}_{\pi^u}(z_j - \bar{y}_j)$ for every $z_j \in Z_j(\bar{y}_j) \cap L_u$, $z_j \geq \{y_j, \bar{y}_j\}$.
- (3) *If for each i , $z'_i \in Z_i(\bar{x}_i) \cap L_u$, $z'_i \leq \{\omega_i, \bar{x}_i\}$, if for each j , $z'_j \in Z_j(\bar{y}_j) \cap L_u$, $z'_j \geq \{0, \bar{y}_j\}$, then*

- (a) $\mathcal{R}_{\pi^u}(\omega - \sum_{i \in I} z'_i + \sum_{j \in J} z'_j) = \sum_{i \in I} \pi_i^u(\bar{x}_i - z'_i) + \sum_{j \in J} \pi_j^u(z'_j - \bar{y}_j)$, and
- (b) *for each i , $\pi_i^u \cdot (\bar{x}_i - z'_i) + \sum_{j \in J} \theta_{ij} \pi_j^u \cdot (z'_j - \bar{y}_j) \geq \mathcal{R}_{\pi^u}(\omega_i - z'_i + \sum_{j \in J} \theta_{ij} z'_j)$.*

Proof. To prove (1), let us assume $x_i \in P_i^u(\bar{x}_i)$ and $z_i \in Z_i(\bar{x}_i) \cap L_u$, $z_i \leq \{x_i, \bar{x}_i\}$. Recall that $P_i(\bar{x}_i) = \hat{P}_i(\bar{x}_i) \cap Z_i(\bar{x}_i)$, thus that $P_i^u(\bar{x}_i) = \hat{P}_i(\bar{x}_i) \cap Z_i(\bar{x}_i) \cap L_u$. We then easily deduce from Proposition 4.1 that:

$$\mathcal{R}_{\pi^u}(x_i - z_i) \geq \pi_i^u \cdot (x_i - z_i) \geq \pi_i^u \cdot (\bar{x}_i - z_i) = \mathcal{R}_{\pi^u}(\bar{x}_i - z_i).$$

The proof of (2) is done symmetrically. The proof of (3) goes as follows.

Using the superadditivity of \mathcal{R}_{π^u} , the last assertion of Proposition 4.1, and assuming that for each i , $z'_i \in Z_i(\bar{x}_i) \cap L_u$, $z'_i \leq \{\omega_i, \bar{x}_i\}$, and for each j , $z'_j \in Z_j(\bar{y}_j) \cap L_u$, $z'_j \geq \{0, \bar{y}_j\}$, we get

$$\begin{aligned} \bar{p}_u \left(\omega - \sum_{i \in I} z'_i + \sum_{j \in J} z'_j \right) &= \mathcal{R}_{\pi^u} \left(\omega - \sum_{i \in I} z'_i + \sum_{j \in J} z'_j \right) \geq \sum_{i \in I} \mathcal{R}_{\pi^u}(\bar{x}_i - z'_i) + \sum_{j \in J} \mathcal{R}_{\pi^u}(z'_j - \bar{y}_j) \\ &= \sum_{i \in I} \pi_i^u(\bar{x}_i - z'_i) + \sum_{j \in J} \pi_j^u(z'_j - \bar{y}_j) = \bar{p}_u \left(\omega - \sum_{i \in I} z'_i + \sum_{j \in J} z'_j \right). \end{aligned}$$

This proves the first assertion of (3).

Finally, recall that $(\bar{x}, \bar{y}, \bar{p}_u)$ is a quasi-equilibrium of \mathcal{E}_u . We thus have for every i ,

$$\bar{p}_u \cdot \bar{x}_i = \bar{p}_u \cdot \omega_i + \sum_{j \in J} \theta_{ij} \bar{p}_u \cdot \bar{y}_j. \quad (4.1)$$

From (4.1), using (1)(c) and (2)(c) of Proposition 4.1, we deduce easily from that for each i ,

$$\begin{aligned} \mathcal{R}_{\pi_u}(\bar{x}_i - z'_i) + \sum_{j \in J} \theta_{ij} \mathcal{R}_{\pi_u}(z'_j - \bar{y}_j) &= \bar{p}_u \cdot (\bar{x}_i - z'_i) + \sum_{j \in J} \theta_{ij} \bar{p}_u \cdot (z'_j - \bar{y}_j) \\ &= \bar{p}_u \cdot (\omega_i - z'_i) + \sum_{j \in J} \theta_{ij} \bar{p}_u \cdot z'_j \geq \mathcal{R}_{\pi_u} \left(\omega_i - z'_i + \sum_{j \in J} \theta_{ij} z'_j \right), \end{aligned}$$

which completes the proof. ■

Let us now recall some notions of properness for functions on L_+ .

Definition 4.4. Let $v \in L_+$ be such that $v > 0$. We say that a function $f: L_+ \rightarrow \mathbb{R}$ is:

- (1) **v -proper** at some $x \in L_+$, if there exists a convex set F such that:
 - (a) $x + v$ is an interior point of F , and
 - (b) $F \cap L_+ = \{y \in L_+ : f(y) > f(x)\}$.
- (2) **v -pointwise proper** at $x \in L_+$, if there exists an open pointed convex cone Γ_x such that:
 - (a) $-v \in \Gamma_x$, and
 - (b) $(x - \Gamma_x) \cap \{y \in L_+ : f(y) > f(x)\} = \emptyset$.

Considering the set $\{y \in L_+ : f(y) > f(x)\}$ as a preferred set for a preference correspondence defined on L_+ by the utility function f , the reader recognizes in these definitions the usual notions of v -pointwise properness as defined by Mas-Colell (1986a) and of v -properness as defined by Tourky (1998). Mas-Colell (1986a) defines uniform properness on a subset X of L_+ as properness at every $x \in X$ with a properness vector and a properness cone independent of x .⁷ The definition of uniform v -properness given here is quite similar.

Definition 4.5. A function $f: L_+ \rightarrow \mathbb{R}$ is **uniformly v -proper** on $X \subset L_+$ if for every $x \in X$ there exist a convex set F_x and a τ -neighborhood V of 0 (independent of x) such that

- (a) $x + v + V \subseteq F_x$, and
- (b) $F_x \cap L_+ = \{y \in L_+ : f(y) > f(x)\}$.

We now introduce the following additional assumption of compatibility between the order structure and the topology τ of the commodity space of our economy.

- (A3) For any finite list $f = (f_1, \dots, f_\ell)$ of continuous linear functionals satisfying $f_k(\omega) > 0$ for each k , the Riesz–Kantorovich functional \mathcal{R}_f is uniformly ω -proper at any point $\omega' \geq \omega$.

Remark 4.6. Condition (A3) is automatically satisfied if the commodity space is a locally convex vector lattice and its dual is a vector sublattice of the order dual. Indeed, in this case, it follows from Theorem 2.1 that the Riesz–Kantorovich functionals are linear and τ -continuous. Assume now that f is a finite list of continuous linear functionals $(f_k)_{k=1}^K$ such that $f_k(\omega) > 0$ for each k (which implies $\mathcal{R}_f(\omega) > 0$). For each $\omega' \in L_+$, define $F_{\omega'} = \{y \in L : g(y) > g(\omega')\}$, where g is the continuous linear functional which coincides with \mathcal{R}_f on L_+ . If V is a τ -neighborhood of 0 such that $g(\omega + V) > 0$, then $g(\omega' + \omega + V) > g(\omega')$, thus $\omega' + \omega + V \subseteq F_{\omega'}$, while $F_{\omega'} \cap L_+ = \{y \in L : \mathcal{R}_f(y) > \mathcal{R}_f(\omega')\}$.

⁷If v is the properness vector, recall that Mas-Colell's uniform properness on L_+ implies v -properness [see Tourky (1998)].

We can provide a simple way of identifying order structures on Banach spaces that satisfy condition **(A3)**. Recall that a subset A of an ordered set **dominates** a point x , in symbols $A \geq x$, if $a \geq x$ holds true for each $a \in A$.

Theorem 4.7. *Let L be an ordered Banach space with a closed generating cone, weakly compact order intervals and admitting a strictly positive linear functional. In this case, condition **(A3)** is satisfied if for any finite list $f = (f_1, \dots, f_\ell)$ of positive (necessarily continuous) linear functionals there exists a weak* compact set $G \subseteq L'$ that dominates each f_k and satisfies*

$$\inf\{g(x): g \in L' \text{ and } g \geq f_k \text{ for all } k = 1, \dots, \ell\} = \min\{g(x): g \in G\},$$

for each $x \in L_+$.

Proof. Let $f = (f_1, \dots, f_\ell)$ be a finite list of continuous linear functionals satisfying $f_k(\omega) > 0$ for all k . Since in ordered normed spaces the norm boundedness of order intervals is equivalent to the normality of the cone, it follows that L'_+ is a generating cone. In particular, there exists some positive linear functional h such that $h \geq -f_k$ for each k . Now if h_1 is a strictly positive linear functional and we let $p = h + h_1$, then $f_k + p$ is strictly positive for each $k = 1, \dots, \ell$. By our assumption there exists a weak* compact set $G \subseteq L'$ that dominates each $f_k + p$ such that

$$\inf\{g(x): g \in L' \text{ and } g \geq f_k + p \text{ for all } k = 1, \dots, \ell\} = \min\{g(x): g \in G\},$$

holds true for each $x \in L_+$.

Let $\mathcal{W} = \{x \in L: g(x) \geq 0 \text{ for all } g \in G\}$. Notice that \mathcal{W} is a closed wedge that contains ω as an interior point. Moreover, we have $L_+ \subseteq \mathcal{W}$. Next, define the mapping $\mathcal{M}: \mathcal{W} \rightarrow \mathbb{R}$ via the formula

$$\mathcal{M}(x) = \min\{g(x): g \in G\}.$$

We show that \mathcal{M} is a norm lower semicontinuous function. Suppose that $\|x_\alpha - x\| \rightarrow 0$, where $\{x_\alpha\}$ is a net of \mathcal{W} such that $\mathcal{M}(x_\alpha) \leq c$. Let $\{g_\alpha\}$ be the net of corresponding minimizers in G . By passing to a subnet (if necessary), we can assume without loss of generality that $g_\alpha \xrightarrow{w^*} g$ in L' . It easily follows that $g_\alpha(x_\alpha) \rightarrow g(x)$. Therefore, $\mathcal{M}(x) \leq g(x) \leq c$.

Clearly, \mathcal{M} is positively homogeneous. That is $\mathcal{M}(\lambda x) = \lambda \mathcal{M}(x)$ for all $\lambda \geq 0$. We show that it is super-additive. Pick $x, y \in \mathcal{W}$ and pick $h_1, h_2 \in G$ such that let $h_1(x) = \mathcal{M}(x)$ and $h_2(y) = \mathcal{M}(y)$ be the respective minimizers. Now for any $g \in G$ we have $g(x+y) \geq h_1(x) + h_2(y)$. Therefore, $\mathcal{M}(x+y) \geq \mathcal{M}(x) + \mathcal{M}(y)$.

We return to the list f . Let $M_f(x) = \min\{g(x): g \in G - p\}$. Notice that for any $x \in \mathcal{W}$ we have $M_f(x) = \mathcal{M}(x) - p(x)$. Therefore, for $x, y \in \mathcal{W}$ we have

$$M_f(x+y) = \mathcal{M}(x+y) - p(x+y) \geq \mathcal{M}(x) + \mathcal{M}(y) - p(x) - p(y) = M_f(x) + M_f(y).$$

That is, M_f is super-additive. Clearly, it is also positively homogeneous. These two facts imply that the upper sections $U_x = \{y \in \mathcal{W}: M_f(y) \geq M_f(x)\}$ are convex sets for each $x \in \mathcal{W}$.

Now we know that $M_f(\omega) = \mathcal{M}(\omega) - p(\omega) > 0$. Since \mathcal{M} is lower semicontinuous, M_f is likewise lower semicontinuous and so there exists an open ball V centered at ω satisfying $V \subseteq \mathcal{W}$ and $\mathcal{M}(v) > p(v)$ for all $v \in V$.

Next, pick any $x \in \mathcal{W}$ and notice that for any $v \in V$ we have $M_f(x+v) \geq M_f(x) + M_f(v)$. Because $M_f(v) = \mathcal{M}(v) - p(v)$, this implies that $M_f(x+v) > M_f(x)$. Therefore, M_f is ω -uniformly proper at each $x \in L_+$.

It is shown in (Aliprantis and Tourky, 2002, Theorem 7.6, p. 2072) that for any $x \in L_+$ that if $\pi = (f_1 + p, f_2 + p, \dots, f_\ell + p)$, then we have $\mathcal{R}_\pi(x) = \mathcal{M}(x)$. Clearly, $\mathcal{R}_f = \mathcal{R}_\pi - p$. Therefore, $\mathcal{R}_f(x) = M_f(x)$ for all $x \in L_+$, and the proof is finished. ■

Examples of closed generating cones that satisfy the assumptions of Theorem 4.7 are the generating polyhedral cones in finite dimensional spaces.

We move to the first application of **(A3)**.

Proposition 4.2. *Under **(A3)**, there exists a price system $\bar{\pi}_u \in L'$ such that $\bar{\pi}_u \cdot \omega > 0$ and $(\bar{x}, \bar{y}, \bar{\pi}_u)$ is a quasi-equilibrium of \mathcal{E}_u .*

Proof. Let us consider the set Ω of all $\omega' = \omega - \sum_{i \in I} z'_i + \sum_{j \in J} z'_j$, where for each i , $z'_i \in Z_i(\bar{x}_i) \cap L_u$, $z'_i \leq \{0, \omega_i, \bar{x}_i\}$, and for each j , $z'_j \in Z_j(\bar{y}_j) \cap L_u$, $z'_j \geq \{0, \bar{y}_j\}$. This set is nonempty in view of the definition of L_u and our assumptions on $Z_i(\bar{x}_i)$ and $Z_j(\bar{y}_j)$. It is directed by the relation

$$\omega'' \geq \omega' \text{ if and only if for each } i \text{ and } j, z''_i \leq z'_i \text{ and } z''_j \geq z'_j.$$

Moreover, if $\omega' \in \Omega$ then $z'_i \leq 0$ for each i and $z'_j \geq 0$ for each j imply $\omega' \geq \omega > 0$. From (3) in Corollary 4.3, it follows that the Riesz–Kantorovich functional \mathcal{R}_{π_u} is exact at each $\omega' \in \Omega$. From Proposition 4.1, it is easily deduced that for each i , either $\pi_i^u = 0$ or $\pi_i^u \cdot \omega > 0$, that for each j , either $\pi_j^u = 0$ or $\pi_j^u \cdot \omega > 0$, that $\mathcal{R}_{\pi_u}(\omega) = \bar{p}_u(\omega) > 0$, thus that at least one of the π_i^u, π_j^u is nonzero. Set $P(\omega') = \{z \in L_+ : \mathcal{R}_{\pi_u}(z) > \mathcal{R}_{\pi_u}(\omega')\}$. Applying Assumption **(A3)**, there exist a 0-neighborhood V and for each $\omega' \in \Omega$, a convex set $\hat{P}(\omega')$ such that $\omega' + \omega + V \subseteq \hat{P}(\omega')$ and $P(\omega') = L_+ \cap \hat{P}(\omega')$.

Note that

$$\mathcal{R}_{\pi_u}(\omega' + \alpha\omega) \geq \mathcal{R}_{\pi_u}(\omega') + \alpha\mathcal{R}_{\pi_u}(\omega) > \mathcal{R}_{\pi_u}(\omega') \text{ every } 0 < \alpha \leq 1,$$

so that ω' belongs to the closure of $\hat{P}(\omega')$. Now, if $z \in \hat{P}(\omega') \cap L_u^+$, then $\mathcal{R}_{\pi_u}(z) > \mathcal{R}_{\pi_u}(\omega') = \bar{p}_u(\omega')$. From $\mathcal{R}_{\pi_u} \leq \bar{p}_u$ on L_u , it follows that $\bar{p}_u \cdot z > \bar{p}_u \cdot \omega'$. If we define $X = \{z \in L_u^+ : \bar{p}_u \cdot z \leq \bar{p}_u \cdot \omega'\}$, the last observation can be rephrased as $X \cap \hat{P}(\omega') = \emptyset$, so that to each $\omega' \in \Omega$, we can associate a nonzero $\bar{\pi}_{\omega'} \in L'$ which separates $\hat{P}(\omega')$ and X , that is,

$$\bar{\pi}_{\omega'} \cdot X \leq \bar{\pi}_{\omega'} \cdot \omega' \leq \bar{\pi}_{\omega'} \cdot \hat{P}(\omega'). \quad (4.2)$$

Since $\omega' + \omega$ is an interior point of $\hat{P}(\omega')$, we have $\bar{\pi}_{\omega'} \cdot \omega > 0$, and we can normalize prices letting

$$\bar{p}_u \cdot \omega = \mathcal{R}_{\pi_u}(\omega) = \bar{\pi}_{\omega'} \cdot \omega = 1. \quad (4.3)$$

Let $L_{\omega'}$ be the ordered vector subspace $L_{\omega'} = \bigcup_{\lambda > 0} \lambda[-\omega', \omega']$. Clearly, $L_{\omega'} \subset L_u$. We first claim that $\bar{\pi}_{\omega'} = \bar{p}_u$ on $L_{\omega'}$. Indeed, for every $z \in L_u^+$, we know that

$$\bar{p}_u \cdot z \leq \bar{p}_u \cdot \omega' \implies \bar{\pi}_{\omega'} \cdot z \leq \bar{\pi}_{\omega'} \cdot \omega'.$$

So, using the existence of Lagrange multipliers for a convex programming problem,⁸ there exist two real numbers $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ not all equal to zero such that $\lambda_1[\bar{\pi}_{\omega'} \cdot z - \bar{\pi}_{\omega'} \cdot \omega'] \leq \lambda_2[\bar{p}_u \cdot z - \bar{p}_u \cdot \omega']$ for every $z \in L_u^+$. From $\bar{\pi}_{\omega'} \cdot \omega = \bar{p}_u \cdot \omega > 0$, we easily deduce that $(\lambda_1, \lambda_2) \gg 0$. Letting successively $z = 0$ and $z = 2\omega'$, we also see that for some $\lambda > 0$, $\bar{\pi}_{\omega'} \cdot \omega' = \lambda\bar{p}_u \cdot \omega'$ and $\bar{\pi}_{\omega'} \cdot z \leq \lambda\bar{p}_u \cdot z$ for every $z \in L_u^+$. The previous inequality holds in particular for every $z \in L_{\omega'}^+$. Since ω' is an interior point (for the order topology) of $L_{\omega'}^+$, it follows that $\bar{\pi}_{\omega'} = \lambda\bar{p}_u$ on $L_{\omega'}$. Recalling that $\bar{\pi}_{\omega'} \cdot \omega = \bar{p}_u \cdot \omega$, it follows that $\lambda = 1$.

⁸A simple proof of the existence of Lagrange multipliers for a convex programming problem can be found in (Barbu, and Precupanu, 1986, Chapter 3, Theorem 1.1). See also Fan et al. (1957) and (Aliprantis and Border, 2006, Chapter 5, Theorem 5.70).

Recalling that for each i , we have $\bar{p}_u \cdot \bar{x}_i = \bar{p}_u \cdot \omega_i + \sum_{j \in J} \theta_{ij} \bar{p}_u \cdot \bar{y}_j$, that is,

$$\bar{p}_u \cdot (\bar{x}_i - z'_i) + \sum_{j \in J} \theta_{ij} \bar{p}_u \cdot (z'_j - \bar{y}_j) = \bar{p}_u \cdot (\omega_i - z'_i) + \sum_{j \in J} \theta_{ij} \bar{p}_u \cdot z'_j,$$

we deduce from our first claim:

$$\bar{\pi}_{\omega'} \cdot (\bar{x}_i - z'_i) + \sum_{j \in J} \theta_{ij} \bar{\pi}_{\omega'} \cdot (z'_j - \bar{y}_j) = \bar{\pi}_{\omega'} \cdot (\omega_i - z'_i) + \sum_{j \in J} \theta_{ij} \bar{\pi}_{\omega'} \cdot z'_j,$$

and, thus, for each i and for every $\omega' \in \Omega$,

$$\bar{\pi}_{\omega'} \cdot \bar{x}_i = \bar{\pi}_{\omega'} \cdot \omega_i + \sum_{j \in J} \theta_{ij} \bar{\pi}_{\omega'} \cdot \bar{y}_j. \quad (4.4)$$

We next claim that for some $\mu > 0$ (depending on ω'), $\mu \bar{\pi}_{\omega'}$ is a supergradient of \mathcal{R}_{π_u} at ω' . Indeed, from $\mathcal{R}_{\pi_u}(z) \geq \mathcal{R}_{\pi_u}(\omega') \implies \bar{\pi}_{\omega'}(z) \geq \bar{\pi}_{\omega'}(\omega')$ for every $z \in L_+$, one deduces, as previously, the existence of two real numbers $\mu_1 \geq 0$ and $\mu_2 \geq 0$ not all equal to zero such that $\mu_1[\bar{\pi}_{\omega'} \cdot z - \bar{\pi}_{\omega'} \cdot \omega'] \geq \mu_2[\mathcal{R}_{\pi_u}(z) - \mathcal{R}_{\pi_u}(\omega')]$ for every $z \in L_+$. As previously, from $\mathcal{R}_{\pi_u}(\omega) = \bar{\pi}_{\omega'} \cdot \omega > 0$, one deduces $(\mu_1, \mu_2) \gg 0$, so that for some $\mu > 0$, $\mu \bar{\pi}_{\omega'} \cdot \omega' = \mathcal{R}_{\pi_u}(\omega')$ and $\mu \bar{\pi}_{\omega'} \cdot z \geq \mathcal{R}_{\pi_u}(z)$ for every $z \in L_+$, which proves the claim.

Fix now $(x, y) \in \prod_{i \in I} P_i^u(\bar{x}_i) \times \prod_{j \in J} Y_j^u$. Let $\omega'_0 = \omega - \sum_{i \in I} z'_{i0} + \sum_{j \in J} z'_{j0}$ where for each i , $z'_{i0} \in Z_i(\bar{x}_i) \cap L_u$, $z'_{i0} \leq \{0, x_i, \omega_i, \bar{x}_i\}$, and for each j , $z'_{j0} \in Z_j(\bar{y}_j) \cap L_u$, $z'_{j0} \geq \{y_j, 0, \bar{y}_j\}$. Once again, this is possible in view of the definition of L_u and our assumptions on $Z_i(\bar{x}_i)$ and $Z_j(\bar{y}_j)$. For every $\omega' \geq \omega'_0$ in Ω , applying Theorem 2.3 and using (1) and (2) of Proposition 4.1, we get:

$$x_i \in P_i^u(\bar{x}_i) \implies \bar{\pi}_{\omega'} \cdot (x_i - z'_i) \geq \bar{\pi}_{\omega'} \cdot (\bar{x}_i - z'_i) \implies \bar{\pi}_{\omega'} \cdot x_i \geq \bar{\pi}_{\omega'} \cdot \bar{x}_i \quad (4.5)$$

and for each i , $\pi_i^u(\bar{x}_i - z'_i) = \bar{\pi}_{\omega'} \cdot (\bar{x}_i - z'_i)$,

$$y_j \in Y_j^u \implies \bar{\pi}_{\omega'} \cdot (z'_j - y_j) \geq \bar{\pi}_{\omega'} \cdot (z'_j - \bar{y}_j) \implies \bar{\pi}_{\omega'} \cdot y_j \leq \bar{\pi}_{\omega'} \cdot \bar{y}_j \quad (4.6)$$

and for each j , $\pi_j^u(z'_j - \bar{y}_j) = \bar{\pi}_{\omega'} \cdot (z'_j - \bar{y}_j)$.

Let V be the τ -neighborhood of 0 referred to in Assumption **A3**. We can assume that V is convex and circled. From (4.2), we deduce that $\bar{\pi}_{\omega'} \cdot V \leq \bar{\pi}_{\omega'} \cdot \omega = 1$, thus that each $\bar{\pi}_{\omega'}$ belongs to V^0 , the polar set of V in L' . Since L is locally convex, it follows from Alaoglu–Bourbaki's theorem that V^0 is τ -equicontinuous, thus $\sigma(L', L)$ -compact. Passing to a subnet if necessary, we can assume that $\bar{\pi}_{\omega'} \xrightarrow{\sigma(L', L)} \bar{\pi}_u \in L'$ such that $\bar{\pi}_u \cdot \omega = 1$. Passing to limit in the relations (4.5), (4.6) and (4.4), we get $\bar{\pi}_u \cdot x_i \geq \bar{\pi}_u \bar{x}_i$, $\bar{\pi}_u \cdot y_j \leq \bar{\pi}_u \bar{y}_j$, and for each i , $\bar{\pi}_u \cdot \bar{x}_i = \bar{\pi}_u \cdot \omega_i + \sum_{j \in J} \theta_{ij} \bar{\pi}_u \cdot \bar{y}_j$, which completes the proof that $(\bar{x}, \bar{y}, \bar{\pi}_u)$ is a quasi-equilibrium of \mathcal{E}_u . ■

To go further, we now consider the family \mathcal{U} of all $u > 0$ in L such that the order interval $[-u, u]$ contains $\bar{x}_i, \omega_i, \bar{y}_j$ for all $i \in I$ and $j \in J$ and notice that \mathcal{U} is a directed set. We will apply the previous result to the economies \mathcal{E}_u defined as above and will pass to limit.

The next proposition is the main result of this paper.

Proposition 4.3. *Under (A3), there exists a price system $\bar{\pi} \in L'$ such that $\bar{\pi} \cdot \omega > 0$ and $(\bar{x}, \bar{y}, \bar{\pi})$ is a quasi-equilibrium of \mathcal{E} . This quasi-equilibrium is non-trivial if for some $\lambda > 0$, $\lambda \omega \in \omega + \sum_{j \in J} Y_j - \sum_{i \in I} X_i$.*

Proof. For each $u \in \mathcal{U}$, in view of the previous proposition and of its proof, let $\bar{\pi}_u \in L'$ such that $\bar{\pi}_u \cdot \omega' \leq \bar{\pi}_u \cdot \widehat{P}(\omega')$ and $(\bar{x}, \bar{y}, \bar{\pi}_u)$ is a quasi-equilibrium of \mathcal{E}_u . Let V be the convex and circled τ -neighborhood of 0 such that $\omega' + \omega + V \subseteq \widehat{P}(\omega')$. As previously, $\bar{\pi}_u \cdot V \leq \bar{\pi}_u \cdot \omega = 1$, and passing to a subnet if necessary, we can assume that $\bar{\pi}_u \xrightarrow{\sigma(L', L)} \bar{\pi} \in L'$ such that $\bar{\pi} \cdot \omega = 1$.

We now claim that $\bar{\pi}$ supports the allocation (\bar{x}, \bar{y}) . To see that, fix now $(x, y) \in \prod_{i \in I} P_i(\bar{x}_i) \times \prod_{j \in J} Y_j$. By construction of \mathcal{U} , all x_i, y_j belong to some L_{u_0} for $u_0 \in \mathcal{U}$ and consequently to any $L_u \supset L_{u_0}$. Passing to limit in the relations $\bar{\pi}_u \cdot x_i \geq \bar{\pi}_u \bar{x}_i$ and $\bar{\pi}_u \cdot y_j \leq \bar{\pi}_u \bar{y}_j$, we get $\bar{\pi} \cdot x_i \geq \bar{\pi} \bar{x}_i$ and $\bar{\pi} \cdot y_j \leq \bar{\pi} \bar{y}_j$.

Passing to limit in the relations $\bar{\pi}_u \cdot \bar{x}_i = \bar{\pi}_u \cdot \omega_i + \sum_{j \in J} \theta_{ij} \bar{\pi}_u \cdot \bar{y}_j$, we get that for each i , $\bar{\pi} \cdot \bar{x}_i = \bar{\pi} \cdot \omega_i + \sum_{j \in J} \theta_{ij} \bar{\pi} \cdot \bar{y}_j$, proving that $(\bar{x}, \bar{y}, \bar{\pi})$ is a quasi-equilibrium of \mathcal{E} .

Finally, assume that for some $\lambda > 0$, $\lambda \omega \in \omega + \sum_{j \in J} Y_j - \sum_{i \in I} X_i$. From $\bar{\pi} \cdot \omega = 1$, we deduce that there exist $x \in \prod_{i \in I} X_i$, $y \in \prod_{j \in J} Y_j$ such that $\bar{\pi} \cdot (\omega + \sum_{j \in J} y_j - \sum_{i \in I} x_i) > 0$. We then have for some i_0 ,

$$\bar{\pi} \cdot x_{i_0} < \bar{\pi} \cdot \omega_{i_0} + \sum_{j \in J} \theta_{i_0 j} \bar{\pi} \cdot y_j \leq \bar{\pi} \cdot \omega_{i_0} + \sum_{j \in J} \theta_{i_0 j} \bar{\pi} \cdot \bar{y}_j = \bar{\pi} \cdot \bar{x}_{i_0},$$

which proves that the quasi-equilibrium is non-trivial. ■

Remark 4.8. The condition for non-triviality is in particular satisfied if inaction is possible as well for consumers ($0 \in X_i$) as for producers ($0 \in Y_j$).

We conclude the paper with our main result.

Theorem 4.9. *If (A1), (A2), and (A3) hold true for the commodity space, then an ω -proper and compact production economy has a non-trivial quasi-equilibrium.*

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