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The Core of a Coalitional Exchange Economy*

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Abstract

In pure exchange economies, a poor attention has been given to how the individual consumption possibilities of the members of a coalition should be represented. It seems economically reasonable that our knowledge and our possibility to make decisions depend on the coalition we belong to. We define a *coalitional exchange economy* by considering a pure exchange economy in which the individual consumption sets of consumers within a coalition depend on the membership of the coalition. Our definition includes as a particular case the classical definition of pure exchange economy. We adapt the core concept to a coalitional exchange economy, and we show the non-emptiness of the core. Finally, we discuss more general setting where individual preferences are also depending on the coalitions.

JEL classification: C71, D50.

Keywords: cooperative game, core, exchange economy, consumption possibility.

Résumé

Dans les économies d′échange, peu d′attention a été donnée à la manière de représenter les possibilités individuelles de consommation des membres d′une coalition. Il semble raisonnable que notre connaissance et notre possibilité de décision dépendent de la coalition à laquelle on appartient. On définit une économie d′échange coalitionnelle en considérant une économie d′échange dans laquelle les ensembles individuels de consommation des membres d′une coalition dépendent de la coalition. Notre définition inclut comme cas particulier la définition d′une économie d′échange. Nous adaptons le concept standard de cœur à une économie d′échange coalitionnelle et nous montrons la non vacuité du cœur. Finalement, nous discutons un cadre plus général où les préférences individuelles dépendent aussi des coalitions.

Journal of Economic Literature Classification Numbers: C71, D50.

Mots-clés: jeu coopératif, cœur, économie d′échange, possibilité individuelle de consommation.

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1 Introduction

The central concept in cooperative game theory is the core, which is the set of feasible payoffs for the grand coalition which no coalition can improve upon. Scarf (1967) provided sufficient conditions for the non-emptiness of the core in a general cooperative game. Then, the author used the result to show that the core of a pure exchange economy with convex preferences is non-empty. While Debreu (1959) considered for each consumer i an *individual consumption set* X_i defined as the set of all *possible* consumptions for consumer i, the analysis of Scarf (1967) in a pure exchange economy simply required to specify *arbitrary redistributions* of the resources of any coalition of consumers, "...It is frequently sufficient to summarize the detailed strategic possibilities open to a coalition by the set of possible utility vectors that can be achieved by the coalition. For example, in a pure exchange economy each coalition will have associated with it the collection of all utility vectors that can be obtained by arbitrary redistributions of the resources of that coalition."

To the best of our knowledge, in Scarf (1967) as well as in the related developments, a poor attention has been given to how the individual consumption possibilities of consumers within a coalition should be represented in pure exchange economies. However, it seems economically reasonable that our knowledge and our possibility to make decisions depend on the coalition we belong to. Importantly, one might willing to distinguish the individual consumption possibilities of *stand alone* agents from the individual consumption possibilities of the members of a coalition.

On production side, coalitional behaviors have been modeled through production sets depending on the membership of the coalition, namely *coalitional production economies*. See in that line Boehm (1974), Champsaur (1974), Border (1984), and Florenzano (1987).

In the spirit of the literature concerning coalitional production economies, we define a *coalitional exchange economy* by considering a pure exchange economy in which the individual consumption sets of consumers within a coalition depend on the membership of the coalition. Then, we adapt the standard definitions of feasibility and core to a coalitional exchange economy.

The main features of this paper are the definition of a coalitional exchange economy and the non-emptiness result of the core in such an economy. Our definition includes as a particular case the classical definition of pure exchange economy, and the non-emptiness result covers the one obtained by Scarf (1967) for the case of a pure exchange economy. Our result relies on the simple idea that our knowledge and our possibility to make decisions increase as the coalition we belong to gets larger. This idea reflects, in terms of possibilities, a "positive effect" for a consumer that belongs to a larger coalition.

The paper is organized as follows. Section 2 is devoted to our basic model and definitions. The non-emptiness result, Theorem 5 whose proof is based on Scarf's theorem on balanced games, is given in Section 3. A remark concludes the paper.

2 The model

A *coalitional exchange economy* is

$$\mathcal{E} := \left(\mathbb{R}^C, \mathcal{N}, \{X_i(S)\}_{i\in N, S\in\mathcal{N}_i}, \{u_i, \omega_i\}_{i\in N}\right)$$

C is the finite number of physical commodities and $\mathbb{R}^C$ is the commodity space. N is the finite set of consumers labeled by subscript i. $\mathcal{N}$ is the family of non-empty subsets of N, and each $S \in \mathcal{N}$ is said coalition. For each $i \in N$, $\mathcal{N}_i := \{S \in \mathcal{N} : i \in S\}$ is the family of coalitions in $\mathcal{N}$ which contain consumer i. For each $i \in N$ and for each $S \in \mathcal{N}_i$, the *individual consumption set*

$X_i(S) \subseteq \mathbb{R}^C$ is defined as the set of all *possible* consumptions for consumer i when S forms, where

$$X_i(S) \subseteq X_i(N) \tag{1}$$

For each $i \in N$, consumer i's preferences are represented by a utility function u_i from $X_i(N)$ to $\mathbb{R}$, and $u_i(x_i) \in \mathbb{R}$ is the consumer i's utility associated with the consumption $x_i \in X_i(N)$. For each $i \in N$, $\omega_i \in \mathbb{R}^C$ is the initial endowment of commodities owned by consumer i.

Remark 1 *In a pure exchange economy, for each $i \in N$, $X_i(S) = X_i$ for every $S \in \mathcal{N}_i$.*

A consumer i is said to be *alone* when coalition $\{i\}$ forms. Observe that in the above definition, the individual consumption set $X_i(\{i\})$ of a stand alone consumer i possibly differs from the individual consumption set $X_i(S)$ of the same consumer i within a coalition $S \supset \{i\}$.

As stressed in Introduction, it seems economically reasonable that for each $i \in N$ and for each S and T in $\mathcal{N}_i$, $S \subseteq T$ implies $X_i(S) \subseteq X_i(T)$. This condition reflects, in terms of possibilities, a "positive effect" for a consumer that belongs to a larger coalition. Furthermore, note that the latter condition is stronger than condition in (1).

Definition 2 *Let $S \in \mathcal{N}$ be a coalition.*

1. *$x^S := (x_i)_{i\in S}$ is a S-feasible bundle of $\mathcal{E}$ if $x_i \in X_i(S)$ for each $i \in S$, and $\sum_{i\in S} x_i \leq \sum_{i\in S} \omega_i$.*
   Let $F^S(\mathcal{E})$ denote the set of S-feasible bundles of $\mathcal{E}$, i.e.

$$F^S(\mathcal{E}) := \left\{ x^S \in \prod_{i\in S} X_i(S) : \sum_{i\in S} x_i \leq \sum_{i\in S} \omega_i \right\} \tag{2}$$

2. *The coalition S blocks a N-feasible bundle $\overline{x} \in F^N(\mathcal{E})$ if there exists $x^S = (x_i)_{i\in S} \in F^S(\mathcal{E})$, such that $u_i(x_i) > u_i(\overline{x}_i)$ for each $i \in S$.*

Definition 3 *The core $\mathcal{C}(\mathcal{E})$ of the coalitional exchange economy $\mathcal{E}$ is defined as the set of all N-feasible bundles of $\mathcal{E}$ that no coalition can block.*

3 Main result

Theorem 5 provides a core non-emptiness result for the coalitional exchange economy defined in Section 2. The proof of Theorem 5 is based on a weak version of the celebrated Theorem 1 of Scarf (1967) for balanced games. Following is the statement of that weak version.

Theorem 4 *(Scarf's Theorem) Let $\Gamma = (N, \{V_S\}_{S\in\mathcal{N}})$ be a balanced game. Γ has non-empty core if the following assumptions are satisfied.*[1]

1. *$V_N \neq \emptyset$, and $V_{\{i\}} \neq \emptyset$ for each $i \in N$;*
2. *for each $S \in \mathcal{N}$, $V_S \neq E^S$ and V_S is a closed set;*

[1]Note that we do not require V_S to be non-empty for $S \neq N$ and $S \neq \{i\}$, nor $\{v = (v_i)_{i\in S} \in V_S : v_i \geq \max V_{\{i\}}, \forall i \in S\}$ to be non-empty for each $S \in \mathcal{N}$. Indeed, observe that the core would be unchanged if we replace the set V_S by $V_S := \{v = (v_i)_{i\in S} \in E^S : v_i \leq \max V_{\{i\}}, \forall i \in S\}$.

3. *for each* $S \in \mathcal{N}$, *if* $v \in V_S$ *and* $w \in E^S$ *with* $w \leq v$, *then* $w \in V_S$;

4. *for each* $S \in \mathcal{N}$, *the set* $\{v = (v_i)_{i \in S} \in V_S : \; v_i \geq \max V_{\{i\}}, \forall i \in S\}$ *is bounded from above.*

It has been shown by Scarf (1967) that a pure exchange economy has a non-empty core. In Theorem 5, we extend his result in the more general setting of coalitional exchange economies.

Theorem 5 *Let* $\mathcal{E}$ *be a coalitional exchange economy. The core* $\mathcal{C}(\mathcal{E})$ *of* $\mathcal{E}$ *is non-empty if for each* $i \in N$, *the following assumptions are satisfied.*

1. $X_i(N)$ *is a convex and bounded from below subset of* $\mathbb{R}^C$;

2. *for each* $S \in \mathcal{N}_i$, $X_i(S)$ *is a closed subset of* $\mathbb{R}^C$;

3. $\omega_i \in X_i(\{i\})$;

4. u_i *is a continuous and quasi-concave function.*

Proof. Define the cooperative game $\Gamma = (N, \{V_S\}_{S \in \mathcal{N}})$, where for each $S \in \mathcal{N}$

$$V_S := \left\{ v = (v_i)_{i \in S} \in E^S : \exists\, x^S \in F^S(\mathcal{E}) \text{ s. t. } v_i \leq u_i(x_i), \forall i \in S \right\}$$

We want to show that the game defined above satisfies the assumptions of Theorem 4.

Claim 1. The game Γ *is balanced.*

Let $\mathcal{B}$ be a balanced family, and $u \in \mathbb{R}^N$ be a vector such that $u^S \in V_S$ for each $S \in \mathcal{B}$. We want to prove $u \in V_N$. For each $S \in \mathcal{B}$, there is $x^S := (x_i^S)_{i \in S} \in \prod_{i \in S} X_i(S)$ such that

$$\sum_{i \in S} x_i^S \leq \sum_{i \in S} \omega_i \text{ and } u_i \leq u_i(x_i^S) \text{ for each } i \in S \tag{3}$$

Since $\mathcal{B}$ is a balanced family, it is possible to find non-negative weights λ_S for each coalition $S \in \mathcal{B}$, such that for each $i \in N$, $\sum_{S \in \mathcal{B}_i} \lambda_S = 1$ where $\mathcal{B}_i := \{S \in \mathcal{B} : i \in S\}$. Now, for each $i \in N$ define $x_i := \sum_{S \in \mathcal{B}_i} \lambda_S x_i^S$. First, observe that $x := (x_i)_{i \in N} \in \prod_{i \in N} X_i(N)$. Indeed, from condition in (1) and the convexity of $X_i(N)$ (see Assumption 1), for each $i \in N$, $x_i \in \sum_{S \in \mathcal{B}_i} \lambda_S X_i(S) \subseteq \sum_{S \in \mathcal{B}_i} \lambda_S X_i(N) \subseteq X_i(N)$. Then, from (3) we get $\sum_{i \in N} x_i \leq \sum_{i \in N} \omega_i$ and for each $i \in N$, $u_i(x_i) = u_i(\sum_{S \in \mathcal{B}_i} \lambda_S x_i^S) \geq \sum_{S \in \mathcal{B}_i} \lambda_S u_i(x_i^S) \geq \sum_{S \in \mathcal{B}_i} \lambda_S u_i = u_i$, since the function u_i is quasi-concave (see Assumption 4). Then, $u \in V_N$.

Claim 2. $V_N \neq \emptyset$, *and* $V_{\{i\}} \neq \emptyset$ *for each* $i \in N$.

From Assumption 3, we get $u_i(\omega_i) \in V_{\{i\}}$ for each $i \in N$. From Assumption 3 and condition in (1), we get $\omega_i \in X_i(\{i\}) \subseteq X_i(N)$ for each $i \in N$. Then, trivially $(u_i(\omega_i))_{i \in N} \in V_N$.

Claim 3. For each $S \in \mathcal{N}$, V_S *is bounded from above and closed.*

Observe that the set $F^S(\mathcal{E})$ defined in (2) is compact. Indeed, from Assumption 2 we deduce that it is closed. From the condition in (1) and the boundedness from below of $X_i(N)$ (see Assumption 1), we have that there exists $b \in \mathbb{R}^C$ such that for each $i \in S$, $b \leq x_i$ for every

$x_i \in X_i(S)$. Let $x^S = (x_i)_{i \in S} \in F^S(\mathcal{E})$, for each $k \in S$ $x_k \leq \sum_{i \in S} \omega_i - (card(S) - 1)b$. Then, $F^S(\mathcal{E})$ is bounded. Therefore, from the continuity of function u_i (see Assumption 4) follows the boundedness from above and the closedness of V_S.

The assumptions of Theorem 4 are satisfied. Then, Γ has non-empty core. Let $u \in V_N$ an element in the core of Γ. By the definition of V_N, there exists $x = (x_i)_{i \in N} \in \prod_{i \in N} X_i(N)$ such that $\sum_{i \in N} x_i \leq \sum_{i \in N} \omega_i$ and $u_i \leq u_i(x_i)$, for each $i \in N$. It easy to see that $x \in \mathcal{C}(\mathcal{E})$. ■

A remark concludes the paper: we analyze a more general setting where utility functions are also depending on the coalitions.

We generalize the definition of coalitional exchange economy as follows,

$$\mathcal{E} := \left(\mathbb{R}^C, \mathcal{N}, \{X_i(S), u_i(\cdot, S)\}_{i \in N, S \in \mathcal{N}_i}, \{\omega_i\}_{i \in N}\right)$$

which differs from the definition given at the beginning of Section 2 since for each $i \in N$ and for each $S \in \mathcal{N}_i$, the utility function of consumer i when S forms is $u_i(\cdot, S)$ from $X_i(S)$ to $\mathbb{R}$. Then, feasibility conditions do not change. A natural adaptation of Definition 2 follows.

Definition 6 *Let $S \in \mathcal{N}$ be a coalition. The coalition S blocks a N-feasible bundle $\overline{x} \in F^N(\mathcal{E})$ if there exists $x^S = (x_i)_{i \in S} \in F^S(\mathcal{E})$ such that $u_i(x_i, S) > u_i(\overline{x}_i, N)$ for each $i \in S$.*

Then, the core non-emptiness result holds replacing Assumption 4 in Theorem 5 with the following assumptions. For a proof, it suffices to use the natural adaptation of the arguments used in the proof of Theorem 5. According to condition in (1), Assumption 7.2 seems economically reasonable.

Assumption 7 *For each $i \in N$,*

1. *$u_i(\cdot, N)$ is a quasi-concave function, and $u_i(\cdot, S)$ is a continuous function for each $S \in \mathcal{N}_i$;*
2. *for each $S \in \mathcal{N}_i$, $u_i(x_i, N) \geq u_i(x_i, S)$ for each $x_i \in X_i(S)$.*

Condition in (1) given in Section 2 is the crucial point to demonstrate that a coalitional exchange economy is representable as a balanced game (see Claim 1, Theorem 5). In that more general setting, an alternative consists of the following notion of *balanced* coalitional economy which is the natural counterpart of the balancedness notion given by Boehm (1974) on the production side. Then, one recovers the core non-emptiness result by assuming in addition that for each $i \in N$ and for each $S \in \mathcal{N}_i$, $X_i(S)$ is bounded from below.

Definition 8 *A coalitional exchange economy $\mathcal{E}$ is said balanced if and only if for every balanced family $\mathcal{B}$ and associated weights $(\lambda_S)_{S \in \mathcal{B}}$, for each $i \in N$*

$$\sum_{S \in \mathcal{B}_i} \lambda_S X_i(S) \subseteq X_i(N)$$

Note that the above condition of balancedness is weaker than condition in (1) if $X_i(N)$ is convex for each $i \in N$.

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