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WALRAS–LINDAHL–WICKSELL: WHAT EQUILIBRIUM CONCEPT FOR PUBLIC GOODS PROVISION? I - THE CONVEX CASE

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Abstract. Despite the large number of its references, this paper is less a survey than a systematic exposition, in an unifying framework and assuming convexity as well on the consumption side as on the production side, of the different equilibrium concepts elaborated for studying provision of public goods. As weak as possible conditions for their existence and their optimality properties are proposed. The general conclusion is that the drawbacks of the different equilibrium concepts lead to founding public economic policy either on direct Pareto improving government interventions or on state enforcement of decentralized mechanisms.

Keywords: Private provision equilibrium, Lindahl–Foley equilibrium, public competitive equilibrium, abstract economies, equilibrium existence, welfare theorems, core

JEL Classification numbers: D 51, D 60, H 41

1. Introduction

What is a public good? Should it be publicly or privately provided? How should be shared the burden of the costs of its production? The aim of this paper is to gather and to present, from an analytical point of view and in relation with a normative theory of public expenditure and public taxation, the answers given to these questions in the framework of the general equilibrium model as defined in 1954 by Arrow–Debreu [1], promptly extended to accommodate public goods, and since then constantly generalized.

We will leave outside this survey "positive" general equilibrium analysis [22, 27, 38, 51, 52, 53] that have as a common feature to study mixed or "second best" economies where the presence and role of a public sector are explicitely modelled and to consider public policy decisions on taxes,

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lump sum transfers and (possibly) the provision of public goods as decisions to be kept separated from the analysis of the competitive functioning of the economy.

Most of the normative theory is done in relation with Samuelson's definition of public goods. A (pure) public good, more precisely a collective consumption good, is formally defined by Samuelson [45, 46] as a good whose each individual's consumption leads to no subtraction from any other individual's consumption. In this definition, individual's consumption is put as well for consumer's consumption of the good as for its use as an input by a producer. The problem dealt with by Samuelson is the research of conditions that guarantee optimality of the public goods provision. The conclusion (the two-folds message delivered by Samuelson's papers) is first that optimum exists, is multiple, depending on the particular form of the social utility function. But, the externality in consumers' preferences, inherent to the definition of public goods, prevents any implementation of their optimal provision by a market mechanism: as often noticed, it is not in the interest of individuals to reveal their preferences, a basic prerequisite for a functioning market solution. Likewise, a planning procedure would require from an omniscient planner to know all consumers' marginal rates of substitution between private and public goods in order to set personalized prices which would allow for financing the chosen optimal public goods provision.[1]

The different characteristics (non-excludability, non-rejectability) assigned by Samuelson to the restrictive definition of (pure) public goods may be combined in more flexible ways [41, 43, 60, 61] with a more specific sharing of the produced public goods, more subtle characteristics may be introduced in the analysis, more sophisticated modellings of externality may be proposed which take in account phenomena of congestion and cost of access to public goods or explicitly introduce transformation technologies of produced public goods into shared consumption goods [47, 59]. Public goods may also be simply modelled as states of the world that affect the utility level of consumers and shape the technological sets of producers. All these different variants in the definition of externality may be accomodated to fit with the optimality and the equilibrium analysis which make the content of equilibrium theories. The basic hypothesis is still that the domain of consumer's sovereignty should be extended to the choice of the amount of public goods to be provided; the basic methodology is that optimum should be shown to be achieved through a decentralized process where public goods are produced by producers and sold to consumers. It is under these basic hypothesis and methodology that the institutional questions raised at the beginning of this introduction aim to finding an answer founded on a purely economic ground.

In order to see how equilibrium theory fulfills this research programme in response to the open questions raised by Samuelson, we will first set the general framework of a competitive private ownership production economy with public goods. Samuelson's definition of (pure) public goods will be used as a first approximation, a benchmark for ulterior study of more complicated consumption phenomena. The objective is to precise, in this canonical framework, existence, properties, significance and limits of two equilibrium concepts refered to by Samuelson: the private provision equilibrium popularized thirty years after Samuelson's articles by the famous paper of Bergstrom, Blume and Varian [3], and the translation by Foley [21] of what is called by Samuelson the equilibrium solution of Lindahl [34, 35].

We will see that, under very mild assumptions, equilibrium exists in the private provision model, is optimal in a sense of "constrained optimality" that we will define, and belongs to the "constrained core". It even belongs (see [14]) to the set of "constrained Edgeworth equilibria" and constrained

[1]The same conclusion will hold with the study of more dynamic planning procedures as [10, 36, 37] which require too much information on individual characteristics for the social planner.

Edgeworth equilibria can be decentralized with prices as equilibria of the original private provision model. While (see [16]) constrained Pareto optimal allocations can be decentralized as equilibria of an economy identical to the original one except for a convenient redistribution of consumers' initial endowments and profit shares. In other words, the whole machinery of general equilibrium theory can be applied to this model which is intended to figurate the case when provision of public goods is done by the way of charities, fondations and other corporate social responsability institutions. Since the welfare of each consumer depends not on his own provision but on the total provision of public goods, the private provision model appears as a particular case of more general equilibrium models where individuals value the consumption of the others, whether it is by altruism, envy or simply because they look at their relative wealth. As far as provision of public goods is the unique individual external concern of agents, private provision equilibrium is sub-optimal for the only optimality notion which makes sense in public goods provision theory. This sub-optimality is considered as the main drawback of the private provision model and calls for solutions to this market failure.

Under the same mild assumptions, Lindahl–Foley equilibrium exists. Optimality of equilibrium is a direct consequence of the definitions. However, decentralizing optimal allocations has not the same interpretation in terms of redistribution of the initial wealth as in the case where all goods are private. A possible solution is in the introduction of a third equilibrium concept, defined by Foley in [20, 21], reminiscent of the Wicksellian [62] principle of unanimity and voluntary consent in the matching of public expenditure and taxation. As in [30], it will be denominated in this paper, Wicksell–Foley public competitive equilibrium.

Lindahl–Foley equilibrium allocations are easily seen to be Wicksell–Foley public competitive equilibrium allocations and, as such, (weakly) Pareto optimal allocations. However their belonging to the core (a property which gives, according to Foley [21], more rationale to considering Lindahl–Foley equilibrium allocations) requires two additional assumptions, made by Foley and many others public goods provision theorists. Namely that consumers' preferences be monotonely increasing with the consumed amount of each public good and that using public goods be unnecessary in the public goods production. Then, exactly as in private goods economies, optimal allocations can be decentralized with Lindahl–Foley prices as Lindahl–Foley equilibria, after redistribution of initial endowments and profit shares of consumers. Moreover, under these two assumptions, it was directly proved in [18] that the core of a public good economy is nonempty, as well as the set of Edgeworth equilibria corresponding to a convenient definition of replication of a public goods economy. Lindahl–Foley equilibria are obtained by decentralizing with Lindahl-Foley prices Edgeworth equilibria. So that the whole machinery of equilibrium theory can be applied to the Lindahl–Foley model, establishing a complete symmetry between Lindahl–Foley equilibrium for public goods economies and Walras equilibrium for private goods economies.

However, the two previous assumptions prevent any application of the model to analysis of negative externalities (public bads) and do not fit with the empirical evidence that most public goods, besides being the extreme case of externalities in consumption, enter also as production factors which influence the firm's ability to produce (think of education, health, research, transportation means, etc...). This paper will show that these unpleasant assumptions are necessary neither for the definition nor for the existence as well of private provision equilibrium as of Lindahl–Foley equilibrium.

Before closing this introduction, it is necessary to stress that the results reported until now strongly depend, as we will see, on convexity assumptions on preferences and production. However, externalities may generate fundamental non-convexities (see [54, 55]). More simply, many so-called collective or public goods are also classical examples of decreasing costs. Non-convexity on the production side requires government intervention for enforcing pricing rules and the design of revenue distribution rules allowing consumers to survive and to finance a possible deficit in the production of public goods. This adds new difficulties in the definition of market mechanisms for public goods provision, To deal with this case and complement this exposition, a companion paper [17] will rely on [26] for conditions of existence of a private provision equilibrium in non-convex production economies, on [5] for existence of Lindahl equilibria, on [33] for the extension of the second welfare theorem in economies with non-convexities and public goods, on [44] for the extension to the non-convex case of the Wicksell–Foley public competitive equilibrium concept. The companion paper will also precise the relations of all results (convex case and nononvex case) with the abundant cost share equilibrium literature [7, 8, 9, 30, 31, 40, 58] that followed in this domain a seminal Mas-Colell paper [39].

The present paper is organized as follows. In Section 2, we set the general model of a competitive private ownership production economy with public goods with its different equilibrium and optimality concepts. In Sections 3 and 4, we give sufficient conditions for equilibrium existence respectively in the private provision and the Lindahl–Foley models. In Sections 5 and 6, coming back to the Samuelson set of questions, we look for decentralisation of optimal allocations and to their relation with the core of the economy. As a conclusion, in Section 7, we show how equilibrium analysis of provision of public goods calls for re-introducing government as an economic agent generally absent from equilibrium models, providing foundations for a theory of economic policy of market economies.

2. The economy and its equilibrium and optimality concepts

We will define equilibrium and optimum concepts in the framework of a canonical private ownership production economy with finitely many agents, a finite set L of private goods and a finite set K of public goods

$$\mathcal{E} = \Big(\langle \mathbb{R}^L \times \mathbb{R}^K, \mathbb{R}^L \times \mathbb{R}^K\rangle, (X_i, P_i, e_i)_{i\in I}, (Y_j)_{j\in J}, (\theta_{ij})_{\substack{i\in I\\ j\in J}}\Big)$$

in which the existence of public goods entering as arguments in the consumers' preferences is the only considered externality.

- $\mathbb{R}^L \times \mathbb{R}^K$, canonically ordered, is the commodity space and price space of the model. As usual, we will denote by $(p, p^g)\cdot(z, z^g) = p\cdot z + p^g \cdot z^g$ the evaluation of $(z, z^g) \in \mathbb{R}^L \times \mathbb{R}^K$ at prices $(p, p^g) \in \mathbb{R}^L \times \mathbb{R}^K$.
- There is a finite set I of consumers who jointly consume private goods and a same amount of public goods that they eventually provide. Each consumer i has a consumption set $X_i \subset \mathbb{R}^L \times \mathbb{R}^K$, a preference correspondence $P_i\colon \prod_{h\in I} X_h \to X_i$ to be precisely defined below and an initial endowment $e_i \in \mathbb{R}^L \times \mathbb{R}^K$. The interpretation of X_i and, consequently, the definition of P_i are different in the private provision model and in the Lindahl–Foley model.
  - In the private provision model (see [56]), for a generic element $(x_i, x_i^g) \in X_i$, x_i is the private commodity consumption of consumer i, while x_i^g denotes his **private provision of public goods**. If π^G denotes the projection onto $\mathbb{R}^K$ of $\mathbb{R}^L \times \mathbb{R}^K$, consumer i's preferences are typically represented by a correspondence $P_i\colon X_i \times \prod_{h\neq i} \pi^G(X_h) \to X_i$ which indicates for each $x = \big((x_i, x_i^g), (x_h^g)_{h\neq i}\big) \in X_i \times \prod_{h\neq i} \pi^G(X_h)$ the set $P_i(x)$ of the elements of X_i that consumer i prefers to (x_i, x_i^g) **taking as**

given the public good provisions of the other consumers whose sum, added to his own provision, determines the amount of public goods he actually enjoys.[2]

– In the Lindahl–Foley model, for a generic element $(x_i, G_i) \in X_i$ of consumer i's choice set, x_i is still the private commodity consumption of consumer i, while the components of the vector G_i denote the amount of each public good that household i claims. Since the existence of public goods is the only considered externality of our economy, $P_i \colon X_i \to X_i$ is simply a correspondence expressing a binary relation on X_i.

- There is a finite set J of producers which jointly produce private and public goods. Each firm is characterized by a production set $Y_j \subset \mathbb{R}^L \times \mathbb{R}^K$. We denote by (y_j, y_j^g) a generic point of Y_j. $Y = \sum_{j\in J} Y_j$ denotes the total production set.
- For every firm j and each consumer i, the firm shares $0 \leq \theta_{ij} \leq 1$ classically represent a contractual claim of consumer i on the profit of firm j when it faces a price $(p, p^g) \in \mathbb{R}^L \times \mathbb{R}^K$. In a core and Edgeworth equilibrium approach, the relative shares θ_{ij} reflect consumer's stock holdings which represent proprietorships of production possibilities and $\theta_{ij} Y_j$ is interpreted as a technology set at i's disposal in Y_i. As usual, $\sum_{i\in I} \theta_{ij} = 1$, for each j.

In a model where consumers are supposed to privately provide an amount of public goods that they all jointly consume, **feasibility of a consumption-private provision allocation** $(x_i, x_i^g)_{i\in I}$ is expressed by the relation

$$\sum_{i\in I} (x_i, x_i^g) \in \left\{ \sum_{i\in I} e_i \right\} + \sum_{j\in J} Y_j.$$

Definition 2.1. *A* ***private provision equilibrium*** *of $\mathcal{E}$ is a t-uple*

$$\big((\overline{x}_i, \overline{x}_i^g)_{i\in I}, (\overline{y}_j, \overline{y}_j^g)_{j\in J}, (\overline{p}, \overline{p}^g) \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j \times (\mathbb{R}^L \times \mathbb{R}^K) \setminus \{0\}$$

such that

(1) *for every $j \in J$, for every $(y_j, y_j^g) \in Y_j$, $(\overline{p}, \overline{p}^g) \cdot (y_j, y_j^g) \leq (\overline{p}, \overline{p}^g) \cdot (\overline{y}_j, \overline{y}_j^g)$,*

(2) *for every $i \in I$, given the provisions $(\overline{x}_h^g)_{h\neq i}$ of the other consimers, $(\overline{x}_i, \overline{x}_i^g)$ is optimal for the correspondence $P_i \colon X_i \times \prod_{h\neq i} \pi^G(X_h) \to X_i$ in the budget set*

$$B_i(\overline{p}, \overline{p}_g) = \big\{(x_i, x_i^g) \in X_i \colon \overline{p} \cdot x_i + \overline{p}^g \cdot x_i^g \leq (\overline{p}, \overline{p}^g) \cdot e_i + \sum_{j\in J} \theta_{ij} (\overline{p}, \overline{p}^g) \cdot (\overline{y}_j, \overline{y}_j^g)\big\},$$

(3) $\sum_{i\in I} (\overline{x}_i, \overline{x}_i^g) = \sum_{i\in I} e_i + \sum_{j\in J} (\overline{y}_j, \overline{y}_j^g)$.

Condition (1) states that each firm maximizes its profit taking as given the vector price $(\overline{p}, \overline{p}^g)$. Condition (2) states that, taking as given prices and the public good provisions of the other consumers, $(\overline{x}_i, \overline{x}_i^g)$ is an optimal choice for consumer i in a budget set where he pays at the

[2]For exemple, if $u^i \colon \mathbb{R}^L \times \mathbb{R}^K \to \mathbb{R}$ denotes the utility function of consumer i (depending on his consumption of private goods and the total provision of public goods), for $\big((x_i, x_i^g), (x_h^g)_{h\neq i}\big) \in X_i \times \prod_{h\neq i} \pi^G(X_h)$, $P_i\big((x_i, x_i^g), (x_h^g)_{h\neq i}\big) = \{(x_i', x_i'^g) \in X_i \colon u_i(x_i', x_i'^g + \sum_{h\neq i} x_h^g) > u_i(x_i, x_i^g + \sum_{h\neq i} x_h^g)\}$.

common market equilibrium price his consumption of private goods and his provision of public goods. Condition (3) states the feasibility of the equilibrium allocation as defined above.

In the Lindahl–Foley model, at a feasible private and public goods consumption allocation all consumers consume a same amount of public goods. For the sake of coherence, we assume that consumers have **no endowment in public goods** and set $e_i = (\omega_i, 0)$ where $\omega_i \in \mathbb{R}^L$ is the private good endowment of consumer i. **Feasibility of a pair** $\big((x_i)_{i\in I}, G\big)$ where for every $i \in I$, $(x_i, G) \in X_i$ is expressed by the relation

$$(\sum_{i\in I} x_i, G) \in \left\{\sum_{i\in I}(\omega_i, 0)\right\} + \sum_{j\in J} Y_j.$$

Definition 2.2. *A **Lindahl–Foley equilibrium** of $\mathcal{E}$ is a t-uple*

$$\big((\overline{x}_i, \overline{G})_{i\in I}, (\overline{y}_j, \overline{y}_j^g)_{j\in J}, (\overline{p}, \overline{p}_i^g)\big) \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j \times \big((\mathbb{R}^L \times \mathbb{R}^{|I|K}) \setminus \{0\}\big)$$

such that:

(1) *for every $j \in J$, for every $(y_j, y_j^g) \in Y_j$, $(\overline{p}, \sum_{i\in I} \overline{p}_i^g) \cdot (y_j, y_j^g) \leq (\overline{p}, \sum_{i\in I} \overline{p}_i^g)) \cdot (\overline{y}_j, \overline{y}_j^g)$,*

(2) *for every $i \in I$, $(\overline{x}_i, \overline{G})$ is optimal for the correspondence $P^i \colon X_i \to X_i$ in the budget set*

$$B_i(\overline{p}, \overline{p}_i^g)) = \big\{(x_i, G) \in X_i \colon \; \overline{p} \cdot x_i + \overline{p}_i^g \cdot G \leq \overline{p} \cdot \omega_i + \sum_{j\in J} \theta_{ij}(\overline{p}, \sum_{i\in I} \overline{p}_i^g) \cdot (\overline{y}_j, \overline{y}_j^g)\big\},$$

(3) $(\sum_{i\in I} \overline{x}_i, \overline{G}) = \sum_{i\in I}(\omega_i, 0) + \sum_{j\in J}(\overline{y}_j, \overline{y}_j^g)$.

If, in the previous definition, we set $\overline{p}^g = \sum_{i\in I} \overline{p}_i^g$, each $\overline{p}_i^g$ can be thought of as a **vector of personalized consumption public good prices** for the consumer i (See for example Foley [21], Milleron [42, Section 3]), while $\overline{p}^g$ is the vector of production public good prices. Then, as in the definition of private provision equilibrium, Condition (1) means that each firm maximizes its profit taking as given the common vector price $(\overline{p}, \overline{p}^g)$. Condition (2) means that each consumer chooses a consumption of private goods and claims an amount of public goods provision, so as to optimize his preferences in his budget set taking as given the common price of private goods and his personalized price vector for public goods. With Condition (3), equilibrium is characterized by feasibility of the allocation, as defined in the Lindahl–Foley model, thus, in particular, by an unanimous consent on the amount of public goods to be produced.

In the next equilibrium definition, feasibility of the equilibrium allocation is defined as in Lindahl–Foley equilibrium, but the vector of personalized public good prices is replaced by a vector of personalized taxes whose sum is equal to the equilibrium cost of the private goods used for producing the equilibrium provision of public goods. In order to understand the equilibrium conditions, let us call **government proposal relative to the price system** (p, p^g) a couple $\big(G, (t_i)_{i\in I}\big)$ of an amount of pubic goods provision with taxes to pay for it. Besides classical market clearing, an additional equilibrium mechanism guarantees, given the equilibrium prices, an unanimous negative consensus on the equilibrium government proposal.

Definition 2.3. *A **Wicksell–Foley public competitive equilibrium** of $\mathcal{E}$ is a t-uple*

$$\big((\overline{x}_i, \overline{G})_{i\in I}, (\overline{y}_j, \overline{y}_j^g)_{j\in J}, (\overline{p}, \overline{p}^g), (\overline{t}_i)_{i\in I}\big) \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j \times \big((\mathbb{R}^L \times \mathbb{R}^K) \setminus \{0\}\big) \times \mathbb{R}^I$$

such that:

(1) *for every* $j \in J$, *for every* $(y_j, y_j^g) \in Y_j$, $(\overline{p}, \overline{p}^g) \cdot (y_j, y_j^g) \leq (\overline{p}, \overline{p}^g)) \cdot (\overline{y}_j, \overline{y}_j^g) := \pi_j(\overline{p}, \overline{p}^g)$,

(2) *for every* $i \in I$, $(\overline{x}_i, \overline{G})$ *is optimal for the correspondence* $P^i \colon X_i \to X_i$ *in the budget set*

$$B_i(\overline{p}, \overline{p}^g, \overline{t}_i)) = \big\{(x_i, \overline{G}) \in X_i \colon \; \overline{p} \cdot x_i + \overline{t}_i \leq \overline{p} \cdot \omega_i\big\},$$

(3) *There is no* $\big((x_i, G)_{i \in I}, (t_i)_{i \in I}\big) \in \prod_{i \in I} X_i \times \mathbb{R}^I$ *such that* $\sum_{i \in I} t_i = \overline{p}^g \cdot G - \sum_j \pi_j(\overline{p}, \overline{p}^g)$ *with for every* $i \in I$, $(x_i, G) \in P_i(\overline{x}_i, \overline{G})$ *and* $\overline{p} \cdot x_i + t_i \leq \overline{p} \cdot \omega_i$.

(4) $(\sum_{i \in I} \overline{x}_i, \overline{G}) = \sum_{i \in I} (\omega_i, 0) + \sum_{j \in J} (\overline{y}_j, \overline{y}_j^g)$ *and* $\sum_{i \in I} \overline{t}_i = \overline{p}^g \cdot \overline{G} - \sum_{j \in J} \pi_j(\overline{p}, \overline{p}^g)$.

In the previous definition, Wicksell–Foley public competitive equilibrium involves profit maximization by producers (Condition (1)), optimization by consumers of their private goods consumption, given the equilibrium provision of public goods, under the after-tax budget constraint (Condition (2)), and the impossibility of finding a new government proposal such that the sum of taxes together with the sum of equilibrium profits finances the provision of public good and that appears to every consumer to leave him better off (Condition (3)). Condition (4) adds to feasibility of the equilibrium allocation the requirement that together with the sum of equilibrium profits, the sum of equilibrium taxes finances the equilibrium value of public goods. In view of Condition (1), it is readily seen that $\sum_{i \in I} \overline{t}_i = -\overline{p} \cdot \sum_{j \in J} \overline{y}_j = \overline{p} \cdot \sum_{i \in I} (\omega_i - \overline{x}_i)$.

Notice that in each equilibrium definition, and in view of feasibility of equilibrium allocation and profit maximization, it is easily seen that the budget constraint of each consumer is bound at equilibrium. The next proposition precises the relation between Lindahl–Foley and Wicksell–Foley public competitive equilibrium.

Proposition 2.1. *Setting* $\overline{p}^g = \sum_{i \in I} \overline{p}_i^g$ *and* $\overline{t}_i = \overline{p}_i^g \cdot G - \sum_{j \in J} \theta_{ij}(\overline{p} \cdot \overline{y}_j + \overline{p}^g \cdot \overline{y}_j^g)$, *a Lindahl–Foley equilibrium* $\big((\overline{x}_i, \overline{G})_{i \in I}, (\overline{y}_j, \overline{y}_j^g)_{j \in J}, (\overline{p}, \overline{p}_i^g)\big)$ *is a Wicksell–Foley public competitive equilibrium.*

Proof. Condition (1) in Definition 2.3 follows from Condition (1) in Definition 2.2. For each $i \in I$, $B_i(\overline{p}, \overline{p}^g, \overline{t}_i)) \subset B_i(\overline{p}, \overline{p}_I^g))$ and $(\overline{x}_i, \overline{G}) \in B_i(\overline{p}, \overline{p}_I^g)) \cap B_i(\overline{p}, \overline{p}^g, \overline{t}_i))$, so that Condition (2) in Definition 2.3 follows from Condition (2) in Definition 2.2. To verify Condition (3) in Definition 2.3, assume by contraposition that $\big((x_i, G)_{i \in I}, (t_i)_{i \in I}\big) \in \prod_{i \in I} X_i \times \mathbb{R}^I$ verifies $\sum_{i \in I} t_i = \overline{p}^g \cdot G - \sum_j \pi_j(\overline{p}, \overline{p}^g)$ with for every $i \in I$, $(x_i, G) \in P_i(\overline{x}_i, \overline{G})$ and $\overline{p} \cdot x_i + t_i \leq \overline{p} \cdot \omega_i$. From Condition (2) in Definition 2.2, we deduce for each $i \in I$, $\overline{p} \cdot x_i + \overline{p}_i^g \cdot G > \overline{p} \cdot \omega_i + \sum_{j \in J} \theta_{ij}(\overline{p}, \sum_{i \in I} \overline{p}_i^g) \cdot (\overline{y}_j, \overline{y}_j^g)$ and summing over i, $\overline{p} \cdot \sum_{i \in I} x_i + \overline{p}^g \cdot G > \overline{p} \cdot \sum_{i \in I} \omega_i + \sum_{j \in J} (\overline{p}, \overline{p}^g) \cdot (\overline{y}_j, \overline{y}_j^g)$. But, in view of the condition on the sum of taxes, one has also: $\overline{p} \cdot \sum_{i \in I} x_i + \overline{p}^g \cdot G \leq \overline{p} \cdot \sum_{i \in I} \omega_i + \sum_{j \in J} (\overline{p}, \overline{p}^g) \cdot (\overline{y}_j, \overline{y}_j^g)$, which yields a contradiction. ■

As usually, the quasiequilibrium definitions keep in each model the profit maximization and feasibility conditions of equilibrium and replace preference optimization in the budget set by the requirement that each consumer binds its budget constraint and could not be strictly better off spending strictly less.

Definition 2.4. *A* ***private provision quasiequilibrium*** *of* $\mathcal{E}$ *is a t-uple*

$$\big((\overline{x}_i, \overline{x}_i^g)_{i \in I}, (\overline{y}_j, \overline{y}_j^g)_{j \in J}, (\overline{p}, \overline{p}^g) \in \prod_{i \in I} X_i \times \prod_{j \in J} Y_j \times (\mathbb{R}^L \times \mathbb{R}^K) \setminus \{0\}$$

verifying conditions (*1*) *and* (*3*) *of Definition 2.1 and*

(2') *for every* $i \in I$, $\overline{p} \cdot \overline{x}_i + \overline{p}^g \cdot \overline{x}_i^g = (\overline{p}, \overline{p}^g) \cdot e_i + \sum_{j \in J} \theta_{ij}(\overline{p}, \overline{p}^g) \cdot (\overline{y}_j, \overline{y}_j^g)$ *and* $(x_i, x_i^g) \in P_i\big((\overline{x}_i, \overline{x}_i^g), (\overline{x}_h^g)_{h \neq i}\big) \Rightarrow \overline{p} \cdot x_i + \overline{p}^g \cdot x_i^g \geq \overline{p} \cdot \overline{x}_i + \overline{p}^g \cdot \overline{x}_i^g$.

If for some $i \in I$, $(x_i, x_i^g) \in P_i\big((\overline{x}_i, \overline{x}_i^g), (\overline{x}_h^g)_{h \neq i}\big)$ *actually implies* $\overline{p} \cdot x_i + \overline{p}^g \cdot x_i^g > \overline{p} \cdot \overline{x}_i + \overline{p}^g \cdot \overline{x}_i^g$, *the private provision quasi-equilibrium is said to be* ***non-trivial***.

Definition 2.5. *A* ***Lindahl–Foley quasiequilibrium*** *of* $\mathcal{E}$ *is a t-uple*

$$\big((\overline{x}_i, \overline{G})_{i \in I}, (\overline{y}_j, \overline{y}_j^g)_{j \in J}, (\overline{p}, (\overline{p}_i^g)_{i \in I})\big) \in \prod_{i \in I} X_i \times \prod_{j \in J} Y_j \times \big((\mathbb{R}^L \times \mathbb{R}^{|I|K} \setminus \{0\}\big)$$

verifying Conditions (1) and (3) of Definition 2.2 and

(2') *for every* $i \in I$, $\overline{p} \cdot \overline{x}_i + \overline{p}_i^g \cdot \overline{G} = \overline{p} \cdot \omega_i + \sum_{j \in J} \theta_{ij}(\overline{p}, \overline{p}^g) \cdot (\overline{y}_j, \overline{y}_j^g)$ *and* $(x_i, G) \in P_i(\overline{x}_i, \overline{G}) \Rightarrow \overline{p} \cdot x_i + \overline{p}_i^g \cdot G \geq \overline{p} \cdot \overline{x}_i + \overline{p}_i^g \cdot \overline{G}$.

If for some $i \in I$, $(x_i, G) \in P_i\big((\overline{x}_i, \overline{G})\big)$ *actually implies* $\overline{p} \cdot x_i + \overline{p}_i^g \cdot G > \overline{p} \cdot \overline{x}_i + \overline{p}_i^g \cdot \overline{G}$, *the Lindahl–Foley quasi-equilibrium is said to be* ***non-trivial***.

As equilibrium concepts, optimality and core concepts are very different in the private provision and in the Lindahl–Foley model or in the public competitive equilibrium model (Recall that Lindahl–Foley and public competitive equilibrium share the same condition for feasibility of an allocation).

Definition 2.6. *A Lindahl–Foley feasible consumption allocation* $\big((\overline{x}_i)_{i \in I}, \overline{G}\big)$ *is (weakly)* ***Pareto optimal*** *if there exists no Lindahl–Foley feasible consumption allocation* $\big((x_i)_{i \in I}, G\big)$ *such that*

$$(x_i, G) \in P_i(\overline{x}_i, \overline{G}) \text{ for each } i \in I.$$

Let $S \subset I$, $S \neq \emptyset$ be a **coalition**.

Definition 2.7. ***Lindahl–Foley feasibility for*** S *of the pair* $\big((x_i)_{i \in S}, G^S\big)$, *where for each* $i \in S$, $(x_i, G^S) \in X_i$, *is defined by*

$$(\sum_{i \in S} x_i, G^S) \in \left\{ \sum_{i \in S} (\omega_I, 0) \right\} + \sum_{i \in S} \sum_{j \in J} \theta_{ij} Y_j.$$

The Lindahl–Foley S*-feasible pair* $\big((x_i)_{i \in S}, G^S\big)$ ***improves upon*** *or* ***blocks*** *the Lindahl–Foley feasible allocation* $\big((\overline{x}_i)_{i \in I}, \overline{G}\big)$ *if*

$$(x_i, G^S) \in P_i(\overline{x}_i, \overline{G}) \text{ for each } i \in S.$$

The ***core*** $\mathcal{C}(\mathcal{E})$ *is the set of all Lindahl–Foley feasible consumption allocations that no coalition can improve upon.*

In the private provision model, preferences of each agent are constrained by the public good private provisions of the other agents. For this reason, we will speak of constrained optimality (some kind of second best concept) and of constrained core.

Definition 2.8. *A feasible consumption allocation* $(\overline{x}_i, \overline{x}_i^g)_{i \in I}$ *of the private provision model is (weakly)* ***constrained Pareto optimal*** *if there is no feasible consumption allocation* $(x_i, x_i^g)_{i \in I}$ *such that*

$$(x_i, x_i^g) \in P_i\big((\overline{x}_i, \overline{x}_i^g), (\overline{x}_h^g)_{h \neq i}\big) \text{ for each } i \in I.$$

If we set $G^S = \sum_{i \in S} x_i^g$, feasibility for the coalition S of $(x_i, x_i^g)_{i \in S} \in \prod_{i \in S} X_i$ in the private provision model corresponds to S-feasibility of $\big((x_i)_{i \in S}, G^S\big)$ in the Lindahl–Foley model.

Definition 2.9. *In the private provision model, the coalition S improves upon the feasible consumption allocation $(\overline{x}_i, \overline{x}_i^g)_{i \in I}$ via the S-feasible allocation $(x_i, x_i^g)_{i \in S} \in \prod_{i \in S} X_i$ if*

$$(x_i, x_i^g) \in P_i\big((\overline{x}_i, \overline{x}_i^g), (\overline{x}_h^g)_{h \neq i}\big) \quad \text{for each } i \in S.$$

*The **constrained core** $\mathcal{C}^c(\mathcal{E})$ is the set of all feasible consumption allocations of the private provision model that no coalition can improve upon.*

It simply follows from the definitions that a private provision equilibrium consumption allocation is (weakly) constrained Pareto optimal and belongs to the constrained core $\mathcal{C}^c(\mathcal{E})$) of the economy. It even belongs to the set of "constrained Edgeworth equilibria" (See [14] for a definition and a study of conditions which allow for a decentralization with prices of a constrained Edgeworth equilibrium allocation as a private provision equilibrium consumption allocation). But **a private provision equilibrium consumption allocation has no reason to lead to a Pareto optimal provision of public goods**. In suitably defined private provision models, one can verify that, unlike Lindahl–Foley equilibrium consumption allocations, consumption–private provision equilibrium allocations do not satisfy Samuelson's first order conditions for optimality and examples abound in the litterature [3, 56] of redistributions of initial endowments that increase the equilibrium public provision of public goods, thus are Pareto improving in case of monotone increasing preferences relative to public goods. With the free-riding problem, sub-optimality of equilibrium is the main drawback of the private provision equilibrium concept.

In counterpart, it also simply follows from the definitions that a Lindahl–Foley equilibrium consumption allocation is (weakly) Pareto optimal but does not necessarily belong to the core $\mathcal{C}(\mathcal{E})$. We will give in Section 4 sufficient conditions for Lindahl–Foley equilibrium allocations to belong to the core. In the two next sections, we study the consistency of the just defined equilibrium concepts by looking for conditions of existence of private provision equilibria as well as of Lindahl–Foley equilibria.

3. Quasiequilibrium and equilibrium existence in the private provision model

In the framework of the private provision model, in order to get an equilibrium result, we will use the strategy of proof of Shafer–Sonnenschein [51] associating to the private provision economy an abstract economy and its conditions for equilibrium existence. However, we depart from [51] in three respects. We look for an equilibrium without disposal (as in Shafer [49]). We also first prove the existence of a quasiequilibrium, before looking for additional conditions under which the quasi-equilibrium is an equilibrium. For this, we use a definition and existence result of quasiequilibrium in abstract economies borrowed from a paper of mine [12], unpublished in 1980, published in french in 1981 in citeFlo81, and used (with all details of its proof) in subsequent published papers (see, for example, [15]). Finally, using ideas of Gale–Mas-Colell [23, 24], we slightly weaken the continuity assumptions on preference correspondences used by Shafer–Sonnenschein.

Let us first give the definition of an abstract economy and its quasi-equilibrium.

Definition 3.1. *An **abstract economy** (or **generalized qualitative game**) is completely specified by*

$$\Gamma = ((X_i, \alpha_i, P_i)_{i \in N})$$

where N is a finite set of agents (players) and, for each $i \in N$,

- *X_i is a **choice set** (or **strategy set**),*

- *the correspondence*
$$\alpha_i \colon \prod_{k \in N} X_k \to X_i$$
*is called **constraint correspondence** (or **feasible strategy correspondence**),*
- *the correspondence*
$$P_i \colon \prod_{k \in N} X_k \to X_i$$
*is a **preference correspondence**.*

Let $X = \prod_{k \in N} X_k$. For each $x \in X$, $\alpha_i(x)$ is interpreted as the set of possible strategies for player i, given the choice $(x_k)_{k \neq i}$ of the other players. For each $x \in X$, under the condition that $x_i \notin P_i(x)$, $P_i(x)$ is interpreted as the set of elements of X_i strictly preferred by player i to x_i when the choice of the other players is $(x_k)_{k \neq i}$.

Now, for each $i \in N$, let $\beta_i \colon \prod_{k \in N} X_k \to X_i$ be a correspondence satisfying for all $x \in \prod_{k \in N} X_k$

$$\beta_i(x) \in \alpha_i(x), \tag{3.1}$$

$$\text{if } \beta_i(x) \neq \emptyset\text{, then } \operatorname{cl}(\beta_i(x)) = \operatorname{cl}(\alpha_i(x)). \tag{3.2}$$

Definition 3.2. *Given $\beta = (\beta_i)_{i \in N}$ as above, $\overline{x} = (\overline{x}_i)_{i \in N} \in X$ is a β-**quasiequilibrium** of Γ if for each for each $i \in N$,*

(1) $\overline{x}_i \in \alpha_i(\overline{x})$
(2) $P_i(\overline{x}) \cap \beta_i(\overline{x}) = \emptyset$.

It is an equilibrium if or each for each $i \in N$,

(1) $\overline{x}_i \in \alpha_i(\overline{x})$
(2) $P_i(\overline{x}) \cap \alpha_i(\overline{x}) = \emptyset$.

Lemma 3.3. *Let $\Gamma = \big((X_i, \alpha_i, P_i)_{i \in N}, X\big)$ be an abstract economy with a finite set N of agents where X is a nonempty closed convex subset of $\prod_{i \in N} X_i$, and $\alpha_i \colon X \to X_i$ and $P_i \colon X \to X_i$ are correspondences respectively interpreted as constraint and preference correspondences for agent i. Let for each $i \in N$, $\beta_i \colon X \to X_i$ satisfying the above relations 3.1 and 3.2. Assume that for every $i \in N$,*

(a) *X_i is a nonempty compact convex subset of some inite dimensional Euclidean vector space,*
(b) *α_i is an upper semicontinuous and nonempty closed convex valued correspondence,*
(c) *β_i is convex valued,*
(d) *P_i is a convex valued correspondence such that for all $x \in \prod_{k \in N} X_k$ $x_i \notin P_i(x)$ and the correspondence $x \to \beta_i(x) \cap P_i(x)$ is lower semicontinuous.*

Then Γ has a β-quasiequilibrium $\overline{x}$. It is an equilibrium (that is, for each $i \in N$, $P_i(\overline{x}) \cap \alpha_i(\overline{x}) = \emptyset$) provided that for every i, $\beta_i(\overline{x}) \neq \emptyset$ and $P_i(\overline{x})$ is open in X_i.

Coming back to the economy $\mathcal{E}$, let

$$A(\mathcal{E}) = \left\{ \big((x_i, x_i^g)_{i \in I}, (y_j, y_j^g)_{j \in J}\big) \in \prod_{i \in I} X_i \times \prod_{j \in J} Y_j \colon \sum_{i \in I} (x_i, x_i^g) = \sum_{i \in I} e_i + \sum_{j \in J} (y_j, y_j^g) \right\}$$

be the set of feasible allocations. Let $\widehat{X}_i$, $\widehat{X}$ and $\widehat{Y}_j$ denote the projections of $A(\mathcal{E})$ respectively on X_i, $\prod_{i \in I} X_i$ and Y_j. We make on $\mathcal{E}$ the following assumptions.

A.1: For each $i \in I$,

(a) X_i is convex and closed, and $\widehat{X}_i$ is compact,

(b) $P_i \colon X_i \times \prod_{h \neq i} \pi^G(X_h) \to X_i$ is lower semicontinuous,

(c) For each $(x_h, x_h^g)_{h \in I} \in \widehat{X}$, $P_i\big((x_i, x_i^g), (x_h^g)_{h \neq i}\big)$ is convex, and $(x_i, x_i^g) \in \operatorname{cl} P_i\big((x_i, x_i^g), (x_h^g)_{h \neq i}\big) \setminus P_i\big((x_i, x_i^g), (x_h^g)_{h \neq i}\big)$,

(d) $e_i \in X_i - \sum_{j \in J} \theta_{ij} Y_j$;

A.2: For each $j \in J$, Y_j is convex and closed, and $\widehat{Y}_j$ is compact.

The following proposition is proved using Lemma 3.3 and standard techniques explained in [49, 16].

Proposition 3.1. *Under Assumptions **A.1** and **A.2**, the private provision economic model $\mathcal{E}$ has a quasiequilibrium $\big((\overline{x}_i, \overline{x}_i^g)_{i \in I}, (\overline{y}_j, \overline{y}_j^g)_{j \in J}, (\overline{p}, \overline{p}^g) \in \prod_{i \in I} X_i \times \prod_{j \in J} Y_j \times (\mathbb{R}^L \times \mathbb{R}^K) \setminus \{0\}$. Under the additional continuity assumption that each correspondence P_i has open (in X_i) values at every $(x_h, x_h^g) \in \widehat{X}$, the quasiequilibrium is nontrivial if $\sum_{i \in I} e_i \in \operatorname{int}\big(\sum_{i \in I} X_i - \sum_{j \in J} Y_j\big)$.*

Note that the quasiequilibrium price $(\overline{p}, \overline{p}^g)$ is nonnull but, obviously, not necessarily positive. Replacing in Assumption **A.1** (b) by

(b') Each correspondence $P_i \colon X_i \times \prod_{h \neq i} \pi^G(X_h) \to X_i$ has open lower sections in $X_i \times \prod_{h \neq i} \pi^G(X_h)$ and open (in X_i) values at every $(x_h, x_h^g) \in \widehat{X}$.

the same quasiequilibrium existence result can be proved using as in [14] the definition of Edgeworth equilibria for this kind of constrained preferences, their existence and their decentralization by nonnull quasiequilibrium prices. If the correspondences P_i are convex valued, Assumption (b') is slightly weaker than the assumption in Shafer–Sonnenschein [50] that the P_i have an open graph, equivalent if correspondences P_i are assumed to have open values for every $(x_h, x_h^g) \in X$ (See a proof of this equivalence in Shafer [48] or in [4]). Our emphasis on the conditions of existence of a non trivial quasiequilibrium is justified by the well-known fact that several irreducibility conditions on the economy guarantee that a non-trivial quasiequilibrium is an equilibrium.

Before closing this section, two remarks are in order.

Remark 3.4. The definition of an abstract economy and the statement of Lemma 3.3 allow for considering more externality in consumers' preferences than the simple existence of public goods. Consumers' preferences may depend more generally on the current consumption and production allocation and on current prices as it was demanded by Arrow–Hahn [2]. In particular, Lemma 3.3 implies equilibrium existence in exchange economies with only private goods where individuals value not only their own consumption but, as in [25], the current consumption allocation.

Remark 3.5. An economy with only private goods and no externality in preferences is a particular case of the private provision model studied in this section. Proposition 3.1 obviously applies to production economies without any public good or externality. This remark will be used in the next section.

4. Quasiequilibrium and equilibrium existence in the Lindahl–Foley model

Economy $\mathcal{E}$ is now the economy associated with a Lindahl–Foley model as explained in the beginning of section 2. We assume in particular that consumers have no endowment in public goods, thus that for each $i \in I$, $e_i = (\omega_i, 0)$ and that the values of $P_i \colon X_i \to X_i$ does not depend on the consumptions of the other consumers. One could get quasiequilibrium existence in the

Lindahl–Foley model using, as for the private provision model, a simultaneous optimization proof based on a slight modification of the definition and the quasiequilibrium existence result for an abstract economy. This is not the strategy generally adopted sincer Foley [20, 21] and that we follow now.

We extend[3] the commodity space by considering each consumer's bundle of public goods as a separate group of commodities. On this $(L+|I|K)$- dimensional commodity space, we associate with $\mathcal{E}$ the production economy

$$\mathcal{E}' = \Big(\langle \mathbb{R}^L \times \mathbb{R}^{|I|K}, \mathbb{R}^L \times \mathbb{R}^{|I|K}\rangle (X'_i, P'_i, e'_i)_{i\in I}, (Y'_j)_{j\in J}, (\theta_{ij})_{\substack{i\in I\\ j\in J}}\Big)$$

defined in the following way:

- For each $i \in I$,
  - $X'_i = \{x'_i = (x_i, 0, \dots, G_i, \dots, 0) \in \mathbb{R}^L \times (\mathbb{R}^K)^{|I|} \colon (x_i, G_i) \in X_i\}$
  - $e'_i = (\omega_i, 0, \dots, 0, \dots, 0)$
  - For each $x'_i = (x_i, 0, \dots, G_i, \dots, 0) \in X'_i$,
    $P'_i(x'_i) = \{\widetilde{x}'_i = (\widetilde{x}_i, 0, \dots, \widetilde{G}_i, \dots, 0) \in X'_i \colon (\widetilde{x}_i, \widetilde{G}_i) \in P_i\big((x_i, G_i)\big)\}$
- For each $j \in J$, $Y'_j = \{y'_j = (y_j, y^g_j, \dots, y^g_j, \dots, y^g_j) \in \mathbb{R}^L \times (\mathbb{R}^K)^{|I|} \colon (y_j, y^g_j) \in Y_j\}$.

Let

$$A(\mathcal{E}) = \left\{ \big((x_i, G)_{i\in I}, (y_j, y^g_j)_{j\in J}\big) \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j \colon \sum_{i\in I} x_i = \sum_{i\in I} \omega_i + \sum_{j\in J} y_j;\ G = \sum_{j\in J} y^g_j \right\}$$

denote the set of feasible Lindahl–Foley allocations of $\mathcal{E}$ and

$$A(\mathcal{E}') = \left\{ \big((x'_i)_{i\in I}, (y'_j)_{j\in j}\big) \in \prod_{i\in I} X'_i \times \prod_{j\in J} Y'_j \colon \sum_{i\in I} x'_i = \sum_{i\in I} e'_i + \sum_{j\in J} y'_j \right\}$$

denote the set of feasible allocations of $\mathcal{E}'$. Under the definitions given above, $\big((x'_i)_{i\in I}, (y'_j)_{j\in j}\big) \in A(\mathcal{E}')$ if and only if $\big((x_i, G)_{i\in I}, (y_j, y^g_j)_{j\in J}\big)$. Let $\widehat{X}_i$ (resp. $\widehat{X}'_i$) and $\widehat{Y}_j$ (resp. $\widehat{Y}'_j$) denote the projections of $A(\mathcal{E})$ (resp. $A(\mathcal{E}')$) on X_i (resp.X'_i) and Y_j (resp. Y'_j).

Economy $\mathcal{E}'$ is an economy with no public good and no externality. In order to apply to this economy the quasiequilibrium existence result obtained in the previous section, we set on $\mathcal{E}$ the following assumptions.

B.1: For each $i \in I$,

(a) $X_i = \mathbb{R}^L_+ \times \mathbb{R}^K_+$ and $\widehat{X}_i$ is compact,

(b) $P_i \colon X_i \to X_i$ is lower semicontinuous,

(c) For each $(x_i, G) \in \widehat{X}_i$, $P_i(x_i, G)$ is convex,
and $(x_i, G) \in \operatorname{cl} P_i(x_i, G) \setminus P_i(x_i, G)$,

(d) $(\omega_i, 0) \in X_i$;

B.2: For each $j \in J$, Y_j is convex and closed, contains $(0,0)$, and $\widehat{Y}_j$ is compact.

It is readily seen that if $\mathcal{E}$ satisfies these assumptions, then $\mathcal{E}'$ satisfies the assumptions of Proposition 3.1. Thus, starting from a quasiequilibrium $\big((\overline{x}'_i)_{i\in I}, (\overline{y}'_j)_{j\in J}, \overline{\pi}\big)$ of $\mathcal{E}'$ with $\overline{\pi} = (\overline{p}, (\overline{p}^g_i)_{i\in I}) \in \mathbb{R}^{L+|I|K} \setminus \{0\}$, and setting $\overline{p}^g = \sum_{i\in I} \overline{p}^g_i$, one deduces from the definition of X'_i and Y'_j that

[3] Independently of Foley, the same strategy was applied in an unpublished paper of F. Fabre-Sender [11], quoted in [42].

there exists $\big((\overline{x}_i,\overline{G}),(\overline{y}_j,\overline{y}^g_j)\big) \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j$ such that $\big((\overline{x}_i,\overline{G}),(\overline{y}_j,\overline{y}^g_j),(\overline{p},(\overline{p}^g_i)_{i\in I})\big)$ is a Lindahl-Foley quasiequilibrium of $\mathcal{E}$. We have thus proved:

Proposition 4.1. *Under the above Assumptions **B.1** and **B.2**, the Lindahl–Foley model $\mathcal{E}$ has a quasiequilibrium* $\big((\overline{x}_i,\overline{G})_{i\in I},(\overline{y}_j,\overline{y}^g_j)_{j\in J},(\overline{p},(\overline{p}^g_i)_{i\in I})\big) \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j \times \big((\mathbb{R}^L \times \mathbb{R}^{|I|K}) \setminus \{0\}\big)$.

The easy proof of the following corollary can be found in [18] and is sketched here for the sake of completeness of the paper .

Corollary 4.1. *Under the additional continuity assumption that each correspondence P_i has open (in X_i) values at every $(x_i,G) \in \widehat{X}_i$, the quasiequilibrium is nontrivial if*

NT: $\sum_{i\in I}(\omega_i,0) \in \mathrm{int}\big(\mathbb{R}^L_+ \times \mathbb{R}^K_+ - \sum_{j\in J} Y_j\big)$.

Proof. Recalling that the quasiequilibrium price $(\overline{p},(\overline{p}^g_i)_{i\in I})$ is nonnull, let $(u,v) \in \mathbb{R}^L \times \mathbb{R}^K$ be such that $(\overline{p},\sum_{i\in I}\overline{p}^g_i)\cdot(u,v) < 0$ and $(\sum_{i\in I},0)+(u,v) \in (\mathbb{R}^L_+ \times \mathbb{R}^K_+) - \sum_{j\in J} Y_j$. one can write for some $(x_i)_{i\in I} \in (\mathbb{R}^L_+)^I$, $G \in \mathbb{R}^K_+$, $(y_j,y^g_j)_{j\in J} \in \prod_{j\in J}$

$$\sum_{i\in I}(\omega_i,0)+(u,v) = \big(\sum_{i\in I} x_i, G\big) - \sum_{j\in J}(y_j,y^g_j).$$

One deduces:

$$\sum_{i\in I}(\overline{p}\cdot x_i + \overline{p}^g_i \cdot G) < \overline{p}\cdot\sum_{i\in I}\omega_i + \sum_{j\in J}(\overline{p},\sum i\in I\overline{p}^g_i)\cdot(y_j,y^g_j)$$
$$\leq \overline{p}\cdot\sum_{i\in I}\omega_i + \sum_{j\in J}(\overline{p},\sum i\in I\overline{p}^g_i)\cdot(\overline{y}_j,\overline{y}^g_j) = \sum_{i\in I}(\overline{p}\cdot\overline{x}_i + \overline{p}^g_i\cdot G).$$

It classically follows that at least one consumer i satisfies at (x_i,G) his budget constraint with a strict inequality, and thus is optimal at $(\overline{x}_i,\overline{G})$ in his budget set as a consequence of the additional continuity assumption made on consumers' preferences. ■

In the following assumption, in order to get informations on respectively the quasiequilibrium price of private goods and the vector of personalized prices of public goods, the local no-satiation assumption contained in Assumption **B.1** (c) is replaced by the stronger assumption of local no-satiation in private goods at any component of a Lindahl–Foley feasible consumption allocation. At every Lindahl–Foley feasible consumption allocation, one assumes in addition global no-satiation in public goods for at least one consumer. This is summarized in the following assumption.

B.3: If $\big((x_h)_{h\in I},G\big)$ is a Lindahl–Foley feasible consumption allocation,
(a) For each $i\in I$, for every neighborhood U of (x_i,G) in X_i, there exists x'_i such that $(x'_i,G)\in U$ and $(x'_i,G)\in P_i((x_i,G))$,
(b) There exists $i\in I$ and G_i such that $(x_i,G_i)\in X_i$ and $(x_i,G_i)\in P_i((x_i,G))$.

Under this assumption, the next proposition completes the previous one. One should note that, besides the fact that **B.3**(a) precises the local no-satiation assumption contained in **B.1**(c), Assumptions **B.3** (a) and (b) are tailored for getting the conclusions (1) and (2) of the next proposition.

Proposition 4.2. *Let $\big((\overline{x}_i,\overline{G})_{i\in I},(\overline{y}_j,\overline{y}^g_j)_{j\in J},(\overline{p},(\overline{p}^g_i)_{i\in I})\big)$ be the quasiequilibrium obtained in Proposition 4.1. If each correspondence P_i has open (in X_i) values at every $(x_i,G)\in\widehat{X}_i$ and if Assumption **B.3** is added to the different assumptions of Proposition 4.1, then*

(1) *The quasiequilibrium price vector for private goods, $\overline{p}$, is nonnull, provided that the quasiequilibrium is non-trivial.*
(2) *If the quasiequilibrium is an equilibrium, then $(\overline{p}_i^g)_{i\in I} \neq 0$.*
(3) *Under some irreducibility condition, a non-trivial quasiequilibrium is actually an equilibrium.*

Remark 4.2. The non-triviality condition **NT** is satisfied under the following mild conditions that the total initial endowment in private goods is strictly positive and that each public good is productible:

- $\omega \gg 0$,
- there is some $(y, y^g) \in Y := \sum_{j\in J} Y_j$ such that $y^g \gg 0$.

Remark 4.3. The role of an irreducibility assumption is to guarantee that, at the quasiequilibrium price, all consumers can satisfy with a strict inequality their budget constraint as soon as this is possible for at least one of them. Such a condition, inspired by Arrow–Hahn [2], was given in [18]:

IR: For any non-trivial partition $\{I_1, I_2\}$ of the set I of consumers and for any Lindahl–Foley feasible consumption allocation $(x_i, G)_{i\in I}$, there exists a consumption allocation $(\widetilde{x}_i, \widetilde{G})_{i\in I}$ and $\widetilde{\omega}$ such that
- $(\widetilde{x}_i, \widetilde{G}) \in \operatorname{cl} P_i(x_i, G)$ $\forall i \in I_1$ with for some $i_0 \in I_1$, $(\widetilde{x}_i, \widetilde{G}) \in P_i(x_i, G)$,
- $(\sum_{i\in I} \widetilde{x}_i, \widetilde{G}) \in \{(\widetilde{\omega}, 0)\} + \sum_{j\in J} Y_j$ with, for each coordinate $\ell \in L$, $\widetilde{\omega}^\ell > \omega^\ell \Rightarrow \sum_{i\in I_2} \omega_i^\ell > 0$.

The obvious interpretation of this condition is that for any partition $\{I_1, I_2\}$ of the set I of consumers into two nonempty subgroups and for each feasible allocation, the group I_1 may be moved to a preferred position feasible with a new vector of total resources in private goods by increasing the total resources of commodities which can be supplied in positive amount by the group I_2. To see how **IR** is an irreducibilty assumption, let $\big((\overline{x}_i, \overline{G})_{i\in I}, (\overline{y}_j, \overline{y}_j^g)_{j\in J}, (\overline{p}, (\overline{p}_i^g)_{i\in I})\big)$ be the quasiequilibrium of $\mathcal{E}$, I_1 be the set of consumers who can verify their budget constraint with a strict inequality, I_2 be the set of consumers for whom this is impossible, a set that we assume to be nonempty. If $(\widetilde{x}_i, \widetilde{G})_{i\in I}$ and $\widetilde{\omega}$ are as in Assumption **IR**, we can write $\widetilde{\omega} - \omega = \alpha \sum_{i\in I_2}(\omega_i - x_i')$ with $\alpha > 0$ and $x_i \in X_i$ $\forall i \in I_2$. Then

$$\overline{p} \cdot \sum_{i\in I_1} (\widetilde{x}_i - \omega_i) + (\sum_{i\in I_1} \overline{p}_i^g) \cdot \widetilde{G} > \sum_{i\in I_1} \theta_{ij} \sum_{j\in J} (\overline{p} \cdot \widetilde{y}_j + \overline{p}^g \cdot \widetilde{y}_j^g),$$

$$(1+\alpha)\big[\overline{p} \cdot \big(\sum_{i\in I_2} \frac{\widetilde{x}_i + \alpha x_i'}{1+\alpha} - \omega_i\big) + (\sum_{i\in I_2} \overline{p}_i^g) \cdot \frac{\widetilde{G}}{1+\alpha}\big] \geq (1+\alpha) \sum_{i\in I_2} \theta_{ij} \sum_{j\in J} (\overline{p} \cdot \widetilde{y}_j + \overline{p}^g \cdot \widetilde{y}_j^g),$$

that is

$$\overline{p} \cdot \big[\sum_{i\in I_2} \widetilde{x}_i - \sum_{i\in I_2} \omega_i + \omega - \widetilde{\omega}\big] + (\sum_{i\in I_2} \overline{p}_i^g) \cdot \widetilde{G} \geq (1+\alpha) \sum_{i\in I_2} \theta_{ij} \sum_{j\in J} (\overline{p} \cdot \widetilde{y}_j + \overline{p}^g \cdot \widetilde{y}_j^g) \geq \sum_{i\in I_2} \theta_{ij} \sum_{j\in J} (\overline{p} \cdot \widetilde{y}_j + \overline{p}^g \cdot \widetilde{y}_j^g),$$

the last inequality following from the positivity of the value of the maximum profit of each producer. Summing in $I \in I$ the first and the third of the previous relations, we get

$$\overline{p} \cdot \sum_{i\in I} (\widetilde{x}_i - \omega_i) + (\sum_{i\in I} \overline{p}_i^g) \cdot \widetilde{G} > \sum_{j\in J} (\overline{p} \cdot \widetilde{y}_j + \overline{p}^g \cdot \widetilde{y}_j^g),$$

which contradicts the feasibility for total resources $\widetilde{\omega}$ of the consumption allocation $(\widetilde{x}_i, \widetilde{G})_{i\in I}$.

Remark 4.4. In the literature, it is sometimes assumed that each consumer $i \in I$ has a strictly positive initial endowment $\omega_i \gg 0$ and a continuous complete preference preorder $\succeq_i$ on his consumptin set X_i, weakly monotone relative to public goods and strictly monotone relative to private goods. Then non-triviality and irreducibilty conditions are of no use and, under the other conditions of Proposition 4.1 and Part (b) of Assumption **B.3**, an equilibrium exists with $\overline{p} \gg 0$ and $(\overline{p}_i^g)_{i\in I} > 0$ (that is ≥ 0 and not equal to 0).

5. Decentralization with prices of (weakly) Pareto optimal Lindahl–Foley feasible consumption allocations

As already noticed, (weak) Pareto optimality of a Lindahl–Foley equilibrium consumption allocation simply follows from the definitions. The same is true for Wicksell–Foley public competitive equilibrium consumption allocations.

Proposition 5.1. *Let* $\big((\overline{x}_i, \overline{G})_{i\in I}, (\overline{y}_j, \overline{y}_j^g)_{j\in J}, (\overline{p}, \overline{p}^g), (\overline{t}_i)_{i\in I}\big)$ *be a Wicksell–Foley public competitive equilibrium. The consumption allocation* $(\overline{x}_i, \overline{G})_{i\in I}$ *is (weakly) Pareto optimal.*

Proof. Assume by contraposition that there is $\big((x_i)_{i\in I}, G\big)$ such that $(x_i, G) \in P_i(\overline{x}_i, \overline{G})$ $\forall i \in I$ and $(\sum_{i\in I} x_i, G) = (\sum_{i\in I} \omega_i, 0) + \sum_{j\in J}(y_j, y_j^g)$. Thus $\sum_{j\in J} y_j = \sum_{i\in I}(x_i - \omega_i)$ and $\sum_{j\in J} y_j^g = G$. In view of Condition (1) in Definition 2.3, for every $j \in J$, $\overline{p} \cdot y_j + \overline{p}^g \cdot y^g \leq \overline{p} \cdot \overline{y}_j + \overline{p}^g \cdot \overline{y}^g = \pi_j(\overline{p}, \overline{p}^g)$ and by summing on i, $\overline{p} \cdot \sum_{j\in J} y_j + \overline{p}^g \cdot G \leq \sum_{j\in J} \pi_j(\overline{p}, \overline{p}^g)$. Let us first set for each i, $t_i = \overline{p} \cdot \omega_i - \overline{p} \cdot x_i$. Summing on i, $\sum_{i\in I} t_i = -\sum_{j\in J} \overline{p} \cdot y_j \geq \overline{p}^g \cdot G - \sum_{j\in J} \pi_j(\overline{p}, \overline{p}^g)$. If $\sum_{i\in I} t_i = \overline{p}^g \cdot G - \sum_{j\in J} \pi_j(\overline{p}, \overline{p}^g)$, the government proposal $(G, (t_i)_{i\in I})$ contradicts Condition (3) of Definition 2.3. If $\sum_{i\in I} t_i > \overline{p}^g \cdot G - \sum_{j\in J} \pi_j(\overline{p}, \overline{p}^g)$, it suffices to define $(t'i)_{i\in I}$ such that $t'_i \leq t_i$ $\forall i \in I$and $\sum_{i\in I} t'_i = \overline{p}^g \cdot G - \sum_{j\in J} \pi_j(\overline{p}, \overline{p}^g)$ to contradict Condition (3) of Definition 2.3,with the government proposal $(G, (t'_i)_{i\in I})$. ■

Weak Pareto optimality of Wicksell–Foley public competitive equilibrium allocations is not surprising. Condition (3) of Definition 2.3 guarantees that, in some sense, the grand coalition cannot block the equilibrium consumption allocation. In view of Proposition 2.1, (weak) Pareto optimality of Lindahl–Foley feasible consumption allocations, obvious from the simple definitions, is also a consequence of the previous proposition and is nothing else than a statement of the first welfare theorem for public goods economies. The purpose of this section is to give converse results.

The next proposition simply extends to convex production sets Theorem of Section 3 in Foley [21].

Proposition 5.2. *Assume* ***B.1****,* ***B.2****,* ***B.3****,* ***NT*** *and* ***IR*** *on* $\mathcal{E}$*. Let* $\big((\overline{x}_i, \overline{G})_{i\in I}, (\overline{y}_j, \overline{y}_j^g)_{j\in J}\big)$ *be a Lindahl–Foley feasible allocation of* $\mathcal{E}$ *such that* $(\overline{x}_i, \overline{G})_{i\in I}$ *is (weakly) Pareto optimal. Then there exists a price system* $(\overline{p}, (\overline{p}_i^g)_{i\in I})$ *such that, setting* $\overline{p}^g = \sum_{i\in I} \overline{p}_i^g$*,*

(1) *for every* $j \in J$*, for every* $(y_j, y_j^g) \in Y_j$*,* $(\overline{p}, \overline{p}^g) \cdot (y_j, y_j^g) \leq (\overline{p}, \overline{p}^g) \cdot (\overline{y}_j, \overline{y}_j^g)\ := \pi_j((\overline{p}, \overline{p}^g)$*,*
(2) *for every* $i \in I$*,* $(x_i, G_i) \in P_i(\overline{x}_i, \overline{G})$ *implies* $\overline{p} \cdot x_i + \overline{p}_i^g \cdot G_i > \overline{p} \cdot \overline{x}_i + \overline{p}_i^g \cdot \overline{G}$*.*

Proof. Let

$$D = \left\{ \big(z, (G_i)_{i\in I}\big) - \big(\sum_{i\in I} \omega_i, 0, \ldots, 0, \ldots, 0\big) \colon z = \sum_{i\in I} x_i \text{ and } (x_i, G_i) \in P_i(\overline{x}_i, \overline{G})\ \forall i \in I \right\},$$

$$F = \left\{ \big(y, (y^g, \dots, y^g, \dots, y^g)\big) \colon (y, y^g) = \sum_{j \in J} (y_j, y_j^g) \text{ with } (y_j, y_j^g) \in Y_j \ \forall j \in J \right\}.$$

The sets D and F are nonempty, convex and it follows from the (weak) Pareto optimality of the allocation that $D \cap F = \emptyset$. From the first separation theorem, there exists $\big(\overline{p}, (\overline{p}_i^g)_{i \in I}\big) \neq 0$ and $\alpha \in \mathbb{R}$ such that for all $j \in J$, for all $(y_j, y_j^g) \in Y_j$, for all $i \in I$, for all $(x_i, G_i) \in P_i(\overline{x}_i, \overline{G})$,

$$\overline{p} \cdot \sum_{i \in I} (x_i - \omega_i) + \sum_{i \in I} \overline{p}_i^g \cdot G_i \geq \alpha \geq \overline{p} \cdot \sum_{j \in J} y_j + (\sum_{i \in I} \overline{p}_i^g) \cdot \sum_{j \in J} y_j^g \ .$$

From local no-satiation at each $(\overline{x}_i, \overline{G})$, and since $\big((\overline{x}_i, \overline{G})_{i \in I}, (\overline{y}_j, \overline{y}_j^g)_{j \in J}\big)$ is a Lindahl–Foley feasible allocation, we get

$$\overline{p} \cdot \sum_{i \in I} (\overline{x}_i - \omega_i) + (\sum_{i \in I} \overline{p}_i^g) \cdot \overline{G} = \alpha = \overline{p} \cdot \sum_{j \in J} \overline{y}_j + (\sum_{i \in I} \overline{p}_i^g) \cdot \sum_{j \in J} \overline{y}_j^g \ .$$

from which we deduce: for every $j \in J$, for every $(y_j, y_j^g) \in Y_j$, $(\overline{p}, \sum_{i \in I} \overline{p}_i^g) \cdot (y_j, y_j^g) \leq (\overline{p}, \sum_{i \in I} \overline{p}_i^g)) \cdot (\overline{y}_j, \overline{y}_j^g)$, and for every $i \in I$, $(x_i, G_i) \in P_i(\overline{x}_i, \overline{G})$ implies $\overline{p} \cdot x_i + \overline{p}_i^g \cdot G_i \geq \overline{p} \cdot \overline{x}_i + \overline{p}_i^g \cdot \overline{G}$.

In view of Assumptions **NT** and **IR**, one shows exactly as in Section 4 that for every $i \in I$, $(x_i, G_i) \in P_i(\overline{x}_i, \overline{G})$ actually implies $\overline{p} \cdot x_i + \overline{p}_i^g \cdot G_i > \overline{p} \cdot \overline{x}_i + \overline{p}_i^g \cdot \overline{G}$. ■

A first corollary of Proposition 5.2 shows that, under the assumptions of the proposition, the set of (weakly) Pareto optimal consumption allocations actually coincides with the set of public competitive equilibrium consumption allocations.

Corollary 5.1. *Assume, as in Proposition 5.2,* ***B.1****,* ***B.2****,* ***B.3****,* ***NT*** *and* ***IR*** *on $\mathcal{E}$ and that $\big((\overline{x}_i, \overline{G})_{i \in I}, (\overline{y}_j, \overline{y}_j^g)_{j \in J}\big)$ is a Lindahl–Foley feasible allocation of $\mathcal{E}$ such that $(\overline{x}_i, \overline{G})_{i \in I}$ is (weakly) Pareto optimal. If $(\overline{p}, (\overline{p}_i^g)_{i \in I})$ is the price vector obtained in Proposition 5.2 and If we set $\overline{p}^g = \sum_{i \in I} \overline{p}_i^g$ and for each $i \in I$, $\overline{t}_i = \overline{p} \cdot \omega_i - \overline{p} \cdot \overline{x}_i$, then $\big((\overline{x}_i, \overline{G})_{i \in I}, (\overline{y}_j, \overline{y}_j^g)_{j \in J}, (\overline{p}, \overline{p}^g), (\overline{t}_i)_{i \in I}\big)$ is a public competitive equilibrium.*

Proof. From (1) in Proposition 5.2, the profit maximization is verified. Moreover, $\sum_{i \in I} \overline{t}_i = \overline{p} \cdot \sum_{i \in I} (\omega_i - \overline{x}_i) = -\overline{p} \cdot \sum_{j \in J} \overline{y}_j = \overline{p}^g \cdot \overline{G} - \sum_{j \in J} \pi_j(\overline{p}, \overline{p}^g)$. If $\sum_{i \in I} t_i = \overline{p}^g \cdot G - \sum_{j \in J} \pi_j(\overline{p}, \overline{p}^g)$ with for each $i \in I$, $(x_i, G) \in P_i(\overline{x}_i, \overline{G})$, one deduces from (2) in Proposition 5.2: $\overline{p} \cdot x_i + \overline{p}_i^g \cdot G > \overline{p} \cdot \overline{x}_i + \overline{p}_i^g \cdot \overline{G}$ $\forall i \in I$, thus $\overline{p} \cdot \sum_{i \in I} x_i + \sum_{i \in I} t_i > \overline{p} \sum_{i \in I} \omega_i$, which proves Condition (3) of Definition2.3. On the other hand, for each $i \in I$, $\overline{p} \cdot \overline{x}_i + t_i = \overline{p} \cdot \omega_i$ and $(x_i, \overline{G}) \in P_i(\overline{x}_i, \overline{G})$ implies $\overline{p} \cdot x_i > \overline{p} \cdot \overline{x}_i$ thus $\overline{p} \cdot x_i + \overline{t}_i > \overline{p} \cdot \omega_i$, which proves Condition (2) of the same definition. ■

To go further on the decentralization with prices of Lindahl–Foley feasible (weakly) Pareto optimal allocations, notice that in Proposition 5.2, $\big((\overline{x}_i, \overline{G})_{i \in I}, (\overline{y}_j, \overline{y}_j^g)_{j \in J}, (\overline{p}, (\overline{p}_i^g)_{i \in I})\big)$ corresponds for the Lindahl–Foley model to what is called **valuation equilibrium** or **equilibrium relative to a price system** in an economy with only private goods. In our public goods economy, with respect to the (weak) Pareto optimal consumption allocation $(\overline{x}_i, \overline{G})_{i \in I}$, prices $(\overline{p}, (\overline{p}_i^g)_{i \in I}, \overline{p}^g = \sum_{i \in I} \overline{p}_i^g)$ have the same normative interpretation as in the usual Second Welfare Theorem. Announced and enforced by a coordinating center, such prices have the property that no consumer will depart from the Pareto optimal consumption allocation $(\overline{x}_i, \overline{G})_{i \in I}$ and that no producer will depart from the corresponding production allocation $(\overline{y}_j, \overline{y}_j^g)_{j \in J}$. The achieved equilibrium corresponds to a vector of individual taxes for public goods equal to $(\overline{p}_i^g)_{i\in I}$ and to a vector of individual lump sum transfers $(\overline{T}_i)_{i\in I}$, each one equal to

$$\overline{T}_i = \overline{p}\cdot\overline{x}_i + \overline{p}_i^g\cdot\overline{G} - \big(\overline{p}\cdot\omega_i + \sum_{j\in J}\theta_{ij}\pi_j(\overline{p},\overline{p}^g)\big),$$

the sum over $i\in I$ of $\overline{T}_i$ being obviously equal to zero.

The interpretation of $(\overline{p},(\overline{p}_i^g)_{i\in I})$ in terms of Lindahl–Foley equilibrium prices for a convenient redistribution of initial endowments and profit shares is more difficult.

A simple corollary of Minkowski–Farkas Lemma (see Corollary 2.3.2. in [19]) gives necessary and sufficient conditions for finding profit shares $(\theta'_{ij})_{\substack{i\in I\\ j\in J}}\geq 0$ such that $\sum_{j\in J}\theta'_{ij}=1\ \forall i\in I$ and for every $i\in I$,

$$\overline{p}\cdot\overline{x}_i + \overline{p}_i^g\cdot\overline{G} = \overline{p}\cdot\omega_i + \sum_{j\in J}\theta'_{ij}(\overline{p}\cdot\overline{y}_j + \overline{p}^g\cdot\overline{y}_j^g) = \overline{p}\cdot\omega_i + \sum_{j\in J}\theta'_{ij}\pi_j(\overline{p},\overline{p}^g).$$

Corollary 5.2. *Let $\big((\overline{x}_i,\overline{G})_{i\in I},(\overline{y}_j,\overline{y}_j^g)_{j\in J}\big)$ be an equilibrium allocation relative to the price system $\big(\overline{p},(\overline{p}_i^g)_{i\in I}\big)$. There exists some (endogenous) system of profit shares $(\theta'_{ij})_{\substack{i\in I\\ j\in J}}$ such that $\big((\overline{x}_i,\overline{G})_{i\in I},(\overline{y}_j,\overline{y}_j^g)_{j\in J},(\overline{p},(\overline{p}_i^g)_{i\in I})\big)$ is a Lindahl–Foley equilibrium of the public goods economy $\mathcal{E}'=\big((\mathbb{R}_+^L\times\mathbb{R}_+^G,P_i,(\omega_i,0))_{i\in I},(Y_j)_{j\in J},(\theta'_{ij})_{\substack{i\in I\\ j\in J}}\big)$ if and only if for every $i\in I$, for every $\alpha\in\mathbb{R}$,*

$$\pi_j(\overline{p},\overline{p}^g)\leq\alpha\ \forall j\in J \implies \overline{p}\cdot(\overline{x}_i-\omega_i)+\overline{p}_i^g\cdot\overline{G}\leq\alpha,$$

$$\pi_j(\overline{p},\overline{p}^g)\geq\alpha\ \forall j\in J \implies \overline{p}\cdot(\overline{x}_i-\omega_i)+\overline{p}_i^g\cdot\overline{G}\geq\alpha.$$

A necessary condition is in particular that for every $i\in I$, $\overline{p}\cdot(\overline{x}_i-\omega_i)+\overline{p}_i^g\cdot\overline{G}\geq 0$.

Corollary 5.3. *If there is only one producer and if for every $i\in I$, $\overline{p}\cdot(\overline{x}_i-\omega_i)+\overline{p}_i^g\cdot\overline{G}\geq 0$, then there exists an endogenous system of profit shares, $(\theta'_i)_{i\in I}$, such that $\big((\overline{x}_i,\overline{G})_{i\in I},(\overline{y},\overline{G}),(\overline{p},(\overline{p}_i^g)_{i\in I})\big)$ is a Lindahl–Foley equilibrium of the public goods economy $\mathcal{E}'=\big((\mathbb{R}_+^L\times\mathbb{R}_+^G,P_i,(\omega_i,0))_{i\in I},Y,(\theta'_i)_{i\in I}\big)$.*

In this case, defining for every $G\in\mathbb{R}_+^K$,

$$c(G)=\min\{-\overline{p}\cdot y\colon (y,G)\in Y\} = -\max\{\overline{p}\cdot y\colon (y,G)\in Y\}$$

and for every $i\in I$, $g_i\colon\mathbb{R}_+^K\to\mathbb{R}$ by

$$g_i(G)=(\overline{p}_i^g-\theta'_i\overline{p}^g)\cdot G+\theta'_i c(G)$$

*then $\big((\overline{x}_i,\overline{G})_{i\in I},(\overline{y},\overline{G}),(g_i(G))_{i\in I}\big)$ is what is called in Mas-Colell–Silvestre [40] a **linear cost share equilibrium** of the economy $\mathcal{E}'$.*

Proof. The first part of the corollary is actually a particular case of the previous one. Recall that, in view of Assumption **B.2**, the maximum profit $\overline{p}\cdot\overline{y}+\overline{p}^g\cdot\overline{G}$ is nonnegative. If $\overline{p}\cdot\overline{y}+\overline{p}^g\cdot\overline{G}>0$, define for each $i\in I$, $\theta'_i=\frac{\overline{p}\cdot(\overline{x}_i-\omega_i)+\overline{p}_i^g\cdot\overline{G}}{\overline{p}\cdot\overline{y}+\overline{p}^g\cdot\overline{G}}$. If $\overline{p}\cdot\overline{y}+\overline{p}^g\cdot\overline{G}=0$, define for each $i\in I$, $\theta'_i=\frac{1}{|J|}$. In both cases, $\big((\overline{x}_i,\overline{G})_{i\in I},(\overline{y},\overline{G}),(\overline{p},(\overline{p}_i^g)_{i\in I})\big)$ is a Lindahl–Foley equilibrium of the economy $\mathcal{E}'=\big((\mathbb{R}_+^L\times\mathbb{R}_+^G,P_i,(\omega_i,0))_{i\in I},Y,(\theta'_i)_{i\in I}\big)$.

Let us now turn to the second part. The cost share system verifies $\sum_{i\in I}(\overline{p}_i^g-\theta'_i\overline{p}^g)=0$ and $\sum_{i\in I}\theta'_i=1$. On the other hand, in view of the profit maximization at $(\overline{y},\overline{G})$, $c(\overline{G})=-\overline{p}\cdot\overline{G}$ and for each $i\in I$, $g_i(\overline{G})=\overline{p}_i^g\cdot\overline{G}-\theta'_i(\overline{p}\cdot\overline{y}+\overline{p}_i^g\cdot\overline{G})$, which implies $\overline{p}\cdot\overline{x}_i+g_i(\overline{G})=\overline{p}\cdot\omega_i$. Now,

$(x_i, G) \in P_i(\overline{x}_i, \overline{G})$ implies $\overline{p} \cdot x_i + \overline{p}_i^g \cdot G > \overline{p} \cdot \omega_i + \theta_i'(\overline{p} \cdot \overline{y} + \overline{p}^g \cdot \overline{G}) \geq \overline{p} \cdot \omega_i + \theta_i'(\overline{p} \cdot y + \overline{p}^g \cdot G)$ for all y such that $(y, G) \in Y$, thus $\overline{p} \cdot x_i + g_i(G) > \overline{p} \cdot \omega_i$. ∎

To sum up, starting from a (weakly) Pareto optimal Lindahl–Foley feasible allocation of the original economy and keeping unchanged the initial endowments of consumers, it is not necessarily possible to redistribute the profit shares so that the (weakly) Pareto optimal allocation is a Lindahl–Foley equilibrium allocation of the new economy. In the next section, under additional assumptions, we will get for public goods economies the same result as in economies with only private goods: (weakly) Pareto optimal Lindahl–Foley feasible allocations are Lindahl–Foley equilibrium allocations of the economy obtained from the original one by redistributing initial endowments and profit shares of consumers.

6. Lindahl–Foley equilibrium and the core

Let us now set the following additional assumptions:

- **B.4:** Each consumer $i \in I$ has a complete and transitive preference ordering $\succeq_i$ on X_i weakly monotone in public goods: $\widetilde{G}_i \geq G_i$ implies $(x_i, \widetilde{G}_i) \succeq (x_i, G_i)\ \forall x_i \geq 0$ (**no public bads**),
- **B.5:** For each producer $j \in J$, $Y_j \subset \mathbb{R}^L \times \mathbb{R}_+^K$ (**public goods are never production inputs**).

The next proposition shows that, under **B.4**, Lindahl–Foley equilibrium consumption allocations belong to the core.

Proposition 6.1. *Assume **B.4** on Economy $\mathcal{E}$. If $\big((\overline{x}_i, \overline{G})_{i\in I}, (\overline{y}_j, \overline{y}_j^g)_{j\in J}, (\overline{p}, (\overline{p}_i^g)_{i\in I})\big)$ is a Lindahl–Foley equilibrium, then for each $i \in I$, $\overline{p}_i^g \geq 0$ and the consumption allocation $(\overline{x}_i, \overline{G})_{i\in I}$ belongs to the core $\mathcal{C}(\mathcal{E})$.*

Proof. We first claim that since $(\overline{p}, (\overline{p}_i^g)_{i\in I})$ is a quasiequilibrium price, $\overline{p}_i^g \geq 0$ holds for each $i \in I$. Indeed, it follows from transitivity and completeness of the preference preorders, monotony in public goods and local no satiation in private goods at $(\overline{x}_i, \overline{G})$ that $G_i > \overline{G} \Rightarrow \overline{p}_i^g \cdot G \geq \overline{p}_i^g \cdot \overline{G}$. Thus the vector price $\overline{p}_i^g$ cannot have a (strictly) negative component.

Then assume that the coalition S blocks $(\overline{x}_i, \overline{G})_{i\in I}$ via the S-feasible pair $\big((x_i)_{i\in S}, G^S\big)$. We have for some $(y_j, y_j^g)_{j\in J} \in \prod_{j\in J} Y_j$:

$$(\sum_{i\in S} x_i, G^S) = \sum_{i\in S}(\omega_i, 0) + \sum_{i\in S}\sum_{j\in J} \theta_{ij}(y_j, y_j^g) \tag{6.1}$$

$$(x_i, G^S) \succ_i (\overline{x}_i, \overline{G}) \text{ for each } i \in S. \tag{6.2}$$

From (6.1) and the equilibrium definition, we deduce:

$$\overline{p}\cdot\sum_{i\in S} x_i + (\sum_{i\in S}\overline{p}_i^g)\cdot G^S > \overline{p}\cdot\sum_{i\in S}\omega_i + \sum_{i\in S}\sum_{j\in J}\theta_{ij}(\overline{p}, \overline{p}^g)\cdot(\overline{y}_j, \overline{y}_j^g) \geq \overline{p}\cdot\sum_{i\in S}\omega_i + \sum_{i\in S}\sum_{j\in J}\theta_{ij}(\overline{p}, \overline{p}^g)\cdot(y_j, y_j^g).$$

From (6.2) and our first claim, it follows:

$$\overline{p}\cdot\sum_{i\in S} x_i + (\sum_{i\in S}\overline{p}_i^g)\cdot G^S \leq \overline{p}\cdot\sum_{i\in S} x_i + \overline{p}^g\cdot G^S = \overline{p}\cdot\sum_{i\in S}\omega_i + \sum_{i\in S}\sum_{j\in J}\theta_{ij}(\overline{p}, \overline{p}^g)\cdot(y_j, y_j^g).$$

We thus have a contradiction. ∎

The last proposition shows, under the assumptions **B.4** and **B.5**, the complete symmetry between concepts and optimality properties of Lindahl–Foley equlibrium for a public goods economy and Walras equilibrium for a private goods economy. However, it should be stressed that these assumptions (made by Foley in [20, 21]) are necessary neither for the existence of Lindahl–Foley equilibrium nor for the decentralization of (weakly) Pareto optimal Lindahl–Foley feasible allocations with personalized taxes and lump sum transfers.

Proposition 6.2. *In addition to the assumptions of Proposition 5.2, assume* ***B.4*** *and* ***B.5*** *on the economy* $\mathcal{E}$. *Let* $\big((\overline{x}_i, \overline{G})_{i\in I}, (\overline{y}_j, \overline{y}^g_j)_{j\in J}\big)$ *be a Lindahl–Foley feasible allocation of* $\mathcal{E}$ *such that* $(\overline{x}_i, \overline{G})_{i\in I}$ *is (weakly) Pareto optimal and let* $(\overline{p}, (\overline{p}^g_i)_{i\in I})$ *the price system whose Proposition 5.2 establishes the existence. The allocation together with the price system is a Lindahl–Foley equilibrium of the economy* $\mathcal{E}' = \big((\mathbb{R}^L_+ \times \mathbb{R}^G_+, P_i, (\omega'_i, 0))_{i\in I}, (Y_j)_{j\in J}, (\theta'_{ij})_{\substack{i\in I\\ j\in J}}\big)$ *for*

$$\theta'_{ij} = \begin{cases} \frac{\overline{p}^g_i\cdot\overline{y}^g_j}{\overline{p}^g\cdot\overline{y}^g_j} & \text{if } \overline{y}^g_j \neq 0 \\ \frac{1}{|I|} & \text{if } \overline{y}^g_j = 0 \end{cases} \qquad \text{and} \qquad \omega'_i = \overline{x}_i - \sum_{j\in J}\theta'_{ij}\overline{y}_j \ .$$

Proof. Recall that, in view of **NT** and **IR**, $\overline{p} \neq 0$ and $(\overline{p}^g_i)_{i\in I} \neq 0$. From **B.4** and **B.5** and the definition of the new profit shares, it follows that $\theta'_{ij} \geq 0$ for every $i \in I$ and for every $j \in J$, with for every $j \in J$, $\sum_{i\in I}\theta'_{ij} = 1$. Then, for every $i \in I$, $\overline{p}\cdot\overline{x}_i + \overline{p}^g_i\cdot\overline{G} = \overline{p}\cdot\omega'_i + \sum_{j\in J}\theta'_{ij}\overline{p}\cdot\overline{y}_j + \overline{p}^g_i\cdot\sum_{j\in J}\overline{y}^g_j = \overline{p}\cdot\omega'_i + \sum_{j\in J}\theta'_{ij}\overline{p}\cdot\overline{y}_j + \sum_{j\in J}\theta'_{ij}\overline{p}^g\cdot\overline{y}^g_j = \overline{p}\cdot\omega'_i + \sum_{j\in J}\theta'_{ij}\pi_j(\overline{p}, \overline{p}^g)$ and the proof is complete. ■

7. Concluding remarks

As Wicksell–Foley public competitive equilibrium is less an equilibrium concept than a characterization of (weak) Pareto optimality, we are left at the end of this presentation with two equilibrium concepts.

The first one according to the analytical point of view adopted in this paper, the second one from an historical point of view since its first elaborations dates back to 1985 [3, 6], is the public good private provision equilibrium. As already noticed, private provision of public goods leads to suboptimal equilibrium allocations of private and public goods. This drawback, the same which was pointed out by Samuelson, is at the origin of a huge literature on Pareto improving government interventions, beginning with [3, 57], that is at the very time of the definition of public goods private provision equilibrium.

The second one, the Lindahl–Foley equilibrium, dating back to 1967–1970 [20, 21], is the translation of the Lindahl solution refered to by Samuelson. However, most of economists, including Samuelson himself, deny to this equilibrium the ability of being implemented by any market mechanism. Lindahl–Foley equilibrium does not satisfy the incentive compatibiity constraint as defined by Hurwicz [32]: equilibrium does not requires that revealing the information necessary for the price mechanism to function be the best strategy of individual consumers. Here also, a huge literature on mechanism design begins in 1977–1980 with the tentative of resolution of the free-rider problem by Groves and Ledyard [28, 29]. This literature will tend to substitute the equilibrium of a suitably defined mechanism to the equilibrium of the original economy and to found on the government enforcement of appropriate mechanisms the public economic policy of public goods market economies.

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