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Condorcet domains and distributive lattices

Bernard MONJARDET

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Condorcet domains and distributive lattices

Bernard Monjardet
CES, Université Paris 1, monjarde@univ-paris1.fr

Abstract

Condorcet domains are sets of linear orders where Condorcet's effect can never occur. Works of Abello, Chameni-Nembua, Fishburn and Galambos and Reiner have allowed a strong understanding of a significant class of Condorcet domains which are distributive lattices –in fact covering distributive sublattices of the permutoèdre lattice- and which can be obtained from a maximal chain of this lattice. We describe this class and we study three particular types of such Condorcet domains.

Key words: acyclic set, alternating scheme, Condorcet effect, distributive lattice, maximal chain of permutations, permutoèdre lattice

1 Introduction

Condorcet domains (called also acyclic or consistent sets) are sets of linear orders where *Condorcet's effect* (called also *voting's paradox*) can never occur. The search for large Condorcet domains has lead to many interesting results and questions. In particular, Abello, Chameni-Nembua and Galambos and Reiner have investigated a class of Condorcet domains which are distributive lattices. Indeed, these Condorcet domains are covering distributive sublattices of the lattice –called the *permutoèdre lattice*- defined on the set of all linear orders on (equivalently, permutations of) a set. Such a Condorcet domain can be defined from a maximal chain in the permutoèdre lattice, and so I shall call it a *CH-Condorcet domain*. In this paper I describe the main results obtained on the class of CH-Condorcet domains and I study more precisely some of them. Section 2 is devoted to notations and preliminaries firstly on distributive lattices (2.1), secondly on the permutoèdre lattice (2.2), thirdly on Condorcet domains (2.3). Section 3 offers a synthesis of the main results obtained on CH-Condorcet domains. Since such a domain is a distributive lattice it can be obtained as the lattice of ideals of a poset and I propose another algorithm to get this poset. In section 4, I describe a maximal chain and the poset generating three significant types of CH-Condorcet domains : those which are minimal, those obtained by Fishburn's *alternating scheme*, and those supplied by Black's *single-peaked* linear orders. The conclusion contains some questions and conjectures.

N.B. All sets considered in this paper are finite.

2.Notations and preliminaries

2.1 Distributive lattices

Let (D,<) be a distributive lattice i.e., a poset such that any two elements x and y of D have a meet $x \wedge y$ and a join $x \vee y$ and such that the meet (respectively, the join) operation is distributive over the join (respectively, the meet) operation.

A *join-irreducible element* of D is an element covering a unique element of D. An *ideal* of a poset (X,<) is a subset I of X such that $x \in I$ and $y < x$ implies $y \in I$. Now, by Birkhoff's duality between distributive lattices and posets, a distributive lattice D is isomorphic to the set ordered by inclusion of all the ideals of the poset J_D of its join-irreducible elements. It is well-known that in this duality the maximal chains of D are in a one-to-one correspondence with the linear extensions of the poset J_D (i.e., with the linear orders containing the partial order between the join-irreducible elements). Indeed, when x_k is covered by x_{k+1} in a maximal chain of D, then there exists a unique join-irreducible element j_k such that $x_{k+1} = x_k \vee j_k$; so the covering relation $x_k \prec x_{k+1}$ can be labelled by j_k and the linear order $j_1 j_2 \ldots j_{|J_D|}$ obtained on J_D is a linear extension of the poset J_D. Since any poset is the intersection of all its linear extensions, one sees that the poset J_D can be obtained as the intersection of all the linear orders on J_D defined by the labelled maximal chains of D. In fact, one need only to use $|\dim J_D|$ such (suitably chosen) chains, where $\dim J_D$, the dimension of J_D, is the minimum number of linear orders of which the intersection is J_D.

2.2 The permutoèdre lattice

A = {1,2,...i,j,k,…..n} is a finite set of n elements denoted by the n first integers (in section 2.3 A will be the set of *alternatives*). A *strict linear order* L on A is an irreflexive, transitive and complete ($x \neq y$ implies xLy or yLx) binary relation on A.

Henceforth, we will omit the qualifier strict and sometimes, when there is no ambiguity, the qualifier linear. Linear orders on A are in a one-to-one correspondence with *permutation*s of A. So if L is a linear order on A one can write it as a permutation $x_1...x_kx_{k+1}...x_n$.Then one says that x_k has *rank* k and is *covered* by x_{k+1} and that x_k and x_{k+1} are *consecutive* in L. I denote by τ_k the *transposition* which exchanges x_k and x_{k+1} in L: $\tau_k(L) = x_1...x_{k+1}x_k...x_n$. The set of all linear orders on A of size n is denoted by $\mathcal{L}_n$, whereas $\mathcal{D}$ denotes any subset of $\mathcal{L}_n$.

Let L be an arbitrary linear order of $\mathcal{L}_n$; it will be convenient to take L = 1<2<….n. For L' $\in \mathcal{L}_n$, one sets InvL' = {{i,j} $\subseteq$ A such that iLj and jL'i} (i.e. the set of pairs {i,j} on which L and L' «disagree»}. For L', L'' $\in \mathcal{L}_n$, one sets L'' $\leq$ L' if InvL' $\subseteq$ InvL''. It has been shown by Guilbaud and Rosenstiehl (1963) that the poset ($\mathcal{L}_n$, $\leq$) -henceforth denoted simply by $\mathcal{L}_n$- is a lattice[1] called the "*permutoèdre*" *lattice* in French tradition (see for instance Barbut et Monjardet 1970). Its maximum element is 1<2<….n denoted by L_u and its minimum element is the dual linear order n<…2<1 denoted by L_0. The lattice $\mathcal{L}_4$ is represented on Figure 1 by a (Hasse) diagram giving its covering relation. The undirected covering relation of this lattice is the *adjacency relation* between linear orders where a linear order is adjacent to another one if they differ on a unique pair of elements. The set of all linear orders endowed with this adjacency relation is called the *permutoèdre graph*.

[1]Some authors attribute this result to Yanagimoto and Okamoto (Partial orderings of permutations and monotonicity of a rank correlation statistic. 1969, *Annals Institute of Statistics* 21: 489-506). One can admit that a paper published in French will be less known that a paper written in English. But Guilbaud and Rosenstiehl's paper which precedes Yanagimoto and Okamoto's paper has been quoted in many english-written papers ; moreover its proof that $\mathcal{L}_n$ is a lattice is reproduced in *Principles of combinatorics* (Berge 1971) and above all Yanagimoto and Okamoto's paper does not contain a real proof of their assertion (read it !). The permutoèdre lattice is not distributive and its properties are studied in Barbut and Monjardet (1970), Le Conte de Poly-Barbut (1990), Duquenne and Cherfouh (1994), Markowsky (1994) and Caspard (2000). The lattice ($\mathcal{L}_n$, $\leq$) is isomorphic to (S_n,$\leq$) the group of the permutations of a n-set ordered by the so-called *weak Bruhat order*. More generally Björner (1984) proved that all finite Coxeter groups partially ordered by the weak Bruhat order are lattices.

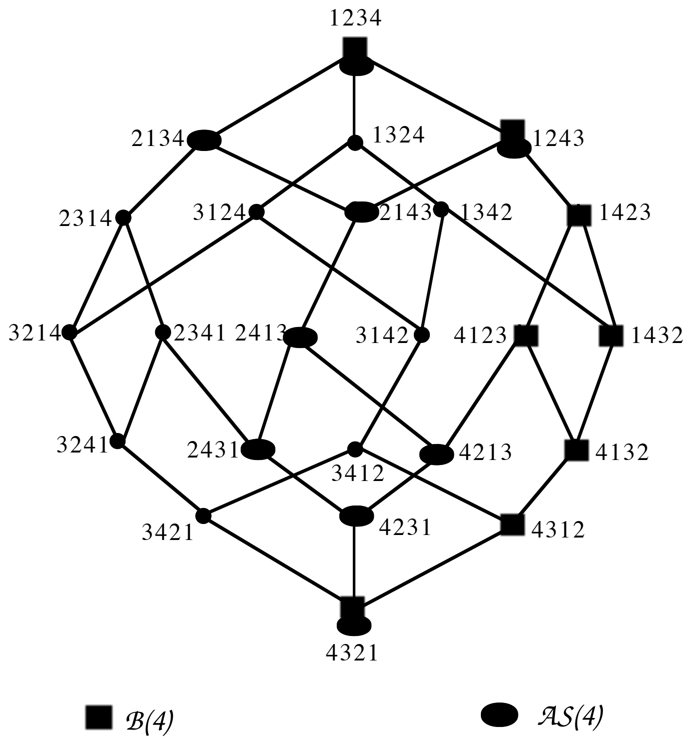


Figure 1

The permutoèdre lattice $\mathcal{L}_n$ and two covering distributive sublattices

2.3 Condorcet domains

The problem to get a collective preference from various voter's preferences on a set A of n alternatives (candidates, issues, decisions, outcomes…) is an old problem dealed on by Condorcet in his 1785 *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix*. He proposed to use the majority rule on the pairs of alternatives: alternative y is preferred by the majority to alternative x -denoted by $xR_{MAJ}y$- if the number of voters preferring y to x is greater than the number of voters preferring x to y. His *Essay* contains the first examples of what has come to be called the "Condorcet effect" (Guilbaud 1952) or the "Voting Paradox": when the voters express their preferences by means of linear orders on the set of alternatives, the majority relation of these orders can contain cycles. The simplest example is obtained with 3 alternatives x, y, z and 3 voters of which the set of preferences is a 3-*cyclic set* like xyz, yzx and zxy[2], since then the majority relation is the 3-cycle $xR_{MAJ}yR_{MAJ}zR_{MAJ}x$.

In 1948 Black initiated a way to escape the Condorcet effect. He proved that this effect cannot occur if the preferences of the voters are restricted to a subset of all possible linear orders, namely the set of the so-called *single-peaked* linear orders. So this set of linear orders was the first example of *Condorcet domains*, i.e., subsets of linear orders where the Condorcet effect can never occur. In fact, the simplest and more general way to prevent Condorcet effect is to forbid 3-cyclic sets in the domain $\mathcal{D}$ of linear orders allowed for preferences'voters: for every 3-set of alternatives, the restrictions of the linear orders of $\mathcal{D}$ to this set must not contain a 3-cyclic set. This condition has been given by Ward (1965) under the name of *Latin-Square-Lessness* and is equivalent (in the case of linear orders) to Sen's *Value Restricted-Preferences* condition (1966). This last condition says that for every 3-set of alternatives there exists an alternative which is either never ranked first or never ranked second or never ranked third in the restrictions of the linear orders of $\mathcal{D}$ to these alternatives. It is useful to write particular cases of this last condition by using Fishburn's notion of Never Condition. For $h \in \{i,j,k\} \subseteq A$ and $r \in \{1,2,3\}$ a set $\mathcal{D}$ of linear orders satisfies the *Never Condition* $hN_{\{i,j,k\}}r$ if h has never rank r in the restrictions to $\{i,j,k\}$ of the linear orders of $\mathcal{D}$. Consider now the linear order $1<2<\ldots n$ on A. A Condorcet domain $\mathcal{D}$ satisfies the *Never Condition hNr* if for every ordered triple $i<j<k$, the same Never Condition $hN_{\{i,j,k\}}r$ is satisfied. For instance $\mathcal{D}$ satisfies jN1 if for every ordered triple $i<j<k$, j has never rank 1 in the restrictions to $\{i,j,k\}$ of the orders of $\mathcal{D}$. In fact, in this case $\mathcal{D}$ is nothing else that the domain of single-peaked linear orders.

When A is a n-set Black's domain of single-peaked linear orders and many other Condorcet domains found in the sixties and seventies contain no more than 2^{n-1} orders (see Arrow and Raynaud 1986). Let us denote by f(n) the maximum size of a Condorcet domain on a n-set. It is not clear when has been raised for the first time the natural question "how large can be Condorcet domains ?" i.e., the problem of determining f(n). This problem has shown daunting (see, for instance, Fishburn 1997 and for an overview Monjardet 2006)[3].

[2] We denote a linear order by a permutation, where xyz means $x<y<z$, and we say that the least preferred alternative x has the first rank, the middle element y the second rank and the best preferred element z the third rank.

[3] Observe that the 1992 Craven conjecture $f(n) = 2^{n-1}$ have been disproved as soon as 1980 in Kim and Roush's book where it is shown that $f(n) \geq 2^{n-1} + 2^{n-3}-1$ ($> 2^{n-1}$ for $n \geq 4$).

Here I will only consider the class of CH-Condorcet domains, a class containing some large Condorcet domains[4]. It has been first studied by Abello which derived such a Condorcet domain from a maximal chain of the permutoèdre lattice $\mathcal{L}_n$. Abello (1984 with Johnson, 1985, 1991) showed that a CH-Condorcet domain is an upper semimodular sublattice of $\mathcal{L}_n$. Independently Chameni-Nembua (1989) showed that *covering distributive sublattices* of $\mathcal{L}_n$ i.e., distributive sublattices of $\mathcal{L}_n$ which keep the covering relation of $\mathcal{L}_n$ are Condorcet domains[5]. Recently Galambos and Reiner (2006) have shown that Abello's lattices are the same that Chameni-Nembua's maximal lattices and that these CH-Condorcet domains are characterized by means of some sets of Never Restrictions. Previously a particular type of CH-Condorcet domain satisfying a set of Never Restrictions (the *alternating scheme*) have been given by Fishburn (1997). This type of CH-Condorcet domain apparently supplies the largest CH-Condorcet domains. In the next section I give the known main results on the CH-Condorcet domains.

3 CH-Condorcet domains (maximal covering distributive sublattices of $\mathcal{L}_n$)

I present here a synthesis of the main results on the CH-Condorcet domains. These results have been obtained by Abello, Chameni-Nembua and Galambos and Reiner (these last two authors using also some more general Ziegler's results on Bruhat orders). It is necessary to give several definitions, notations and preliminary results.

For $L = x_1...x_kx_{k+1}...x_n$ a linear order of $\mathcal{L}_n$,I denote by $t_3(L)$ the set of ordered triples $x_i…x_j...x_k$ contained in L. For instance $t_3(2431) = \{243,241,231, 431\}$. I denote by C a maximal chain of the lattice $\mathcal{L}_n$ and I set $t_3(C) = \cup\{ t_3(L), L \in C\}$. So $t_3(C)$ is the set of all ordered triples occurring in the orders of the maximal chain C and it is easy to see that $|t_3(C)| = 4n(n-1)(n-2)/6$. On the other hand another easy observation is that the set of ordered triples contained in the orders of a Condorcet domain $\mathcal{D}$ of $\mathcal{L}_n$ has size at most $4n(n-1)(n-2)/6$ (if not $\mathcal{D}$ contains a 3-cyclic set). So when one adds to a Condorcet domain $\mathcal{D}$ all the linear orders which don't increase the set of ordered triples already

[4] But Fishburn (1997) have shown that at least for $n \geq 16$ there exist larger Condorcet domains which - contrary to the CH-Condorcet domains - are not connected subgraphs of the permutoèdre graph.

[5] For his thesis Chameni-Nembua answered some of my questions raised by Guilbaud's observation in his 1952 paper: the set of Black's single-peaked linear orders has a distributive lattice structure (other such examples are in Frey and Barbut's 1971 book).

present in $\mathcal{D}$ one gets a maximal acyclic domain. More generally the map which adds to an arbitrary set of linear orders all the linear orders which don't increase the set of ordered triples is a closure operator on the subsets of $\mathcal{L}_n$[6]. So by applying this closure operator to the maximal chain C one obtains a maximal Condorcet domain that I call a *CH-Condorcet domain* and that I denote by $\mathcal{D}(C)$.

I denote by $P^2(n)$ the set of the n(n-1)/2 ordered pairs (i<j) -written simply ij-of A = {1<2<…n}. Every maximal chain $C = L_0 \prec ... \prec L_p \prec L_{p+1} ... \prec L_u$ of the lattice $\mathcal{L}_n$ induces a linear order λ_C (written only λ if there is no ambiguity) on $P^2(n)$: the order of the first apparition of the ordered pair i<j in an order of C. More formally $\lambda_C = (ij)_1 \prec ... \prec (ij)_p \prec (ij)_{p+1} \prec ... (ij)_{n(n-1)/2}$ where if $L_p = x_1 ... x_k x_{k+1} ... x_n$, then $L_{p+1} = x_1 ... x_{k+1} x_k ... x_n$ with $(x_{k+1}, x_k) = (ij)_{p+1}$ and $x_k > x_{k+1}$, see an example below).

A linear order on $P^2(n)$ is called *admissible* if it is induced by a maximal chain of $\mathcal{L}_n$. From a more general Ziegler's result one gets: a linear order λ on $P^2(n)$ is admissible if and only if for every ordered triple i<j<k –henceforth written simply ijk- the three ordered pairs ij, ik and jk are ordered by λ either lexicographically ((ij)λ(ik)λ(jk)) or dually lexicographically ((jk)λ(ik)λ(ij)). I denote by Λ_n the set of all admissible linear orders on $P^2(n)$ and for $\lambda \in \Lambda_n$ I set:

$LEX_2\lambda = \{\{(ij),(ik),(jk)\} : (ij)\lambda(ik)\lambda(jk)\}$

$ALEX_2\lambda = \{\{(ij),(ik),(jk)\} : (jk)\lambda(ik)\lambda(ij)\}$

$LEX_3\lambda = \{ijk : \{(ij),(ik),(jk)\} \in LEX_2\lambda\}$

$ALEX_3\lambda = \{ijk : \{(ij),(ik),(jk)\} \in ALEX_2\lambda\}$

So the ordered triple ijk belongs to $LEX_3\lambda$ (respectively, to $ALEX_3\lambda$) if the set of the three ordered pairs ij, ik and jk is lexicographically (respectively, dually lexicographically) ordered by λ.

I define also a partial order $<_\lambda$ on $P^2(n)$ contained in the linear order λ:

$<_\lambda$ = transitive closure of the binary relation $\cup\{((ij)<(ik)<(jk)), ijk \in LEX_3\lambda\} \cup \{(jk)<(ik) < (ij), ijk \in ALEX_3\lambda\}$.

I illustrate these definitions for $C = 4321 \prec 4231 \prec 4213 \prec 2413 \prec 2143 \prec 2134 \prec 1234$ a maximal chain of $\mathcal{L}_4$. The associated linear order on $P^2(4)$ is $\lambda = 23 \prec 13 \prec 24 \prec 14 \prec 34 \prec 12$.

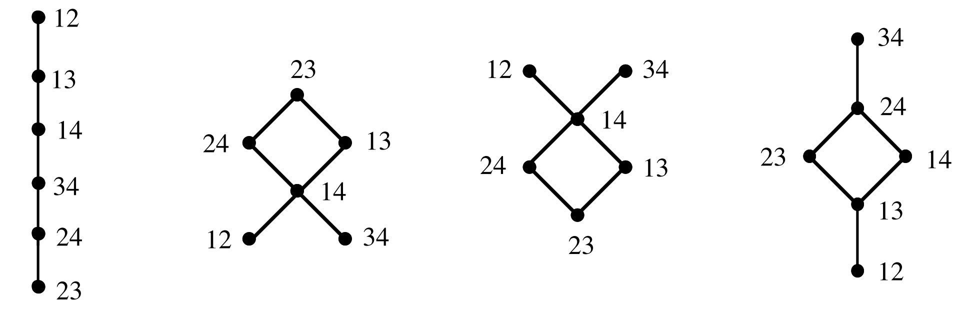
One gets:

$LEX_2\lambda = \{\{13,14,34\}, \{23,24,34\}\}$; $LEX_3\lambda = \{134,234\}$

$ALEX_2\lambda = \{\{23,13, 12\}, \{24,14,12\}\}$; $ALEX_3\lambda = \{123,124\}$

The partial order $<_\lambda$ is represented on Figure 2c by its diagram.

[6] This closure operator appears already in Kim and Roush's 1980 book (see Definition 5.12)



2a 2b 2c 2d

Figure 2

The four types of posets on $P^2(4)$ associated to the CH-Condorcet domains of $\mathcal{L}_4$

Let $\mathcal{D}$ (= $\mathcal{D}(C)$) be a CH-Condorcet domain. By point 1 of the below theorem $\mathcal{D}$ is a lattice. An (admissible) linear order on $P^2(n)$ is associated to each maximal chain of $\mathcal{D}$. I denotes by $\Lambda(\mathcal{D}) \subseteq \Lambda_n$ the set of all these linear orders on $P^2(n)$.

One can now gather together the main results on the CH-Condorcet domains in the following theorem.

Theorem 1

Let C be a maximal chain of the lattice $\mathcal{L}_n$ and λ the associated linear order on $P^2(n)$:

1. The closure $\mathcal{D} = \mathcal{D}(C)$ of C is a maximal Condorcet domain and a maximal covering distributive sublattice of $\mathcal{L}_n$. One goes from a maximal chain of $\mathcal{D}$ to another one by a sequence of «quadrangular transformations» of linear orders: let L = $x_1 \ldots x_k x_{k+1} \ldots x_i x_{i+1} \ldots x_n$ be a linear order such that x_k, x_{k+1}, x_i and x_{i+1} are four different alternatives ; then L is transformed into L' = $x_1 \ldots x_{k+1} x_k \ldots x_{i+1} x_i \ldots x_n$ (= $\tau_i\tau_k(L)$ = $\tau_k\tau_i(L)$).

2. The poset $J_{\mathcal{D}}$ of the join-irreducible elements of the distributive lattice $\mathcal{D}$ is isomorphic to the poset ($P^2(n)$, $<_\lambda$). Any order in $\mathcal{D}$ corresponds to an ideal of ($P^2(n)$, $<_\lambda$) obtained by applying to $L_0 = n<\ldots 2<1$ all the transpositions of the ordered pairs belonging to this ideal.

3. $\mathcal{D}$ is the set of all linear orders satisfying the following Never Conditions:

jN1, $\forall$ i<j<k with ijk $\in$ $LEX_3\lambda$

jN3, $\forall$ i<j<k with ijk $\in$ $ALEX_3\lambda$.

4. $\forall\ \lambda' \in \Lambda(\mathcal{D})$, $\forall$ p = 2,3 $\mathrm{LEX}_p\lambda' = \mathrm{LEX}_p\lambda$, $\mathrm{ALEX}_p\lambda' = \mathrm{ALEX}_p\lambda$ and

$<_{\lambda'} = <_{\lambda} = \cap\{\lambda' \in \Lambda(\mathcal{D})\}$.

One goes from a linear order in $\Lambda(\mathcal{D})$ to another one by a sequence of interchanges of two ordered pairs (ij) and (kl) which are *disjoint* ($\{i,j\}\cap\{k,l\} = \varnothing$) and consecutive in the linear order.

One can illustrate these results on the case of the maximal chain of $\mathcal{L}_4$ already considered above C = 4321≺4231≺4213≺2413≺2143≺2134≺1234. One checks that $\mathcal{D}(C)$ contains two more linear orders 2431 and 1243. The nine orders in $\mathcal{D}(C)$ form the distributive lattice marked by black ellipsoids on Figure 1 and they correspond to the nine ideals of the poset of Figure 2c (for instance, the order 2413 corresponds to the ideal {23,24,13}). One one can easily check all the other properties of the theorem, for instance one has $3N_{\{134\}}1$ and $2N_{\{124\}}3$.

There are two natural equivalence relations associated to the previous notions. Two maximal chains C and C' of $\mathcal{L}_n$ are equivalent if they have the same closure: $\mathcal{D}(C) = \mathcal{D}(C')$. Two admissible linear orders in Λ_n are equivalent if $\mathrm{LEX}_{\acute{e}}\lambda' = \mathrm{LEX}_{\acute{e}}\lambda$ (equivalently, $\mathrm{ALEX}\,\lambda' = \mathrm{ALEX}_3\lambda$ or $<_{\lambda} = <_{\lambda'}$) or still if they corresponds to two maximal chains of the same CH-Condorcet domain $\mathcal{D}$. Points 1 and 3 of the above theorem give a constructive way to determine the corresponding equivalence classes. Each of these classes corresponds to a CH-Condorcet domain and to a partial order on $P^2(n)$. For instance, the 16 maximal chains of $\mathcal{L}_4$ are partitionned into 8 equivalence classes, two classes containing 4 chains, two containing 2 chains and four containing one unique chain. The corresponding partial orders on $P^2(4)$ are represented on Figure 1 (by only 4 of these partial orders, the 4 others beeing isomorphic). I shall present the three types of CH-Condorcet domain generalizing those corresponding to Figure 1 in section 4.

As recalled in section 2.1 the poset of join-irreducible elements of a distributive lattice is obtained by the intersection of some labelled maximal chains of this lattice. It is rather intriguing that in the case of the distributive lattice $\mathcal{D}$ which is a CH-Condorcet domain any maximal chain of this lattice allows to get its poset of join-irreducible. Indeed, this poset is isomorphic to $(P^2(n), <_{\lambda})$ where λ is the linear order on $P^2(n)$ associated to this maximal chain. From point 4 of theorem 1 the partial order $<_{\lambda}$ depends only of $\mathcal{D}$ and not of the particular linear order λ in $\Lambda(\mathcal{D})$, and so it will be also denoted by $<_{\mathcal{D}}$. Observe that one obtains also $\mathcal{D}$ from a single of its maximal chains (since elements of $\mathcal{D}$ correspond to the ideals of ($P^2(n)$, $<_{\mathcal{D}}$).

An interesting algorithmic problem is to construct $\mathcal{D}(C)$ from a maximal chain C. Abello gives an algorithm which constructs a sequence of Condorcet domains from L_0 (the least element of C) to $\mathcal{D}(C)$. As just said another way to construct $\mathcal{D}(C)$ is to construct first the poset ($P^2(n)$, $<_{\mathcal{D}}$) (= $<_\lambda$ for the linear order λ on $P^2(n)$ associated to C), then to consruct the distributive lattice of the ideals of this poset. Galambos and Reiner give a representation of ($P^2(n)$, $<_{\mathcal{D}}$) in terms of *arrangement of pseudolines* which don't give in general an explicit construction of this poset. I propose here an algorithm to construct this poset based on the observation that if (i,j) is covered by (k,l) in $<_{\mathcal{D}}$ then (ij) and (kl) *intersect* ($\{i,j\}\cap\{k,l\} \neq \emptyset$) and i = k or j = l.

Algorithm to get $P^2(n)$, $<_{\mathcal{D}}$)

Let $\lambda = (ij)_1 \prec \ldots \prec (ij)_p \prec (ij)_{p+1} \prec \ldots (ij)_{n(n-1)/2}$ be the admissible linear order on $P^2(n)$ associated to the maximal chain C of $\mathcal{D}$. One constructs a sequence $<_\lambda(0)$ $<_\lambda(1)$..... $<_\lambda(p)$... $<_\lambda(n(n-1)/2) = <_{\mathcal{D}}$ of partial orders on $P^2(n)$ by setting:

$<_\lambda(0) = \emptyset$

and for p = 0,1,2....... n(n-1)/2 – 1,

$<_\lambda(p+1)$ = Transitive closure of $[<_\lambda(p) \cup \{(ij, (ij)_{p+1})$, ij maximal element of $<_\lambda(p)$ intersecting $(ij)_{p+1}\}]$, if there exists such an ij in $<_\lambda(p)$

$= [<_\lambda(p) \cup \{(ij)_{p+1}\}]$, if not.

So at first step $<_\lambda(1)$ contains only the ordered pair $(ij)_1$. At second step $<_\lambda(2)$ is the 2-element chain $(ij)_1 \prec (ij)_2$ if these two ordered pairs intersect and $<_\lambda(2)$ is the 2- element antichain $\{(ij)_1,(ij)_2\}$ if not.

4 Three types of CH-Condorcet domains

4.1 Minimal CH-Condorcet domains

By definition a CH-Condorcet domain $\mathcal{D}$ is equal to $\mathcal{D}(C)$ with C maximal chain of $\mathcal{L}_n$. Is it possible that this chain C be a maximal Condorcet domain i.e. that $\mathcal{D}(C) = C$ (what means that C is a closed set in the closure operator defined on the subsets of $\mathcal{L}_n$)? The positive answer is easy to give. Let $\Lambda(\mathcal{D})$ be the set of linear orders on $P^2(n)$ associated to the maximal chains of $\mathcal{D}$. Point 4 of theorem 1 says that one goes from the linear order λ in $\Lambda(\mathcal{D})$ to another one in $\Lambda(\mathcal{D})$ by a sequence of interchanges of two ordered pairs (ij) and (kl) which are *disjoint* ($\{i,j\}\cap\{k,l\} = \emptyset$) and consecutive in the linear order λ. Then, $\Lambda(\mathcal{D}) = \{\lambda\}$ (and $\mathcal{D}(C) = C$) if and only if the linear order λ does not contain consecutive and disjoint ordered pairs.

For n = 5 here is an example of such a linear order: $\lambda = 34\prec24\prec23\prec25\prec35\prec45\prec15\prec14\prec13\prec12$.

The corresponding maximal chain of $\mathcal{L}_5$ is obtained from 12345 by the sequence of transpositions exchanging successively the ranks of 1 and 5, then the ranks of 2 and 4. Since $LEX_3\lambda = \{235,245,345\}$ and $ALEX_3\lambda = \{125,135,145,123,124, 134,234\}$, one sees (point 3 of theorem 1) that the set of Never Conditions defining this Condorcet domain is jN1 $\forall$ i<j<k with ijk $\in\{235,245,345\}$ and jN3 $\forall$i<j<k with ijk $\in\{125,135,145,123,124,134,234\}$.

More generally one considers a maximal chain of $\mathcal{L}_n$ where the sequence of transpositions from $L_u = 1<2<....n$ exchanges successively for i = 1,2...... $\lfloor n/2\rfloor$ the ranks of i and n-1+i. By computing $LEX_3\lambda$ and $ALEX_3\lambda$ for the linear order λ on $P^2(n)$ corresponding to such a maximal chain, one gets the corresponding Never Conditions.

Proposition 2

The set of following Never Conditions defines a maximal CH-Condorcet domain which is a maximal chain of $\mathcal{L}_n$:

jN1 $\forall$ i<j<k with

k $\in\{n,n-1,....(n+t)/2\}$ where t = 4 (respectively,3) for n even (respectively, n odd) and i > n+1-k.

jN3 $\forall$ i<j<k with

i $\in\{1,2......\lfloor(n-1)/2\rfloor\}$ and k < n+2-i.

The above maximal chain of $\mathcal{L}_n$ is not the only example of maximal chain which is a maximal CH-Condorcet domain of $\mathcal{L}_n$. For instance, for n = 5, one can take the maximal chain where the associated linear order on $P^2(n)$ is $23\prec24\prec34\prec14\prec13\prec12\prec15\prec25\prec35\prec45$, which from L_u = 12345 exchanges first the ranks of 5 and 1, then the ranks of 4 and 2. And more generally one can take the sequence of transpositions from $L_u = 1<2...<n$ exchanging successively for i = 1,2...... $\lfloor n/2\rfloor$ the ranks of n-1+i and i. It is possible to show that for any pair of two consecutive elements (i,i+1) of L_u one can find such a sequence of transpositions

beginning by the transposition of i and i+1 and one can ask if such a sequence is unique.

4.1 CH-Condorcet domains given by Fishburn's alternating scheme

Let $1 < 2 \ldots\ldots < n$ be the linear order L_u on A. A Condorcet domain $\mathcal{D}$ of $\mathcal{L}_n$ satisfies the *alternating scheme (*Fishburn, 1977), if for all $i<j<k$

either (1) jN3 if j is even and   or (2) jN1 if j is even and
jN1 if j is odd.   jN3 if j is odd

I will consider here only the domain defined by (1) and I will denote it by $\mathcal{AS}$(n) (the other domain is its dual i.e., is formed by the dual orders of the first).

Since $\mathcal{AS}$(n) is defined by the Never Conditions jN1 and jN3 it is a CH-Condorcet domain (Galambos and Reiner, 2006) and a maximal covering distributive sublattice of $\mathcal{L}_n$. Figure 3 shows $\mathcal{AS}$(6)[7]. These Condorcet domains are especially significant, since it is conjectured that they are the CH-Condorcet domains of maximum size. Indeed, Fishburn has proved this conjecture for $n \leq 6$ (and the fact that in this case they are also the Condorcet domains of maximum size). There are two interesting questions about $\mathcal{AS}$(n): what is the poset (P^2(n), $<_{\mathcal{AS}(n)}$) and what are the maximal chains of $\mathcal{L}_n$ of which the closure is $\mathcal{AS}$(n). The poset (P^2(n), $<_{\mathcal{AS}(n)}$) has a very regular structure observed by Galambos and Reiner and used by them to compute the number of its ideals, i.e. the size of $\mathcal{AS}$(n). I complete their result by giving the expression of the covering pairs of this poset (for n odd ; there is a similar expression for n even). The linear extensions of this poset are the admissible linear orders on P^2(n) associated to the maximal chains of $\mathcal{AS}$(n). I also give an inductive procedure to get such a linear order and so a maximal chain of $\mathcal{AS}$(n) (which corresponds to the left maximal chain in Figure 3).

[7] I found the lattice $\mathcal{AS}$(6) when I was director of Chameni-Nembua's thesis and it is the last figure of Chameni-Nembua's 1989 paper (where one also finds the distributive lattice $\mathcal{AS}$(5)). I was pretty sure that there was a general construction to get such large Condorcet domains but since I didn't find it I sent these examples to Fishburn who was already working on the topic and (obviously) found quickly the above general characterization by Never Conditions.

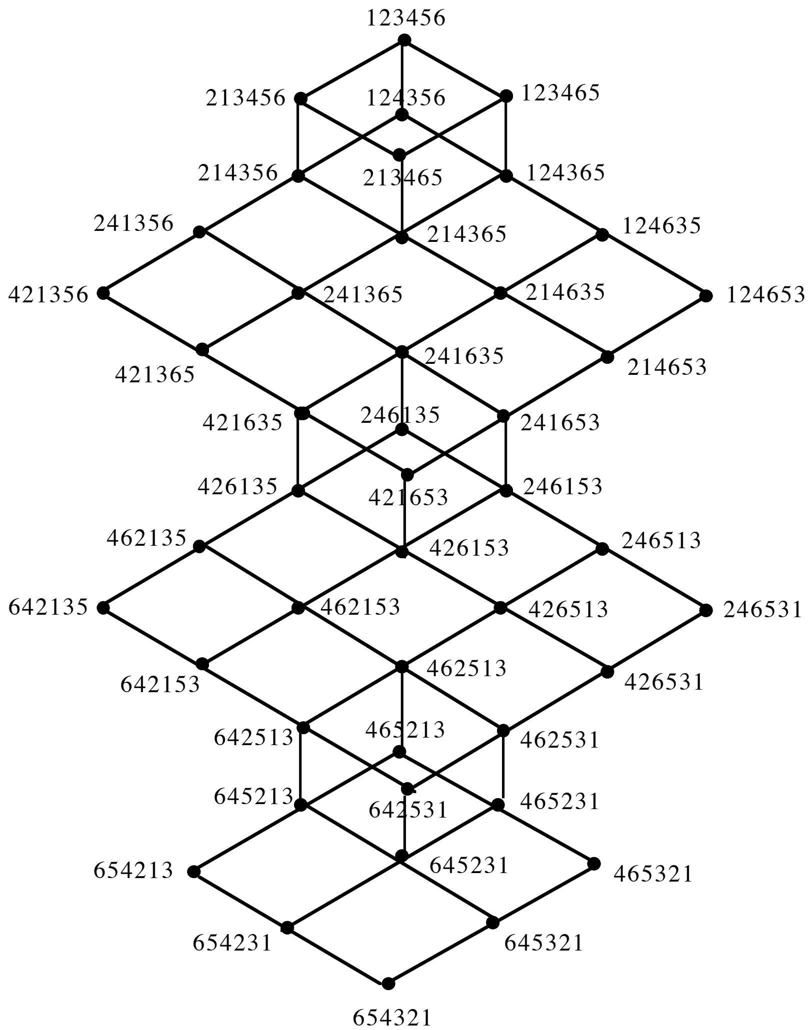


FIGURE 3 The distributive lattice $\mathcal{AS}(6)$

<u>Proposition 3</u>

Let $\mathcal{AS}$(n) be the CH-Condorcet domain given by Fishburn's alternating scheme.

Then for n odd, the covering pairs (i,j)$\prec$(k,l) ($1\leq i<j\leq n$) of the poset (P^2(n), $<_{\mathcal{AS}(n)}$) are given by :

$\forall$ 2 < j, (1,j)$\prec$(2,j) $\qquad$ $\forall$ i < n-1, (i,n-1)$\prec$(i,n)

For i even < j-2, (i,j)$\prec$(i+2,j) $\qquad$ For i odd > 2, (i,j)$\prec$(i-2,j)

For j even < n-2, $(i,j)\prec(i,j+2)$ For j odd > i+2, $(i,j)\prec(i,j-2)$

One gets a linear order $\lambda(n)$ on $P^2(n)$ associated to a maximal chain of $\mathcal{AS}(n)$ by the following inductive procedure:

$\lambda(2) = 12 = \lambda^d(2)$

$\lambda(n = 2p) = \lambda^g(n)\prec\lambda^d(n)$ with

$\lambda^g(n) = 23\prec........(n-3,n-1)$ and $\lambda^d(n) = (n-2,n)\prec........ 12$

$\lambda(n = 2p+1) = \lambda^g(n-1)\prec[(n-1,n) \prec.... (2,n) \prec(1,n) \prec....(n-4,n) \prec(n-2,n)] \prec \lambda^d(2p)$

$= \lambda^g(n)\prec\lambda^d(n)$ with

$\lambda^g(n) =23\prec........(n-2,n)$ and $\lambda^d(n) =(n-3,n-1)\prec.......12 = \lambda^d(n-1)$

$\lambda(n = 2p+2) = \lambda^g(n-1)\prec[(n-2,n)\prec(n-4,n) \prec....(2,n) \prec(1,n) \prec....(n-3,n) \prec(n-1,n)] \prec$

$\lambda^d(n-1)$

So, $\lambda(3) = 23\prec13\prec12$, $\lambda(4) = 23\prec13\prec24\prec14\prec34\prec12$, $\lambda(5) = 23\prec13\prec45\prec25\prec15\prec35\prec24\prec14\prec34\prec12$ etc.

One sees that the linear order $\lambda(n+1)$ is given by an insertion procedure where the new covering pairs of $\lambda(n+1)$ are inserted between two sequences of $\lambda(n)$.

There is a corresponding insertion procedure to build the poset $(P^2(n+1), <_{\mathcal{AS}(n+1)})$ from the poset $(P^2(n), <_{\mathcal{AS}(n)})$. Figure 4 shows how the sequence $45\prec25\prec15\prec35$ (respectively, $46\prec26\prec16\prec36\prec56$) is inserted into $(P^2(4), <_{\mathcal{AS}(4)})$ (respectively, $(P^2(5), <_{\mathcal{AS}(5)})$ to get $(P^2(5), <_{\mathcal{AS}(5)})$ (respectively, $(P^2(6), <_{\mathcal{AS}(6)})$). Observe that the above linear orders are the linear extensions of $(P^2(n), <_{\mathcal{AS}(n)})$ obtained by concatening chains of this poset in a traversal of the diagram from left to right.

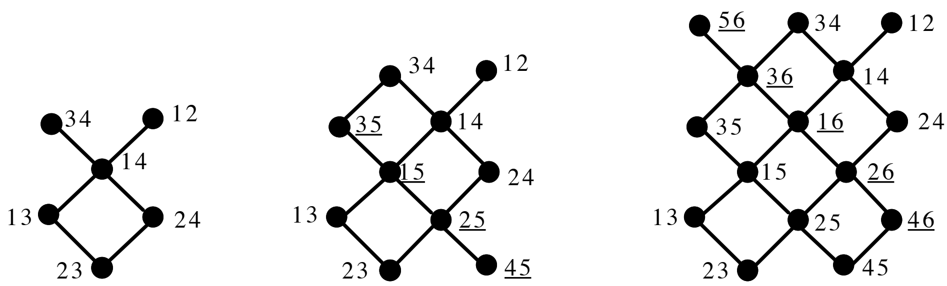


Figure 4 Posets corresponding to $\mathcal{AS}(n)$, n = 4,5,6

4.3 CH-Condorcet domains given by Black's single-peaked orders

As already said the Condorcet domain given by Black's single-peaked orders on $\mathcal{L}_n$ -denoted here by $\mathcal{B}(n)$- is obtained by the set of Never Conditions:

for all i<j<k jN1

Then, as already observed by Guilbaud (1952), $\mathcal{B}(n)$ is a (maximal covering) distributive sublattice of $\mathcal{L}_n$ and it is a CH-Condorcet domain. I give below the expression of the covering pairs of the poset $(P^2(n),<_{\mathcal{B}(n)})$. The linear extensions of this poset are the admissible linear orders on $P^2(n)$ associated to the maximal chains of $\mathcal{B}(n)$. I also give such a linear order and so a maximal chain of $\mathcal{B}(n)$.

<u>Proposition 3</u>

Let $\mathcal{B}(n)$ be the CH-Condorcet domain given by Black's single-peaked orders.

The poset $(P^2(n),<_{\mathcal{B}(n)})$ is a lattice of which the covering relation is given by:

$(i,j)\prec(k,h)$ $(1\leq i<j\leq n)$ if $i = k$ and $h = j+1$, or if $k = i+1$ and $j = h$. The join and meet operations of this lattice are:

$(i,j)\vee(k,h) = (\max(i,k), \max(j,h))$ and $(i,j)\wedge(k,h) = (\min(i,k), \min(j,h))$.

A maximal chain of $\mathcal{B}(n)$ is: $12\prec....\prec1n\prec23\prec....\prec2n\prec34\prec.....\prec3n\prec.....\prec1n$.

In fact the poset $(P^2(n),<_{\mathcal{B}(n)})$ is the restriction to $P^2(n)$ of the lattice direct product of the linear order $1<2....<n$ by itself and it inherits its covering relation and its lattice operations. The diagram of $(P^2(n), <_{\mathcal{B}(4)})$ is represented on Figure 2d.

5 Conclusion

The search for large Condorcet domains has lead to discover the class of CH-Condorcet domains which are distributive lattices, covering sublattices of the permutoèdre lattice. This proves once more the interest of –especially distributive-lattice structures in problems of social choice (another significant example is the lattice theory of the median ; see Barthélemy and Monjardet 1981 or Day and McMorris 2005). And this other connexion between social choice theory and lattice theory raises interesting questions. For example, is it possible to characterize the distributive lattices (or the corresponding posets) which are isomorphic to a CH-Condorcet domain ?

We end this paper by a conjecture and a problem. As subgraph of the permutoèdre graph a CH-Condorcet domain of $\mathcal{L}_n$ has diameter n(n-1)/2 i.e., the maximum length of a shortest path between two of its vertices is n(n-1)/2.

Let g(n) be the maximum size of a connected Condorcet domain of diameter n(n-1)/2 contained in $\mathcal{L}_n$.

Conjecture Galambos and Reiner 2006)

$g(n) = |\mathcal{AS}(n)|$

This conjecture is true for $n \leq 6$ since in this case Fishburn has shown that the maximum size of a Condorcet domain is $|\mathcal{AS}(n)|$ and Galambos and Reiner proved it for n = 7.

Remark

One can observe that $\mathcal{L}_4$ contains maximal covering distributive sublattice of size 7 (the minimum size obtained by the maximal chains characterized in Proposition 1), 8 and 9 (= g(4) the maximum size). One could suppose that $\mathcal{L}_n$ contains maximal covering distributive sublatticcs of size i, for any integer i in the interval [1+ n(n-1)/2, g(n)]. But this is false for n = 5. In fact, in this case there maximal covering distributive sublattices of $\mathcal{L}_5$ of sizes 11, 12, 14,15, 16, 17, 19 and 20 but not of sizes 13 and 18.

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