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Towards an Explanation of the Exponential Distribution of Firm Growth Rates*

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Abstract

A robust feature of the corporate growth process is the exponential distribution of firm growth rates. This striking empirical regularity has been found to hold for a number of different datasets and at different levels of aggregation. In this paper, we propose a simple theoretical model capable of explaining this observed exponential distribution. We do not attempt to generalize on where growth opportunities come from, but rather we focus on how firms build upon growth opportunities. We borrow ideas from the self-organizing criticality literature to explain how the interdependent nature of discrete resources may lead to the triggering off a series of additions to a firm's resources. In a formal model we consider the case of employment growth in a hierarchy, and observe that growth rates follow an exponential distribution.

A LA RECHERCHE D'UNE EXPLICATION DE LA DISTRIBUTION EXPONENTIELLE DES TAUX DE CROISSANCE DES FIRMES

Résumé: Les taux de croissance des firmes sont généralement distribués selon une loi exponentielle. Cette régularité empirique a été vérifiée pour plusieurs bases de données et à plusieurs niveaux d'agrégation. Dans ce papier, nous proposons un modèle théorique simple qui est capable de reproduire cette distribution exponentielle. Nous ne tentons aucune généralisation relative aux causes d'apparition des opportunités de croissance, mais nous nous attachons plutôt au comment les firmes réagissent et internalisent ces opportunités. Nous utilisons des concepts issus de la littérature sur la 'criticité' auto-organisatrice pour expliquer comment la nature interdépendante de ressources discrètes peut déclencher toute une série d'additions aux ressources d'une firme. Dans un modèle formel, nous considérons le cas de la croissance en termes d'emploi dans le contexte d'une hiérarchie, et nous observons que les taux de croissance suivent une loi exponentielle.

JEL codes: L1, C1
Keywords: Firm Growth Rates, Exponential Distribution, Hierarchy
Mots clés: Croissance des firmes, distribution exponentielle, hiérarchie

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1 Introduction

It has long been known that the distribution of firm growth rates is fat-tailed. In an early contribution, Ashton (1926) considers the growth patterns of British textile firms and observes that "In their growth they obey no one law. A few apparently undergo a steady expansion...With others, increase in size takes place by a sudden leap..." (Ashton 1926:572-573). Little dedicates a section of his 1962 empirical study to the distribution of growth rates, and also finds that the distribution is fat-tailed. However, he concludes the section without proposing any theoretical explanation: "I do not know what plausible hypothesis explains the highly leptokurtic nature of the distributions" (Little 1962:408). Recent empirical research into industrial dynamics has discovered that the distribution of firm growth rates closely follows an exponential distribution. Using the Compustat database of US manufacturing firms, Stanley et al. (1996) and Amaral et al. (1997) observe a 'tent-shaped' distribution on log-log plots that corresponds to the symmetric exponential, or Laplace distribution. The quality of the fit of the empirical distribution to the Laplace density is quite remarkable. The Laplace distribution is also found to be a rather useful heuristic when considering growth rates of firms in the worldwide pharmaceutical industry (Bottazzi et al. 2001). Giulio Bottazzi and coauthors extend these findings by considering the Laplace density in the wider context of the family of Subbotin distributions. They find that, for the Compustat database, the Laplace is indeed a suitable distribution for modelling firm growth rates, at both aggregate and disaggregated levels of analysis (Bottazzi and Secchi 2003a). The exponential nature of the distribution of growth rates also holds for other databases, such as Italian manufacturing (Bottazzi et al. (2003)). In addition, the exponential distribution appears to hold across a variety of firm growth indicators, such as Sales growth, employment growth or Value Added growth (Bottazzi et al. 2003). The growth rates of French manufacturing firms have also been studied, and roughly speaking a similar shape was observed, although it must be said that the empirical density was noticeably fatter-tailed than the Laplace (Bottazzi et al. 2005).[1] To summarize, then, growth rates appear to have a distribution that is much fatter tailed than the Gaussian. The symmetric exponential, or Laplace distribution, seems to be a suitable candidate for describing the empirical distribution. In Figure 1, we use the Compustat database to show that annual employment growth is approximately exponentially distributed for large US firms. [2] In this short paper, we propose that it would be fruitful to conceive firms as being composed of discrete, interrelated resources, that are subject to local interactions, and susceptible to containing some degree of organizational slack. In section 2 we review and discuss previous models of industry growth, and in section 3 we present a formal model. Section 4 concludes.

2 A Discussion of Previous Models

There is something of a tradition in Industrial Organization modelling to represent growth processes in purely stochastic terms. Ijiri and Simon (1977) offered an explanation of the skewed firm size distribution in terms of a random process in which the probability of a firm

[1] i.e. the observed subbotin b parameter (the 'shape' parameter) is significantly lower than the Laplace value of 1. This highlights the importance of following Bottazzi et al. (2002) and considering the Laplace as a special case in the Subbotin family of distributions. The model presented here can nonetheless be modified to explain the particularly fat-tailed distribution of growth rates in France, by allowing for the parameter α (i.e. the span of control) to diminish at higher levels of the hierarchy.

[2] We calculate annual employment growth rates for the periods 1980-1, 1990-1 and 2000-1, obtaining 5256, 5931 and 7948 observations respectively. For each period, firms are sorted into 100 bins.

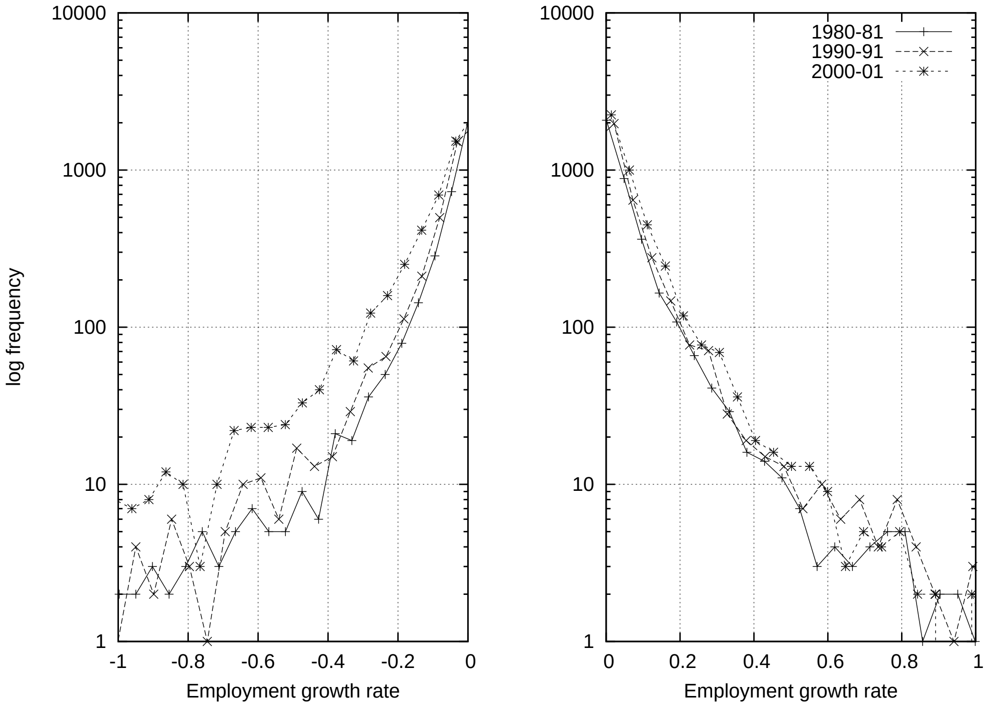

Figure 1: The exponential distribution of employment growth rates, using the Compustat dataset for large US firms. Note the log scale on the y-axis.

taking up an additional business opportunity is conditional upon its size. This model, dubbed the 'island' model because of the independence of growth opportunities, has been widely accepted, and interest in it was recently revived by Sutton (1998).

Previous models that focused on the exponential distribution of firm growth rates have also taken the approach of stochastic explanations. Amaral et al. (1997) develop a model in which the emergence of the distribution rests on a particular specification of the functional form of the stochastic growth process. However, there is little justification of the choice of such a functional form, and so we argue that their model is more of a tautology than an explanation. The model of Bottazzi and Secchi (2003b, 2005) also conceives of firm growth as a random process - "in our model luck is the principal factor that finally distinguishes winners from losers among the contenders" (Bottazzi and Secchi 2005:4). The allocation of growth opportunities is governed by the 'Polya Urn' statistics, and as a justification of such mathematical apparatus they evoke the principle of 'increasing returns to growth' in the competitive process (more in the sense of Brian Arthur rather than in that of the Kaldor-Verdoorn 'dynamic increasing returns'). Within the period of a calendar year, they suppose that the probability of a growth opportunity being taken up depends positively on the number of growth opportunities already taken up that year. This hypothesis, however, is difficult to reconcile with the existence of a small *negative* year-on-year autocorrelation of growth rates observed in empirical studies (see for example Bottazzi et al. (2005)). Furthermore, given that the resulting exponential distribution is determined by the (perhaps *ad hoc*) choice of the underlying random process, there are limits to how much such models can actually explain.

The choice of stochastic models to describe industrial evolution bears witness to a reluctance to generalize across firms. Firms grow for a wide variety of different reasons, they are indeed heterogeneous, and it is believed that the best or only way to model growth may be by treating it as purely stochastic. To move beyond describing industry dynamics in terms of purely random shocks, we need to address the following question: "Can we generalize across firms?" Our answer is: "Yes we can, to some degree". Without attempting to resuscitate the idea of a 'representative firm', we maintain that there are some general features that are present in all firms. We propose a theoretical explanation that is based on the 'resource-based approach', which views firms as being composed of discrete, complementary resources (Penrose 1959). In addition, we allow for the possibility of growth being accommodated by organizational slack. Organizational slack is a widely-recognized characteristic of business firms - indeed, a firm's resources will not be fully utilized at any given time for a number of reasons.[3] However, managers will seek to use a firms resources efficiently, to have them as close as possible to 'full utilization'. If a firms resources are under-utilized, then growth can feed off these slack resources. On the other hand, if resources are already more or less fully employed, then growth will only be possible with the addition of new resources. In the former case, growth requires no additional investment, whilst in the latter case, firm growth will be accompanied by potentially wide-scale investment. This depiction of firm growth can be expressed in terms of self-organizing criticality. The firm can be seen as a system which tends to a 'critical state' of full utilization of its resources, as managers strive to organize the firms resources efficiently within the firm's hierarchical framework. Depending upon the criticality of the system, the addition of an activity during growth will result in a (marginally) increased strain for many associated resources, thus potentially triggering off a chain reaction of subsequent growth across the whole of the organization. In this vein, Dixon (1953:50) comments on the criticality of a firm at a more general level: "the later addition of one person to regular activities can bring into operation a chain of reactions in the form of salaried employee increases, salary increases, and fixed asset additions." The 'avalanche' will only stop if there is sufficient organizational slack to absorb the extra workload associated with the additional resources.

To illustrate this idea, we consider the special case of the propagation of employment growth throughout the various levels of a firm's hierarchy. The organization of production in a hierarchy is indeed a general feature of all firms - in fact, in the Transaction-Cost-Economics literature, the words 'firm' and 'hierarchy' are used almost interchangeably. In this context, a firm may grow by adding an additional worker on the factory shopfloor, who will require the attention of a supervisor. It may occur, however, that all of the current supervisors are already too busy to take on this extra burden of supervision. With a small probability, then, the addition of this supplementary worker requires that the firm hire another supervisor. Furthermore, the addition of a supervisor may then increase the administrative workload of the central office, such that this latter also needs to hire a supplementary worker. An analogy with the classic 'sandpile' model[4] can therefore be drawn, as the addition of a supplementary worker can lead to a 'snowball effect' of hiring of employees at higher levels of the hierarchy.

[3]Here are a few possible examples. Slack may be present because indivisibilities of key inputs may prevent a firm from attaining perfect productive efficiency. Also, slack may creep in as the learning-by-doing effects that increase a workers productivity are not counterbalanced by increasing demands made of the worker. Furthermore, slack may be necessary because firms must be able to adapt and act flexibly in response to unforeseen contingencies and the changing market environment.

[4]See Bak and Chen, 1991; see also Bak et al. 1993 for an economic application. In the 'sandpile' model, grains of sand are dropped on top of each other until a sandpile is formed. "[R]andomly dropping on additional sand will result in the slope of the pile increasing to a critical slope, at which point avalanches of all sizes (limited only by the size of the pile) can occur in response to the dropping of a single additional grain of sand." (Bak et al. 1993:7)

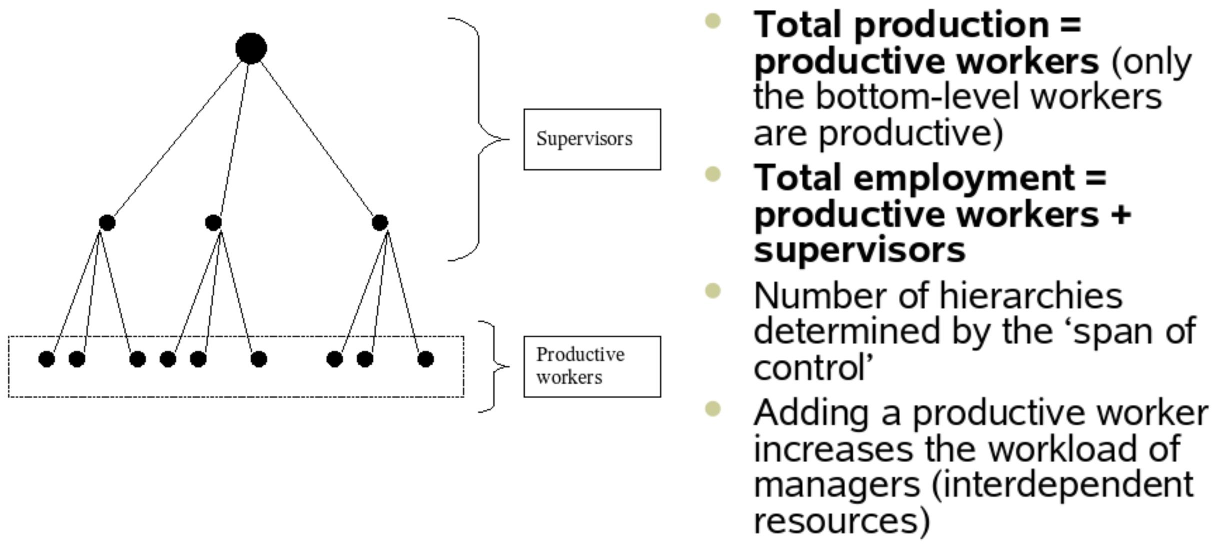

Figure 2: A stylized depiction of employment in a firm.

3 The Model

In the previous discussion, we argued that firms grow by adding discrete resources to a complex of interdependent resources that they already possess. However, a verbal discussion is not methodologically sufficient to prove our point that discrete, interdependent resources organized within a firm bring about an explicitly exponential distribution of growth rates. As stated by Leonardo da Vinci, "no human investigation can be called true science without passing through mathematical tests." In this section, therefore, we present an analytical model capable of reproducing the required functional form by exploiting a few simplifying assumptions.

We characterize a firm as being composed of a relatively large number of hierarchies.[5] The bottom layer of the firm (i.e. the very lowest hierarchical level) is composed exclusively of productive workers, whilst all of the other levels are composed of managers whose task is to supervise either productive workers or subordinate managers. A firm grows by adding a productive worker. The number of managers is determined by the number of productive workers and also by limits on the efficient span of control, α, which correspond to the maxumum number of subordinates that a manager can effectively supervise. "At executive levels [the span of control] is seldom less than three, and seldom more than ten, and usually lies within narrower bounds - particularly if we take averages over all executives in an organization at a given level." (Simon 1957:32). In this model, though, we do not need to attribute any specific numerical value to α and so we leave it in algebraic form.[6] For analytical simplicity, however, we do assume that α is a constant and does not vary either within a hierarchical level or across levels (for a discussion of the plausibility of this assumption, see Williamson (1967:128)). For the purposes of this model, we also must assume that adjustment of the firm's hierarchical organization to additional productive workers occurs within one time period. Finally, we assume that the firm is initially at a stable steady state, such that it is already efficiently

[5]We do not need to define the number 'large' nor define what happens at the very top of the hierarchy. Also, we do not need to suppose that the number of hierarchies tends to infinity, because we only want to explain the distribution of growth rates for a certain limited range.

[6]However, it is computationally helpful and also theoretically meaningful to assume that α is a whole number that is strictly greater than unity (i.e. $\alpha \in \mathbb{N}^+$, $\alpha > 1$).

organized in the sense that it is not possible for it to employ fewer managers given the number of productive workers and its given value of α (i.e. the limit on the efficient span of control). The reader may notice major similarities between the model developed here and the executive compensation model of Simon (1957) and the information flows model of Williamson (1967). The fact that the same hierarchical model has been applied in quite different contexts lends credibility to its use here - indeed, we cannot be accused of having conclusions that emerge from *ad hoc* modelling assumptions.

A summary understanding can be obtained by looking at Figure 2. Two important points should be emphasized. First, there is a distinction between changes in total production (which corresponds to changes in the number of productive workers, n) and changes in total employment (which corresponds to changes in numbers of both productive workers and supervisors, x). Furthermore, it should be noted that we do not attempt to generalize on the sources of growth opportunities, but rather we focus on how firms build upon given growth opportunities. We argue that the fat-tailed distribution of growth rates does not come from the distribution of opportunities available to firms, but rather on the reactions of firms to growth stimuli. The model is admittedly a gross simplification and does not take into account such factors as the interdependence of growth rates between firms, liquidity constraints that limit growth, or limits on the availability of suitable workers. Nonetheless, its simplicity will make it clear to what properties we owe the emergence of the exponential distribution.

Let us begin with the simplest possible case, considering one firm that grows by adding just one productive worker (i.e. $n = 1$). It is possible that all of the managers in the second hierarchical level (i.e. those that supervise the productive workers) are already fully occupied. This will occur when the number of productive workers (before adding the new one) is exactly a multiple of α. If this is the case, the arrival of the supplementary worker will require that one supplementary manager be hired at the next hierarchical level. This scenario will occur with probability $1/\alpha$. However, the arrival of this new manager at the second level may add to the workload of managers on the third hierarchical level. The probability that the addition of a productive worker leads to *two* managers being hired at two successive levels is $1/\alpha \times 1/\alpha = 1/\alpha^2$. We can continue with this reasoning to end up with an exponential distribution of employment growth:

Prob.(growth $\geq 1|n = 1) = 1$
Prob.(growth $\geq 2|n = 1) = 1/\alpha$
Prob.(growth $\geq 3|n = 1) = 1/\alpha^2$
Prob.(growth $\geq 4|n = 1) = 1/\alpha^3$
...

and so on. Formally, we have an exponential distribution with the following functional form:

$$P(X \geq x|n = 1) = \alpha^{1-x} \tag{1}$$

or, expressed differently,

$$P(X = x|n = 1) = \alpha^{1-x}(1 - 1/\alpha) \tag{2}$$

where x is a particular realization of the distribution X of the total employment growth rate (of workers and managers combined), with $x \geq 1$. We therefore observe that the growth rate density of a firm that grows by adding one productive worker will follow an exponential distribution.

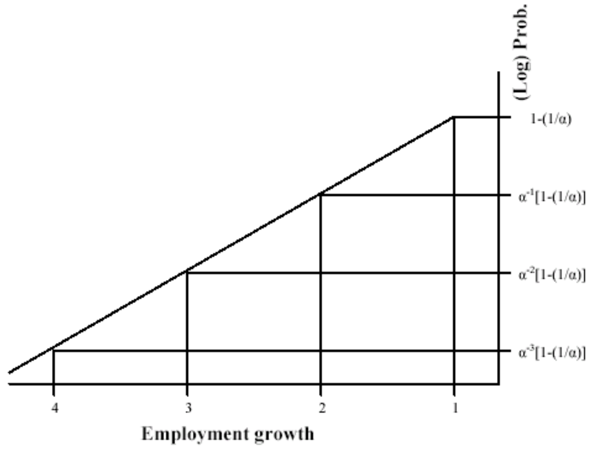

Figure 3: The growth rate density of total employment if a firm shrinks by $n = 1$ (see Proposition 2).

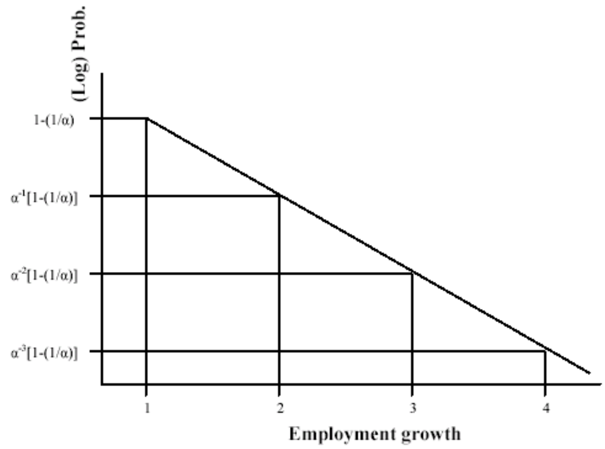

Figure 4: The growth rate density of total employment if a firm grows by $n = 1$ (see Proposition 1).

It is also possible to generalize for the case where a firm grows by adding $n \in \mathbb{N}^+$ productive workers (with, of course, $x \geq n$). For $n < \alpha$, we obtain the following distribution:

$$\begin{aligned} P(X = x|n) &= 1 - n/\alpha && \text{if} \quad x = n \\ P(X = x|n) &= n \cdot \alpha^{n-x}(1 - 1/\alpha) && \text{if} \quad x > n \end{aligned} \tag{3}$$

where equation (2) corresponds to the special case where $n = 1$.

Thus, we have:

Proposition 1 *The growth rate density of total employment of a firm that adds n productive workers will follow an exponential distribution.*

An illustration is offered in Figure 4.

Analogical reasoning can be applied to the case where a firm shrinks in size. Consider a firm that shrinks by n units - at least one supervisor will no longer be needed when, after shrinking, the number of productive workers is an exact multiple of α. Formally, we can still use Equation (3), where a firm shrinks by x employees as a response to shedding n production workers (with $n < \alpha$), i.e.:

$$\begin{aligned} P(X = x|n) &= 1 - n/\alpha && \text{if} \quad x = n \\ P(X = x|n) &= n \cdot \alpha^{n-x}(1 - 1/\alpha) && \text{if} \quad x > n \end{aligned} \tag{4}$$

Proposition 2 *The growth rate density of total employment of a firm that shrinks by n productive workers will follow an exponential distribution.*

An illustration is given in Figure 3.

4 Conclusion

In a formal model we considered employment growth in the case of a hierarchical organization, in which a limit has been placed on the efficient span of control. We do not attempt to generalize upon where growth opportunities come from, but instead we consider how firms build upon growth opportunities. Adding an extra worker at the bottom of the hierarchy will marginally increase the workload at higher levels of the hierarchy, which may trigger off to a potentially large hiring of supervisors. It is observed that a firm's growth rate of total employment (productive workers and supervisors combined) will follow an exponential distribution.

The model is admittedly far too simple to be realistic, yet it's simplicity makes for greater visibility of the source of the emergence of the exponential distribution. The model can be seen as a 'special case' in a family of possible models that view firms as coherent collections of resources that are complementary and discrete. These latter are subject to localised interactions and embedded in an organization that tends to a critical state of full utilization of its resources. In this context, a small growth stimulus working through local interaction channels can be transmitted throughout a firm to produce potentially large-scale effects. We argue that it is these properties that explain the emergence of the observed fat-tailed growth rate distributions.

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