# Flexible Time Series Models for Subjective Distribution Estimation with Monetary Policy in View[*]

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**Abstract:** In this paper, we introduce a new approach to estimate the subjective distribution of the future short rate from the historical dynamics of futures, based on a model generated by a Normal Inverse Gaussian distribution, with dynamical parameters. The model displays time varying conditional volatility, skewness and kurtosis and provides a flexible framework to recover the conditional distribution of the future rates. For the estimation, we use maximum likelihood method. Then, we apply the model to Fed Fund futures and discuss its performance in recovering the expected stance of monetary policy in the US.

**Keywords:** Subjective Distribution, Autoregressive Conditional Density, Generalized Hyperbolic Distribution, Fed Funds futures contracts.

**JEL Codes:** C51, E44

# 1 Introduction

Risk neutral and subjective distributions are linked through the aggregated risk aversion that is implicit in financial markets. To detect the term structure of the subjective distribution, most of literature use the latent information in option prices while making assumption on the shape of risk aversion. We propose here to use a time series model for the historical distribution of the futures on interest rates in order to estimate the subjective distribution. This model is based on a GARCH volatility structure and is conditionally distributed as a Normal Inverse Gaussian distribution. The model thus encompasses time varying volatility, skewness and kurtosis. We estimate the model on a dataset made of several Fed Fund futures contracts and we discuss the empirical results obtained.

The subjective distribution is the distribution that reflects the market participants' perception of the future value of a financial asset. This distribution carries important information regarding the market participants' risk perception over states and maturities. Contrary to the subjective one, the risk neutral distribution[1] is supposed to make market participant neutral toward risk. This latter distribution should therefore carry no risk aversion component. Since these distributions are equivalent in the probabilistic sense, they are solely related through a risk aversion correction. Following Leland (1980), this relation can be roughly stated as follow:

$$\text{Risk Neutral Probability} \ = \ \text{Subjective Distribution} \ \times \ \text{Risk Aversion Adjustment}. \quad (1)$$

This relation is well-known for empirical finance applications: see the results presented in Ait-Sahalia and Lo (2000), Jackwerth (2000) and Rosenberg and Engle (2002). However, it has scarcely been used to analyze the link between monetary policy and the bond market. For most of the developed countries the Central Bank directly controls the short rate. In this perspective, the subjective distribution of the short rate process yields pieces of information on the market participants' perception of the future stance of monetary policy. Monetary policy makers need to measure the perception of the Central Bank policy by financial markets. For example, the information disclosure following the regular meetings of the Federal Reserve Board is known to be followed by a reduction of the bond market volatility. However, the financial market assessment of the monetary policy goes far beyond the sole changes in conditional variance: the estimation of the conditional distribution is thus essential to Central Bankers. On the market participants side, monitoring the changes in "market beliefs" is also important for asset management and risk control purposes: monetary policy is known to be the main risk factor in the government bond market. For all these reasons, the design of a model of the kind that is to be discussed here is one of the cornerstone of empirical

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[1]The risk neutral distribution is the distribution used to give a price to financial assets.

monetary policy.

Until now, several estimation strategies for the subjective distribution have been proposed. On the one hand, several articles proposed to use the risk neutral distribution estimated from option prices as a proxy for the subjective one: see Mandler (2002), Brière (2006) and the survey presented in Mandler (2003). On the other hand, using the relation presented in equation (1), Jackwerth (2000), Ait-Sahalia and Lo (1998) and Rosenberg and Engle (2002) used the historical distribution of index returns as a proxy for the subjective distribution. The conclusion of the last stream of literature is that the risk neutral and the subjective distribution proxy are very different.

However, these attempts were mainly designed for equity assets, and little attention has been devoted to fixed income securities. For such markets, the additional problem is the term structure aspect of the subjective distribution. We thus need a kind of financial asset whose historical dynamic carries information about this term structure. We propose to use the existing future contracts to do so: they are actively traded and used to make bets on the future stance of monetary policy. In order to have a reactive estimate of the subjective distribution, we use here a dynamical Normal Inverse Gaussian distribution, building a flexible time series model. This methodology allows for a direct mapping from the past observations space into the parameters' space, without any a priori knowledge about the dependence between them. The estimation of this model can be performed by maximum likelihood. When confronted to the data, this model is accepted. An event study shows that the results obtained with this model are close to what is expected.

The paper is organized as follow. Section 2 presents several theoretical and empirical arguments to use the historical dynamic of future rates for the estimation of the subjective distribution. Section 3 presents a new time series model using a dynamic NIG distribution for the estimations. Finally, the Section 4 discusses the empirical results. Section 5 concludes.

## 2   Methodology and framework

One of the main novelties of this paper is to show how to use the dynamics of the futures contract prices to estimate the subjective distribution of the futures instantaneous rate for various maturities. In this section we discuss the reasons why we propose such a strategy, both from an empirical and theoretical point of view. We first present the main hypothesis used in this article. Then we provide details regarding the Fed fund contracts and the datasets used in this paper. Finally, we present empirical facts

regarding the dataset and supporting the model presented in the next section.

## 2.1  Main assumption and notations

In this section, we review the main notations and hypothesis that are used in this paper. We provide essential insights within the theoretical link between monetary policy and the bond market, through future rates.

Let $R(t,T)$ be the spot rate of maturity $T$ at time $t$. The short or instantaneous spot rate $r_t$ on date $t$ is given by:

$$r_t = \lim_{T \to t} R(t,T). \tag{2}$$

When $t$ is different from today, this rate becomes the future instantaneous spot rate, that is thus unknown at time $t$ and assumed to be a random variable. We denote $F(t,T)$ the instantaneous future rate known at time $t$ for a maturity $T$ from the future contracts. Under no arbitrage restrictions, the link between spot rates and instantaneous forward rates is given by:

$$R(t,T) = \frac{\int_{\tau=t}^{T} F(t,\tau)d\tau}{T - t} \tag{3}$$

The spot yield curve being an average of the forward rates can be used to recover the instantaneous forward rates (using a spline model for example, see e.g. Svensson (1994)). Now, we know that[2]:

$$E^S[r_T|\mathcal{F}_t] = F(t,T), \tag{4}$$

where $E^S[.]$ denotes the expectation under the subjective distribution $S$, and $\mathcal{F}_t$ being the filtration produced by the information set at time $t$. We will define define this filtration in the next section: it will basically result from the past evolutions of the futures. This relation states that the forward rates can be considered as a market forecast of the future short rate. In many countries, this rate is directed by the Central Banks: in this perspective, the forward rates can be used to investigate the financial markets' understanding of monetary policy and the market perception of the risk associated to the upcoming Central Bankers meetings[3]. This link between the bond market and

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[2]This is one of the presentation of the expectation hypothesis. See e.g. Jarrow (2002) for more details.

[3]Recent papers nonetheless empirically showed that these forecasts may be biased: Piazzesi and Swanson (2004) showed that the difference between the realized target rate and the market forecast implicit in the yield curve is statistically significant and even linked to the economic momentum. This bias in the forecast is often referred to as *term premium*. Even though this premium can be very well explained *in sample*, the out of sample performances are really poor (and biased again). This is why we choose to discard this problem for the time being, considering that given the information on the current date $t$, the market forecast for the date $T > t$ is *conditionally* unbiased. We hope being able to circumvent in a more documented fashion this difficulty in a future paper.

monetary policy has already been used in e.g. Svensson (1994), in order to measure the market forecast over several central bank decision meetings.

One of the main novelty of this paper is to develop a time series model to recover the distribution of $r_T$ conditionally upon $\underline{F(t,T)} = \{F(t,T), F(t-1,T), ...\}$, for each maturity $T_i$ of interest (those of the different future contracts or those of the Central Bank decision meetings). Here, we assume that the conditional historical distribution of $F(t,T)$ is close enough to the conditional subjective distribution of $r_T | \mathcal{F}_t$, hence they can be treated as equal. The main hypothesis of our work is then:

**Hypothesis 1.** *Let $F_t^S(.)$ be the cumulative distribution function associated to the subjective distribution of $r_T$ conditionally upon the information available at time $t$. Let $F_t^H(.)$ be the cumulative distribution function associated to the historical distribution of the future rate $F(t,T)$, conditionally upon the information available at time $t$. Then, the two distributions are equal on any point of their common support, i.e. $F_t^S = F_t^H$.*

Most of the existing literature assumes that the historical distribution of equity returns is a consistent proxy for the subjective distribution: see e.g. Jackwerth (2000), Ait-Sahalia and Lo (2000) and Rosenberg and Engle (2002). This hypothesis is valid as long as the working purpose is to recover the short term subjective distribution for equity indexes. Here, we are interested in recovering the term structure of the subjective distribution from the yield curve: this is why we propose to use the futures instead of the short rate. We document this hypothesis in the following subsection.

## 2.2 Dataset description

In this section, we present the Fed fund futures contracts that are used in this article. We first review the motivations to use these future contracts, before we detail the building of our own dataset.

There exist several types of future contracts that may be used to recover market probabilities of future rates hikes and cuts. Most of them are three-months contracts, which is not convenient for monetary policy analysis: Central Bank meetings are closer to a monthly frequency. For this main reason, we focus on the the Federal Fund Future contracts, that are monthly contracts.

This contract has a price that is equal to

$$P(t,T) = 100 - 100 \times \frac{1}{n} \sum_{i=T-n+1}^{T} r_i, \tag{5}$$

that is 100 minus the one-month average of the future refinancing rate $r_i$ – that is the future target rate plus a daily cash premium – over a one month period (with $n$ days).

The Fed Fund futures are a useful and now quite liquid asset to extract monetary policy stance expectations from the bond market. It has already been widely used in empirical research: see e.g. Krueger and Kuttner (1996), Robertson and Thornton (1997), Kuttner (2000), and Carlson et al. (2003). These contracts are widely used by practitioners to measure the current market feeling about the future monetary policy stance. The extraction of market based monetary policy expectations requires some preliminary calculations that are detailed e.g. in Kuttner (2000). These are not reported here.

The global dataset used in this paper is made of 77 Fed Fund future contracts over their whole lifespan. The first contract's maturity is November 2000 and the final contract's is March 2007. The future prices time series are observed on a daily basis. The prices are converted into rates using the relationship mentioned before. This dataset includes rate hikes and cuts periods, which is important for the purpose of this paper, and especially for the examination of the information implicit in these future rates.

Nevertheless, for the ease of the presentation of the empirical results in Section 4, we chose to focus on 4 particular future contracts. These contracts have the following maturities: December 2006, January 2007, February 2007 and March 2007. We retained these contracts because of the period covered. Over the lifespan of these contracts, monetary policy started to change, with the end of a rate hikes period and the starting of an upcoming rate cuts period. This period should result in a dramatic change in the shape of the subjective distribution across the sample that is used here. At least, we should note a progressive change in the skew of the distribution over the period that is considered here: moving from a positive one – which is the sign of upcoming rate hikes – to a negative one, as financial markets will start believing in upcoming rate cuts.

This sub-sample starts on the $1^{st}$ of August 2006 and ends on the $30^{th}$ of October 2006. It is necessary to suppress at least the last two months of the lifespan of the future contracts for each of them in so far the level of uncertainty for these months is quickly decreasing: thus, during these last two months the volatility is likely to decrease quickly as expectations about the future stance of monetary policy are getting more and more accurate. This decreasing conditional volatility usually leads to diagnose non stationarity.

In the remaining of the paper, we generally refer to the first sample of 77 contracts as the "*full-sample*" and to the latter restricted sample as the "*sub-sample*". The full-sample is mainly used in this section and the sub-sample will be used for the empirical results presented in Section 4.

6

## 2.3   Stylized facts and informative content of the dataset

From a practical point of view, the future contracts are actively used by traders to take bets over the upcoming Central Bank decisions. On this point see the references presented in the previous subsection. Thus, their historical dynamics should reflect the changes in the market perception of the future monetary policy.

We propose to test this hypothesis by the following statistical analysis. Building on the previous notations, $r_{T_i}$ is the spot rate on date $T_i$, that should be forecast – accordingly to our assumption – by $F(t, T_i)$. If the dynamics of the corresponding future rate is linked to the realized rates[4], then it should be possible to relate the distribution – and thus the moments – of $F(t, T_i)$ to $r_{T_i}$. We denote

$$\Delta f(t, T_i) = \log \frac{F(t, T_i)}{F(t-1, T_i)}, \tag{6}$$

the log increment of the future rate, for a given maturity. This transformation ensures the second order stationarity of $\Delta f(t, T_i)$. Now, the descriptive statistics of $\Delta f(t, T_i)$ are denoted as $m_1(T_i)$ for the expectation, $m_2(T_i)$ for the volatility, $m_3(T_i)$ for the skewness and $m_4(T_i)$ for the kurtosis. Their estimation is done using the corresponding sample statistics.

We propose to check whether the historical distribution of the futures, proxied by the previous moments – expectation, volatility, skewness and kurtosis – can be explained by any combination of the realized corresponding spot rates. It would support the fact that the historical dynamic of the future rates carries information regarding the market view of the future realized rates.

We adopt the following notations: $L(T_i) = \log r_{T_i}$ is the log of the realized rate: it is meant to capture a level effect. $\Delta L(T_i) = L(T_i) - L(\tau_i)$ is the variation of the spot rate over the windows defined by the lifespan of the future contracts, i.e. the time elapsed between the issuance $\tau_i$ and the delivery $T_i$ dates. $|\Delta L(T_i)| = |L(T_i) - L(\tau_i)|$ represents the total variation over the duration of the future contract. Its absolute value accounts for the realized volatility over the future lifetime. Finally, $S(T_i) = |L(T_i) - \overline{L}|$ captures the effect of turning points of the rates' dynamics, when $L_T$ is way above or below its historical average $\overline{L}$. It is bound to have a particular effect regarding the historical dynamic of the futures, given the well documented mean reverting property of interest rates.

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[4]The subjective distribution is a forecasting density of $r_{T_i}$.

We adjust on the sample moment $m_j(.)$ the following linear model:

$$\hat{m}_j(T_i) = \alpha_0 + \alpha_1 L(T_i) + \alpha_2 \Delta L(T_i) + \alpha_3 |\Delta L(T_i)| + \alpha_4 S(T_i) + \epsilon(T_i), \forall i \qquad (7)$$

where $\epsilon(T_i)$ is a centered white noise and $\hat{m}_j$ the sample moment. The estimation has been performed on the full sample dataset, using ordinary least squares. The estimated $R^2$ are presented on figure 1. The $R^2$ obtained are very high for a model with 4 explanatory variables, indicating that the moments of the historical distribution of the futures are statistically linked to the realized rates. It seems now natural to relate this historical distribution to market expectations: the subjective distribution is by construction a forecasting density and should verify empirical properties of this kind.

Second, we briefly discuss the stylized facts of the futures dataset used in this subsection, for time series modeling purposes.

Financial datasets and especially financial returns are known to display time varying volatility and higher order moments. The log increments of future rates display the same characteristics. On the previous dataset, we computed the first four moments estimator for each contract. Figure 6 presents the results obtained: each of the moments dramatically vary across the contracts. Thus, we performed $T \times R^2$ tests for autoregressive moments of order 1 to 4 for each contract in the dataset (for this test, see Engle (1982) and for its application to higher order modeling see Jondeau et al. (2006)). Figure 6 present the test statistics along with the $\chi^2$ quantile for testing first order autoregressive patterns for each moment across the contracts: for most moments and contracts, there is an autoregressive component that is statistically significant. This supports the idea that the first four moments are time varying.

Finally, we propose here to test the adequacy of four distributions to the data at hand. These distributions are the skewed t-distribution, developed in Hansen (1994), the skewed Laplace distribution (see Kotz et al. (2001) and the references within), the Normal Inverse Gaussian distribution, introduced by Barndorff-Nielsen (1997) and the Gaussian distribution, used as a benchmark. Before testing the adequation of these distributions, the log-returns were first filtrated using a GARCH(1,1) model introduced in Bollerslev (1986), so as to take into account the possible second order dependency in the data at hand. We performed Kolmogoroff-Smirnoff tests on the GARCH residuals. The results obtained are presented in the table 1. The NIG distribution yields the best results, once compared to the other distributions, up to a 5% risk level. Thus, in the remaining of the paper, we will only retain the NIG distribution to model the distribution of the future rates. The adequation tests lead to accept this distribution for 95% of the samples: the regularity of the acceptation is a very interesting feature for our purposes. We want to find a distribution that is likely to fit any future sample. The

8

fact that this distribution has at least one more parameter that the other competitors may explain the quality of the fit. However, this is not a drawback for our approach: what we favor here is the flexibility of the distribution for estimation purposes. For all these reasons, we will only consider the NIG distribution for the models that are to be presented in the following section.

# 3 Flexible time series models

On the basis of the previous preliminary empirical work, we propose a series of NIG-based new time series models allowing for time varying volatility, skewness and kurtosis by letting the more sophisticated version of the model have dynamic parameters controlling the skewness and the kurtosis. We first present the general settings of the models and then discuss the estimation methodology.

## 3.1 General settings of the model

The Normal Inverse Gaussian distribution (NIG hereafter) has been introduced by Barndorff-Nielsen (1997) and used successfully in many financial applications (see e.g. Jensen and Lunde (2001)). This distribution accommodates basic stylized fact of financial time series such as asymmetry and excess kurtosis greater than 0. In this section, we build nested time series models based on this distribution to provide estimates of the subjective distribution.

A stochastic variable $X$ is said to be normal inverse Gaussian distributed – $X \sim NIG(\alpha, \beta, \delta, \mu)$ – if it has a probability density function of the following form:

$$f_X(x) = \frac{\alpha\delta}{\pi} \frac{\exp[p(x)]}{q(x)} K_1[\alpha q(x)], \tag{8}$$

with $K_1(x)$ the modified Bessel function of the second kind, with index 1; $p(x) = \delta\sqrt{\alpha^2 - \beta^2} + \beta(x - \mu)$, $q(x) = \sqrt{(x-\mu)^2 + \delta^2}$ with $\alpha > 0$, $|\beta| < \alpha$ and $\delta > 0$. The expectation, variance, skewness and kurtosis of $X$ have a closed form expression and are:

$$E[X] = \mu - \frac{\delta\beta}{\sqrt{\alpha^2 - \beta^2}} \qquad V[X] = \frac{\delta\alpha^2}{\sqrt{\alpha^2 - \beta^2}} \tag{9}$$

$$S_k[X] = \frac{3\beta}{\alpha} \frac{1}{\sqrt{\delta\sqrt{\alpha^2 - \beta^2}}} \qquad K_u[X] = 3\left(1 + 4\left(\frac{\beta}{\alpha}\right)^2\right)\left(\frac{1}{\delta\sqrt{\alpha^2 - \beta^2}}\right). \tag{10}$$

The characteristic function of $X$ is given by:

$$\phi(\omega) = \exp\left\{i\omega\mu + \delta\left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + i\omega)^2}\right)\right\}, \tag{11}$$

and the moment generating function of $X$ is:

$$\psi(\omega) = \exp\left\{\omega\mu + \delta\left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + \omega)^2}\right)\right\}. \tag{12}$$

The Laplace transform is defined whenever $\omega \in ]\beta - \alpha : \beta + \alpha[$.

Here, we use this distribution to model the conditional distribution of the log-increments of the future rates $\Delta f(t, T)$ introduced in equation (6). It is noteworthy that by doing so we have positive rates at any time with probability 1, which is an essential feature of any interest rates model.

In the following, we work with a fixed maturity, thus we use the simplified notation $\Delta f_t$, henceforth. Now, the general settings are:

$$\Delta f_t = \sigma_t \epsilon_t,$$

and we consider the following models:

Model 1: $\sigma_t = \sigma$ and $\epsilon_t|\sigma_t \sim NIG(\alpha, \beta, \delta, \mu)$.

Model 2: $\sigma_t = \sqrt{\omega_0 + \omega_1\sigma_{t-1}^2 + \omega_2\Delta f_{t-1}^2}$ and $\epsilon_t|\sigma_t \sim NIG(\alpha, \beta, \delta, \mu)$

Model 3: $\sigma_t = \sqrt{\omega_0 + \omega_1\sigma_{t-1}^2 + \omega_2\Delta f_{t-1}^2}$ and $\epsilon_t|\sigma_t \sim NIG(\alpha_t, \beta_t, \delta, \mu)$, with:

$$\alpha_t = \kappa_0 + \kappa_1\alpha_{t-1} + \kappa_2\exp\{\chi\epsilon_{t-1}\}$$
$$\beta_t = \gamma_0 + \gamma_1\beta_{t-1} + \gamma_2\exp\{\chi\epsilon_{t-1}\}$$

Model 4: $\sigma_t = \sqrt{\omega_0 + \omega_1\sigma_{t-1}^2 + \omega_2\Delta f_{t-1}^2}$ and $\epsilon_t|\sigma_t \sim NIG(\alpha_t, \beta_t, \delta, \mu)$, with:

$$\alpha_t = \kappa_0 + \kappa_1\alpha_{t-1} + \kappa_2\exp\{\chi_1\epsilon_{t-1}\}$$
$$\beta_t = \gamma_0 + \gamma_1\beta_{t-1} + \gamma_2\exp\{\chi_2\epsilon_{t-1}\},$$

where $\epsilon_t|\sigma_t$ means that $\epsilon_t$ is taken conditionally upon $\sigma_t$. $\epsilon_t$ conditionally on $\underline{\epsilon_{t-1}} = \{\epsilon_{t-1}, \epsilon_{t-2}, ...\}$ is assumed to be a second order stationary process. These models are thus nested, displaying richer and richer dynamics. Model 1 is an homoscedastic NIG model. Model 2 is GARCH(1,1) model with NIG innovations. Model 3 has a GARCH(1,1) variance dynamics and innovations with varying coefficients with the same parameter $\chi$ relating the dataset to the parameters' space. Finally, model 4 has $\chi_1 \neq \chi_2$. In both these models, the parameters of the innovations work conditionally upon $\sigma_t$. This way, we are able to focus the modeling work on the third and fourth moments, given that the $\alpha$ parameter of the NIG controls the kurtosis and $\beta$ the skewness.

In an approach with dynamic parameters, the main problem is to choose which power (or functional form) of $\underline{\epsilon_{t-1}}$ is related to each parameter. In model 3 and 4, we propose

to relate the parameters to the conditional spectral moments of $\epsilon_{t-1}$, i.e. $\exp\{\chi.\epsilon_{t-1}\}$. This allows a direct mapping of the past information $\underline{\epsilon_{t-1}}$ into the parameters space and requires no a priori knowledge on the proper moment to match. What is more, it may highlight the fact that each parameter of the distribution that is modeled may depend on a different spectral moment, when testing between model 3 and 4 whether it is possible to impose $\chi_1 = \chi_2 = \chi$. In the NIG case, this dependence of the parameters on different values can be documented. For example, Hanssen and Oigard (2001) showed how the NIG parameters are differently related to the cumulant generating function – and thus to the moment generating function. It should be possible to make this dependency over past observation richer, relating $\alpha_t$ and $\beta_t$ to $\epsilon_{t-1}$ and $\epsilon_{t-2}$ for example. Due to numerical complexity and to the number of parameters to estimate, we discarded this possibility. Our work is far from the approach used by Jondeau and Rockinger (2003). They propose several ways to map $\underline{\epsilon_{t-1}}$ into the parameter space for the Skewed student t distribution of Hansen (1994) in a GARCH framework again, using several moments or truncated moments; but their approach requires to have preliminary intuitions about the way to truncate series or about the power of the series to consider, which is hardly the case with NIG distribution.

Let us quickly discuss stationarity conditions. Model 1 and 2 yield second order stationary processes whenever the usual conditions on the existence and stationarity of NIG and GARCH processes are fulfilled. In particular, we need to impose that $\omega_0, \omega_1, \omega_2$ are positive and that $|\omega_1 + \omega_2| < 1$. The conditions for model 3 and 4 are derived using the conditional moments of order 1 and 2 of each dynamic parameter. We thus need to impose $|E[\beta_t]| < E[\alpha_t]$. $E[\beta]$ does not need to be positive. More precisely[5], for model 3, the conditions are: $|\omega_1 + \omega_2| < 1$, $\omega_0, \omega_1, \omega_2 > 0$, $E[e^{\chi\epsilon_{t-1}}]$ exists[6], $\kappa_1 \neq 1$, $\gamma_1 \neq 1$, $sgn(\kappa_0 + \kappa_2\psi(\chi)) = sgn(1 - \kappa_1)$ and $|E[\beta_t]| < E[\alpha_t]$. Similar conditions can be obtained for model 4: $|\omega_1 + \omega_2| < 1$, $\omega_0, \omega_1, \omega_2 > 0$, $E[e^{\chi_1\epsilon_{t-1}}]$ and $E[e^{\chi_2\epsilon_{t-1}}]$ exists, $\kappa_1 \neq 1$, $\gamma_1 \neq 1$, $sgn(\kappa_0 + \kappa_2\psi(\chi_1)) = sgn(1 - \kappa_1)$, and $|E[\beta_t]| < E[\alpha_t]$.

These conditions are obtained when computing the unconditional expectation of $\alpha_t$ and $\beta_t$. We chose to let $\kappa_0, \kappa_1, \kappa_2, \gamma_0, \gamma_1, \gamma_2$ to be either positive or negative, provided that the above constraints are satisfied, in so far as it is hard to have a precise idea of the domain of definition of these parameters. Constraining these parameters to be positive may be of no harm for the final results, but some of the estimated parameters may be negative without threatening both the conditional and unconditional existence of the process.

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[5]$sgn(x)$ is the sign function: -1 if $x < 0$ and 1 if not.

[6]The conditions of existence for the moment generating function can be found in e.g. Hanssen and Oigard (2001).

## 3.2 Estimation methodology

In this section, we present the estimation methodology used to fit the model on a real dataset. We propose to use a sequential maximum likelihood approach to estimate the parameters of the dynamic models presented in the previous section. We first consistently estimate the GARCH parameters using a pseudo maximum likelihood approach, assuming that the innovations are Gaussian. Then, once the GARCH parameters have been estimated, we estimate the remaining parameters using the conditional maximum likelihood method, using the filtrated GARCH innovations estimated before. The likelihood function is obtained by assuming that the residuals are NIG distributed. The adequation tests performed in Section 2 clearly point toward the use of this distribution.

The conditional log likelihood function associated to the NIG distribution is:

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\begin{aligned}
\ln L =& (T-1)\ln\delta + \sum_{t=2}^{T}\ln\alpha_t + \sum_{t=1}^{T}\delta\sqrt{\alpha_t^2 - \beta_t^2} + \sum_{t=2}^{T}K_1\left(\alpha_t\sqrt{\delta^2 + (\epsilon_t - \mu)^2}\right) \\
&+ \sum_{t=2}^{T}\beta_t\left(\frac{\epsilon_t}{\sigma_t} - \mu\right) - (T-1)log(\pi) - \frac{1}{2}\sum_{t=2}^{T}\ln\left(\delta\sigma_t^2 + (\epsilon_t - \mu\sigma_t)^2\right),
\end{aligned}
$$

with $K_1(.)$ being the previously mentioned Bessel function of the second kind with index 1 and $\pi$ being approximatively equal to 3.14159. We assume that the usual regularity conditions for the maximum likelihood method to be both consistent and asymptotically Gaussian are verified. The latter expression is maximized using a simulated annealing method, as presented in Belisle (1992), given the possible existence of several local optima.

With models 3 and 4, the derivatives of the log likelihood with respect to the parameters are rather involved, because of the Bessel function. We thus propose to estimate the estimates' variance covariance matrix $\Sigma$ using bootstrap. This matrix will be estimated using the block bootstrap approach. This bootstrap procedure is described in Efron and Tibshirani (1993).

## 4 Results

This section is devoted to the analysis of the empirical results: first, we discuss the bulk results: the significativity of the estimates and the log-likelihood ratio tests that were performed for model selection purposes. Then, we present an event study so as to test the ability of the model to capture known features of the subjective distribution.

## 4.1 Main estimation results

The estimation results are presented in table 2 for model 1, in table 3 for model 2, in table 4 for model 3 and in table 5 for model 4. Most of the parameters estimated are significant up to a 5% risk level. The parameters relating $\alpha_t$ and $\beta_t$ to past observations are significant, either for models 3 and 4. This is a corner stone for any model with time varying parameters of the kind that is presented here: should it be 0, the parameters $\alpha_t$ and $\beta_t$ would be degenerated.

The tables 6 and 7 present Likelihood ratio tests, for model selection purposes. We briefly recall the methodology: for example, let model 3 be the constrained model, with log likelihood denoted $lnL_c$ and model 4 be the unconstrained model, with a log-likelihood denoted $lnL_u$. The null hypothesis $\mathbb{H}_0$ $\chi_1 = \chi_2$ assumes that the constraint in model 3 can statistically be imposed. The test statistics is then:

$$LR = 2(lnL_c - lnL_u), \tag{13}$$

with the previous notations. Under the null hypothesis, this statistic has a Chi-square distribution, with a degree of freedom equal to the number of constraints imposed in the constraint model. For model 3 vs. model 4, there is only one constraint: $\chi_1 = \chi_2(= \chi)$. For model 2 vs. model 3, there are 5 constraints: $\kappa_1 = \kappa_2 = \gamma_1 = \gamma_2 = \xi = 0$. Globally, model 4 is always the one that is favored by the test, bringing support to our approach. As expected, the parameters driving $\alpha_t$ and $\beta_t$ are not linked to same spectral moments, in so far as $\chi_1$ is statistically different from $\chi_2$. What is more, the spectral moment to which they are related is close to 0. This finding is not totally surprising: around 0, the derivatives of the moment generating function are known to deliver information regarding any moments of the distribution.

Like when selecting the distribution, we favor again the model that contains the greater number of parameters. This is however not a real problem since we are not interested in the forecasting ability of the model. Our problem is basically a filtration problem: the conditional distribution of the log increments of the futures is unobservable and we propose to estimate it from the time series at hand. Thus, our concern is the in-sample quality of the fit. In this perspective, the result that the more parameter we have, the better the obtained fit is rather intuitive. However, we cannot increase the number of parameters at will for the numerical feasibility of the estimation strategy: this is why we do not propose a more complex model.

Finally, we propose to use the fourth model to perform an event study so as to assess the global ability of the model to capture key facts about the market perception of monetary policy and back the aforementioned empirical results.

## 4.2 Event study

We propose here to observe the changes in probabilities for the maturities available in the sub-sample dataset during two particular Central Bank decision meetings. These meetings are the ones that occurred on January, $31^{st}$ 2006 and August, $8^{th}$ 2006. During these days, the Fed announced the new target rate applying until the next Central Bank meeting, along with an economic justification of this decision. In the meantime, Mr. Bernanke delivered speeches concerned with the future evolution of the macroeconomic figures and thus of the stance of the future American monetary policy. These two decision meetings took place in an interesting period, that matches the end of a tightening period, i.e. of a rising target rate period. During the meeting of January, no rate cut were forecast by the bond market, and progressively, over the year, a rate cut forecast progressively appeared. It is very important to assess the ability of our model to capture these changes in the market perception of monetary policy.

The change in the Fed's target rate process is very well documented (see Goodfriend (1990) for a detailed analysis stylized facts related to the Fed target rate). The target rate is made of multiple of 25 basis points, i.e. 25/100 percents, and is thus a discrete process. The discreteness of the process is an important feature of monetary policy (see e.g. Guégan and Ielpo (2006) and the references within). We propose here to compute the estimated probabilities associated to these discrete outcomes implied by the continuous and time varying distributions. The computation of these discrete probabilities is made by integrating numerically the continuous distribution over 25 bps sets, using the conditional parameters estimated with model 4. Thus if $R_T$ is the future target rate for the maturity date $T$, then its support $S_T$ is such that:

$$S_T = \{0.25, 0.5, \ldots, 4, 4.25, \ldots\}. \tag{14}$$

Thus $P(R_T = k)$, with $k \in S_T$ can be computed as:

$$P(R_T = k) = \int_{k-0.125}^{k+0.125} \hat{f}(x)dx, \tag{15}$$

were $\hat{f}(.)$ is the subjective distribution density estimated though the historical distribution of the futures rates of maturity $T$, following our hypothesis 1. Given that the NIG cumulative distribution function has no closed form expression, the expression in equation (15) must be numerically computed (we used trapezoidal integration).

The probabilities for these different meetings are presented in figures 4 and 5. Each figure presents the implicit discrete probabilities before and after the meetings, along with the changes in the probabilities over these decision meetings. Several observations can be made on the basis of this event study:

14

– First, it appears that the uncertainty is reduced by the meetings, which is a very classical result of the literature dedicated to the impact of "Central Bank transparency". On this point see e.g. Brière (2006) and the references within. The announcement of the Central Bank decision, along with the chairman's speech brings about a reduction of the volatility in the market. This can be observed by noting that some events close to the mode of the distribution are getting higher probabilities whereas others have lower ones, after the meeting. The subjective distribution is thus getting more concentrated around its expectation. This is true for the two decision meetings that are observed.

– Second, over the two meetings, the skew of the distributions changed progressively. In January, the skew of the subjective distributions is positive, underlining the fact that the market is still believing in the increase of the Central Bank target rate over the futures' maturities that are considered. On the contrary, on the August decision meeting, the skew turned out to be negative. At this time, the Central Bank communication made the market believe that no more rate increases were to be feared. Rate cuts were now forecast by the market, underlining the change in economic conditions in the US economy: a slowing economic growth and a slowing inflation.

Beyond this event study, an economic interpretation of several parameters is also possible. First, the parameter controlling the level of persistence of the $\alpha_t$ process (that is $\kappa_1$) is higher than that of the $\beta_t$ (that is $\gamma_1$) process over every sample: the kurtosis is a more persistent process than the skewness one. More, for both $\alpha_t$ and $\beta_t$ the estimated value of $\kappa_1$ and $\gamma_1$ are remarkably stable across samples. The signs associated to the estimated $\gamma_0$ and $\kappa_0$ yield interesting pieces of information: $\kappa_0$ (resp. $\gamma_0$) is a function of the unconditional expectation of $\alpha_t$ (resp. $\beta_t$). Over each dataset, $\kappa_0$ remains positive and it is increasing from December to March, revealing a growing level of uncertainty in the financial market perception of the upcoming Fed's decisions: $\alpha_t$ is explicitly related to the thickness of the tails of the subjective distribution and thus to the probability of appearance of extreme values, that is extreme monetary policy decisions. On the contrary, $\gamma_0$ changes sign when comparing the December-January estimations to the February-March period, moving from a positive sign to a negative one. This is consistent with the change in the skewness sign remarked during the event study: over months, upcoming rate cuts seemed to be more and more likely to happen. These results are globally confirmed by the observation of the time varying conditional moments obtained with model 4. The computation of these moments can be made using the moments closed form expressions previously mentioned. We present it on figure 3: the red mark signals the two Central Bank decision meetings studied earlier. The previous observations still holds when observing the figures: the evolution of the skewness changes after the August meeting, confirming what has been presented ear-

15

lier. Thus, by looking at these results, the approach developed here seems to be likely to provide an accurate measure of the financial markets' view of the future stance of monetary policy. A tool of this kind should be useful to both monetary policy makers and to market analysts.

# 5    Conclusion

In this paper, we developed a time series model to infer the subjective distribution from the time series of the fed fund future contracts. To do so, we use the link that we introduced between the dynamic parameters and the moment generating function. The empirical results obtained confirm the interest of this approach. What is more, the assumption that the historical distribution of futures should be close to the subjective distribution of the future short rate is empirically realistic and the model proposed presents empirical performances that validate it. This assumption can thus be used for more general study around the estimation of the subjective distribution, and even extended to the econometrics of stochastic discount factors, given that most of the futures are backed to American options.

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# 6 Appendices

|  | Skewed Student | Skewed Laplace | NIG | Gaussian |
|---|---|---|---|---|
| # of test acceptation | 39 | 65 | 73 | 36 |
| # of test rejection | 38 | 12 | 4 | 41 |
| Total | 77 | 77 | 77 | 77 |

Table 1: **P-value associated to a Kolmogorov Smirnoff test.**
The Kolmogorov Smirnoff tests were performed over the Fed fund future contracts for the Skewed Student, the Skewed Laplace, the NIG and the Normal distributions, for Fed fund future log daily increments with maturities ranging from November 2000 until June 2007.

| Contracts | $\alpha$ | $\beta$ | $\delta$ | $\mu$ |
|---|---|---|---|---|
| **December 2006** | 81.99929 | -24.2365 | 0.00238 | 0.00025 |
| *(St. Dev.)* | (8.33558) | (11.43168) | (0.00107) | (0.00017) |
| **January 2007** | 80.81522 | -23.80958 | 0.00243 | 0.00023 |
| *(St. Dev.)* | (5.32102) | (13.59215) | (0.00109) | (0.00024) |
| **February 2007** | 82.61353 | -27.50026 | 0.00294 | 0.00046 |
| *(St. Dev.)* | (4.039) | (7.34675) | (0.00126) | (0.00032) |
| **March 2007** | 82.51478 | -25.77417 | 0.00339 | 0,00040 |
| *(St. Dev.)* | (3.96988) | (8.94021) | (0.00135) | -0,00030 |

Table 2: **Subjective distribution: Estimation Result for model 1.**
The model 1 is defined by $\sigma_t = \sigma$ and $\epsilon_t|\sigma_t \sim NIG(\alpha, \beta, \delta, \mu)$. The standard deviation of the estimates are reported between brackets under the estimates. Standard deviations have been computed by block bootstrap.

| Contracts | $\alpha$ | $\beta$ | $\delta$ | $\mu$ |
|---|---|---|---|---|
| **December 2006** | 19.72028 | -1.28814 | 0.00196 | 0.00014 |
| *(St. Dev.)* | (5.42572) | (2.41243) | (0.00073) | (0.00013) |
| **January 2007** | 19.63776 | -2.47544 | 0.00214 | 0.00013 |
| *(St. Dev.)* | (2.30486) | (1.53736) | (0.00077) | (0.00012) |
| **February 2007** | 25.95418 | -2.30276 | 0.00267 | 0.00022 |
| *(St. Dev.)* | (17.8399) | (2.95125) | (0.00094) | (0.00021) |
| **March 2007** | 28.02034 | -2.18745 | 0.00281 | 9,00E-05 |
| *(St. Dev.)* | (15.72817) | (2.47819) | (0.00118) | (0.00026) |

Table 3: **Subjective distribution: Estimation Result for model 2.**

The model 2 is defined by $\sigma_t = \sqrt{\omega_0 + \omega_1\sigma_{t-1}^2 + \omega_2\Delta f_{t-1}^2}$ and $\epsilon_t|\sigma_t \sim NIG(\alpha, \beta, \delta, \mu)$. The standard deviation of the estimates are reported between brackets under the estimates. Standard deviations have been computed by block bootstrap.

| Contracts | $\kappa_0$ | $\kappa_1$ | $\kappa_2$ | $\gamma_0$ | $\gamma_1$ | $\gamma_2$ | $\delta$ | $\mu$ | $\chi$ |
|---|---|---|---|---|---|---|---|---|---|
| **December 2006** | 0.71189 | 0.68914 | 0.71187 | -0.08465 | -0.09915 | -0.08472 | 0.00202 | 0.00011 | 0.04968 |
| *(St. Dev.)* | (0.32016) | (0.24499) | (0.32013) | (0.07901) | (0.01465) | (0.07896) | (0.00079) | (0.00011) | (0.00111) |
| **January 2007** | 0.70322 | 0.64884 | 0.70319 | -0.08771 | -0.09917 | -0.08781 | 0.00206 | 9,00E-05 | 0.04927 |
| *(St. Dev.)* | (0.54862) | (0.24375) | (0.54854) | (0.06636) | (0.00915) | (0.06619) | (0.00082) | (0.00013) | (0.00428) |
| **February 2007** | 0.60165 | 0.58186 | 0.60154 | -0.11875 | -0.09543 | -0.11884 | 0.00235 | 0.00016 | 0.04702 |
| *(St. Dev.)* | (0.30081) | (0.27026) | (0.30055) | (0.2056) | (0.04276) | (0.20608) | (0.00096) | (2.00E-04) | (0.02441) |
| **March 2007** | 0.83497 | 0.7271 | 0.835 | -0.12828 | -0.06981 | -0.12823 | 0.0028 | 8,00E-05 | 0.05253 |
| *(St. Dev.)* | (0.46218) | (0.22909) | (0.46254) | (0.19541) | (0.16773) | (0.19513) | (0.00113) | (0.00028) | (0.03069) |

Table 4: **Subjective distribution: Estimation Result for model 3.**

The model 3 is defined by $\sigma_t = \sqrt{\omega_0 + \omega_1 \sigma_{t-1}^2 + \omega_2 \Delta f_{t-1}^2}$ and $\epsilon_t | \sigma_t \sim NIG(\alpha_t, \beta_t, \delta, \mu)$, with:

$$\alpha_t = \kappa_0 + \kappa_1 \alpha_{t-1} + \kappa_2 \exp\{\chi \epsilon_{t-1}\}$$
$$\beta_t = \gamma_0 + \gamma_1 \beta_{t-1} + \gamma_2 \exp\{\chi \epsilon_{t-1}\}.$$

The standard deviation of the estimates are reported between brackets under the estimates. Standard deviations have been computed by block bootstrap.

| Contracts | $\kappa_0$ | $\kappa_1$ | $\kappa_2$ | $\gamma_0$ | $\gamma_1$ | $\gamma_2$ | $\delta$ | $\mu$ | $\chi_1$ | $\chi_2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| **December 2006** | 0.67249 | 0.64738 | 0.67247 | 0.02928 | 0.08713 | 0.02922 | 0.00195 | 0.00013 | 0.04963 | 0.04995 |
| *(St. Dev.)* | (0.32288) | (0.2438) | (0.32285) | (0.04142) | (0.00798) | (0.04146) | (0.00074) | (9,00E-05) | (0.00099) | (5,00E-05) |
| **January 2007** | 0.7397 | 0.68358 | 0.73969 | 0.03475 | 0.0818 | 0.03463 | 0.00217 | 0.00014 | 0.04976 | 0.04988 |
| *(St. Dev.)* | (0.485) | (0.29177) | (0.48499) | (0.12857) | (0.03975) | (0.12839) | (0.00076) | (0.00013) | (0.00069) | (4,00E-04) |
| **February 2007** | 0.81814 | 0.81833 | 0.81811 | -0.04199 | 0.09453 | -0.04204 | 0.00273 | 0.00017 | 0.04957 | 0.05001 |
| *(St. Dev.)* | (0.38033) | (0.14075) | (0.38032) | (0.14463) | (0.09314) | (0.14464) | (0.00094) | (0.00022) | (0.001) | (6,00E-05) |
| **March 2007** | 0.79315 | 0.696 | 0.79314 | -0.0342 | 0.09559 | -0.03421 | 0.00287 | 9,00E-05 | 0.04989 | 0.05014 |
| *(St. Dev.)* | (0.48094) | (0.27547) | (0.48095) | (0.18922) | (0.09249) | (0.18928) | (0.00116) | (0.00029) | (0.00207) | (0.00114) |

Table 5: **Subjective distribution: Estimation Result for model 4.**

The model 4 is defined by $\sigma_t = \sqrt{\omega_0 + \omega_1 \sigma_{t-1}^2 + \omega_2 \Delta f_{t-1}^2}$ and $\epsilon_t | \sigma_t \sim NIG(\alpha_t, \beta_t, \delta, \mu)$, with:

$$\alpha_t = \kappa_0 + \kappa_1 \alpha_{t-1} + \kappa_2 \exp\{\chi_1 \epsilon_{t-1}\}$$
$$\beta_t = \gamma_0 + \gamma_1 \beta_{t-1} + \gamma_2 \exp\{\chi_2 \epsilon_{t-1}\}.$$

The standard deviation of the estimates are reported between brackets under the estimates. Standard deviations have been computed by block bootstrap.

| Contracts | LL - model 2 | LL - model 3 | T stat | $\chi^2$ quantile | P-value |
|---|---|---|---|---|---|
| December 2006 | -1406,6721 | -399,4014 | 2014,5413 | 11,0705 | 0,0000 |
| January 2007 | -2121,7703 | -563,9566 | 3115,6274 | 11,0705 | 0,0000 |
| February 2007 | -2939,1257 | -590,9558 | 4696,3399 | 11,0705 | 0,0000 |
| March 2007 | -3648,5262 | -1039,1681 | 5218,7162 | 11,0705 | 0,0000 |

Table 6: **Likelihood Ratio tests for model 2 vs. model 3.**

| Contracts | LL - model 3 | LL - model 4 | T stat | $\chi^2$ quanti le | P-value |
|---|---|---|---|---|---|
| December 2006 | -399,40145 | -353,67602 | 91,45086 | 3,84146 | 0,00000 |
| January 2007 | -630,25184 | -563,95661 | 132,59046 | 3,84146 | 0,00000 |
| February 2007 | -1181,20967 | -590,95577 | 1180,50780 | 3,84146 | 0,00000 |
| March 2007 | -1039,16812 | -934,45203 | 209,43219 | 3,84146 | 0,00000 |

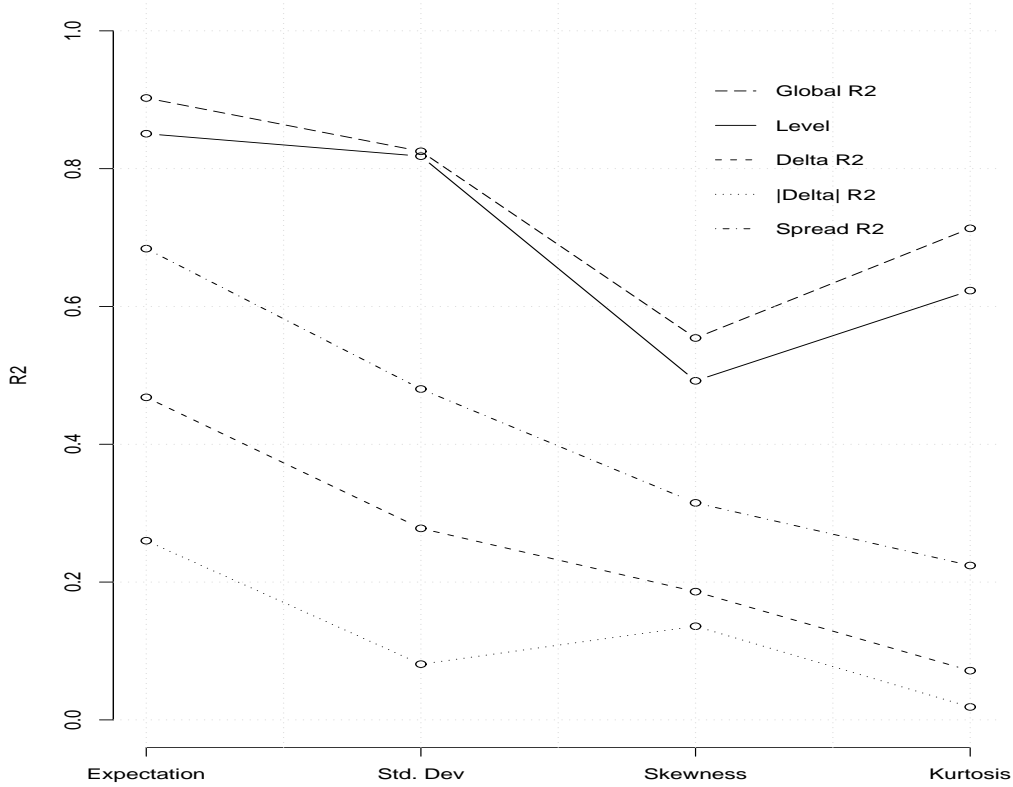Table 7: **Likelihood ratio tests for model 3 vs. model 4.**

22

Figure 1: $R^2$ **study for Fed fund futures contracts, using 77 contracts of maturities ranging from November 2000 until March 2007.**

The $R^2$ is obtained when estimating the linear model defined by equation (7) by Ordinary Least Squares with the following explanatory variables:

$$L(T_i) = \log \widehat{R}(T_i) \tag{16}$$

$$\Delta L(T_i) = L(T_i) - L(\tau_i) \tag{17}$$

$$|\Delta L(T_i)| = |L(T_i) - L(\tau_i)| \tag{18}$$

$$S(T_i) = \left| L(T_i) - \overline{L} \right|. \tag{19}$$

Global R2 presents the $R^2$ obtained when using the four variables defined in equations (16), (17), (18) and (19). Level presents the $R^2$ obtained when using only the explanatory variable presented in (16). Delta R2 presents the $R^2$ obtained when using only the explanatory variable presented in (17). |Delta R2| presents the $R^2$ obtained when using only the explanatory variable presented in (18). Spread R2 presents the $R^2$ obtained when using only the explanatory variable presented in (19).
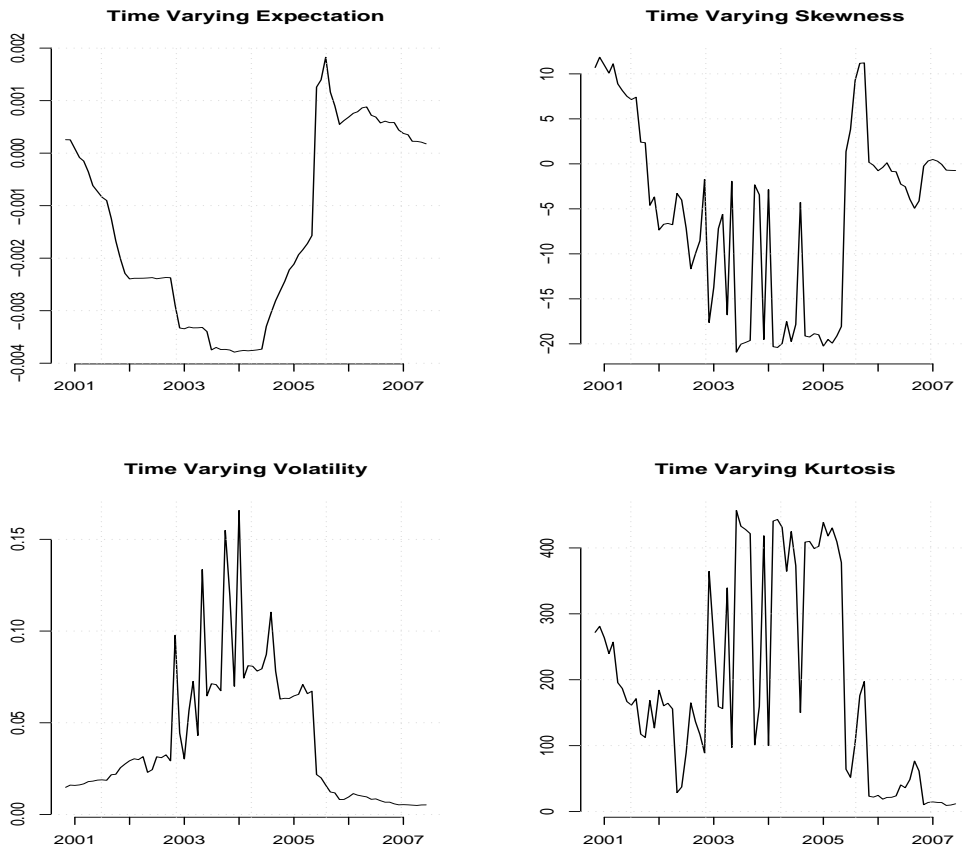
23

Figure 2: **Average (annualized), volatility (annualized), skewness and kurtosis of the log-returns of full sample.**

The full sample includes 77 Fed fund future contracts, with maturities ranging from November 2000 until March 2007. The descriptive statistics are estimated using the sample moments.
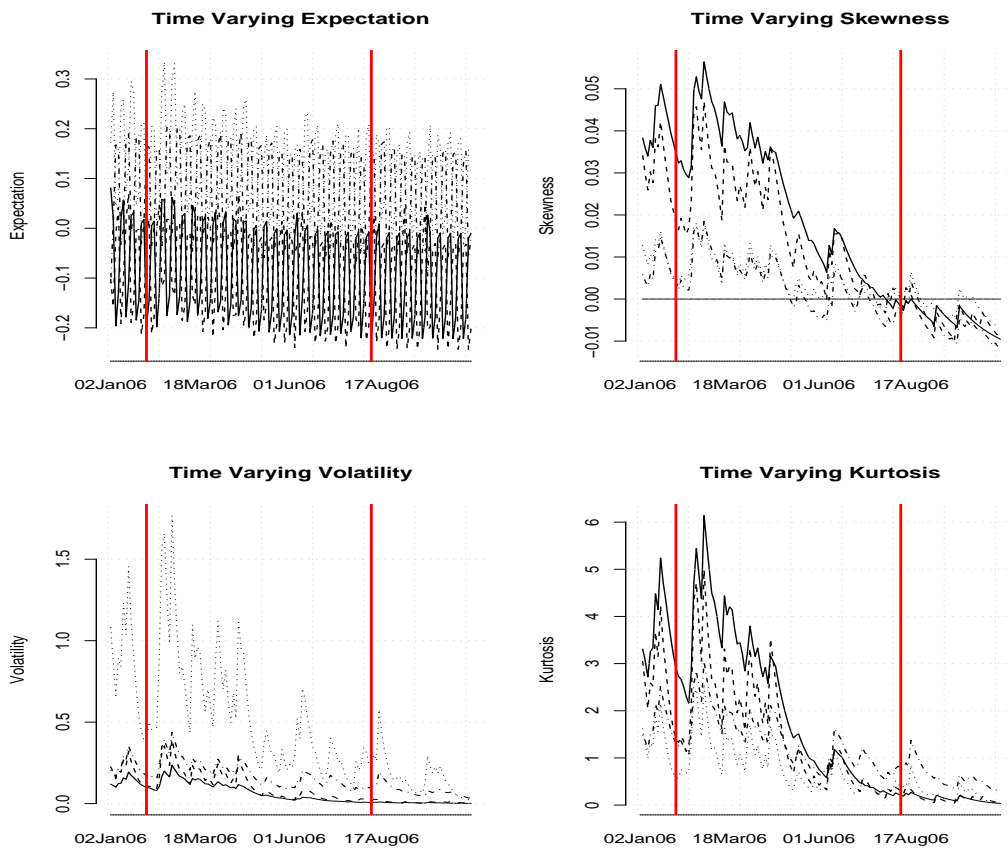
**Time Varying Expectation**   **Time Varying Skewness**

**Time Varying Volatility**   **Time Varying Kurtosis**

Figure 3: **Term structure of the time varying moments estimated using model 4.**

The Fed fund futures contracts maturities are: December 2006 (plain line), January 2007 (dashed line), February 2007 (dashed and dotted line) and March 2007 (dotted line), with quotations from January, $2^{nd}$ 2006 until October, $30^{th}$ 2006. The red lines indicates the Central Bank decision meetings of January, $31^{st}$ 2006 and of August, $8^{th}$ 2006.
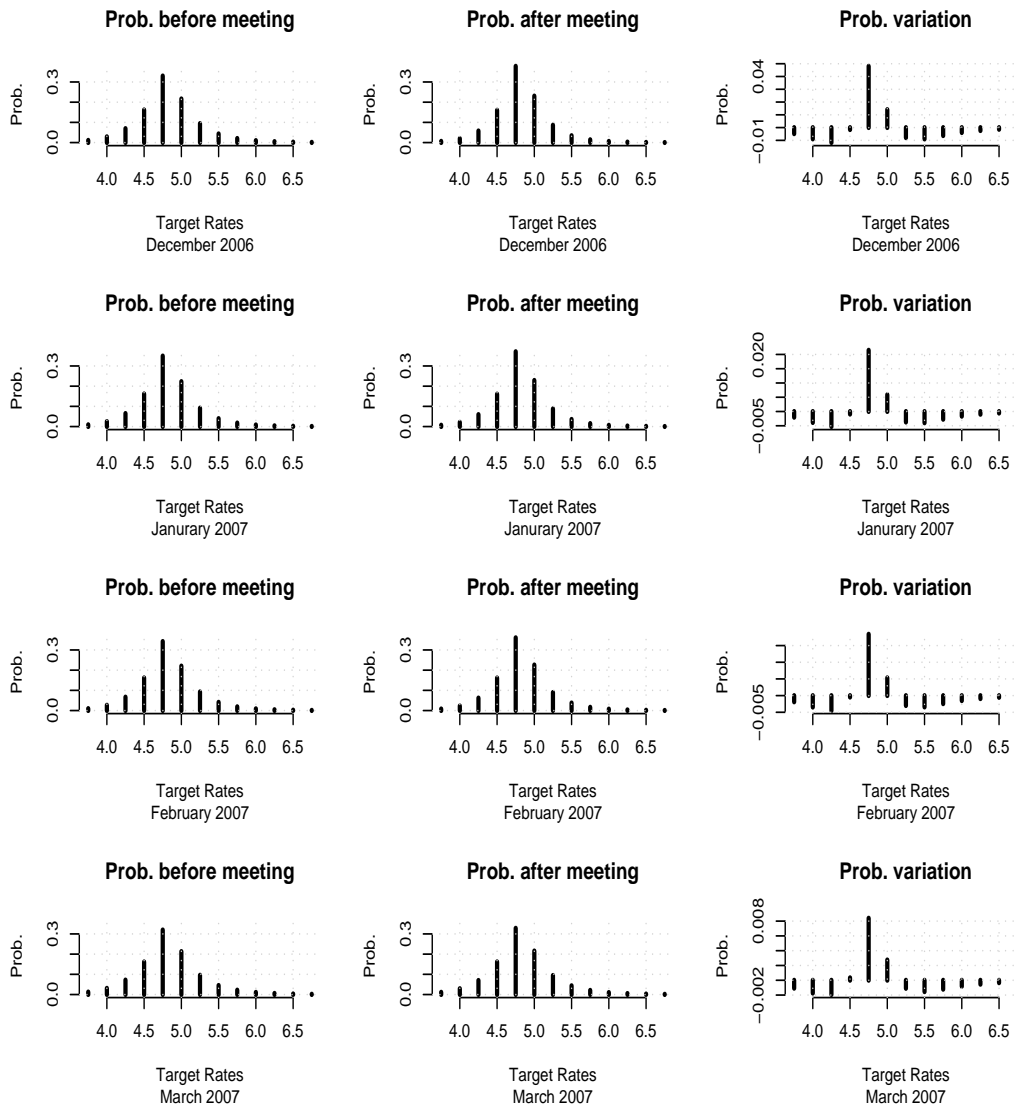
Figure 4: **Implicit monetary policy scenario probabilities around January, $31^{st}$ 2006.**

The probabilities are computed before (left) and after (center) the Central Bank meeting on the January, $31^{st}$ 2006, estimated using model 4. The figure on the right presents the variation of the probabilities over the event. These probabilities are presented for the following maturities: December 2006, January, February and March 2007.
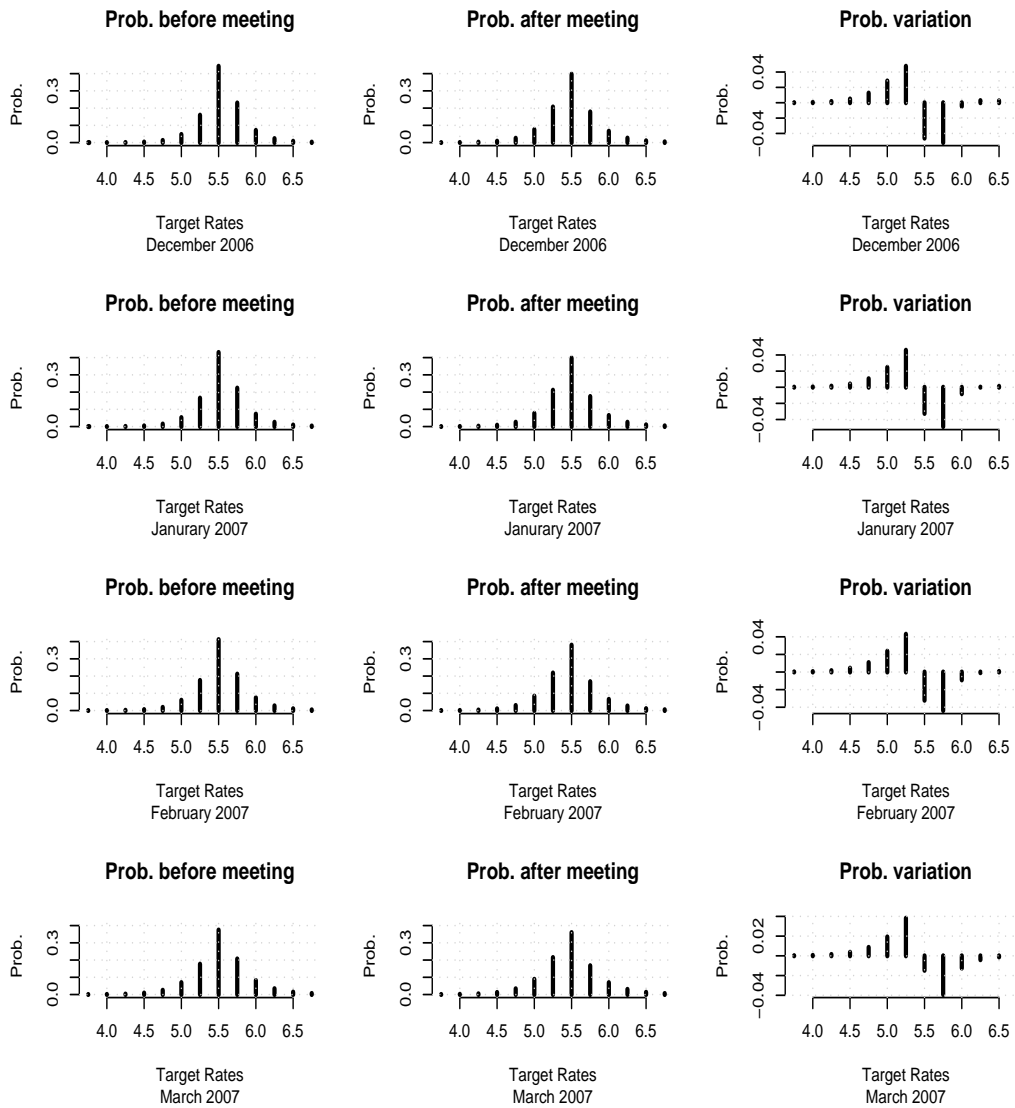
Figure 5: **Implicit monetary policy scenario probabilities around August, $8^{th}$ 2006.**

The probabilities are computed before (left) and after (center) the Central Bank meeting on the August, $8^{th}$ 2006, estimated using model 4. The figure on the right presents the variation of the probabilities over the event. These probabilities are presented for the following maturities: December 2006, January, February and March 2007.
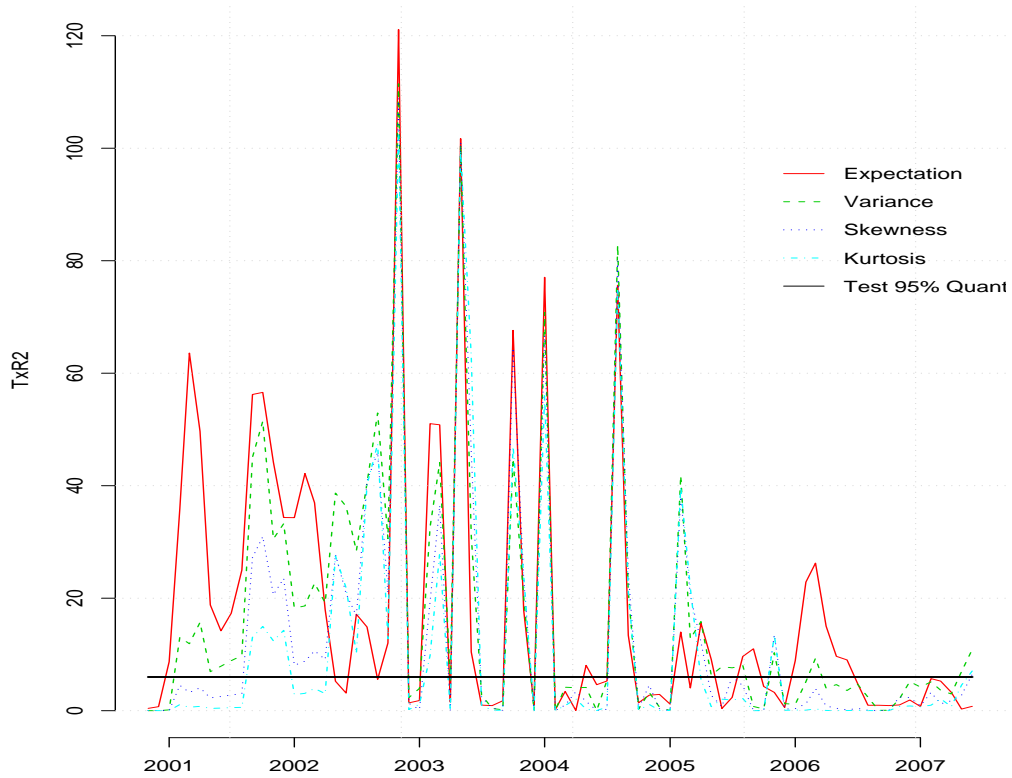
Figure 6: **Engle (1982)'s $T \times R^2$ test for autoregressive moments.**

This figure presents the $T \times R^2$ test for autoregressive higher order moments. It presents the test statistics obtained for each Fed fund futures contracts, with maturity ranging from November 2000 until March 2007. The test statistics were computed over the whole lifespan of the future contracts but the last two months of it, due to stationarity concerns. The plain black line indicates the $\chi^2$ quantile used to test the null hypothesis.