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# Uniform Payoff Security and Nash Equilibrium in Metric Games<sup>\*</sup>

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## Abstract

We introduce a condition, *uniform payoff security*, for games with separable metric strategy spaces and payoffs bounded and measurable in players' strategies. We show that if any such metric game  $G$  is uniformly payoff secure, then its mixed extension  $\bar{G}$  is payoff secure. We also establish that if a uniformly payoff secure metric game  $G$  has compact strategy spaces, and if its mixed extension  $\bar{G}$  has reciprocally upper semicontinuous payoffs, then  $G$  has a Nash equilibrium in mixed strategies. We provide several economic examples of metric games satisfying uniform payoff security.

**JEL** Classification: C72

*Keywords:* uniform payoff security, Nash equilibrium, discontinuous games, mixed extension

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## 1 Introduction

In his paper on Nash equilibrium in discontinuous games, Reny ([8]) introduced the notion of better-reply security and showed that any game with compact, convex strategy spaces and payoffs at least quasiconcave in each

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player's strategy has a Nash equilibrium if, in addition, the game is better-reply secure.<sup>2</sup> Moreover, Reny showed that better-reply security is implied by two conditions, reciprocal upper semicontinuity and payoff security, both of which are easily checked in applications.<sup>3</sup> Here, we show that if a game  $G$  with separable metric strategy spaces and payoffs bounded and measurable in players' strategies (i.e., a metric game) satisfies *uniform payoff security*, then its mixed extension  $\overline{G}$  is payoff secure.<sup>4</sup> The payoff security of  $G$  neither implies nor is implied by the payoff security of  $\overline{G}$ . We present an example from [3] of a payoff secure zero-sum game with a mixed extension that is not payoff secure. Our result shows that if payoff security of  $G$  is strengthened to uniform payoff security, then uniform payoff security of  $G$  does imply payoff security of  $\overline{G}$ . Uniform payoff security of  $G$ , is often times easy to check in applications - whereas the same cannot be said for checking payoff security of the mixed extension  $\overline{G}$ .

An immediate corollary of our result is that if a compact metric game  $G$  (i.e., a game  $G$  with compact metric strategy spaces and bounded measurable payoffs) is uniformly payoff secure, then its mixed extension  $\overline{G}$  has a Nash equilibrium if, in addition,  $\overline{G}$  is reciprocally upper semicontinuous. Another immediate corollary is that if a compact metric game  $G$  is uniformly payoff secure and upper semicontinuous, then its mixed extension  $\overline{G}$  has a Nash equilibrium. The example of [3] is reciprocally uppersemicontinuous and has no Nash equilibrium in mixed strategies. In contrast uniformly payoff secure zero-sum games have Nash equilibrium in mixed strategies.

## 2 Metric Games

A metric game,

$$G = (X_i, u_i)_{i=1}^N,$$

consists of  $N$  players indexed by  $i$ , where each player has a separable metric strategy space  $X_i$  and a bounded measurable payoff function

$$u_i : X \rightarrow \mathbb{R}.$$

<sup>2</sup> A game is said to be better-reply secure if for every nonequilibrium strategy,  $x^*$ , and every payoff vector limit  $u^*$ , generated by strategies approaching  $x^*$  some player has a strategy yielding a payoff strictly above  $u_i^*$  even if other players deviate slightly from  $x^*$ .

<sup>3</sup> Stated loosely, a game is reciprocally upper semicontinuous if, whenever some player's payoff jumps down, some other player's payoff jumps up (see [4], [8] and [9]). Reciprocal upper semicontinuity is implied by upper semicontinuity (but the converse does not hold in general).

<sup>4</sup> See Definition 2 below.

Here  $X := \prod_{i=1}^N X_i$ . We shall denote by  $X_{-i}$  the Cartesian product  $\prod_{j \neq i} X_j$ . The metric game  $G = (X_i, u_i)_{i=1}^N$  is upper semicontinuous if the payoff functions  $u_i(\cdot)$  are upper semicontinuous on  $X$ .

If  $G$  is a metric game and  $\mathcal{M}_i$  is the set of Borel probability measures on  $X_i$ ,  $1 \leq i \leq N$  we define

$$U_i(\mu) = \int_X u_i(x) d\mu(x), \mu = (\mu_1, \dots, \mu_N) \in \mathcal{M} := \prod_{i=1}^N \mathcal{M}_i$$

The set  $\mathcal{M}_i$  is a separable metric space for the weak\* convergence of measures.<sup>5</sup>

The metric game,

$$\bar{G} = (U_i, \mathcal{M}_i)_{i=1}^N,$$

is the mixed extension of the game  $G$ .

### 3 Uniform Payoff security

Let  $G = (X_i, u_i)_{i=1}^N$  be a metric game.

**Definition 1** (*Payoff Security*) *The game  $G$  is payoff secure if for every  $x = (x_i, x_{-i}) \in X$  and  $\epsilon > 0$  there is for each player  $i$  a strategy  $\bar{x}_i \in X_i$  and a neighborhood  $\mathcal{N}(x_{-i}) \subset X_{-i}$  of  $x_{-i}$  such that*

$$y \in \mathcal{N}(x_{-i}) \Rightarrow u_i(\bar{x}_i, y) \geq u_i(x_i, x_{-i}) - \epsilon.$$

Thus, a game  $G$  is payoff secure if starting at any strategy profile  $x = (x_i, x_{-i}) \in X$  each player has a strategy  $\bar{x}_i \in X_i$  he can move to in order to secure a payoff of  $u_i(x_i, x_{-i}) - \epsilon$  against deviations by other players in some neighborhood of  $x_{-i} \in X_{-i}$  (i.e., for each strategy profile  $x = (x_i, x_{-i}) \in X$  each player has a strategy  $\bar{x}_i \in X_i$  that provides security at that profile).

**Definition 2** (*Uniform Payoff Security*) *The game  $G$  is uniformly payoff secure if for every  $x_i \in X_i$  and every  $\epsilon > 0$  there is for each player  $i$  a strategy  $\bar{x}_i \in X_i$  such that for every  $y \in X_{-i}$  there exists a neighborhood  $\mathcal{N}(y)$  of  $y$  such that*

$$z \in \mathcal{N}(y) \Rightarrow u_i(\bar{x}_i, z) \geq u_i(x_i, y) - \epsilon.$$

Thus, a game  $G$  is *uniformly* payoff secure if each player starting at any strategy  $x_i \in X_i$  has a strategy  $\bar{x}_i \in X_i$  he can move to in order to secure a

<sup>5</sup> See [2] Theorem 14.12.

payoff of  $u_i(x_i, y) - \epsilon$  against deviations by other players in some neighborhood of  $y \in X_{-i}$  for all strategy profiles  $y \in X_{-i}$  (i.e., for each player  $i$  and each strategy  $x_i \in X_i$  there is a strategy  $\bar{x}_i \in X_i$  that provides security for all  $y$ ).

## 4 Examples

Let us give a few economic examples of uniformly payoff secure metric games. We begin by considering a complete information, all-pay auction.

**Example 1 (all-pay auction)** *There are  $N$  bidders, indexed by  $i \in I := \{1, 2, \dots, N\}$ , competing for a object with a known value equal to 1. The highest bidder wins and every bidder pays his bid. Ties are broken randomly with equal probabilities. Thus, if bids are given by the  $N$ -tuple*

$$b = (b_1, b_2, \dots, b_N) \in [0, 1]^N,$$

the winning bid is given by

$$b^* = \max_{j \leq N} b_j.$$

Let  $H = \{i \in I : b_i = b^*\}$  denote the set of bidders whose bid equals the winning bid,  $b^*$ . Bidder  $i$ 's payoff is then given by

$$u_i(b) = \begin{cases} \frac{1}{|H|} - b_i & \text{if } b_i = b^*; \\ -b_i & \text{if } b_i < b^*, \end{cases}$$

where  $|H|$  is the number of bidders whose bid equals  $b^*$ . This sealed bid auction game is uniformly payoff secure. To prove this suppose  $\epsilon > 0$  and  $b \in [0, 1]^N$ . There are 3 cases to consider:

- a.  $b_i = 0$ . In this case let  $\bar{b}_i := \epsilon$ . Suppose first that  $\bar{b}_i \neq 0$ . Since  $u_i(b) = 0$  and the maximum that a bid loses is his payment, then for every  $b_{-i}$ ,  $u_i(\bar{b}_i, b'_{-i}) \geq -\bar{b}_i = -\epsilon = u_i(b) - \epsilon$ . Now if  $b_{-i} = 0$  then  $u_i(0) = \frac{1}{N}$  and  $u_i(\bar{b}_i, b'_{-i}) = 1 - \epsilon$  for any  $b'_{-i}$  in a  $\epsilon$  neighborhood of 0.
- b.  $1 > b_i > 0$ . Here increasing his bid to  $\bar{b}_i = b_i + \min\{\epsilon, 1 - b_i\}$ , provides bidder  $i$  with uniform security: Thus if  $b^* \leq b_i$  we choose  $\mathcal{N}(b) = \{b'_{-i}; \max_{j \neq i} b'_j < \bar{b}_i\}$ . If  $b^* > b_i$ ,  $\mathcal{N}(b) = \{b'_{-i}; \max_{j \neq i} b'_j > b_i\}$ .
- c.  $1 = b_i$ . In this case, bidder  $i$ 's payoff is 0 or negative. Thus, if  $\bar{b}_i = 0$  bidder  $i$ 's payoff is secure in any neighborhood of  $b_{-i}$  for any  $b_{-i}$ .

**Remark 1** *It is easy to check that the example above has no pure strategy equilibria (e.g., see Baye, Kovenock, and de Vries [1] for a detailed treatment of complete information, all-pay auctions). It is also easy to check that the*

sum of bidders' payoffs is continuous. Thus, the all-pay auction game above is reciprocally upper semicontinuous.

We now present an auction example with incomplete information.

**Example 2 (sealed-bid, first-price auction)** Consider two bidders competing for an object in a sealed-bid, first-price auction. Letting  $x$  denote bidder 1's valuation and  $y$  bidder 2's valuation, suppose the joint distribution of  $(x, y) \in [0, 1]^2$  is given by

$$f(x, y) = \frac{\theta + 1}{2} (x^\theta + y^\theta) \text{ for } \theta > -1.$$

Note that this density is not affiliated (this density is taken from example 2 in Monteiro and Moreira (2005)). Letting bidder 1's and bidder 2's strategy spaces be given by

$$X_1 = X_2 = C[0, 1] := \{b(\cdot) : [0, 1] \rightarrow \mathbb{R}_+, b(\cdot) \text{ is continuous}\},$$

payoffs are then given by

$$u_1(b_1, b_2) = \int_{[0,1]} (x - b_1(x)) \left( E \left[ I_{[b_2(y) < b_1(x)]} | x \right] + \frac{E \left[ I_{[b_2(y) = b_1(x)]} | x \right]}{2} \right) f_1(x) dx$$

and

$$u_2(b_1, b_2) = \int_{[0,1]} (y - b_2(y)) \left( E \left[ I_{[b_1(x) < b_2(y)]} | y \right] + \frac{E \left[ I_{[b_2(y) = b_1(x)]} | y \right]}{2} \right) f_2(y) dy,$$

where  $f_1(x) = \int_{[0,1]} f(x, y) dy$  and  $f_2(y) = \int_{[0,1]} f(x, y) dx$  are the marginal densities. Here,

$$E \left[ I_{[b_2(y) < b_1(x)]} | x \right] + \frac{E \left[ I_{[b_2(y) = b_1(x)]} | x \right]}{2}$$

is bidder 1's conditional probability of winning or tying given that his valuation is  $x$  (similar remarks apply to the corresponding expression for bidder 2). For example, written out explicitly, given bidder 1's conditional density

$$h(y | x) = \frac{f(x, y)}{f_1(x)} = \frac{(\theta + 1)(x^\theta + y^\theta)}{(\theta + 1)x^\theta + 1},$$

$$E \left[ I_{[b_2(y) < b_1(x)]} | x \right] = \int_{[0,1]} I_{[b_2(y) < b_1(x)]}(x, y) h(y | x) dy,$$

is bidder 1's condition probability of winning given valuation  $x$ . Here,

$$I_{[b_2(y) < b_1(x)]}(x, y) = \begin{cases} 1 & \text{if } b_2(y) < b_1(x) \\ 0 & \text{otherwise.} \end{cases}$$

A Nash equilibrium for this auction game  $G = (X_i, u_i)_{i=1}^2$  is a pair of bidding functions  $(b_1(\cdot), b_2(\cdot)) \in X_1 \times X_2$ . Equipped with the sup metric given by

$$d(b', b'') = \sup_{0 \leq x \leq 1} |b'(x) - b''(x)|,$$

this metric game is uniformly payoff secure.<sup>6</sup>

**Remark 2** The auction game above is neither upper semicontinuous nor reciprocally upper semicontinuous. However, given uniform payoff security, it is possible to show that this auction game is better reply secure.

The next example is taken from Carbonell-Nicolau and Ok (2005).

**Example 3 (electoral competition)** Carbonell-Nicolau and Ok consider a zero-sum, two-party voting game in which each party - whose objective is to maximize the net plurality - proposes a tax function from a given set of admissible tax functions (that raise an exogenously given amount of revenue) and voters vote selfishly for the tax function that taxes them less. The population of voters have incomes  $x \in [0, 1]$  distributed according to a continuous and strictly increasing function  $F : [0, 1] \rightarrow [0, 1]$ . The income distribution  $F$  is right-skewed, that is,

$$F^{-1}\left(\frac{1}{2}\right) < m_F := \int_0^1 x dF(x).$$

A tax function  $t(\cdot) \in \mathcal{T}$  is a continuous function  $t : [0, 1] \rightarrow [0, 1]$  such that  $t(x) \leq x$ ,  $t$  is increasing, and  $x - t(x)$  is increasing. A tax function  $t(\cdot) \in \mathcal{T}$  is admissible if

$$\int_0^1 t(x) dF(x) \geq r,$$

where  $r$  is the exogenously given required revenue. Letting  $\mathcal{T}_{(F,r)}$  denote the set of admissible tax functions,  $\mathcal{T}_{(F,r)}$  is a compact subset of the metric space  $\mathbf{C}[0, 1]$  of continuous functions defined on  $[0, 1]$  equipped with the sup metric (see Lemma 2 in Carbonell-Nicolau and Ok (2005)). If  $t$  and  $\tau$  are tax functions offered by party 1 and party 2 respectively, the share of individuals that strictly prefer party 1's tax proposal  $t$  over party 2's tax proposal  $\tau$  is given by

$$\omega(t, \tau) = \int_{[t < \tau]} dF = \Pr(t < \tau).$$

If the objective of each party is to maximize the net plurality, then party 1's payoff function is given by

$$u_1(t, \tau) = \omega(t, \tau) - \omega(\tau, t),$$

<sup>6</sup> It is well-know that under the sup metric  $C[0, 1]$  is a complete metric space. It is also separable (see [2] Lemma 3.85).

while party 2's payoff function is given by

$$u_2(t, \tau) = -u_1(t, \tau) = \omega(\tau, t) - \omega(t, \tau).$$

The metric game  $G = (X_i, u_i)_{i=1}^2$ , with  $X_1 = X_2 = \mathcal{T}_{(F,r)}$  and payoff functions  $u_i$  as defined above, is a compact metric game. In their proof of Lemma 3, Carbonell-Nicolau and Ok show that for any  $t \in \mathcal{T}_{(F,r)}$  there exists a tax profile  $t_\epsilon \in \mathcal{T}_{(F,r)}$  such that for  $\tau \in \mathcal{T}_{(F,r)}$  there is a neighborhood  $\mathcal{N}(\tau)$  of  $\tau$  such that

$$f \in \mathcal{N}(\tau) \Rightarrow \omega(t_\epsilon, f) - \omega(f, t_\epsilon) > \omega(t, \tau) - \omega(\tau, t) - \epsilon.$$

This is uniform payoff security. Thus, the zero-sum, compact metric game,  $(\mathcal{T}_{(F,r)}, u_1, u_2)$  of tax competition is uniformly payoff secure.

Our final example is taken from [7].

**Example 4 (catalog games)** Page and Monteiro (2003) consider a common agency contracting game in which firms compete for the business of an agent of unknown type  $t \in T$ , where  $T$  is a Borel space. Types are distributed according to a probability measure  $\mu$  defined on  $B(T)$ , the Borel  $\sigma$ -field in  $T$ . Suppose now that there are two firms, indexed by  $i$  or  $j = 1, 2$ , and that firms compete simultaneously in prices and products (broadly defined). Let  $X$  be a compact metric space representing the set of all products each firm can offer and let  $D = [0, \bar{d}]$  be the set of prices that can be charged. We will assume that  $X$  contains an element 0 which denotes no contracting. We will also assume that the agent can only contract with one firm or can choose to abstain from contracting altogether. Let  $X_i$  be a closed subset of  $X$  and let

$$K_i := X_i \times D$$

be the feasible set of products and prices that firm  $i = 1, 2$  can offer. In order to allow for the possibility that the agent chooses to abstain from contracting altogether, we will assume that there is a fictitious firm  $i = 0$  with feasible set products and prices given by<sup>7</sup>

$$K_0 := \{(0, 0)\}$$

Firms compete, by offering the agent a non-empty, closed subset  $C_i \subset K_i$ ,  $i = 0, 1, 2$ , of products and prices called a catalog.<sup>8</sup> Thus each firm's strategy space is given by  $P_f(K_i)$ , the compact metric space of catalogs equipped with the Hausdorff metric ([2], section 3.15). A type  $t$  agent who chooses  $(i, x, p)$ ,  $(x, p) \in C_i$  has utility  $v_t(i, x, p) = 0$  if  $i = 0$  and has utility  $v_t(i, x, p) = u_t(i, x) - p$  if  $i = 1, 2$ . In [7] it is assumed that utility is measurable in type  $t$  and

<sup>7</sup> Thus, the agent chooses to abstain from contracting by choosing to contract with firm  $i = 0$ .

<sup>8</sup> Fictitious firm  $i = 0$ , of course, can only offer catalog  $K_0 = \{(0, 0)\}$ .



continuous in contract choice  $(i, x, p)$ . If firms offer catalog profile  $(C_1, C_2)$ , then the agent's choice set is given by

$$\Gamma(C_1, C_2) = \{(i, x, p) : (x, p) \in C_i, i \in \{0, 1, 2\}\},$$

and the agent's choice problem is given by

$$\max \{v_t(i, x, p) : (i, x, p) \in \Gamma(C_1, C_2)\}.$$

Letting

$$v^*(t, C_1, C_2) = \max \{v_t(i, x, p) : (i, x, p) \in \Gamma(C_1, C_2)\},$$

and

$$\Phi(t, C_1, C_2) = \{(i, x, p) \in \Gamma(C_1, C_2) : v_t(i, x, p) = v^*(t, C_1, C_2)\},$$

the function  $v^*(t, \cdot, \cdot)$  specifies a type  $t$  agent's induced preferences over catalogs, while the correspondence  $\Phi(t, \cdot, \cdot)$  is the type  $t$  agent's best response mapping.<sup>9</sup> The  $j$ th firm's profit function is given by

$$\pi_j(i, x, p) = (p - c_j(x)) I_j(i)$$

where the cost function  $c_j(\cdot)$  is bounded and lower semicontinuous and where

$$I_j(i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Letting

$$\pi_j^*(t, C_1, C_2) := \max \{\pi_j(i, x, p) : (i, x, p) \in \Phi(t, C_1, C_2)\},$$

firm  $j$ 's expected payoff under catalog profile  $(C_1, C_2)$  is given by

$$\Pi_j(C_1, C_2) = \int_T \pi_j^*(t, C_1, C_2) d\mu(t), \quad j = 1, 2.$$

The catalog game,  $G = (P_f(K_i), \Pi_i)_{i=1}^2$ , is an upper semicontinuous, compact metric game. It is proved in Theorem 5, page 96 of [7] that this game is uniformly payoff secure.

<sup>9</sup> It is shown in [7] that the induced utility function is measurable in types and continuous in catalog profiles, while the best response mapping is jointly measurable in types and catalog profiles and upper semicontinuous in catalog profiles.

## 5 Main Result

Before proving that a uniformly payoff secure game has a payoff secure mixed extension we present an example<sup>10</sup> of a payoff secure game with a mixed extension that is not payoff secure. And even more can be said: the mixed extension does not have a Nash equilibrium.

**Example 5 (Carmona(2005))** *There are two players. The set of strategies is  $X_i = [0, 1]$ . The game is zero-sum and the payoff of player 1 is*

$$u(x, y) = \begin{cases} 1 & \text{if } y \in [0, x) \\ 0 & \text{if } y = x \text{ or } y = x + \frac{1}{2} \\ -1 & \text{if } x < y < x + \frac{1}{2} \\ 1 & \text{if } x + \frac{1}{2} < y \leq 1. \end{cases}$$

To show that the game is payoff secure note that since  $u(x, z)$  is continuous for  $z \neq x$  and  $z \neq x + \frac{1}{2}$  it is automatically payoff secure in this region. Suppose now that  $y = x$ . Just define  $\bar{x} = x + \epsilon$  if  $x < 1, \epsilon < 1 - x$ . If  $x = 1$  the payoff is secure in any neighborhood of  $y$ . Suppose now that  $y = x + \frac{1}{2}$ . In this case to increase  $x$  is not a good idea. However if we decrease  $x$  everything is fine. If  $x = 0$  then choose  $\bar{x} = 1$ . An analogous reasoning shows that the payoff is secure for player 2. We now show that the mixed extension is not payoff secure. Let  $\lambda_1^* = \delta_0$  and  $\lambda_2^* = \frac{2}{3}\delta_1 + \frac{1}{3}\delta_{1/2}$ . Thus  $u(\lambda^*) = \frac{2}{3}$ . Suppose we could secure the payoff with  $\bar{\lambda}_1$  and the neighborhood  $\mathcal{W} \ni \lambda_{-i}^*$ . Thus if  $\nu_n := \frac{2}{3}\delta_1 + \frac{1}{3}\delta_{1/2-1/n} \rightarrow \lambda_2^*$  we have that  $\nu_n \in \mathcal{W}$  for  $n$  large enough. Thus

$$u(\bar{\lambda}_1, \nu_n) \leq \bar{\lambda}_1(0) \frac{1}{3} + \bar{\lambda}_1\left(\left(0, \frac{1}{2}\right]\right) \frac{1}{3} + \bar{\lambda}_1\left(\frac{1}{2}\right) 0 + \bar{\lambda}_1\left(\left(\frac{1}{2}, 1\right)\right) \frac{-1}{3} + \bar{\lambda}_1(1) \frac{1}{3} < \frac{2}{3}.$$

We now proceed with the main result.

**Theorem 1** *If a game  $G$  is uniformly payoff secure, then its mixed extension  $\bar{G}$  is payoff secure.*

Before going into the proof we need the following:

**Lemma 1** *Let  $Z$  be a topological space. If  $v : Z \rightarrow \mathbb{R}$  is bounded then there exists a lower semicontinuous function  $\phi : Z \rightarrow \mathbb{R}$  such that if  $v \geq \phi$  and if  $v \geq g$  and  $g$  is lower semicontinuous then  $\phi \geq g$ .*

<sup>10</sup> We thank W. Daher for bringing [3] to our attention

**Proof of Lemma 1:** Let

$$\mathcal{L} = \{g : Z \rightarrow \mathbb{R} : v \geq g \text{ and } g \text{ is lower semicontinuous}\}.$$

This set is non-empty since the constant function  $\inf v(Z)$  belongs to  $\mathcal{L}$ . Now if we define  $\phi(z) = \sup \{g(z) ; g \in \mathcal{L}\}$  we have that  $\phi$  is lower semicontinuous since it is the pointwise supremum of lower semicontinuous functions ([2], Lemma 2.38, page 42). ■

The function  $\phi$  is called the lower semicontinuous hull of  $v$ . We now prove theorem 1.

**Proof of Theorem 1:** Suppose  $\lambda^* = (\lambda_1^*, \dots, \lambda_N^*) \in \mathcal{M}$  is a vector of mixed strategies. Suppose also that  $\epsilon^* > 0$  is given. We have to find a mixed strategy  $\lambda'_i$  and a  $\delta > 0$  such that

$$d_{-i}^*(\lambda_{-i}, \lambda_{-i}^*) < \delta \Rightarrow \int_X u_i(x_i, y) d\lambda'_i(x_i) d\lambda_{-i}(y) \geq \int_X u_i(x) d\lambda^*(x) - \epsilon^*,$$

where  $d_{-i}^*$  denotes the metric on  $\mathcal{M}_{-i}$ . To proceed, let  $\epsilon = \frac{\epsilon^*}{2}$ . Since  $\lambda_i^*$  is a probability measure there exists a  $\tilde{x}_i \in X_i$  such that

$$\int_{X_{-i}} u_i(\tilde{x}_i, y) d\lambda_{-i}^*(y) \geq \int_X u_i(x_i, y) d\lambda_i^*(x_i) d\lambda_{-i}^*(y) = U_i(\lambda^*). \quad (1)$$

We now apply the uniform payoff security property to find for the given  $\tilde{x}_i$  a  $\bar{x}_i \in X_i$  such that for every  $y \in X_{-i}$  there exists a neighborhood  $\mathcal{N}(y)$  of  $y$  such that

$$z \in \mathcal{N}(y) \Rightarrow u_i(\bar{x}_i, z) \geq u_i(\tilde{x}_i, y) - \epsilon. \quad (2)$$

Let  $\phi(\cdot)$  be the lower semicontinuous hull of  $u_i(\bar{x}_i, \cdot)$ . Since  $\phi(\cdot)$  is lower semicontinuous,  $\int_{X_{-i}} \phi(y) d\mu_{-i}(y)$  is lower semicontinuous in  $\mu_{-i}$  ([2], Theorem 14.5, page 479). Thus there is a neighborhood  $\mathcal{N}(\lambda_{-i}^*)$  of  $\lambda_{-i}^*$  such that for every  $\mu_{-i} \in \mathcal{N}(\lambda_{-i}^*)$ ,

$$\int_{X_{-i}} \phi(y) d\mu_{-i}(y) > \int_{X_{-i}} \phi(y) d\lambda_{-i}^*(y) - \epsilon.$$

Therefore,

$$\int_{X_{-i}} u_i(\bar{x}_i, y) d\mu_{-i}(y) \geq \int_{X_{-i}} \phi(y) d\mu_{-i}(y) > \int_{X_{-i}} \phi(y) d\lambda_{-i}^*(y) - \epsilon \quad (3)$$

For each  $y \in X_{-i}$  define the function

$$\psi^y : X_{-i} \rightarrow R$$

as follows:

$$\psi^y(z) = \begin{cases} u_i(\bar{x}_i, y) - \epsilon & \text{if } z \in \mathcal{N}(y) \\ \inf u_i(X) & \text{if } z \notin \mathcal{N}(y). \end{cases} \quad (4)$$

Thus, for all  $y \in X_{-i}$ ,  $\psi^y$  is lower semicontinuous with  $u_i(\bar{x}_i, x) \geq \psi^y(x)$  for  $x \in X_{-i}$ . Because  $\phi$  is the lower semicontinuous hull of  $u_i(\bar{x}_i, \cdot)$ , it is true that for all  $x \in X_{-i}$

$$\phi(x) \geq \sup_y \psi^y(x) \geq \psi^x(x) = u_i(\bar{x}_i, x) - \epsilon. \quad (5)$$

Given (3) and (5) we have

$$\left. \begin{aligned} \int_{X_{-i}} u_i(\bar{x}_i, y) d\mu_{-i}(y) &\geq \int_{X_{-i}} \phi(y) d\mu_{-i}(y) > \int_{X_{-i}} \phi(y) d\lambda_{-i}^*(y) - \epsilon \\ &\geq \int_{X_{-i}} (u_i(\bar{x}_i, y) - \epsilon) d\lambda_{-i}^*(y) - \epsilon \\ &= \int_{X_{-i}} u_i(\bar{x}_i, y) d\lambda_{-i}^*(y) - 2\epsilon \\ &\geq \int u_i(x) d\lambda^*(x) - \epsilon^*. \end{aligned} \right\} \quad (6)$$

The last inequality in (6) follows from (1). Thus, the mixed extension  $\bar{G}$  is payoff secure. ■

**Remark 3** *The use of the lower semicontinuous hull together with the function  $\psi^y(\cdot)$  defined in (4) allows us to greatly streamline the proof that for metric games uniform payoff security implies payoff security of the mixed extension. In an earlier version of this paper we showed that for compact metric games uniform payoff security implies payoff security of the mixed extension. The proof of our earlier result was longer and relied on compactness and known results about weak convergence of probability measures. Here, using the lower semicontinuous hull and the function  $\psi^y(\cdot)$  we obtain a more general result with a much shorter proof. Reny ([8]) uses the lower semicontinuous hull of the payoff function in his proof of Theorem 3.1 (see p. 1036), and Carbonell-Nicolau and Ok (2005) use the lower semicontinuous hull of the payoff function in their proof of Lemma 3 (p. 23).<sup>11</sup>*

## 6 A Corollary on Existence

The following corollary is an immediate consequence of Theorem 1 above and Corollary 5.2 in [8].

**Corollary 1** *Suppose that  $G = (X_i, u_i)_{i=1}^N$  is a compact metric game. If  $G$  is uniformly payoff secure and  $\bar{G}$  is reciprocally upper semicontinuous, then  $G$  possesses a mixed strategy Nash equilibrium.*

<sup>11</sup> We thank the referees for drawing our attention to the paper by Carbonell-Nicolau and Ok (2005), and for suggesting ways to shorten our proof.

If  $G$  is upper semicontinuous, then  $\overline{G}$  is also upper semicontinuous - and therefore reciprocally upper semicontinuous. Thus, it follows from the Corollary that if  $G$  is uniformly payoff secure and upper semicontinuous, then  $G$  possesses a mixed strategy Nash equilibrium. The catalog game discussed in example 4 is an example of an upper semicontinuous, compact metric game. Thus, it follows from the Corollary above that catalog games possess mixed strategy Nash equilibria (also see Theorems 4, 5, and 6 in [7]).

## References

- [1] M. R. Baye, D. Kovenock, C. G. de Vries, The all-pay auction with complete information, *Econ. Theory* 8 (1996), 291-305.
- [2] C. Aliprantis, Kim Border, *Infinite Dimensional Analysis, A Hitchhiker's Guide*, second edition, Springer, 1999.
- [3] G. Carmona, On the existence of equilibria in discontinuous games: three counterexamples, *Int. J. Game Theory* 33 (2005), 181-187.
- [4] P. Dasgupta, E. Maskin, The existence of equilibrium in discontinuous economic games, I: theory, *Rev. Econ. Stud.* 43 (1986), 1-26.
- [5] O. Carbonell-Nicolau, Efe A. Ok, Voting over income taxation, Working Paper (2005), Rutgers University.
- [6] P. K. Monteiro, H. Moreira, First-price auctions without affiliation, Working Paper (2005), forthcoming in *Econ. Letters*.
- [7] P. K. Monteiro, Frank. H. Page Jr., Three principles of competitive nonlinear pricing, *J. Math. Econ.* 39 (2003), 63-109.
- [8] P. J. Reny, On the existence of pure and mixed strategy Nash equilibria in discontinuous games, *Econometrica* 67 (1999), 1029-1056.
- [9] L. Simon, Games with discontinuous payoffs, *Rev. Econ. Stud.* 54 (1987), 569-597.
- [10] M. Sion, P. Wolfe, On a Game without a Value, In: M. Drescher, M. Tucker, P. Wolfe (eds) *Contributions to the Theory of Games III*, Princeton University Press, 1957