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Existence of Competitive Equilibrium in a Single-Sector Growth Model with Heterogeneous Agents and Endogenous Leisure

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Abstract

We prove the existence of competitive equilibrium in a single-sector dynamic economy with heterogeneous agents and elastic labor supply. The method of proof relies on exploiting the existence of Lagrange multipliers in infinite dimensional spaces and the link between Pareto-optima and competitive equilibria.

Keywords: Optimal Growth Model, Lagrange Multipliers, Single-sector growth model, Competitive equilibrium, Elastic labor supply

JEL Classification: *C61,C62, D51, E13, O41*

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1 Introduction

This paper deals with existence of a competitive equilibrium in a one-sector growth model with heterogeneous agents and endogenous leisure. The issue of endogenous labor supply in intertemporal models have been analyzed before. These models focused on the existence and uniqueness of stationary equilibrium paths in stochastic infinite horizon economies with elastic labor that are subject to externalities, taxes or other distortions. See, for example, Greenwood and Huffman [1995], Coleman(1997), and Datta et al. [2002]. Although models with elastic labor have provided the basic framework for a substantial body of applied work in macroeconomics, there are few studies that are concerned with the question of existence of equilibria, especially for models with heterogeneous agents and endogenous leisure.

The question of existence of a competitive equilibrium in a one-sector growth model with heterogeneous agents with inelastic labor supply have been studied by Le Van and Vailakis [2003] in which they used the Pareto-optimum problem involving individual weights in a social value function and constituted a price equilibrium with transfers. Recently, Nguyen and Nguyen Van [2005] have proved the existence of a competitive equilibrium in a version of a Ramsey model for only one agent in which leisure enters the utility function by exploiting the existence of Lagrange multipliers in infinite dimensional spaces and their representation as a summable sequence that relies on some results in LeVan and Saglam [2004]. To develop these above methods for the heterogeneous-agents Ramsey model studied in Le Van-Vailakis[2003] is extended to include an endogenous non-reproducible factor such as labor, this paper exploits the existence of Lagrange multipliers in infinite dimensional spaces and the link between Pareto-optima and competitive equilibria for studying the existence of competitive equilibrium without attempting to impose the usual Inada conditions.

Following the pioneer work of Debreu (1954), Bewley [1972] studied the existence of equilibrium in an economy in which l^∞ is the commodity space and the method of using the limit of equilibria of finite dimensional economies. The most important development since Bewley's work was provided by Mas-Collel [1986], by using Negishi's approach when the commodity space is a topological vector lattice. Many others works can be found in Florenzano [1983], Aliprantis et al. [1990], Mas-Collel and Zame [1991], Dana and Le Van [1991],...Their methods yield a general results but require a high level of abstraction. Our simple approach uses the existence of Lagrange multipliers in infinite dimensional spaces and their representation as a summable sequence to show that the necessary conditions of the planner's problem guarantee the existence of a set of multipliers or shadow prices which together with the optimal allocations satisfy the sufficient conditions for the optimization problems of the consumer and the firm. The existence of a competitive equilibrium is carried out by exploiting the link between Pareto-optima and competitive equilibria (Negishi method) in which the Browder fixed-point theorem for multivalued maps with boundary condition is used.

The paper is organized as follows. Section 2 describes the economic environment and characterizes the competitive equilibrium for this economy. In section 3 we describe the Pareto-optimum problem and prove existence of Lagrange multipliers in ℓ^1_+. Section 4 proves the existence of a competitive equilibrium.

2 Characterization of Equilibrium

We consider an intertemporal one-sector model with $m \geq 1$ consumers and one firm. The preferences of each consumer take additively form: $\sum_{t=0}^{\infty} \beta_i^t u_i(c_t^i, l_t^i)$ where $\beta_i \in (0,1)$ is the discount factor. At date t, agent i consumes the quantity c_t^i, spends a quantity of leisure l_t^i and supplies a quantity of labor L_t^i. Production possibilities are presented by gross production function F and a physical depreciation $\delta \in (0,1)$

For any initial condition $k_0 \geq 0$, when a sequence $\mathbf{k} = (k_0, k_1, ..., k_t, ...)$ such that $0 \leq k_{t+1} \leq F(k_t, m) + (1-\delta)k_t$ for all t, we say it is feasible from k_0 and we denote the class of feasible capital paths by $\Pi(k_0)$. Let $c_t = (c_t^1, c_t^2, ...c_t^m)$ denote the $m-$vector of consumptions and $l_t = (l_t^1, l_t^2, ...l_t^m)$ denote m-vector of leisure of all agents at date t. A pair of consumption-leisure sequences $(\mathbf{c}, \mathbf{l}) = ((c_0, l_0),(c_1, l_1), ...)$ is feasible from $k_0 \geq 0$ if there exists a sequence $\mathbf{k} \in \Pi(k_0)$ that satisfies

$$\sum_{i=1}^{m} c_t^i + k_{t+1} \leq F(k_t, \sum_{i=1}^{m}(1-l_t^i)) + (1-\delta)k_t \text{ and } 0 \leq l_t^i \leq 1 \forall t.$$

The set of feasible from k_0 consumption-leisure is denoted by $\sum(k_0)$.

We make the following assumptions:

A1: *For* $i = 1, ..., m$, $u_i(c,l) \in R_+$ *if* $(c,l) \in R_+^2$, $u_i(c,l) = -\infty$ *if* $(c,l) \notin R_+^2$, u_i *is strictly increasing, strictly concave and continuous in* R_+^2. *Moreover,* $u_i(0,0) = 0$, *and* $\forall i$, $\lim_{\varepsilon \to 0} \frac{u_i(\varepsilon,1)}{\varepsilon} < +\infty$.

A2: *The gross production function* $F(k,L) \in R_+$ *if* $(k,L) \in R_+^2$, $F(k,L) = -\infty$ *if* $(k,L) \notin R_+^2$, F *strictly increasing, strictly concave and continuous in* R_+^2. *Moreover,* $F(0,L) = F(k,0) = 0$, $F_k(0,m) > \delta$ *and* $\lim_{k \to +\infty} F_k(k,m) = 0$.

Observe that $u_i(0,0) = 0$ in **A1** is weaker than than assuming $u_i(0,l) = u_i(c,0) = 0$ and there is no need to impose Inada conditions on utility functions. In **A2,** we only require the capital's marginal to be greater than the depreciation rate that is weaker than $F_k(0,m) > \frac{1}{\min \beta_i} - 1 + \delta$. Neither Inada conditions nor homogeneous property required on production function.

Define $f(k_t, L_t) = F(k_t, L_t) + (1-\delta)k_t$. Assumption **A2** implies that

$$\begin{aligned} f_k^{'}(+\infty, m) &= F_k(+\infty, m) + (1-\delta) = 1-\delta < 1 \\ f_k^{'}(0, m) &= F_k(0, m) + (1-\delta) > 1. \end{aligned}$$

From above, it follows that there exists $\overline{k} > 0$ such that: (i) $f(\overline{k}, m) = \overline{k}$, (ii) $k > \overline{k}$ implies $f(k, m) < k$, (iii) $k < \overline{k}$ implies $f(k, m) > k$. Therefore for any $\mathbf{k} \in \Pi(k_0)$, we have $0 \leq k_t \leq \max(k_0, \overline{k})$. Thus, a feasible sequence $\mathbf{k} \in l_+^\infty$ which in turn implies a feasible sequence $(\mathbf{c}, \mathbf{l}) \in l_+^\infty \times [0,1]^\infty$.

Now, we give the characterization of Equilibrium. For each consumer i, denote:

A sequence of prices $(p_0, p_{1,...)} \in l_+^1 \backslash \{0\}$,a price $r > 0$ for the initial capital stock.

A consumption allocation $\mathbf{c}^i = (c_0^i, c_1^i, ...c_t^i, ...)$ where c_t^i denote the quantity which agent i consumes at date t.

A sequence of capital stocks $\mathbf{k} = (k_0, k_1, ...k_t, ...)$ where k_0 is the initial endowment of capital.

Denote $\alpha^i > 0$ is the share the profit of the firm owned by consumer i, $\sum\limits_{i=1}^m \alpha^i = 1$.

Denote $\vartheta^i > 0$ is the share of initial endowment owned by consumer i, $\sum\limits_{i=1}^m \vartheta^i = 1$ and $\vartheta^i\ k_0$ is the endowment of consumer i.

A sequence of leisure $\mathbf{l}^i = (l_0^i, l_1^i, ..., l_t^i, ...)$, a sequence of labor supply $\mathbf{L}^i = (L_0^i, L_1^i, ..., L_t^i, ...)$ with $L_t^i = 1 - l_t^i$.

A sequence of wage $\mathbf{w} = (w_0, w_1, ..., w_t, ...)$.

In what follows we show that with an allocation $\{\mathbf{c}^{i*}, \mathbf{k}^*, \mathbf{l}^{i*}, \mathbf{L}^{i*}\}$,one can associate a price sequence $\mathbf{p}^*$ for consumption good, a wage sequence $\mathbf{w}^*$ for labor and a price r for the initial capital stock k_0 such that

i)

$$\mathbf{c}^* \in l_+^\infty, \mathbf{l}^{i*} \in l_+^\infty, \mathbf{L}^{i*} \in l_+^\infty, \mathbf{k}^* \in l_+^\infty, \mathbf{p}^* \in l_+^1, \mathbf{w}^* \in l_+^1, r > 0.$$

ii)For every i, $(\mathbf{c}^{i*}, \mathbf{l}^{i*})$ is a solution to the problem

$$\begin{aligned} &\max \textstyle\sum_{t=0}^\infty \beta_i^t u_i(c_t^i, l_t^i) \\ \text{s.t} \quad &\sum_{t=0}^\infty p_t^* c_t^i + \sum_{t=0}^\infty w_t^* l_t^i \ \leq \sum_{t=0}^\infty w_t^* + \vartheta^i r k_0 + \alpha^i \boldsymbol{\pi}^* \end{aligned}$$

where $\boldsymbol{\pi}^*$ is the maximum profit of the single firm.

iii) $(\mathbf{k}^*, \mathbf{L}^*)$ is a solution to the firm's problem:

$$\begin{aligned}\boldsymbol{\pi}^* = \max \sum_{t=0}^{\infty} p_t^* F(k_t, L_t) - \sum_{t=0}^{\infty} p_t^* (k_{t+1} - (1-\delta)k_t) \\ - \sum_{t=0}^{\infty} w_t^* L_t - r k_0 \\ st \quad 0 \quad \leq \quad k_{t+1} \leq F(k_t, L_t) + (1-\delta)k_t), 0 \leq L_t, \forall t\end{aligned}$$

iv)Markets clear: $\forall t$,

$$\sum_{i=0}^{m} c_t^{i*} + k_{t+1}^* - (1-\delta)k_t^* = F(k_t^*, \sum_{i=1}^{m} L_t^{i*}),$$

$$l_t^{i*} + L_t^{i*} = 1, L_t^* = \sum_{i=1}^{m} L_t^{i^*} \text{ and } k_0^* = k_0.$$

3 Lagrange Multipliers for Pareto-Optimum Problem

We prove the existence of a competitive equilibrium by studying the Pareto optimum problem. Let $\Delta = \{\eta_1, \eta_2, ..., \eta_m | \eta_i \geq 0 \text{ and } \sum_{i=0}^{m} \eta_i = 1\}$.

Define the Pareto problem

$$\max \sum_{i=1}^{m} \eta_i \sum_{t=0}^{\infty} \beta_i^t u_i(c_t^i, l_t^i) \tag{P}$$

subject to

$$\sum_{i=1}^{m} c_t^i + k_{t+1} \leq F(k_t, \sum_{i=1}^{m} (1 - l_t^i)) + (1-\delta)k_t \quad \forall t \geq 0$$

$\forall i = 1...m$, $\forall t \geq 0$

$$\begin{aligned} -c_t^i &\leq 0 \\ -k_t &\leq 0 \\ -l_t^i &\leq 0 \\ l_t^i - 1 &\leq 0 \\ k_0 &\geq 0 \quad \text{is given.} \end{aligned}$$

Note that, for all $k_0 \geq 0$, $0 \leq k_t \leq \max(k_0, \overline{k}) = A$, then $0 \leq c_t^i \leq f(A, m) \quad \forall t$, $\forall i = 1...m$.

Define the sequence

$$u_i^n(\mathbf{c}^i, \mathbf{l}^i) = \sum_{i=1}^{n} \beta_i^t u_i(c_t^i, l_t^i).$$

Since this sequence is increasing and bounded, it converges, and we can write

$$\sum_{i=1}^{m} \eta_i \sum_{t=0}^{\infty} \beta_i^t u_i(c_t^i, l_t^i) = \sum_{t=0}^{\infty} \sum_{i=1}^{m} \eta_i \beta_i^t u_i(c_t^i, l_t^i)$$

Denote $\mathbf{c} = (\mathbf{c}^1, \mathbf{c}^2, ..., \mathbf{c}^i, ..., \mathbf{c}^m)$ where $\mathbf{c}^i = (c_0^i, c_1^i, ...c_t^i, ...)$,
$\mathbf{l} = (\mathbf{l}^1, \mathbf{l}^2, ..., \mathbf{l}^i, ..., \mathbf{l}^m)$ where $\mathbf{l}^i = (l_0^i, l_1^i, ...l_t^i, ...)$,
$\mathbf{x} = (\mathbf{c}, \mathbf{k}, \mathbf{l}) \in (l_+^\infty)^m \times l_+^\infty \times (l_+^\infty)^m$.Define

$$\begin{aligned}
\mathcal{F}(\mathbf{x}) &= -\sum_{t=0}^{\infty} \sum_{i=1}^{m} \eta_i \beta_i^t u_i(c_t^i, l_t^i) \\
\Phi_t^1(\mathbf{x}) &= \sum_{i=1}^{m} c_t^i + k_{t+1} - F(k_t, \sum_{i=1}^{m} (1 - l_t^i)) - (1-\delta) k_t \ \forall t \\
\Phi_t^{2i}(\mathbf{x}) &= -c_t^i, \forall t, \forall i = 1...m \\
\Phi_t^3(\mathbf{x}) &= -k_t, \ \forall t \\
\Phi_t^{4i}(\mathbf{x}) &= -l_t^i, \forall t, \forall i = 1...m \\
\Phi_t^{5i}(\mathbf{x}) &= l_t^i - 1, \forall t, \forall i = 1...m \\
\Phi_t &= (\Phi_t^1, \Phi_t^{2i}, \Phi_{t+1}^3, \Phi_t^{4i}, \Phi_t^{5i}), \ \forall t, \forall i = 1...m
\end{aligned}$$

The Pareto problem can be written as:

$$\begin{aligned}
& \min \mathcal{F}(\mathbf{x}) \\
\text{s.t.} \Phi(\mathbf{x}) \ \leq \ & \mathbf{0}, \mathbf{x} \in (l_+^\infty)^m \times l_+^\infty \times (l_+^\infty)^m
\end{aligned}$$

where:

$$\begin{aligned}
\mathcal{F} \ &: \ (l_+^\infty)^m \times l_+^\infty \times (l_+^\infty)^m \to \mathbb{R} \cup \{+\infty\} \\
\Phi \ &= \ (\Phi_t)_{t=0...\infty} : (l_+^\infty)^m \times l_+^\infty \times (l_+^\infty)^m \to \mathbb{R} \cup \{+\infty\} \\
\text{Let } C \ &= \ dom(\mathcal{F}) = \{\mathbf{x} \in (l_+^\infty)^m \times l_+^\infty \times (l_+^\infty)^m | \mathcal{F}(\mathbf{x}) < +\infty\} \\
\Gamma \ &= \ dom(\Phi) = \{\mathbf{x} \in (l_+^\infty)^m \times l_+^\infty \times (l_+^\infty)^m | \Phi_t(\mathbf{x}) < +\infty, \ \forall t\}.
\end{aligned}$$

The following theorem follows from Theorem1 and Theorem2 in Le Van and Saglam [2004].

Theorem 1 *Let* $\mathbf{x}$, $\mathbf{y} \in (l_+^\infty)^m \times l_+^\infty \times (l_+^\infty)^m$, $T \in N$.
Define $x_t^T(\mathbf{x}, \mathbf{y}) = \begin{cases} x_t & if \ \ t \leq T \\ y_t & if \ \ t > T \end{cases}$
Suppose that two following assumptions are satisfied:

T1: *If* $\mathbf{x} \in C$, $\mathbf{y} \in (l_+^\infty)^m \times l_+^\infty \times (l_+^\infty)^m$ *satisfy* $\forall T \geq T_0$, $\mathbf{x}^T(\mathbf{x},\mathbf{y}) \in C$ *then* $\mathcal{F}(\mathbf{x}^T(\mathbf{x},\mathbf{y})) \to \mathcal{F}(\mathbf{x})$ *when* $T \to \infty$.

T2: *If* $\mathbf{x} \in \Gamma$, $\mathbf{y} \in \Gamma$ *and* $\mathbf{x}^T(\mathbf{x},\mathbf{y}) \in \Gamma$, $\forall T \geq T_0$.

Then,

$$a)\ \Phi_t(\mathbf{x}^T(\mathbf{x},\mathbf{y})) \to \Phi_t(\mathbf{x}) as\ T \to \infty$$
$$b)\ \exists M s.t. \forall T \geq T_0, \|\Phi_t(\mathbf{x}^T(\mathbf{x},\mathbf{y}))\| \leq M$$
$$c)\ \forall N \geq T_0, \lim_{t\to\infty}[\Phi_t(\mathbf{x}^T(\mathbf{x},\mathbf{y})) - \Phi_t(\mathbf{y})] = 0$$

Let $\mathbf{x}^*$ *be a solution to (**P**) and* $\bar{\mathbf{x}} \in C$ *satisfy the Slater condition:*

$$\sup_t \Phi_t(\bar{\mathbf{x}}) < 0.$$

Suppose $\mathbf{x}^T(\mathbf{x}^*,\bar{\mathbf{x}}) \in C \cap \Gamma$. *Then, there exists* $\mathbf{\Lambda} \in l_+^1$ *such that*

$$\mathcal{F}(\mathbf{x}) + \mathbf{\Lambda}\Phi(\mathbf{x}) \geq \mathcal{F}(\mathbf{x}^*) + \Lambda\Phi(\mathbf{x}^*),\ \forall \mathbf{x} \in (C \cap \Gamma)$$

and $\Lambda\Phi(\mathbf{x}^*) = 0$.

Obviously, for any $\eta \in \Delta$, an optimal path will depend on η. In what follows, we will suppress η and denote by $(\mathbf{c}^{i*}, \mathbf{k}^*, \mathbf{L}^{i*}, \mathbf{l}^{i*})$ any optimal path for each agent i if possible.

Proposition 1 *If* $\mathbf{x}^* = (\mathbf{c}^{i*}, \mathbf{k}^*, \mathbf{l}^{i*})$ *is a solution to the following problem:*

$$-\min \sum_{t=0}^{\infty}\sum_{i=1}^{m} \eta_i \beta_i^t u_i(c_t^i, l_t^i)$$

$$\begin{aligned}
s.t. \sum_{i=1}^m c_t^i + k_{t+1} &\leq F(k_t \sum_{i=1}^m (1-l_t^i)) + (1-\delta)k_t\ \forall t \geq 0 \\
-c_t^i &\leq 0, \forall i = 1...m, \forall t \geq 0 \\
-k_t &\leq 0, \forall t \geq 0 \\
-l_t^i &\leq 0, \forall i = 1...m, \forall t \geq 0 \\
l_t^i - 1 &\leq 0, \forall i = 1...m, \forall t \geq 0 \\
k_0 &\geq 0 \textit{ is given.}
\end{aligned}$$

Then there exists, $\forall i = 1...m$, $\boldsymbol{\lambda} = (\boldsymbol{\lambda}^1, \boldsymbol{\lambda}^{2i}, \boldsymbol{\lambda}^3, \boldsymbol{\lambda}^{4i}, \boldsymbol{\lambda}^{5i}) \in l_+^1 \times\ (l_+^1)^m \times l_+^1 \times$

$(l^1_+)^m \times (l^1_+)^m$ *such that:* $\forall \mathbf{x} = (\mathbf{c}, \mathbf{k}, \mathbf{l}) \in (l^\infty_+)^m \times l^\infty_+ \times (l^\infty_+)^m$

$$\begin{aligned}
&\sum_{t=0}^{\infty}\sum_{i=1}^{m}\eta_i\beta_i^t u_i(c_t^{i*}, l_t^{i*}) - \sum_{t=0}^{\infty}\lambda_t^1(\sum_{i=1}^{m} c_t^{i*} + k_{t+1}^* - F(k_t^*, \sum_{i=1}^{m}(1-l_t^{i*}))) \\
&-(1-\delta)k_t^*) + \sum_{t=0}^{\infty}\lambda_t^{2i}c_t^{i*} + \sum_{t=0}^{\infty}\lambda_t^3 k_t^* + \sum_{t=0}^{\infty}\lambda_t^{4i}l_t^{i*} + \sum_{t=0}^{\infty}\lambda_t^{5i}(1-l_t^{i*}) \\
\geq\ &\sum_{t=0}^{\infty}\sum_{i=1}^{m}\eta_i\beta_i^t u_i(c_t^{i}, l_t^{i}) - \sum_{t=0}^{\infty}\lambda_t^1(\sum_{i=1}^{m} c_t^{i} + k_{t+1} - F(k_t, \sum_{i=1}^{m}(1-l_t^{i}))) \\
&-(1-\delta)k_t) + \sum_{t=0}^{\infty}\lambda_t^{2i}c_t^{i} + \sum_{t=0}^{\infty}\lambda_t^3 k_t + \sum_{t=0}^{\infty}\lambda_t^{4i}l_t^{i} + \sum_{t=0}^{\infty}\lambda_t^{5i}(1-l_t^{i}) \qquad (1)
\end{aligned}$$

$$\lambda_t^1[\sum_{i=1}^{m} c_t^{i*} + k_{t+1}^* - F(k_t^*, \sum_{i=1}^{m} L_t^{i*}) - (1-\delta)k_t^*] = 0 \qquad (2)$$

$$\lambda_t^{2i}c_t^{i*} = 0, \forall i = 1...m \qquad (3)$$

$$\lambda_t^3 k_t^* = 0 \qquad (4)$$

$$\lambda_t^{4i}l_t^{i*} = 0, \forall i = 1...m \qquad (5)$$

$$\lambda_t^{5i}(1-l_t^{i*}) = 0, \forall i = 1...m \qquad (6)$$

$$0 \in \eta_i\beta_i^t\partial_1 u_i(c_t^{i*}, l_t^{i*}) - \{\lambda_t^1\} + \{\lambda_t^{2i}\}, \forall i = 1...m \qquad (7)$$

$$0 \in \eta_i\beta_i^t\partial_2 u_i(c_t^{i*}, l_t^{i*}) - \lambda_t^1\partial_2 F(k_t^*, L_t^*) + \{\lambda_t^{4i}\} - \{\lambda_t^{5i}\}, \forall i = 1...m \qquad (8)$$

$$0 \in \lambda_t^1\partial_1 F(k_t^*, L_t^*) + \{(1-\delta)\lambda_t^1\} + \{\lambda_t^3\} - \{\lambda_{t-1}^1\} \qquad (9)$$

where, $L_t^* = \sum_{i=1}^{m} L_t^{i*} = \sum_{i=1}^{m}(1-l_t^{i*})$, $\partial_j u(c_t^{i*}, l_t^{i*}), \partial_j F(k_t^*, L_t^*)$ *respectively denote the projection on the* j^{th} *component of the subdifferential of function* u *at* (c_t^{i*}, l_t^{i*}) *and the function* F *at* (k_t^*, L_t^*).

Proof: We show that the Slater condition holds. Since $f'_k(0,m) > 1$, then for all $k_0 > 0$, there exists some $0 < \widehat{k} < k_0$ such that: $0 < \widehat{k} < f(\widehat{k}, m)$ and $0 < \widehat{k} < f(k_0, m)$.Thus, there exists two small positive numbers ε, ε_1 such that:

$$0 < \widehat{k} + \varepsilon < f(\widehat{k}, m - \varepsilon_1) \text{ and } 0 < \widehat{k} + \varepsilon < f(k_0, m - \varepsilon_1).$$

Denote $\bar{\mathbf{x}} = (\bar{\mathbf{c}}, \bar{\mathbf{k}}, \bar{\mathbf{l}})$ such that $\bar{\mathbf{c}} = (\bar{\mathbf{c}}^1, \bar{\mathbf{c}}^2, ..., \bar{\mathbf{c}}^i, ..., \bar{\mathbf{c}}^m)$, where

$$\bar{\mathbf{c}}^i = (\bar{c_t}^i)_{t=0,...\infty} = (\frac{\varepsilon}{m}, \frac{\varepsilon}{m}, \frac{\varepsilon}{m}, ...)$$

$\bar{\mathbf{l}} = (\bar{\mathbf{l}}^1, \bar{\mathbf{l}}^2, ..., \bar{\mathbf{l}}^i, ..., \bar{\mathbf{l}}^m)$, where

$$\bar{\mathbf{l}}^i = (\bar{l_t}^i)_{t=0,...\infty} = (\frac{\varepsilon_1}{m}, \frac{\varepsilon_1}{m}, \frac{\varepsilon_1}{m}, ...).$$

and $\bar{\mathbf{k}} = (k_0, \widehat{k}, \widehat{k}, ...)$. We have

$$\begin{aligned}
\Phi_0^1(\bar{\mathbf{x}}) &= \sum_{i=0}^{m} c_0^i + k_1 \leq F(k_0, \sum_{i=1}^{m}(1-l_0^i)) + (1-\delta)k_0 \\
&= \varepsilon + \widehat{k} - f(k_0, m - \varepsilon_1) < 0 \\
\Phi_1^1(\bar{\mathbf{x}}) &= \sum_{i=0}^{m} c_1^i + k_2 \leq F(k_1, \sum_{i=1}^{m}(1-l_1^i)) + (1-\delta)k_1 \\
&= \varepsilon + \widehat{k} - f(\widehat{k}, m - \varepsilon_1) < 0 \\
\Phi_t^1(\bar{\mathbf{x}}) &= \varepsilon + \widehat{k} - f(\widehat{k}, m - \varepsilon_1) < 0, \ \forall t \geq 2
\end{aligned}$$

$$\Phi_t^{2i}(\bar{\mathbf{x}}) = -\bar{c_t}^i = -\frac{\varepsilon}{m} < 0, \ \forall t \geq 0, \forall i = 1...m$$

$$\begin{aligned}
\Phi_0^3(\bar{\mathbf{x}}) &= -k_0 < 0; \\
\Phi_t^3(\bar{\mathbf{x}}) &= -\widehat{k} < 0 \ \ \forall t \geq 1.
\end{aligned}$$

$$\Phi_t^{4i}(\bar{\mathbf{x}}) = -\frac{\varepsilon_1}{m} < 0, \ \forall t \geq 0, \forall i = 1...m$$

$$\Phi_t^{5i}(\bar{\mathbf{x}}) = \frac{\varepsilon_1}{m} - 1 < 0, \ \forall t \geq 0, \forall i = 1...m$$

Therefore the Slater condition is satisfied. Now, it is obvious that, $\forall T$, $\mathbf{x}^T(\mathbf{x}^*, \bar{\mathbf{x}})$ belongs to $(l_+^\infty)^m \times l_+^\infty \times (l_+^\infty)^m$.

As in Le Van-Saglam 2004, Assumption **T2** is satisfied. We now check Assumption **T1**.

For any $\widetilde{\mathbf{x}} \in C, \widetilde{\widetilde{\mathbf{x}}} \in (l_+^\infty)^m \times l_+^\infty \times (l_+^\infty)^m$ such that for any T, $\mathbf{x}^T(\widetilde{\mathbf{x}}, \widetilde{\widetilde{\mathbf{x}}}) \in C$ we have

$$\mathcal{F}(\mathbf{x}^T(\widetilde{\mathbf{x}}, \widetilde{\widetilde{\mathbf{x}}})) = -\sum_{t=0}^{T}\sum_{i=1}^{m}\eta_i\beta_i^t u_i(\widetilde{c_t^i}, \widetilde{l_t^i}) - \sum_{t=T+1}^{\infty}\sum_{i=1}^{m}\eta_i\beta_i^t u_i(\widetilde{\widetilde{c_t^i}}, \widetilde{\widetilde{l_t^i}}).$$

As $\widetilde{\widetilde{\mathbf{x}}} \in (l_+^\infty)^m \times l_+^\infty \times (l_+^\infty)^m$, $\sup_t |\widetilde{\widetilde{c_t}}| < +\infty$, there exists $a > 0, \forall t, |\widetilde{\widetilde{c_t}}| \leq a$. Since $\beta \in (0,1)$,as $T \to \infty$ we have

$$0 \leq |\sum_{t=T+1}^{\infty}\sum_{i=1}^{m}\eta_i\beta_i^t u_i(\widetilde{\widetilde{c_t^i}}, \widetilde{\widetilde{l_t^i}})| \leq u(a,1)\sum_{t=T+1}^{\infty}\sum_{i=1}^{m}\eta_i\beta_i^t = \sum_{i=1}^{m}\sum_{t=T+1}^{\infty}\eta_i\beta_i^t \to 0 \ .$$

Hence, $\mathcal{F}(\mathbf{x}^T(\widetilde{\mathbf{x}}, \widetilde{\widetilde{\mathbf{x}}})) \to \mathcal{F}(\widetilde{\mathbf{x}})$ when $T \to \infty$.Taking account of the Theorem 1, we get (1) - (6).

Obviously, $\cap_{i=1}^m ri(dom(u_i)) \neq \emptyset$ where $ri(dom(u_i))$ is the relative interior of $dom(u_i)$. It follows from the Proposition 6.5.5 in Florenzano and Le Van (2001), we have

$$\partial\sum_{i=1}^{m}\eta_i\beta_i^t u_i(c_t^{i*}, l_t^{i*}) = \eta_i\beta_i^t\sum_{i=1}^{m}\partial u_i(c_t^{i*}, l_t^{i*})$$

We then get (7) - (9) from the Kuhn-Tucker first-order conditions. ■

4 Existence of a Competitive Equilibrium

With the optimal path $\mathbf{c}^*(\eta)$, $\mathbf{k}^*(\eta), \mathbf{l}^*(\eta)$, $\mathbf{L}^*(\eta)$ we have proved that there exists the Lagrange multipliers

$\boldsymbol{\lambda}(\eta) = (\boldsymbol{\lambda}^{\mathbf{1}}(\eta), \boldsymbol{\lambda}^{\mathbf{2i}}(\eta), \boldsymbol{\lambda}^{\mathbf{3}}(\eta), \boldsymbol{\lambda}^{\mathbf{4i}}(\eta), \boldsymbol{\lambda}^{\mathbf{5i}}(\eta)) \in l^1_+ \times (l^1_+)^m \times l^1_+ \times (l^1_+)^m \times (l^1_+)^m$, $i = 1...m$, for the Pareto problem. As in the previous section we will suppress η whenever it is impossible.

We will prove that there exits $f_t^2(k_t^*, L_t^*) \in \partial_2 F(k_t^*, L_t^*)$,where $L_t^* = \sum_{i=1}^m L_t^{i*}$, then one can associate a sequence of prices p_t^*, a sequence of wages w_t^* defined as

$$\begin{aligned} p_t^* &= \lambda_t^1 \ \forall t \\ w_t^* &= \lambda_t^1 f_t^2(k_t^*, L_t^*) \ \forall t \end{aligned}$$

and a price $r > 0$ for the initial capital stock k_0 such that ($\mathbf{c}^*, \mathbf{k}^*, \mathbf{l}^*, \mathbf{L}^*, \mathbf{p}^*, \mathbf{w}^*, r$) is a price equilibrium with transfers. That means

i)

$$\mathbf{c}^* \in (l^\infty_+)^m, \mathbf{l}^* \in (l^\infty_+)^m, \mathbf{k}^* \in l^\infty_+, \mathbf{p}^* \in l^1_+, \mathbf{w}^* \in l^1_+, \ r > 0$$

ii) For every $i = 1...m$, $(\mathbf{c}^{i*}, \mathbf{l}^{i*})$ is a solution to the problem

$$\begin{aligned} &\max \sum_{t=0}^\infty \beta_i^t u_i(c_t^i, l_t^i) \\ &st \quad \sum_{t=0}^\infty p_t^* c_t^i + \sum_{t=0}^\infty w_t^* l_t^i \leq \sum_{t=0}^\infty p_t^* c_t^{i*} + \sum_{t=0}^\infty w_t^* l_t^{i*} \end{aligned}$$

iii) $(\mathbf{k}^*, \mathbf{L}^*)$ is a solution to the firm's problem:

$$\begin{aligned} \boldsymbol{\pi}^* &= \max \sum_{t=0}^\infty p_t^* F(k_t, L_t) - \sum_{t=0}^\infty p_t^*(k_{t+1} - (1-\delta)k_t) \\ &\quad - \sum_{t=0}^\infty w_t^* L_t - r k_0 \\ st \quad 0 &\leq k_{t+1} \leq F(k_t, L_t) + (1-\delta)k_t), 0 \leq L_t, \forall t \end{aligned}$$

iv)Markets clear

$$\begin{aligned} \forall t, \ \sum_{i=0}^m c_t^{i*} + k_{t+1}^* - (1-\delta)k_t^* = &\ F(k_t^*, \sum_{i=1}^m L_t^{i*}), \\ L_t^* = \sum_{i=1}^m L_t^{i*}, \ l_t^{i*} = 1 - L_t^{i*} \qquad &\text{and } k_0^* = k_0 \end{aligned}$$

Lemma 1 *Let $k_0 > 0$. The sequence of wages w_t^* defined as*

$$w_t^* = \lambda_t^1 f_t^2(k_t^*, L_t^*) \ \forall t \ \textit{where} \ f_t^2(k_t^*, L_t^*) \in \partial_2 F(k_t^*, L_t^*)$$

is a sequence which belong to l^1_+.

Proof: Consider $\boldsymbol{\lambda}(\eta) = (\boldsymbol{\lambda^1}, \boldsymbol{\lambda^{2i}}, \boldsymbol{\lambda^3}, \boldsymbol{\lambda^{4i}}, \boldsymbol{\lambda^{5i}})$ of Proposition 1. Conditions (7),(8),(9) in Proposition 1 show that $\forall i = 1...m$, $\partial u_i(c_t^{i*}, l_t^{i*})$ and $\partial F(k_t^*, L_t^*)$ are nonempty. Moreover,$\forall t, \forall i = 1...m$, there exists $u_t^{1i}(c_t^{i*}, l_t^{i*}) \in \partial_1 u_i(c_t^{i*}, l_t^{i*})$, $u_t^{2i}(c_t^{i*}, l_t^{i*}) \in \partial_2 u_i(c_t^{i*}, l_t^{i*})$, $f_t^1(k_t^*, L_t^*) \in \partial_1 F(k_t^*, L_t^*)$and $f_t^2(k_t^*, L_t^*) \in \partial_2 F(k_t^*, L_t^*)$ such that

$$\eta_i \beta_i^t u_t^{1i}(c_t^{i*}, l_t^{i*}) - \lambda_t^1 + \lambda_t^{2i} = 0, \forall i = 1...m \tag{10}$$

$$\eta_i \beta_i^t u_t^{2i}(c_t^{i*}, l_t^{i*}) - \lambda_t^1 f_t^2(k_t^*, L_t^*) + \lambda_t^{4i} - \lambda_t^{5i} = 0, \forall i = 1...m \tag{11}$$

$$\lambda_t^1 f_t^1(k_t^*, L_t^*) + (1-\delta)\lambda_t^1 + \lambda_t^3 - \lambda_{t-1}^1 = 0 \tag{12}$$

We have

$$+\infty > \sum_{t=0}^{\infty} \beta_i^t u_i(c_t^{i*}, l_t^{i*}) - \sum_{t=0}^{\infty} \beta_i^t u_i(0,0) \geq$$

$$\sum_{t=0}^{\infty} \beta_i^t u_t^{1i}(c_t^{i*}, l_t^{i*}) c_t^{i*} + \sum_{t=0}^{\infty} \beta_i^t u_t^{2i}(c_t^{i*}, l_t^{i*}) l_t^{i*}, \forall i = 1...m$$

which implies

$$\sum_{t=0}^{\infty} \beta_i^t u_t^{2i}(c_t^{i*}, l_t^{i*}) l_t^{i*} < +\infty, \forall i = 1...m \tag{13}$$

and for any i,

$$+\infty > \sum_{t=0}^{\infty} \lambda_t^1 F(k_t^*, L_t^*) - \sum_{t=0}^{\infty} \lambda_t^1 F(0, L_t^* - L_t^{i*}) \geq$$

$$\sum_{t=0}^{\infty} \lambda_t^1 f_t^1(k_t^*, L_t^*) k_t^* + \sum_{t=0}^{\infty} \lambda_t^1 f_t^2(k_t^*, L_t^*) L_t^{i*}$$

which implies

$$\sum_{t=0}^{\infty} \lambda_t^1 f_t^2(k_t^*, L_t^*) L_t^{i*} < +\infty. \tag{14}$$

Given T, we multiply (11), for each i, by L_t^{i*} and sum from 0 to T. We then obtain

$$\forall T,\ \sum_{t=0}^{T} \eta_i \beta_i^t u_t^{2i}(c_t^{i*}, l_t^{i*}) L_t^{i*} = \sum_{t=0}^{T} \lambda_t^1 f_t^2(k_t^*, L_t^*) L_t^{i*} \tag{15}$$

$$- \sum_{t=0}^{T} \lambda_t^{4i} L_t^{i*} + \sum_{t=0}^{T} \lambda_t^{5i} L_t^{i*}, \forall i = 1...m$$

Observe that

$$0 \leq \sum_{t=0}^{\infty} \lambda_t^{5i} L_t^{i*} \leq \sum_{t=0}^{\infty} \lambda_t^{5i} < +\infty, \forall i = 1...m \tag{16}$$

$$0 \leq \sum_{t=0}^{\infty} \lambda_t^{4i} L_t^{i*} \leq \sum_{t=0}^{\infty} \lambda_t^{4i} < +\infty, \forall i = 1...m \tag{17}$$

Thus, since $L_t^{i*} = 1 - l_t^{i*}, \forall i = 1...m$, from (15), we get

$$\sum_{t=0}^{T}\eta_i\beta_i^t u_t^{2i}(c_t^{i*}, l_t^{i*}) = \sum_{t=0}^{T}\eta_i\beta_i^t u_t^{2i}(c_t^{i*}, l_t^{i*})l_t^{i*} + \sum_{t=0}^{T}\lambda_t^1 f_t^2(k_t^*, L_t^*)L_t^{i*} + \sum_{t=0}^{T}\lambda_t^{5i}L_t^{i*} - \sum_{t=0}^{T}\lambda_t^{4i}L_t^{i*}$$

Using (13),(14),(16),(17) and letting $T \to \infty$, we obtain

$$0 \le \sum_{t=0}^{\infty}\eta_i\beta_i^t u_t^{2i}(c_t^{i*}, l_t^{i*}) = \sum_{t=0}^{\infty}\eta_i\beta_i^t u_t^{2i}(c_t^{i*}, l_t^{i*})l_t^{i*} +$$

$$\sum_{t=0}^{\infty}\lambda_t^1 f_t^2(k_t^*, L_t^*)L_t^{i*} + \sum_{t=0}^{\infty}\lambda_t^{5i}L_t^{i*} - \sum_{t=0}^{\infty}\lambda_t^{4i}L_t^{i*} < +\infty$$

Consequently, from (11),

$$\sum_{t=0}^{\infty}w_t^* = \sum_{t=0}^{\infty}\lambda_t^1 f_t^2(k_t^*, L_t^*) < +\infty \text{ i.e. } \mathbf{w}^* \in l_+^1.$$

Theorem 2 *Let (* $\mathbf{k}^*, \mathbf{c}^*, \mathbf{L}^*, \mathbf{l}^*$*) solve Problem* $(\mathbf{P})$*. Take*

$$p_t^* = \lambda_t^1,\ w_t^* = \lambda_t^1 f_t^2(k_t^*, L_t^*) \text{ for any } t$$
$$\text{and } r = \frac{\lambda_0^1(1-\delta)}{2} > 0.$$

Then $\{\mathbf{c}^*, \mathbf{k}^*, \mathbf{L}^*, \mathbf{p}^*, \mathbf{w}^*, r\}$ *is a price equilibrium with transfers .*

Proof: From Proposition 1 and Lemma 1, we get

$$\mathbf{c}^* \in (l_+^\infty)^m, \mathbf{l}^* \in (l_+^\infty)^m, \mathbf{k}^* \in l_+^\infty, \mathbf{p}^* \in l_+^1, \mathbf{w}^* \in l_+^1, r > 0.$$

We now show that $(\mathbf{c}^{i*}, \mathbf{l}^{i*})$ solves the consumer's problem. Let $(\mathbf{c}^i, \mathbf{l}^i)$ satisfies

$$\sum_{t=0}^{\infty}p_t^* c_t^i + \sum_{t=0}^{\infty}w_t^* l_t^i \le \sum_{t=0}^{\infty}p_t^* c_t^{i*} + \sum_{t=0}^{\infty}w_t^* l_t^{i*}.$$

By the concavity of u_i, we have:

$$\sum_{t=0}^{\infty}\beta_i^t u_i(c_t^{i*}, l_t^{i*}) - \sum_{t=0}^{\infty}\beta_i^t u_i(c_t^i, l_t^i)$$

$$\ge \sum_{t=0}^{\infty}\beta_i^t u_t^{1i}(c_t^{i*}, l_t^{i*})(c_t^{i*} - c_t^i) + \sum_{t=0}^{\infty}\beta_i^t u_t^{2i}(c_t^{i*}, l_t^{i*})\ (l_t^{i*} - l_t^i).$$

Combining (3),(6),(10),(11) yields that

$$\Delta \ge \sum_{t=0}^{\infty}\frac{(\lambda_t^1 - \lambda_t^{2i})}{\eta_i}(c_t^{i*} - c_t^i) + \sum_{t=0}^{\infty}\frac{(\lambda_t^1 f_t^2(k_t^*, L_t^*) - \lambda_t^{4i} + \lambda_t^{5i})}{\eta_i}(l_t^{i*} - l_t^i)$$

$$
\begin{aligned}
&= \sum_{t=0}^{\infty}\frac{\lambda_t^1}{\eta_i}(c_t^{i*}-c_t^i)+\sum_{t=0}^{\infty}\frac{\lambda_t^{2i}}{\eta_i}c_t^i-\sum_{t=0}^{\infty}\frac{\lambda_t^{2i}}{\eta_i}c_t^{i*}+\sum_{t=0}^{\infty}\frac{\lambda_t^1 f_t^2(k_t^*,L_t^*)}{\eta_i}(l_t^{i*}-l_t^i)\\
&\quad+\sum_{t=0}^{\infty}\frac{\lambda_t^{5i}}{\eta_i}(l_t^{i*}-l_t^i)-\sum_{t=0}^{\infty}\frac{\lambda_t^{4i}}{\eta_i}l_t^{i*}+\sum_{t=0}^{\infty}\frac{\lambda_t^{4i}}{\eta_i}l_t^i
\end{aligned}
$$

$$
\begin{aligned}
&\geq \sum_{t=0}^{\infty}\frac{\lambda_t^1}{\eta_i}(c_t^{i*}-c_t^i)+\sum_{t=0}^{\infty}\frac{\lambda_t^1 f_t^2(k_t^*,L_t^*)}{\eta_i}(l_t^{i*}-l_t^i)+\sum_{t=0}^{\infty}\frac{\lambda_t^{5i}(1-l_t^i)}{\eta_i}\\
&\geq \sum_{t=0}^{\infty}\frac{\lambda_t^1}{\eta_i}(c_t^{i*}-c_t^i)+\sum_{t=0}^{\infty}\frac{\lambda_t^1 f_t^2(k_t^*,L_t^*)}{\eta_i}(l_t^{i*}-l_t^i)
\end{aligned}
$$

$$
=\sum_{t=0}^{\infty}\frac{p_t^*}{\eta_i}(c_t^{i*}-c_t^i)+\sum_{t=0}^{\infty}\frac{w_t^*}{\eta_i}(l_t^{i*}-l_t^i)\geq 0.
$$

This means $(\mathbf{c}^{i*},\mathbf{l}^{i*})$ solves the consumer's problem.

We show that $(\mathbf{k}^*,\mathbf{L}^*)$ is solution to the firm's problem. Since $p_t^*=\lambda_t^1$, $w_t^*=\lambda_t^1 f_t^2(k_t^*,L_t^*)$, we have

$$
\begin{aligned}
\boldsymbol{\pi}^* &= \sum_{t=0}^{\infty}\lambda_t^1[F(k_t^*,L_t^*)+(1-\delta)k_t^*-k_{t+1}^*]\\
&\quad-\sum_{t=0}^{\infty}\lambda_t^1 f_t^2(k_t^*,L_t^*)\ L_t^*-rk_0.
\end{aligned}
$$

Let :

$$
\begin{aligned}
\Delta_T &= \sum_{t=0}^{T}\lambda_t^1[F(k_t^*,L_t^*)+(1-\delta)k_t^*-k_{t+1}^*]-\sum_{t=0}^{T}\lambda_t^1 f_t^2(k_t^*,L_t^*)\ L_t^*-rk_0\\
&\quad-\left(\sum_{t=0}^{T}\lambda_t^1[F(k_t,L_t)+(1-\delta)k_t-k_{t+1}]-\sum_{t=0}^{T}\lambda_t^1 f_t^2(k_t^*,L_t^*)L_t-rk_0\right)
\end{aligned}
$$

By the concavity of F , we get

$$
\begin{aligned}
\Delta_T &\geq \sum_{t=1}^{T}\lambda_t^1 f_t^1(k_t^*,L_t^*)(k_t^*-k_t)+(1-\delta)\sum_{t=1}^{T}\lambda_t^1(k_t^*-k_t)-\\
&\quad\sum_{t=0}^{T}\lambda_t^1(k_{t+1}^*-k_{t+1})=[\lambda_1^1 f_t^1(k_1^*,L_1^*)+(1-\delta)\lambda_1^1-\lambda_0^1](k_1^*-k_1)+...\\
&\quad+[\lambda_T^1 f_t^1(k_T^*,L_T^*)+(1-\delta)\lambda_T^1-\lambda_{T-1}^1](k_T^*-k_T)-\lambda_T^1(k_{T+1}^*-k_{T+1}).
\end{aligned}
$$

By (4) and (12), we have: $\forall t=1,2,...,T$

$$
[\lambda_t^1 f_t^1(k_t^*,L_t^*)+(1-\delta)\lambda_t^1-\lambda_{t-1}^1](k_t^*-k_t)=-\lambda_t^3(k_t^*-k_t)=\lambda_t^3 k_t\geq 0. \quad (18)
$$

Thus,

$$\Delta_T \geq -\lambda_T^1(k_{T+1}^* - k_{T+1}) = -\lambda_T^1 k_{T+1}^* + \lambda_T^1 k_{T+1} \geq -\lambda_T^1 k_{T+1}^*.$$

Since $\boldsymbol{\lambda}^1 \in l_+^1$, $\sup\limits_T k_{T+1}^* < +\infty$, we have

$$\lim_{T\to+\infty} \Delta_T \geq \lim_{T\to+\infty} -\lambda_T^1 k_{T+1}^* = 0.$$

We have proved that the sequences $(\mathbf{k}^*, \mathbf{L}^*)$ maximize the profit of the firm. ■

The appropriate transfer to each consumer is the amount that just allows the consumer to afford the consumption stream allocated by the social optimization problem. Thus, for given weights $\eta \in \Delta$, the required transfers are:

$$\phi_i(\eta) = \sum_{t=0}^{\infty} \lambda_t^1(\eta) c_t^{i*}(\eta) + \sum_{t=0}^{\infty} w_t^*(\eta) l_t^{i*}(\eta) - \sum_{t=0}^{\infty} w_t^*(\eta) - \vartheta^i r k_0 - \alpha^i \boldsymbol{\pi}^*(\eta)$$

where

$$\begin{aligned}\boldsymbol{\pi}^*(\eta) &= \sum_{t=0}^{\infty} \lambda_t^1(\eta) F(k_t^*(\eta), L_t^*(\eta)) - \sum_{t=0}^{\infty} \lambda_t^1(\eta)(k_{t+1}^*(\eta) - (1-\delta)k_t^*(\eta)) \\ &\quad - \sum_{t=0}^{\infty} w_t^*(\eta) L_t^*(\eta) - r k_0.\end{aligned}$$

A competitive equilibrium for this economy corresponds to a set of welfare weights $\eta \in \Delta$ such that these transfers equal to zero.

Proposition 2 *i) Let $k_0 > 0$. Then for any $\eta \in \Delta$, $\boldsymbol{\pi}^*(\eta) > 0$.*

ii) If $\eta_i = 0$ then $\forall t$, $c_t^{i*} = 0$, $l_t^{i*} = 0$.

Proof: i) Let $(k_0, 0, 0, ...) \in \Pi(k_0)$. Then

$$\begin{aligned}\boldsymbol{\pi}^*(\eta) &\geq \lambda_0^1(\eta)[F(k_0, 0) + (1-\delta)k_0] - r k_0 \\ &= \lambda_0^1(\eta)(1-\delta)k_0 - r k_0 > 0\end{aligned}$$

ii) Let $\eta_i = 0$. Suppose for simplicity that $c_0^{i*} > 0$.

Let j satisfies $\eta_j > 0$. Define $c_0^{i**} = 0$, $c_0^{j**} = c_0^{j*} + c_0^{i*}$. We have

$$\eta_i u_i(c_0^{i**}, l_0^{i*}) = \eta_i u_i(c_0^{i*}, l_0^{i*}) = 0, \eta_j u_j(c_0^{j**}, l_0^{j*}) > \eta_j u_j(c_0^{j*}, l_0^{j*}).$$

Hence we get new utility is greater than the optimum which leads to contradiction. Now, assume that $l_0^{i*} > 0$.

Let j satisfies $\eta_j > 0$. Define

$$\begin{aligned}c_0^{j**} &= F(k_0, m - \sum_{k\neq i} l_0^k) + (1-\delta)k_0 - k_1 - \sum_{k\neq j} c_0^{k*} \\ l_0^{i**} &= 0\end{aligned}$$

We have $c_0^{j**} > c_0^{j*}$ and

$$\eta_i u_i(c_0^{i*}, l_0^{i**}) = \eta_i u_i(c_0^{i*}, l_0^{i*}) = 0,\ \eta_j u_j(c_0^{j**}, l_0^{j*}) > \eta_j u_j(c_0^{j*}, l_0^{j*}).$$

that also leads to contradiction. ■

Theorem 3 *For every i, $\phi_i(\eta)$ is compact valued, upper semi-continuous and convex.*

Proof: It is easy to check that, for given $\eta \in \Delta$,

$$U(\boldsymbol{\eta}, \mathbf{k}, \mathbf{c}, \mathbf{l}) = \sum_{t=0}^{\infty}\sum_{i=1}^{m} \eta_i \beta_i^t u_i(c_t^i, l_t^i)$$

is continuous over $\Delta \times \Pi(k_0)\times \sum(k_0)$, $\Pi(k_0)\times \sum(k_0)$ are compact, it follows from Berge's Theorem that $c_t^{i*}(\eta), k_t^*(\eta), l_t^{i*}(\eta)$ are continuous functions of η for the product topology.

Let $\eta^n \in \Delta$ and $\eta^n \to \eta$.

i) Assume that $c_t^{i*}(\eta^n) \to c_t^{i*}(\eta) = 0$.

It follows from Assumption **A1** that

$$\forall i \in I = \{i|\eta_i^n > 0\}, \lim_{\varepsilon \to 0} \frac{u_i(\varepsilon, 1)}{\varepsilon} < +\infty,$$

we have

$$\begin{aligned} \eta_i\beta_i^t u_i(c_t^{i*}(\eta^n), l_t^{i*}(\eta^n)) - \eta_i\beta_i^t u_i(0, l_t^{i*}(\eta^n)) \\ \geq \eta_i\beta_i^t u_t^{1i}(c_t^{i*}(\eta^n), l_t^{i*}(\eta^n))c_t^{i*}(\eta^n) \end{aligned}$$

$$= \left[\lambda_t^1(\eta^n) - \lambda_t^{2i}(\eta^n)\right] c_t^{i*}(\eta^n) = \lambda_t^1(\eta^n)c_t^{i*}(\eta^n). \tag{19}$$

Thus

$$0 \leq \lambda_t^1(\eta^n) \leq \eta_i^n\beta_i^t \frac{u_i(c_t^{i*}(\eta^n), l_t^{i*}(\eta^n))}{c_t^{i*}(\eta^n)} \leq \eta_i^n\beta_i^t \frac{u_i(c_t^{i*}(\eta^n), 1)}{c_t^{i*}(\eta^n)} \leq 2\alpha,$$

when η^n is close to η, where $\alpha = \lim_{\varepsilon\to 0}\frac{u_i(\varepsilon,1)}{\varepsilon}$. This shows that $\lambda_t^1(\eta^n)$ is bounded from above.

Moreover, $\sum\limits_{t=0}^{\infty}\lambda_t^1(\eta^n) \leq \alpha\sum\limits_{t=0}^{\infty}\eta_i^n\beta_i^t < +\infty$ when η^n is close to η. Hence there exists a subsequence of $\{\lambda_t^1(\eta^n)\}$, denoted again by $\{\lambda_t^1(\eta^n)\}$, say $\lambda_t^1(\eta^n) \to \lambda_t^1(\eta) \in l_+^1$.

Furthermore, from (19) we get

$$\sum_{t=0}^{\infty} u_t^{1i}(c_t^{i*}(\eta^n), l_t^{i*}(\eta^n)) < +\infty$$

It implies from (10) $\lambda_t^{2i}(\eta^n) \in l_+^1$ when $\eta^n \to \eta$.

ii) Assume that $l_t^{i*}(\eta^n) \to l_t^{i*}(\eta) = 0$. We shall prove that

$$w_t^*(\eta^n) = \lambda_t^1(\eta^n) f_t^2(k_t^*(\eta^n), L_t^*(\eta^n)) \to \lambda_t^1(\eta) f_t^2(k_t^*(\eta), L_t^*(\eta)) \in l_+^1.$$

For any $\nu \in (0, \varepsilon)$, $\lim_{\varepsilon\to 0} \frac{u_i(\nu,\varepsilon)}{\varepsilon} < \lim_{\varepsilon\to 0} \frac{u_i(\varepsilon,1)}{\varepsilon} < +\infty$ which implies

$$\lim_{\varepsilon\to 0} \frac{u_i(0,\varepsilon)}{\varepsilon} < \lim_{\varepsilon\to 0} \frac{u_i(\nu,\varepsilon)}{\varepsilon} < +\infty.$$

We have

$$\begin{aligned} & \eta_i\beta_i^t u_i(c_t^{i*}(\eta^n), l_t^{i*}(\eta^n)) - \eta_i\beta_i^t u_i(c_t^{i*}(\eta^n), 0) \\ \geq\; & \eta_i\beta_i^t u_t^{2i}(c_t^{i*}(\eta^n), l_t^{i*}(\eta^n)) l_t^{i*}(\eta^n) \\ =\; & [w_t^*(\eta^n) - \lambda_t^{4i}(\eta^n) + \lambda_t^{5i}(\eta^n)] l_t^{i*}(\eta^n) \\ =\; & [w_t^*(\eta^n) + \lambda_t^{5i}(\eta^n)] l_t^{i*}(\eta^n) \geq w_t^*(\eta^n) l_t^{i*}(\eta^n) \end{aligned}$$

This implies

$$\frac{w_t^*(\eta^n)}{\eta_i^n\beta_i^t} \leq \frac{u_i(c_t^{i*}(\eta^n), l_t^{i*}(\eta^n))}{l_t^{i*}(\eta^n)} \leq M < +\infty \text{ as } \eta^n \to \eta.$$

It follows that

$$\sum_{t=0}^{\infty} w_t^*(\eta^n) \leq \sum_{t=0}^{\infty} \eta_i^n\beta_i^t M < +\infty \text{ when } \eta^n \to \eta.$$

Similarly, we get

$$\frac{u_t^{2i}(c^{i*}(\eta^n), l^{i*}(\eta^n)) l^{i*}(\eta^n)}{\eta_i^n\beta_i^t} \leq M_1 < +\infty\ ,$$

$$\frac{\lambda_t^{5i}(\eta^n)}{\eta_i^n\beta_i^t} \leq M_2 < +\infty \text{ as } \eta^n \to \eta$$

$$\text{then } \sum_{t=0}^{\infty} u_t^{2i}(c^{i*}(\eta^n), l^{i*}(\eta^n)) < +\infty \text{ and } \sum_{t=0}^{\infty} \lambda_t^{5i}(\eta^n) < +\infty \text{ as } \eta^n \to \eta.$$

Thus, it follows from (11) that $\lambda_t^{4i}(\eta^n)$, $\lambda_t^{5i}(\eta^n)$ belong to l_+^1 when $\eta^n \to \eta$.

iii) If $c_t^{i*}(\eta^n) \to c_t^{i*}(\eta) > 0$, we have

$$\begin{aligned} & \eta_i\beta_i^t u_i(c_t^{i*}(\eta^n), l_t^{i*}(\eta^n)) - \eta_i\beta_i^t u_i(0, l_t^{i*}(\eta^n)) \\ & \geq \eta_i\beta_i^t u_t^{1i}(c_t^{i*}(\eta^n), l_t^{i*}(\eta^n)) c_t^{i*}(\eta^n) \\ & = \left[\lambda_t^1(\eta^n) - \lambda_t^{2i}(\eta^n)\right] c_t^{i*}(\eta^n) = \lambda_t^1(\eta^n) c_t^{i*}(\eta^n). \end{aligned}$$

or

$$\begin{aligned} 0 \;&\leq\; \lambda_t^1(\eta^n) \leq \eta_i^n\beta_i^t \frac{u_i(c_t^{i*}(\eta^n), l_t^{i*}(\eta^n))}{c_t^{i*}(\eta^n)} \leq 2M \text{ when } \eta^n \to \eta. \\ \text{where } M \;&=\; \frac{u_i(c_t^{i*}(\eta), l_t^{i*}(\eta))}{c_t^{i*}(\eta)} < +\infty \end{aligned}$$

This means $\lambda_t^1(\eta^n)$ contains a convergent subsequence, denoted again by $\lambda_t^1(\eta^n)$, say $\lambda_t^1(\eta^n) \to \lambda_t^1(\eta)$ and $\sum\limits_{t=0}^{\infty}\lambda_t^1(\eta) \leq M\sum\limits_{t=0}^{\infty}\eta_i\beta_i^t < +\infty$.

iv) If $l_t^{i*}(\eta^n) \to l_t^{i*}(\eta) > 0$. By the same argument above, we get $w_t^*(\eta^n) \to w_t^*(\eta) \in l_+^1$.

We have proved that $\lambda_t^1(\eta^n)$, $\lambda_t^{2i}(\eta^n), \lambda_t^{4i}(\eta^n), \lambda_t^{5i}(\eta^n), w_t^*(\eta^n)$ belong to l_+^1 when $\eta^n \to \eta$.We shall show that $\lambda_t^3(\eta^n) \in l_+^1$ when $\eta^n \to \eta$. Indeed, we have :

$$+\infty > \sum_{t=0}^{\infty}\lambda_t^1(\eta^n)F(k_t^*(\eta^n), L_t^*(\eta^n)) - \sum_{t=0}^{\infty}\lambda_t^1(\eta^n)F(0,0) \geq$$

$$\sum_{t=0}^{\infty}\lambda_t^1(\eta^n)f_t^1(k_t^*(\eta^n), L_t^*(\eta^n))k_t^*(\eta^n) + \sum_{t=0}^{\infty}\lambda_t^1(\eta^n)f_t^2(k_t^*(\eta^n), L_t^*(\eta^n))L_t^*(\eta^n)$$

This implies that

$$\sum_{t=0}^{\infty}\lambda_t^1(\eta^n)f_t^1(k_t^*(\eta^n), L_t^*(\eta^n))k_t^*(\eta^n) < +\infty \quad \forall t \tag{20}$$

Since $f_k'(0,m) > 1$, then for all $k_0 > 0$, there exists some $0 < \widehat{k} < k_0$ such that:

$$\begin{aligned} 0 &< \widehat{k} < F(k_0, m) + (1-\delta)k_0. \\ 0 &< \widehat{k} < F(\widehat{k}, m) + (1-\delta)\widehat{k} \end{aligned}$$

Take a feasible sequences from k_0 :

$$\begin{aligned} \mathbf{k} &= (k_0, \widehat{k}, \widehat{k}, ...) \\ (\mathbf{c}, \mathbf{l}) &= ((0,0), (0,0), ...) \end{aligned}$$

It follows from (18) that

$$\begin{aligned} &[\lambda_t^1(\eta^n)f_t^1(k_t^*(\eta^n), L_t^*(\eta^n)) + (1-\delta)\lambda_t^1(\eta^n) - \lambda_{t-1}^1(\eta^n)]k_t^*(\eta^n) \geq \\ &\qquad [\lambda_t^1(\eta^n)f_t^1(k_t^*(\eta^n), L_t^*(\eta^n)) + (1-\delta)\lambda_t^1(\eta^n) - \lambda_{t-1}^1(\eta^n)]\widehat{k}. \end{aligned}$$

This implies

$$\sum_{t=0}^{\infty}\lambda_t^1(\eta^n)f_t^1(k_t^*(\eta^n), L_t^*(\eta^n))\widehat{k} \leq \sum_{t=0}^{\infty}\lambda_t^1 f_t^1(k_t^*(\eta^n), L_t^*(\eta^n))k_t^*(\eta^n) +$$

$$\sum_{t=0}^{\infty}[(1-\delta)\lambda_t^1(\eta^n) - \lambda_{t-1}^1(\eta^n)]k_t^*(\eta^n) + \sum_{t=0}^{\infty}[\lambda_{t-1}^1(\eta^n) - (1-\delta)\lambda_t^1(\eta^n)]\widehat{k}.$$

It follows from (20), $0 \leq k_t^* \leq \max(k_0, \bar{k})$ and $\lambda_t^1(\eta^n) \to \lambda_t^1(\eta) \in l_+^1$, by using a convergent subsequence, we can say that

$$\sum_{t=0}^{\infty}\lambda_t^1(\eta^n)f_t^1(k_t^*(\eta^n), L_t^*(\eta^n)) < +\infty \text{ when } \eta^n \to \eta.$$

Therefore, from (12), we conclude $\lambda_t^3(\eta) \in l_+^1$.

Now, take $y(\eta^n) \in \phi_i(\eta^n)$, then there exists $\lambda_t^1(\eta^n)$ such that

$$y^n = \sum_{t=0}^{\infty} \lambda_t^1(\eta^n) c_t^{i*}(\eta^n) + \sum_{t=0}^{\infty} w_t^*(\eta^n) l_t^{i*}(\eta^n)$$
$$- \sum_{t=0}^{\infty} w_t^*(\eta^n) - \vartheta^i r k_0 - \alpha^i \boldsymbol{\pi}^*(\eta^n)$$

where

$$\boldsymbol{\pi}^*(\eta^n) = \sum_{t=0}^{\infty} \lambda_t^1(\eta^n) F(k_t^*(\eta^n), L_t^*(\eta^n)) - \sum_{t=0}^{\infty} \lambda_t^1(\eta^n)(k_{t+1}^*(\eta^n)$$
$$-(1-\delta)k_t^*(\eta^n)) - \sum_{t=0}^{\infty} w_t^*(\eta^n) L_t^*(\eta^n) - \frac{\lambda_0^1(\eta^n)(1-\delta)}{2} k_0$$

We may suppose that $y(\eta^n) \to y(\eta)$. It follows from the continuity of $c_t^{i*}(\eta^n)$, $k_t^*(\eta^n), l_t^{i*}(\eta^n), F(k_t^*(\eta^n), L_t^*(\eta^n))$ that

$$\forall t, \lambda_t^1(\eta^n) \to \lambda_t^1(\eta) \in l_+^1, w_t^*(\eta^n) \to w_t^*(\eta) \in l_+^1.$$

By taking $\eta^n \to \eta$ in all the inequalities and equalities from (1) to (6) in the Proposition 1,all $\lambda_t^1(\eta), \lambda_t^{2i}(\eta), \lambda_t^3(\eta), \lambda_t^{4i}(\eta), \lambda_t^{5i}(\eta)$ satisfy conditions (1) $-$ (6). This means $\boldsymbol{\lambda}(\eta)$ is Lagrange Multipliers of the Pareto-Optimum problem.

Therefore, we have $y(\eta) \in \phi_i(\eta)$. Moreover, $\phi_i(\eta)$ is bounded. Hence $\phi_i(\eta)$ is compact valued and upper semi-continuous.

We shall prove that $\phi_i(\eta)$ is convex .

For given weights $\eta \in \Delta$, each consumer's utility function is strictly concave, the Pareto-optimum problem will have unique solution for each $\eta \in \Delta$. Thus, the maps

$$(c_t^{i*}(\eta), k_t^*(\eta), l_t^{i*}(\eta)) = \arg\max U(\boldsymbol{\eta}, \mathbf{k}, \mathbf{c}, \mathbf{l}) \text{ over } \Pi(k_0) \times \sum(k_0)$$

is well-defined on Δ.

Let $y \in \phi_i(\eta)$, $y' \in \phi_i(\eta)$, there exists two sequences $\{\lambda_t^1\}, \{\lambda_t'^1\}$ such that $y \in \phi_i(\lambda_t^1(\eta))$, $y' \in \phi_i(\lambda_t'^1(\eta))$.

We have, for all $\alpha \in [0,1]$,

$$\alpha y_1 + (1-\alpha) y' =$$
$$\sum_{t=0}^{\infty} [\alpha \lambda_t^1(\eta) + (1-\alpha)\lambda_t'^1(\eta)] c_t^{i*}$$

$$+ \sum_{t=0}^{\infty} [\alpha \lambda_t^1(\eta) + (1-\alpha)\lambda_t'^1(\eta)] f_t^2(k_t^*, L_t^*) l_t^{i*}$$
$$- \sum_{t=0}^{\infty} [\alpha \lambda_t^1(\eta) + (1-\alpha)\lambda_t'^1(\eta)] f_t^2(k_t^*, L_t^*) - \vartheta^i r k_0$$

$$- \alpha^i[\sum_{t=0}^{\infty}(\alpha\lambda_t^1(\eta) + (1-\alpha)\lambda_t^{,1}(\eta))F(k_t^*, L_t^*)$$

$$-\sum_{t=0}^{\infty}(\alpha\lambda_t^1(\eta) + (1-\alpha)\lambda_t^{,1}(\eta))(k_{t+1}^* - (1-\delta)k_t^*)$$

$$-\sum_{t=0}^{\infty}(\alpha\lambda_t^1(\eta) + (1-\alpha)\lambda_t^{,1}(\eta))f_t^2(k_t^*, L_t^*)L_t^* - rk_0]$$

Since $\lambda_t^1(\eta)$ and $\lambda_t^{,1}(\eta)$ satisfy the conditions $(1)-(6)$, it is easy to check that $\alpha\lambda_t^1(\eta) + (1-\alpha)\lambda_t^{,1}(\eta)$ satisfies $(1)-(6)$. Thus, $\alpha\lambda_t^1(\eta) + (1-\alpha)\lambda_t^{,1}(\eta)$ is also Lagrange multipliers for Pareto-Optimum problem.

Therefore, $\alpha y_1 + (1-\alpha)y^{,} \in \phi_i(\eta)$ or $\phi_i(\eta)$ is convex . ■

We now use the Browder Fixed-Point Theorem for Multivalued Maps with Boundary Condition to prove there exists an equilibrium.

Theorem 4 *(See Zeidler[1992], Theorem 9.C) Suppose that*

(i) the map $T(\eta) : \Delta \to 2^X$ *is upper semi-continuous, and that* Δ *is a nonempty, compact, convex set in a locally convex space* X*;*

(ii) the set $T(\eta)$ *is nonempty, closed, and convex for all* $\eta \in \Delta$;

(iii) one of the following two boundary conditions is satisfied:

For every $\eta \in \partial\Delta$ *there are points* $\zeta \in T(\eta)$ *and* $\xi \in \Delta$, *and a number* $a > 0$ *such that* $\zeta = \eta + a(\xi - \eta)$;

For every $\eta \in \partial\Delta$ *there are points* $\zeta \in T(\eta)$ *and* $\xi \in \Delta$, *and a number* $a < 0$ *such that* $\zeta = \eta + a(\xi - \eta)$;

Then $T(\eta)$ *has a fixed point.*

Lemma 2 *(The inward boundary condition)*

For given $\eta \in \Delta, k_0 > 0$ *and* $\phi(\eta) = (\phi_1(\eta), \phi_2(\eta), ..., \phi_m(\eta)) \in R^m$. *If* $\eta_i = 0$ *then for all* $y \in \phi(\eta)$, $y_i < 0$.

Proof: Indeed, if there exists i such that $\eta_i = 0$ then, by the Proposition 2(ii), $c_t^{i*} = 0$, $l_t^{i*} = 0$ for all t.

Then, if $y \in \phi(\eta)$,

$$y_i \in \phi_i(\eta)$$
$$= \{\sum_{t=0}^{\infty} p_t^*(\eta)c_t^{i*}(\eta) + \sum_{t=0}^{\infty} w_t^*(\eta)l_t^{i*}(\eta) - \sum_{t=0}^{\infty} w_t^*(\eta) - \vartheta^i rk_0 - \alpha^i\boldsymbol{\pi}^*(\eta)\}$$
$$= \{-\sum_{t=0}^{\infty} w_t^*(\eta) - \vartheta^i rk_0 - \alpha^i\boldsymbol{\pi}^*(\eta)\}.$$

It follows from Proposition 2(i) that $y_i < 0$. ■

Theorem 5 *Let $k_0 > 0$. Then there exists $\bar{\eta} \in \Delta, \bar{\eta} >> 0$, such that $0 \in \phi_i(\bar{\eta}), \forall i$, that means there exist an equilibrium.*

Proof: Let $T(\eta) = \eta + \phi(\eta)$, $T(\eta) = (T_1(\eta), T_2(\eta), ..., T_m(\eta))$.It follows from Lemma 2 that $\eta_i = 0$ implies $y_i \in \phi_i(\bar{\eta}) \subset R_{--}$ or $y_i \in T_i(\eta)$ and $y_i < 0$. From the Theorem 4, there exists $\bar{\eta}$ such that $\bar{\eta} \in T(\bar{\eta})$. This implies that $0 \in \phi(\bar{\eta})$ or $0 \in \phi_i(\bar{\eta})$ for all $i = 1...m$. ∎

References

[1] Aliprantis C. D. and D. J. Brown and O. Burkinshaw, [1990], *Existence and Optimality of Competitive Equilibria*, Berlin Heidelberg NewYork, Springer.

[2] Bewley T. F., [1972], Existence of Equilibria in Economies with Infinitely Many Commodities, *Journal of Economic Theory*, 4, pp. 514-540.

[3] Berge C., [1963], *Espaces Topologiques*, Paris, Dunod.

[4] Dana R-A. and Le Van C, [1991], Equilibria of a Stationary Economy with Recursive Preferences, *Journal of Optimization Theory and Applications*, 71, pp. 289-313.

[5] Datta M. and L. J. Mirman and K. L. Reffett, [2002], Existence and Uniqueness of Equilibrium in Distorted Dynamic Economies with Capital and Labor, *Journal of Economic Theory*, 103, pp. 377-410.

[6] Debreu G. (1954) "Valuation, equilibrium and Pareto-optimum", *Mathematical Economics: Twenty Papers of Gerard Debreu*, Cambridge University Press 1983.

[7] Coleman W.J., [1997], Equilibria in distorted infinite-horizon economies with capital and labor, *Journal of economic Theory*, 103, 446-461.

[8] Florenzano M and Le Van C, [2001], *Finite Dimensional Convexity and Optimization*, Springer-Verlag.

[9] Greenwood G. and G. W. Huffman, [1995], On the Existence of Non-Optimal Equilibria in Dynamic Stochastic Economies, *Journal of Economic Theory*, 65, pp. 611-623.

[10] Le Van C. and Y. Vailakis, [2003], Existence of a Competitive Equilibrium in a One Sector Growth Model with Heterogeneous Agents and Irreversible Investment, *Economic Theory*, 22, pp. 743-771.

[11] Le Van C. and H. Saglam, [2004], Optimal Growth Models and the Lagrange Multiplier, *Journal of Mathematical Economics*, 40, pp. 393-410.

[12] Mas-Collel A and W. Zame [1991], Equilibrium Theory in Infinite dimensional spaces, in W.Hildenbrand and H.Sonnenschein (eds), *Hand-book of Mathematical Economics*, Vol.IV, Amsterdam, North Holland.

[13] Nguyen M.H and S. Nguyen Van, [2005], The Lagrange Multipliers and Existence of Competitive Equilibrium in an Intertemporal Model with Endogenous Leisure, *Cahiers de la MSE*, $N^0$2005.41.

[14] Zeidler E.[1992], *Nonlinear Functional Analysis and Its Application*, Springer.

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