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SOME ORDER DUALITIES IN LOGIC, GAMES AND CHOICES

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Abstract

We first present the concept of duality appearing in order theory, i.e. the notions of dual isomorphism and of Galois connection. Then we describe two fundamental dualities, the duality extension/intention associated with a binary relation between two sets, and the duality between implicational systems and closure systems. Finally we present two "concrete" dualities occuring in social choice and in choice functions theories.

Keywords: antiexchange closure operator, closure system, Galois connection, implicational system, Galois lattice, path-independent choice function, preference aggregation rule, simple game.

JEL classification: C00, D71

1. Introduction

In the index of the Encyclopaedia of Mathematics (Kluwer, 1987) the terms duality or dual appear about 120 times. Here we will only consider the duality occurring in order theory. Why order duality matters ? for a very simple reason. When one can show that two sets of objects are two dual posets, then one can automatically transfer any result on one of these posets to the other. We will begin by recalling the notion of dual posets and the notion of Galois connection,

a useful tool to obtain dual posets. Then we will present two significant dualities obtained through a Galois connection: the "extension/intention" duality associated to a binary relation and the duality between implicational systems and closure systems. The first one is applied to an example in social choice theory. The second one generalizes the well known duality between posets and T_0-topologies (or distributive lattices). We will finally present a duality between path-independent choice functions and anti-exchange closure operators. Some other uses of order duality are mentionned in the conclusion.

N.B. Unless explicitly stated, all the sets considered in this paper are finite.

2. Order Duality and Galois Connection

A *partially ordered set* (or *poset*) $P = (X, \leq_P)$ is a set X equipped with a reflexive, transitive and antisymmetric binary relation $\leq_P$. Often, we shall only denote such a poset by $(P, \leq)$ or simply by P. Moreover, we write $x \in P$ or $F \subseteq P$ rather than $x \in X$ or $F \subseteq X$.

The *dual* of the poset $P = (X, \leq)$ is the poset $P^d = (X, \geq)$, where $x \geq y$ if and only if $y \leq x$.

A map f from a poset $P = (X, \leq_P)$ to a poset $Q = (Y, \leq_Q)$ is *isotone* (respectively *antitone*) if $x \leq_P y$ implies $f(x) \leq_Q f(y)$ (respectively $x \leq_P y$ implies $f(x) \geq_Q f(y)$).

Two posets P and Q are *isomorphic* if there exists an *isomorphism* beetwen them, *i.e.* a bijective map f from P to Q which is isotone as well as its inverse f^{-1} (equivalently, a bijective map f from P to Q atisfying $x \leq_P y$ if and only if $f(x) \leq_Q f(y)$).

Definition 1

Two posets P and Q are *dual* if there exists a *dual isomorphism* beetwen them, i.e. a bijective map f from P to Q satisfying $x \leq_P y$ if and only if $f(x) \geq_Q f(y)$.

Observe that the posets $P = (X, \leq)$ and $P^d = (X, \geq)$ are dual since the identity map is a dual isomorphism between them, and that P and Q are dual if and only if there exists an isomorphism between P and Q^d.

Many interesting dual posets can be obtained through a Galois connection between two posets, a notion that we define below, after recalling the notions of closure operator and closure system.

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Definition 2

A *closure operator* ϕ on a poset P is a map from P to itself which is isotone, *extensive* and *idempotent* ($x \leq \phi(x) = \phi^2(x)$). The fixed points of ϕ are called the *closed elements* of P.

A *closure sytem* C on a poset P is a subset C of P such that for each x in P, the set $\{y \in C : x \leq y\}$ has a least element denoted by $\phi_C(x)$.

It is well known that the set of all closure operators and the set of all closure systems defined on a poset are in a one-to-one correspondence (by the two inverse bijective maps associating to a closure operator the set of its fixed points and to a closure system the application $x \rightarrow \phi_C(x)$).

Definition 3

A *Galois connection* between two posets P and Q is an ordered pair (f,g) of maps

$$f: P \rightarrow Q \quad g: Q \rightarrow P$$

satisfying the following condition:

$$\text{for all } x \in P \text{ and } y \in Q, \qquad x \leq g(y) \text{ if and only if } y \leq f(x).$$

An equivalent definition uses the composite maps fog denoted by $fg(fg(x) = f(g(x)))$ and gof denoted by gf: (f,g) is a Galois connection if and only if the maps f and g are antitone, and the maps fg and gf are extensive.

Now one can state the easily proved but significant result:

Theorem 4

Let (f,g) be a Galois connection between two posets P and Q. Then

1) gf is a closure operator on P and fg is a closure operator on Q ;

2) the two closure systems $gf(P)$ on P and $fg(Q)$ on Q are two dual posets.

One can add that $g(Q) = gf(P)$, $f(P) = fg(Q)$ and that the two inverse dual isomorphims between $gf(P)$ and $fg(Q)$ are given by the restrictions of f and g to the two posets $g(Q)$ and $f(P)$.

A special case of this result occurs when P and Q are two lattices. Recall that a *meet-semilattice* (respectively a *join-semilattice*) is a poset where any two elements x and y have a *meet* $x \wedge y$, *i.e.* a

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greatest lower bound (respectively a *join, i.e.* a lowest upper bound $x \vee y$) and that a *lattice* $(L, \leq)$ is both a meet-semilattice and a join-semilattice. Then one has:

Theorem 5

Let (f,g) be a Galois connection between two lattices L and L'. Then

1) gf is a closure operator on L and fg is a closure operator on L',

2) the two closure systems $gf(L)$ and $fg(L')$ are two dual lattices.

One can add that the lattice $gf(L)$ (respectively $fg(L')$) is a meet-subsemilattice of L (respectively of L') and that its least element is $f(1_L)$ (respectively $g(1_{L'})$).

This result can be in particular applied when $L = (2^E, \subseteq)$ (respectively $L' = (2^F, \subseteq)$) is the Boolean lattice of all the subsets of a set E (respectively of a set F) ordered by set inclusion. We begin by recalling what becomes the definition of a closure system in the case of the lattice $L = (2^E, \subseteq)$. Note that we say that such a closure system is defined on E although it is in fact a subset of 2^E.

Definition 6

A closure system on E (is a family C of subsets of E satisfying the two following conditions:

1) $E \in C$,

2) $C_1, C_2 \in C \Rightarrow C_1 \cap C_2 \in C$.

Then, $(C, \subseteq)$ is a lattice whose the meet operation $\wedge$ is the set intersection, whereas the join operation $\vee$ is given by $C_1 \vee C_2 = \cap\{C \in C : C_1 \cup C_2 \subseteq C\}$.

Remarks

1) The ("abstract") notion of Galois connection (or connexion) is due to Öre (Öre,1944). The term Galois connection refers to the existence of such a connection (between subfields and subgroups) in the Galois theory of equations. Before Öre, Birkhoff (1940) has considered the ("concrete") Galois connection associated to a binary relation and defined in the next section (in fact Öre showed that any Galois connection can be obtained as such a Galois connection).

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2) Proofs of Theorems 4 and 5 can be found in, for example, Barbut and Monjardet (1970) and Szasz (1963).

3. The Galois Connection associated with a Binary Relation and the Extension/Intention Duality

When R is a binary relation between the two sets E and F, we say that the triple (E, F, R) is a *bigraph*. The following theorem, due to Birkhoff, is an example of an easily proven but fundamental result.

Theorem 7

Let (E, F, R) be a bigraph. One defines a Galois connection (f,g) between the lattices $(2^E, \subseteq)$ and $(2^F, \subseteq)$ by setting for $X \subseteq E$ and $Y \subseteq F$

$$f(X) = \{y \in F : xRy \text{ for every } x \in X\} \quad ; \quad g(Y) = \{x \in E : xRy \text{ for every } y \in Y\}.$$

Let (X,Y) be such that $X \subseteq E$ (respectively $Y \subseteq F$) is a closed set in the associated closure system $gf(2^E)$ on E (respectively $fg(2^F)$ on F) and such that $X = g(Y)$ (and $Y = f(X)$). Then, it follows from Theorem 7 that

1) for all $x \in X$, $y \in Y$, xRy,

2) for all $x \notin X$, there exists $y \in Y$ such that xR^Cy (i.e. $(x,y) \notin R$),

3) for all $y \notin Y$, there exists $x \in X$ such that xR^Cy.

These facts induce the below definition:

Definition 8

Let (E, F, R) be a bigraph. The ordered pair (X,Y), where $X = g(Y)$ and $Y = f(X)$ are two corresponding closed sets in the Galois connection associated with the bigraph is called a *concept* of (E, F, R). X is called the *extension* (or the *extent*) of the concept (X,Y) and Y is called its *intention* (or its *intent*).

Indeed, when E is a set of objects, F a set of attributes and R the relation "the object x has the attribute y", a concept is a set of objects and a set of attributes such that these objects are the only ones satisfying all these attributes and these attributes are the only ones satisfied by all

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these objects. So, it is not inadequate to use the traditional terms used in logic of *extension* and *intention*.

Ordered by set-inclusion the two sets of extensions and intentions are two posets. It results from the properties of a Galois connection between two lattices (Theorem 5) that one has the following result:

Corollary 9

Let (*E*, *F*, *R*) be a bigraph. The two posets of extensions and intentions associated with the Galois connection induced by this bigraph are two dual lattices.

Then, if one defines an "order of generality" between two concepts (*X*,*Y*) and (*X*',*Y*') by:

$$(X,Y) \leq (X',Y') \text{ if } X \subseteq X' \text{and } Y \supseteq Y'$$

(*i.e.* if the extension of the first concept is smaller and its intention bigger), one gets that the sets of concepts (ordered by this order) is a lattice isomorphic to the lattice of extensions and dually isomorphic to the lattice of intentions. This lattice called **t**he *Galois* (or the *concept*) *lattice* of the bigraph (*E, F, R*) has many applications for instance in data analysis, formal concept analysis and knowledge representation theory (see for instance Barbut and Monjardet, 1970, Ganter and Wille, 1999, Valtchev, 1999 or Valtchev & *al*, 2000).

In section 5, we will present two examples of a Galois connection associated with a bigraph, occuring in social choice theory and game theory.

Remark

According to Barbut's result (Barbut, 1965) any lattice L can be represented as the Galois lattice of a bigraph (see Barbut and Monjardet, 1970).

4.The Duality between Implicational Systems and Closure Systems

We consider now a fundamental duality arising from a Galois connection between binary relations between subsets of a set and families of subsets of this set. In fact, it is a generalization of the well-known duality between posets and T_0-topologies (or distributive lattices), arising from a Galois connection between binary relations on a set and families of subsets of this set. We first begin by giving a name to a binary relation between subsets of a set.

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Definition 10

An *implicational system* Σ on a set S is a binary relation on 2^S:

$$\Sigma \subseteq 2^S \times 2^S.$$

$(X,Y) \in \Sigma$ will be denoted by $X \rightarrow Y\ (\Sigma)$, and one says that (X,Y) is an *implication* and that X *implies* Y (w.r.t. Σ).

One defines a Σ-*closed set* by:

$C \subseteq S$ is a Σ-closed set if for every $X \subseteq S$, $[X \subseteq C$ and $X \rightarrow Y\ (\Sigma)] \Rightarrow [Y \subseteq C]$.

We set $f(\Sigma) = C_\Sigma$ = {Σ-closed sets C}.

Let now $\mathcal{C}$ be a *set system* on S (*i.e.* a family of subsets of S). One defines an implicational system on S, denoted by $\Gamma_{\mathcal{C}}$, by :

$X \rightarrow Y\ (\Gamma_{\mathcal{C}})$ if $Y \subseteq C$ for every $C \in \mathcal{C}$ such that $X \subseteq C$.

We set $g(\mathcal{C}) = \Gamma_{\mathcal{C}}$.

Then we have the following result due to Doignon and Koppen (1989) and Muller (1989) (see also Doignon and Falmagne, 1999).

Theorem 11

Let $(2^{2^S \times 2^S}, \subseteq)$ and $(2^{2^S}, \subseteq)$, be the Boolean lattices of all implicational systems and of all set systems on S. The ordered pair (f,g), with $f(\Sigma) = C_\Sigma$ and $g(\mathcal{C}) = \Gamma_{\mathcal{C}}$, is a Galois connection between these two lattices.

The implicational systems closed for the associated closure operator gf on $2^S \times 2^S$ are the so-called full implicational systems, defined below:

Definition 12

A *full implicational system* is an implicational system Γ satisfying the three following properties : for all X, $Y, Z \subseteq S$,

1) $X \supseteq Y \Rightarrow X \rightarrow Y\ (\Gamma)$

2) $X \rightarrow Y\ (\Gamma),\ Y \rightarrow Z\ (\Gamma) \Rightarrow X \rightarrow Z\ (\Gamma)$

3) $X \rightarrow Y\ (\Gamma), Z \rightarrow T\ (\Gamma) \Rightarrow X \cup Z \rightarrow Y \cup T\ (\Gamma)$.

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On the other hand, it is easy to check that the set systems closed for the associated closure operator *fg* on 2^S are the closure sytems on *S*. So, it results from the properties of a Galois connection that:

Corollary 13

The poset of all full implicational systems on *S* and the poset of all closure systems on S are two dual lattices.

Remarks

1) In early works of Tarski (for instance Tarski, 1930, see also Martin and Pollard's book, 1996) the consequence relation of a *logical deductive system* is defined as a closure operator on an infinite set *S* satisfying a "finitary" axiom. Other logicians (like Hertz,1929, Scott, 1974) defined it as a binary relation between sets and, later, a one-to-one correspondence between Scott's "informations systems" and "algebraic $\cap$-structures" has been displayed (see Davey and Priestley, 1990). In the finite case, this correspondence becomes exactly the one-to-one correspondence between full implicational systems and closure systems. This last one has been first shown by Armstrong (1974), who, in the context of relational data bases, called a full implicational system *a full family of functional dependencies.*

2) Particular cases of the above duality are the well-known dualities between preorders and topologies, and between partial orders and T_0-topologies (or distributive lattices). See for instance, Barbut and Monjardet, 1970, or Davey and Priestley, 1990. This duality between preorders and topologies plays important roles in many situations, for instance in the study of the structure of the (distributive) lattices of stable marriages (see Gusfield and Irving's book, 1989).

3) There exists significant results on the *bases* of full implicational systems (or of closure operators), *i.e.* on the minimal sets of implications allowing to recover all the implications of the system (or, equivalently, all the closed sets). In particular, the existence of a *canonical basis* has been shown by Maier, 1983, and Guigues and Duquenne, 1986 (see also Caspard and Monjardet, 2003, where this result is presented in the context of the study of the lattice of all closure systems).

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5.The Duality between Profiles of Preferences and Simple Games

We consider here the aggregation preference problem. There is a set A of alternatives, a set N of n voters and one assumes that the preference of a voter on the alternatives is given by a *linear order* (*l.e.* a complete partial order) L belonging to the set $\mathcal{L}$ of all linear orders on A. $P = \mathcal{L}^N$ denotes the set of all possible profiles $\pi = (L_1,...,L_n)$ of preferences of voters. In order to define a class of aggregation rules, we recall the notion of (monotonic) simple game.

Definition 14

A *simple game* on N is a (non-empty) set $\mathcal{F}$ of subsets of N satisfying:

$$[T \in \mathcal{F} \text{ and } T \subseteq U] \Rightarrow [U \in \mathcal{F}].$$

We denote by G the set of all simple games on N.

Now one can associate with any simple game a preference aggregation function, *i.e.* a map from the set P of profiles of linear orders to the set of "collective" preference relations:

Definition 15

The *preference aggregation function* $f_{\mathcal{F}}$ associated with the simple game $\mathcal{F}$ is given by:

$$\text{for every } \pi \in P,\ f_{\mathcal{F}}(\pi) = R_{\mathcal{F}}(\pi), \text{ where}$$

$$\text{for all } x,y \in A,\ [xR_{\mathcal{F}}(\pi)y] \Leftrightarrow [\{i \in N : xL_iy\} \in \mathcal{F}].$$

Thus, x is collectively preferred to y according to the preference aggregation function $f_{\mathcal{F}}$ if and only if the set of voters preferring x to y in the profile π belongs to the simple game $\mathcal{F}$. Then $R_{\mathcal{F}}(\pi)$ is a complete and antisymmetric relation (a so-called *tournament*), but since it is not non necessarily transitive, it is not necessarily a linear order.

Now we consider the binary relation between profiles and simple games which describes the "good" case where the collective preference relation is transitive, *i.e.* is a linear order.

Definition 16

Let P be the set of all profiles of linear orders and G the set of all simple games. One defines a binary relation R between P and G by :

$$\pi R \mathcal{F} \quad \Leftrightarrow \quad R_{\mathcal{F}}(\pi) \in \mathcal{L}.$$

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Thus, one can apply the results of section 3 on the Galois connection associated to a binary relation defined between two sets. So, there is a Galois connection *(f,g)* between the lattices $(2^P, \subseteq)$ and $(2^G, \subseteq)$ and two dual lattices of closed sets on P and G. The least elements of these two lattices, i.e. the closed sets $f(2^P)$ and $g(2^G)$ are characterized in the below result (Monjardet, 1978).

Theorem 17

Let (*f*,*g*) be the Galois connection associated to the binary relation *R* defined between the Boolean lattices $(2^P, \subseteq)$ of all sets of preference profiles and $(2^G, \subseteq)$ of all sets of simple games. The least element of the lattice $gf(2^P)$ of closed sets on *P* is the set of profiles without *cyclical subprofile* (like, for instance *xyz*, *yzx* and *zxy*), and the least element of the lattice $fg(2^G)$ of closed sets on *G* is the set of *ultrafilters* on *N*, *i.e.* the set of simple games $\mathcal{F}_i = \{T \subseteq N : i \in T\}$, $i \in N$.

This characterization result shows the duality between Guilbaud's Arrovian theorem (1952, see also Monjardet, 2004) and Sen's possibility theorem (1966) for profiles of linear orders: the only preference aggregation rules defined by simple games providing a transitive collective preference for all profiles of linear orders are the dictatorial rules, and the only profiles for which all such rules provide always a linear order are the "value-restricted"profiles. (One will find other results on this duality in Monjardet, 1978).

Remark

Preference aggregation functions associated with simple games are a particular case of preference aggregation functions associated with *effectivity functions*, i.e. maps (satisfying some mild conditions) from the set of all subsets of the set *N* to the set of all set systems on the set *S*. (in the case of simple games the image of a subset of *N* is either the empty set or the set of all non-empty subsets of *S*). But such a map is the same than a binary relation between 2^N and 2^S and so one can consider the Galois connection and the Galois lattice associated to this relation. It is exactly what is made by Vannucci in several papers (1999,2002,2003) where he studies this Galois lattice and uses it to get new "structural" representations and *desirabilty relations* for effectivity functions.

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6.The duality between convex geometries and path-independent choice operators

We consider now a significant duality between a class of closure operators and a class of choice operators which is not induced by a a Galois connection.

Definition 18

A closure operator ϕ on S is an *antiexchange* closure operator if it satisfies the following two properties:

- $\phi(\emptyset) = \emptyset$,

- $x,y \notin \phi(X)$, $x \neq y$ and $y \in \phi(X + x)$ imply $x \notin \phi(X + y)$**.**

It is easy to check that the set of all antiexchange closure operators on S, ordered by the point-wise order between maps, is a meet-semilattice (without greatest element).

An antiexchange closure operator ϕ_P is associated with any poset $P = (X, \leq)$. It associates to any subset A of P, the set of all elements below or equal to an element of A ; formally $\phi_{P(}A) = \{y \in S$: there exists $x \in A$ with $y \leq x\}$. We call such an antiexchange closure operator an *order closure operator*, and in particular, we call it a *linear order closure operator* when P is linearly ordered. The closed sets of an order closure operator ϕ_P are the *order ideals* (called also *down sets*) of P, *i.e.* the subsets X of P such that $y \in X$ and $x < y$ imply $x \in S$.

Definition 19

A *choice operator* σ on a set S is a map associating to any subset A of S a non empty subset $\sigma(A)$ contained in A. A choice operator is *path independent* if it satisfies the following property: for all A, B subsets of S

$$\sigma(A \cup B) = \sigma(\sigma(A) \cup \sigma(B)).$$

It is easy to check that the set of all path-independent choice operators on S, ordered by the pointwise order between maps, is a join-semilattice (without lowest element).

A path-independent choice operator σ_P is associated with any poset $P = (X, \leq)$. It is obtained by associating to any subset A of P the set of its maximal elements. Formally, $\sigma_{P(}A) = \{x \in A$: there does not exist exists $y \in A$ with $x<y\}$. We call such a path-independent choice operator an *order choice operator*, and in particular, a *linear order choice operator*, if P is linearly ordered.

The chosen sets by an order choice operator σ_P are the *antichains* of *P*, *i.e.* the subsets of *P* which do not contain two elements *x* and *y* with $x<y$.

It is easy to understand that the obvious one-to-one correspondence between order ideals and antichains of a poset induce a one to one correspondence between order closure operators and order choice operators. But this is a particular case of a general result (Monjardet and Raderanirina, 2001):

Theorem 20

The join-semilattice of all the path-independent choice operators on a set S is dual of the meet-semilattice of all the antiexchange closure operators on S.

Then, we have for instance the consequence that the representation result of any antiexchange closure operator as a meet of linear closure operators obtained by Edelman and Saks (1988, see also Matalon, 1965, P.96) is the same (under the duality) that the representation result of any path-independent choice operator as a join of linear choice operators obtained by Aizerman and Malishevski (1981). Moreover the use of the duality allows to get interesting precisions on the way to get such representations (see Monjardet and Raderanirina, 2001).

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Remarks

1) The notion of path-independent choice operator introduced in Plott, 1973, have been previously studied by Afriat (1967). Aizerman and Malishevski's representation result quoted just above amounts to say that choices by such an operator can be obtained by the so-called "extremal choice" mechanism, where the agent collects the best choices obtained under different -linearly ordered- criteria (see Aizerman and Aleskerov, 1995).

2) The fact that there exists a one-to-one correspondence between antiexchange closure operators and path-independent choice operators has been independently obtained by Koshevoy (1999) and Johnson and Dean (2001). It would be interesting to find a Galois connection inducing this correspondence.

3) The closure systems corresponding to the antiexchange closure operators are called the *convex geometries* and are characterized by two properties: the empty set Ø is closed, and for every closed set C ($\neq S$) there exists $x \notin C$ such that $C \cup \{x\}$ is closed. Since the poset of all

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convex geometries on S (ordered by set inclusion) is dual of the meet-semilattice of all the antiexchange closure operators on *S*, it is a join-semilattice isomorphic to the join-semilattice of all path-independent choice operators on *S*.

7. Conclusion

As said in our introduction order duality matters since as soon as two sets of objects are shown be two dual posets, then any result on one of these posets can be tranferred to the other. We still illustrate this point by a last (rather trivial) duality: the meet-semilattice of all asymmetric binary relations on a set *S* is dual of the join-semilattice of all complete relations on S (by the map $R \rightarrow R^{cd} = \{(x,y) : (y,x) \notin R\}$). Then to work, for instance in social choice, with asymmetric prefences is totally equivalent to work with complete preferences, a fact that is not universally well understood.

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