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A Note On Self-Similarity For Discrete Time Series.

Dominique Guégan[a], Zhiping LU[b, *]

[a] Centre d'Economie de la Sorbonne, UMR 8147, Université Paris 1 Panthéon Sorbonne, 75647 Paris. E-mail: dguegan@univ-paris1.fr
[b] Department of Mathematics, East China Normal University, 3663, Zhongshan Road North, Shanghai, China, 200062; Centre d'Economie de la Sorbonne Antenne de Cachan, Ecole Normale Supérieure de Cachan, 61 avenue du président Wilson, 94235, Cachan, France. E-mail: camping9@gmail.com

Abstract
The purpose of this paper is to study the self-similar properties of discrete-time long memory processes. We apply our results to specific processes such as GARMA processes and GIGARCH processes, heteroscedastic models and the processes with switches and jumps.

Keywords : Long Memory Processes, Short Memory Processes, Self-similar, Asymptotically Second-order Self-similar.

1 Introduction

In the past decades there has been a growing interest in studying self-similar processes and asymptotically self-similar processes, which has been a subject in literature (eg. Beran 1994, Samorodnitsky and Taqqu 1994). However, most of the works concern continuous-time processes. In this paper, we are devoted to study the relationship between long memory and self-similarity for discrete-time series and we present some new results. We prove that if a

*Corresponding author: E-mail Adress: camping9@gmail.com (Z.P. LU), Centre d'Economie de la Sorbonne Antenne de Cachan, Ecole Normale Supérieure de Cachan, 61 avenue du président Wilson, 94235, Cachan, France.

process is both covariance stationary and long memory, then it is asymptotically second-order self-similar. On the other hand, if a process is covariance stationary and short memory, then it is not asymptotically self-similar. We apply these results to the k-factor GARMA process and the k-factor GIGARCH process, the ARMA process, the GARCH process, the model with switches and the model with breaks, etc. What's more, for the processes with breaks and jumps, although they are theoretically short memory, they are empirically long memory processes. Consequently, theoretically they are not asymptotically second-order self-similar, while empirically they are asymptotically second-order self-similar. We propose in this paper a new notion to describe this kind of property which is the "spurious asymptotically second-order self-similar" behavior.

The paper is organized as follows. In section two, we recall some concepts and give some new results. Section three is devoted to the study of the self-similar properties for discrete-time series.

2 Concepts and main results

In this part, we recall some definitions and give some new results on self-similarity.

2.1 Concepts of long memory and short memory

Definition 2.1 *A covariance stationary process $(X_t)_t$ is called short memory process, if its autocorrelation function $\rho(k)$ satisfies $\Sigma_{k=0}^{\infty}\rho(k) < \infty$.*

Definition 2.2 *A covariance stationary process $(X_t)_t$ is called long memory process if it has an autocorrelation function $\rho(k)$ which behaves like a power function decaying to zero hyperbolically as*

$$\rho(k) \sim C_\rho(k) \cdot k^{-\alpha}, \ as \quad k \to \infty, \, 0 < \alpha < 1, \tag{1}$$

where $\sim$ represents the asymptotic equivalence, and $C_\rho(k)$ is a function which changes slowly to infinity, i.e. for all $a \in \mathbb{R}$, $C_\rho(k)(ax)/C_\rho(k)(x) \to 1$, when $x \to \infty$ (or $x \to 0$).

These concepts have been studied a lot in literature, for example, by Beran (1994), Guégan (2005).

2.2 Concepts of self-similarity

Self-similarity provides an elegant explanation and interpretation for an empirical law that is commonly referred as the Hurst's effect. Let $(X_t)_t$ be a covariance stationary time series. Denote

$$X_t^{(m)} = 1/m \sum_{k=(t-1)m+1}^{tm} X_k, \qquad k = 1, 2, ... \tag{2}$$

the corresponding aggregated sequence with level of aggregation m (> 1). Denote $\rho^{(m)}(k)$ the autocorrelation function of the process $(X_t^{(m)})_t$.

Definition 2.3 *A strictly stationary stochastic process $(X_t)_t$ is exactly self-similar (or asymptotically self-similar) if for all t,*

$$X_t =^d m^{1-H} X_t^{(m)}$$

holds for all m (or as $m \to \infty$), where $(X_t^{(m)})_t$ is defined as in (2) and $1/2 < H < 1$.

Definition 2.4 *Let $(X_t)_t$ be a covariance stationary process,*
(1) The process $(X_t)_t$ is called exactly second-order self-similar, or s.o.s.s, if $m^{1-H}X_t^{(m)}$ has the same autocorrelation as X, for all m and all t. Thus we have $Var(X^{(m)}) = Var(X)m^{2H-2}$ and $\rho^{(m)}(k) = \rho(k)$ where $1/2 < H < 1$, $m > 1$, $k = 0, 1, 2, \cdots$, and $\rho(k) \sim Ck^{2H-2}$, as $k \to \infty$.
(2) The process $(X_t)_t$ is called asymptotically second-order self-similar, or a.s.o.s.s, if

$$\lim_{m\to\infty} \rho^{(m)}(k) = \frac{1}{2}[(k+1)^{2H} - 2k^{2H} + (k-1)^{2H}], \forall\ k > 0. \tag{3}$$

So the autocorrelation function of the process $(X_t)_t$ is such that :

$$\rho(k) = \frac{1}{2}[(k+1)^{2H} - 2k^{2H} + (k-1)^{2H}], \quad \forall k > 0, \tag{4}$$

We can also use the relationship(3) and (4) as the definition of second-order self-similarity processes. The notion of exactly (asymptotically) self-similar concerns all the finite-dimensional distributions of a strict stationary process, while the notion of exact (asymptotical) second-order self-similarity concerns only the variance and autocorrelation function of a covariance stationary process.

2.3 Main Results

Lemma 2.5 *Let $(X_t)_t$ be a covariance stationary process, if this process is short memory as is defined by definition 2.1, then it is not asymptotically second-order self-similar.*

Proof : For a short memory process $(X_t)_t$, its autocorrelation function decays exponentially to zero, so it does not satisfy the equation (4), which means that the process $(X_t)_t$ is not asymptotically second-order self-similar. □

Lemma 2.6 *Let $(X_t)_t$ be a covariance stationary long memory process with $\frac{1}{2} < H < 1$, then this process is asymptotically second-order self-similar. Furthermore, under Gaussianity, the process is asymptotically self-similar.*

Proof : For a covariance stationary process $(X_t)_t$, its autocorrelation function decays hyperbolically, i.e. $\lim_{k\to\infty} \frac{\rho(k)}{k^{2H-2}} = c,\ \frac{1}{2} < H < 1$. According to the results of Tsybakov and Georganas (1997), we deduce that $\lim_{m\to\infty} \rho^{(m)}(k) = \frac{1}{2}[(k+1)^{2H} - 2k^{2H} + (k-1)^{2H}],\ \forall\ k = 1, 2, \cdots$. Thus the process is asymptotically second-order self-similar following the definition 2.4. □

From the lemma 2.6, the following result is straightforward :

Lemma 2.7 *Let $(X_t)_t$ be a covariance stationary long memory process with $\frac{1}{2} < H < 1$, if the spectral density of the process blows up at the origin, then the process is asymptotically second-order self-similar. Under Gaussianity, it is asymptotically self-similar.*

3 Examples of self-similar processes

In this part, we investigate the classical discrete-time processes and their self-similar properties. The study concerns the long memory processes under stationarity, for example, fGn, $k-$factor GARMA process, $k-$factor GIGARCH process, and some short memory processes such as processes with switches, processes with breaks, and also processes with threshold.

3.1 Fractional Gaussian noise (fGn)

Definition 3.1 *A process $(X_t)_{t\in\mathbb{Z}}$ is called a fractional Gaussian noise, or fGn, if it satisfies, for all $t \in \mathbb{Z}$, $X_t = B_H(t) - B_H(t-1)$, where $\{B_H(t)\}_{t\in\mathbb{R}}$ is a fractional Brownian motion.*

The fractional Gaussian noise (fGn) is the unique stationary Gaussian process which is exactly self-similar process with zero mean, which has been studied, for instance, by Beran (1994), Samorodnitsky and Taqqu (1994).

3.2 k-factor GARMA process

Definition 3.2 *A stationary process $(X_t)_t$ is called a k-factor GARMA process, if it has the following representation*

$$\phi(B)\prod_{i=1}^{k}(1 - 2u_iB + B^2)^{d_i}X_t = \theta(B)\varepsilon_t \tag{5}$$

where k is a finite integer, $|u_i| \leq 1$ for all $i = 1, \cdots, k$, $(\varepsilon_t)_t$ is a white noise with mean zero and variance σ_ε^2, and $\phi(B)$ and $\theta(B)$ are polynomials of order p and q respectively, $d_i \in \mathbb{R}$, B is the backshift operator satisfying $BX_t = X_{t-1}$.

Woodward, Cheng and Gray (1998) proved that a stationary k-factor GARMA process exhibits long memory behavior, if u_i are distinct, all the roots of the polynomials $\phi(B)$ and $\theta(B)$ are distinct and lie outside the unit circle and if (i) $0 < d_i < \frac{1}{2}$ and $|u_i| < 1$ or if (ii) $0 < d_i < \frac{1}{4}$ and $|u_i| = 1$, for $i = 1, \cdots, k$.

Proposition 3.3 *Let $(X_t)_t$ be a covariance stationary and long memory k-factor GARMA process, then it is asymptotically second-order self-similar. Furthermore, under Gaussianity, it is also asymptotically self-similar.*

Proof : Due to lemma 2.6, a covariance stationary and long memory k-factor GARMA process is asymptotically second-order self-similar. The conclusion is the same for particular k-factor GARMA process like GARMA process, Gegenbauer process and FARIMA process. □

3.3 Heteroscedastic Processes

3.3.1 k-factor GIGARCH processes

Definition 3.4 *A process $(X_t)_t$ is called a k-factor GIGARCH process, if it has the following representation*

$$\phi(B)\Pi_{i=1}^{k}(I - 2u_iB + B^2)^{d_i}(X_t - \mu) = \theta(B)\varepsilon_t \tag{6}$$

where $\varepsilon_t = \xi_t\sigma_t$ with $(\xi_t)_{t\in\mathbb{Z}}$ a white noise process with unit variance and mean zero, $\sigma_t^2 = a_0 + \Sigma_{i=1}^{r} a_i\varepsilon_{t-i}^2 + \Sigma_{j=1}^{s} b_j\sigma_{t-j}$, μ the mean of the process $(X_t)_{t\in\mathbb{Z}}$, $\phi(B)$ and $\theta(B)$ polynomials in B of order p and q respectively, B the backshift operator satisfying $BX_t = X_{t-1}$, $d = (d_1, ..., d_k)$ the memory parameters and $u = (u_1, ..., u_k)$ the frequency location parameters, $d_i \in \mathbb{R}$, $|u_i| \leq 1$, $i = 1, \cdots, k$.

Guégan (2000, 2003) proposed this model and has given out the corresponding covariance stationary condition and long memory conditions.

Proposition 3.5 *Let $(X_t)_t$ be a covariance stationary and long memory k-factor GIGARCH process, then it is asymptotically second-order self-similar. Furthermore, under Gaussianity, it is also asymptotically self-similar.*

Proof : For a covariance stationary and long memory k-factor GIGARCH process, according to lemma 2.6, a k-factor GIGARCH process is asymptotically second-order self-similar. □

3.3.2 GARCH processes and related processes

Definition 3.6 *A process $(X_t)_t$ is a generalized autoregressive conditional heteroscedastic process with order p and q respectively, or a GARCH(p,q) process, if it has the following representation $X_t = \sigma_t\varepsilon_t$ and*

$$\sigma_t^2 = \alpha_0 + \Sigma_{i=1}^{q}\alpha_i X_{t-i}^2 + \Sigma_{j=1}^{p}\beta_j\sigma_{t-j}^2 = \alpha_0 + a(B)X_t^2 + b(B)\sigma_t^2 \tag{7}$$

where $\alpha_0 > 0$, $\alpha_i \geq 0, i = 1, \cdots, q$, $\beta_j \geq 0, j = 1, \cdots, p$, $\{\varepsilon_t\} \sim IID(0, 1)$, and ε_t is independent of $\{X_{t-k}, k \geq 1\}$ for all t, B is the backshift operator, $a(B)$ and $b(B)$ are polynomials in B of order q and p respectively.

This model has been introduced by Bollerslev (1986) and has several particular cases : if $\beta_j = 0$ in (7), we get an ARCH process (Engle 1982). If $\sum_{i=1}^{q} \alpha_i + \sum_{j=1}^{p} \beta_j = 1$, we get an IGARCH process (Bollerslev 1988).

Proposition 3.7 *Let $(X_t)_t$ the covariance stationary GARCH process introduced in (7), then it is not asymptotically second-order self-similar.*

Proof : If the GARCH model is covariance stationary, then it is short memory process. Thus, following the lemma 2.5, it is not asymptotically second-order self-similar. The conclusion is the same for particular GARCH models like ARCH and IGARCH models. □

3.4 Processes with switches and jumps

Structural breaks have been observed in many economic and financial time series. A lot of models have been proposed in order to capture the existence of structural changes and complex dynamic patterns. Let $(X_t)_t$ be a process whose recursive scheme is

$$X_t = \mu_{s_t} + \varepsilon_t \tag{8}$$

where $(\mu_{s_t})_t$ is a process we specify below and $(\varepsilon_t)_t$ is a strong white noise, independent of $(\mu_{s_t})_t$. With respect to the process $(\mu_{s_t})_t$, we distinguish two cases :

1. $(\mu_{s_t})_t$ depends on a hidden ergodic Markov chain $(s_t)_t$,
2. If $(\mu_{s_t})_t = \mu_t$, then we will assume that this process depends on a probability p.

The first class of models considered above includes “models with switches" and the second class includes “models with breaks". For most of these processes, they are covariance stationary and short memory. Nevertheless, some processes exhibit empirically a kind of long memory behavior when we observe the behavior of the sample autocorrelation function.

3.4.1 Processes with switches

The processes with switches have been studied in literature by Hamilton (1988), Diebold and Inoue (2001), Guégan (2003, 2006), etc. We consider a two-state Markov Switching model $(X_t)_t$ defined by the following equations :

$$X_t = \mu_{s_t} + \phi_{s_t} X_{t-1} + \sigma_{s_t} \varepsilon_t, \tag{9}$$

where μ_{s_t}, ϕ_{s_t} and σ_{s_t} $(i = 1, 2)$ are real parameters, σ_i are positive and $(\varepsilon_t)_t$ is a strong white noise with mean $m \in \mathbb{R}$ and variance $\sigma^2 \in \mathbb{R}_+^*$, and $(\varepsilon_t)_t$ is independent of the hidden ergodic Markov chain $(s_t)_t$, which is characterized by its transition probabilities p_{ij}, defined by :

$$P[s_t = j | s_{t-1} = i] = p_{ij} \tag{10}$$

with $0 \leq p_{ij} \leq 1$ and $\sum_{j=1}^{2} p_{ij} = 1$, $i = 1, 2$.

Proposition 3.8 *Let* $\{X_t\}_t$ *be a process defined by* (9) *and* (10)*, then if* $\max_{i=1,2}\{p_{i1}|\phi_1|^2 + p_{i2}|\phi_2|^2\} < 1$*, the model is not asymptotically second-order self-similar.*

Proof : Yang (2000) proved that under the above conditions, the process $\{X_t\}_t$ is covariance stationary . Under stationarity, this model is known to be theoretically short memory. Thus, following lemma 2.5, it is not asymptotically second-order self-similar. □

Actually, there are many other interesting models with switches contained in equation (8), for example : the mean switching model($X_t = \mu_{s_t} + \varepsilon_t$), the mean variance switching model ($X_t = \mu_{s_t} + \sigma_{s_t}\varepsilon_t$), the sign model ($X_t = sign(X_{t-1}) + \varepsilon_t,\ where\ \varepsilon_t \sim^{i.i.d} N(0, \sigma^2)$). For all of these processes, if they are covariance stationary, then they are theoretically short memory processes, although empirically they exhibit long memory behavior. Likewise, they are not asymptotically second-order self-similar theoretically.

3.4.2 Processes with breaks

For processes with breaks, many people have investigated their properties, for example, Engle and Smith (1999), Granger and Hyung (2004), Hyung and Franses (2005), Guégan (2003, 2006). Assume the process $(X_t)_t$ is defined by

$$X_t = \mu_t + \varepsilon_t \tag{11}$$

where the process $(\mu_{s_t})_t = \mu_t$ depends on a probability p. Different dynamics of the process $(\mu_t)_t$ correspond to different break models, for example : the Binomial model, the random walk model with a Bernouilli process, the STOPBREAK model, the stationary random level shift model, the mean-plus-noise model, etc.

Under stationarity, these models are short memory and cannot be asymptotically second-order self-similar theoretically.

3.4.3 Processes with threshold

Consider the general form of the processes with threshold :

$$X_t = f(X_{t-1})(1 - G(X_{t-d}, \gamma, c)) + g(X_{t-1})G(X_{t-d}, \gamma, c) + \varepsilon_t \tag{12}$$

where the function f and g can be any linear or nonlinear functions of the past values of X_t or ε_t. The process $(\varepsilon_t)_t$ is a strong white noise and G is an indicator function or some continuous function. For a given threshold c and the position of the random variable X_{t-d} with respect to this threshold c, the process $(X_t)_t$ contains different models, for example, the SETAR model, the STAR model (Tong 1990).

Proposition 3.9 *For the process with threshold defined as in* (12)*, if the functions* f *and* g *correspond to short memory process, then it is not asymptotically second-order self-similar.*

Proof : If the functions f and g correspond to short memory process, then the process is short memory. Thus, following lemma 2.5, it is not asymptotically second-order self-similar.□

Now we consider the long memory SETAR model, defined as follows :

$$X_t = (1-B)^{-d}\varepsilon_t^{(1)} I_t(X_{t-d} \le c) + \varepsilon_t^{(2)}[1 - I_t(X_{t-d} \le c)], \tag{13}$$

and the assumptions :
(H_5) : the process $(\varepsilon_t^{(i)})_t$ $(i = 1, 2)$ is a sequence of independent identically distributed random variables.
(H_6) : the long memory parameter d is such that $0 < d < 1/2$.

Proposition 3.10 *Under the assumptions (H_5) and (H_6), the process $(X_t)_t$ defined by the relation(13) is globally stationary, and then it is asymptotically second-order self-similar. Furthermore, under Gaussianity, it is asymptotically self-similar.*

Proof : Under the assumptions (H_5) and (H_6), the stationary model defined in (13) is long memory in the covariance sense. According to lemma 2.6, it is asymptotically second-order self-similar. □

3.4.4 A New Concept

For the models with switching and models with breaks introduced above, theoretically they are not asymptotically second-order self-similar. However, empirically, they exhibit long memory behavior with the sample autocorrelation function which decreases in an hyperbolic way towards zero. Since we often call this long memory behavior the "Spurious long memory" behavior, we can also call this self-similar behavior the "spurious asymptotical second-order self-similar" behavior.

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