More pessimism than greediness: a characterization of monotone risk aversion in the Rank-Dependent Expected Utility model*

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Summary. This paper studies monotone risk aversion, the aversion to monotone, mean-preserving increase in risk (Quiggin [21]), in the Rank Dependent Expected Utility (RDEU) model. This model replaces expected utility by another functional, characterized by two functions, a utility function u in conjunction with a probability-perception function f. Monotone mean-preserving increases in risk are closely related to the notion of comparative dispersion introduced by Bickel & Lehmann [3, 4] in Non-parametric Statistics. We present a characterization of the pairs $(u; f)$ of monotone risk averse decision makers, based on an index of greediness G_u of the utility function u and an index of pessimism P_f of the probability perception function f: the decision maker is monotone risk averse if and only if $P_f \geq G_u$. The index of greediness (non-concavity) of u is the supremum of $u'(x)/u'(y)$ taken over $y \leq x$. The index of pessimism of f is the infimum of $\frac{1-f(v)}{1-v} / \frac{f(v)}{v}$ taken over $0 < v < 1$. Thus, $G_u \geq 1$, with $G_u = 1$ iff u is concave. If $P_f \geq G_u$ then $P_f \geq 1$, i.e., f is majorized by the identity function. Since $P_f = 1$ for Expected Utility maximizers, $P_f \geq G_u$ forces u to be concave in this case; thus, the characterization of risk aversion as $P_f \geq G_u$ is a direct generalization from EU to RDEU. A novel element is that concavity of u is not necessary. In fact, u must be concave only if $P_f = 1$.

Keywords and Phrases: Risk aversion, pessimism, greediness, Rank-dependent Expected Utility.

JEL Classification Numbers: D81

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* We are most grateful to Mark Machina, Peter Wakker and two anonymous referees for very helpful suggestions and comments.

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1 Introduction

Under the expected utility (EU) model, a decision maker (DM) is characterized by a single function u, called the utility function. The crucial criticism of the EU model by Allais [1] from a theoretical point of view is that in such a model, the same function u characterizes two distinct behaviors - satisfaction from certain wealth and attitude to risk - that have no reason to be linked: a DM with diminishing marginal utility on certain wealth may be risk-seeking, but this is precluded by the EU model.

The rank-dependent expected utility (RDEU) model (see [20, 32, 11, 28, 21, 6]) has been built in part as an attempt to answer the criticism of Allais. A RDEU DM is characterized by two functions: a utility function u on outcomes and a probability-perception function $f : [0;1] \to [0;1]$. A RDEU decision maker compares a lottery with cumulative distribution function F with another by the expected utility of the lottery evaluated as if the lottery had distribution function $1 - f(1 - F)$ instead of F (see (4)). Thus, f below (above) the identity function indicates pessimism (optimism), while the diagonal case $f(p) = p$ is the perception-neutral attitude of EU maximizers.

A risk averse DM is usually defined as one that, for every bounded random variable, prefers the expectation of the random variable to the random variable itself. This notion will be called weak risk aversion. Aversion to mean-preserving increase in risk (MPIR) in the sense of Rothschild & Stiglitz (see [14, 24]) will be called strong risk aversion.

In Yaari's Dual Theory (RDEU with linear u), weak risk aversion is characterized by f majorized by the identity function (henceforth pessimism). Monotone mean-preserving increase in risk (M-MPIR), introduced by Quiggin [20, 21] and properly defined in the sequel, was then obtained ([5, 9], implicitly in [32]) as the kind of added risk to which weakly risk averse Dual Theory decision makers are averse.

It is tautologically true that strong risk aversion implies monotone risk aversion, which implies weak risk aversion. In the EU model, weak risk aversion is characterized by concavity of u. MPIR was then obtained ([14, 24]) mathematically as the kind of added risk to which weakly risk averse EU decision makers are averse. Thus, in EU theory, weak, monotone and strong risk aversion are equivalent. In Dual Theory, weak and monotone risk aversion are equivalent, while strong risk aversion requires f to be convex ([32]).

The situation in the RDEU model is more flexible due to the trade-off between risk aversion implications of utility and probability perception. However, if the utility on outcomes u is concave, the situation is much like for the Dual Theory. Weak risk aversion under all concave utilities u is characterized by perception functions displaying pessimism (see [21, 32]), while strong risk aversion under all concave utilities u requires the stricter condition that f be convex (see Chew, Karni & Safra [11]). Chew, Karni & Safra [11] proved, more generally, that concavity of u and convexity of f are necessary and sufficient for strong risk aversion. Quiggin [20, 21] also brought to light that M-MPIR is the kind of increase in risk to which RDEU weakly risk averse decision makers with concave utility (not only linear) are averse.

Similarly to the definition where G is a MPIR with respect to F if there exist random variables X and Z with X distributed F, $X + Z$ distributed G and Z is "noise" around X ($E[Z|X] = 0$ a.s.), the notion of G being a M-MPIR requires instead that Z, with mean zero, be comonotone with X. Because EU (under all concave utilities) is a special case of RDEU (under all concave utilities and convex probability perception functions), M-MPIR is in particular MPIR.

The notion of monotone risk aversion is model-free; it has been proved to be useful in EU (see Section 2.3) and is well fitted to RDEU theory (see [11, 20, 28, 32]), where comonotonicity plays a fundamental part at the axiomatic level. The above analysis restricted u to the case of concave functions. Consistent with Allais' criticism, it is of interest to study whether a decision maker can be averse to risk without u being concave. In a previous paper, Chateauneuf & Cohen [7] proved that pessimism of f is a necessary condition for weak risk aversion, while concavity of u is not, but did not succeed to fully characterize weak risk aversion. This is the subject of our current research, by which we know that when non-concave u are allowed, weak risk aversion does not imply monotone risk aversion.

In this paper we characterize the class of pairs $(u; f)$ of utility and probability perception function that model aversion to M-MPIR.

More specifically, for an increasing probability perception function f, let

(1) $$P_f = \inf_{0<v<1}\left[\frac{1-f(v)}{1-v} \Big/ \frac{f(v)}{v}\right]$$

be its index of pessimism (see Section 3.1). The probability perception function displays pessimism, i.e., is majorized by the identity function, if and only if $P_f \geq 1$.

For a strictly increasing utility function u, define

(2) $$G_u = \sup_{x_1 < x_2 \le x_3 < x_4} \left[\frac{u(x_4) - u(x_3)}{x_4 - x_3} \Big/ \frac{u(x_2) - u(x_1)}{x_2 - x_1} \right]$$

to be its index of non-concavity. This index (also called greediness, see Section 3.2) satisfies $G_u \ge 1$ as well, and the value 1 corresponds exclusively to concavity. When u is differentiable, it is the supremal value of $u'(x)/u'(y)$ taken over $x \ge y$.

The characterization of monotone risk aversion in the RDEU model will be based on the comparison of these two indices.

In Section 2 we recall the basic definitions and some basic properties of M-MPIR and monotone risk aversion, and justify the interest in M-MPIR, irrespective of the model of decision under risk used. The corresponding introductory material on the RDEU model is contained in Section 3. Section 4 states the main result (Theorem 1) characterizing monotone risk aversion in the RDEU model and its corollary that handles the case of monotone risk-seeking attitudes, illustrates in the RDEU context the monotone risk seeking behavior of a DM with diminishing marginal utility and compares monotone and strong risk averse attitudes.

2 Monotone mean-preserving increase in risk and monotone risk aversion

2.1 Notation

We assume that risk prevails and describe it through a set of states of nature $\Omega = [0;1]$ endowed with the uniform probability measure P on the Borel σ-field. Let $\mathcal{V} = \{X; Y; \cdots\}$ be the set of bounded $\mathbb{R}$-valued random variables on Ω.

For any X in $\mathcal{V}$, we denote by F_X (respectively, $\overline{F}_X = 1 - F_X$) the cumulative (decumulative) distribution function of X and by $E(X)$ the expected value of X. Let $\mathcal{D}$ be the set of cumulative probability distribution functions on $\mathbb{R}$.

The distribution (or law) of a discrete random variable Z will be denoted by

(3) $$\mathcal{L}(Z) = (x_1; p_1; \ldots; x_k; p_k; \ldots; x_n; p_n)$$

with $x_1 < x_2 < \cdots < x_n$, $p_i \ge 0$ and $\sum p_i = 1$.

2.2 Comonotonicity and monotone mean-preserving increase in risk

Two functions X and Z: $\Omega \to \mathbb{R}$ are comonotone if each is non-decreasing in the other. Z may be non-decreasing in X: $\Omega \to \mathbb{R}$, that is, $X(\omega) > X(\omega')$ implies that $Z(\omega) \ge Z(\omega')$, without

Z being a function of X at all, that is, without the further requirement that $X(\omega) = X(\omega')$ imply $Z(\omega) = Z(\omega')$. If Z is non-decreasing in X then X is clearly non-decreasing in Z as well, so this notion is symmetric[1]. The common formal definition (see [26, 27, 32, 13]) of this notion of functions that are non-decreasing in each other, or comonotone functions, is:

Definition 1 Two real-valued functions X and Z on Ω are comonotone if for any ω and $\omega' \in \Omega$, $[X(\omega) - X(\omega')][Z(\omega) - Z(\omega')] \geq 0$.

Comonotonicity is not a transitive relation because constant functions are comonotone with any function. Consistent with the usual conventions, random variables are said to be comonotone if they are comonotone functions on some sure event.

Quiggin [21] introduced the following notion of monotone mean-preserving increase in risk (M-MPIR).

Definition 2 For $X, Y \in V$, Y is a monotone mean-preserving increase in risk M-MPIR of X if there exists a random variable $Z \in V$ with $E(Z) = 0$ such that X and Z are comonotone and Y has the same probability distribution as $X + Z$. Equivalently, X will be said to be a monotone mean-preserving reduction in risk of Y.

This is a very intuitive notion of increase in risk: Y could be said to be a "progressive stretch" of X: if an X-mass at x is shifted to become a Y-mass at y, then all X-masses to the right of x will be shifted by at least $y - x$. In particular, their distribution functions single cross. For illustrative intuition, observe that net income and tax are comonotone because each is a non-decreasing function of gross income. Thus, taxation (centered so as to display mean zero) accomplishes a monotone mean-preserving reduction in risk. The notion of price band stabilization, used in the theory of the firm, is also a particular case of monotone reduction in risk.

Monotone increases or reductions in risk are assumed to be mean preserving. Except for this requirement, they coincide with the notion of comparative dispersion introduced in a statistical framework by Bickel & Lehmann (see [3, 4]):

Definition 3 For F and G in D, F is Bickel & Lehmann less dispersed than G if for every $0 < x < y < 1$, $F^{-1}(y) - F^{-1}(x) \leq G^{-1}(y) - G^{-1}(x)$.

[1] Equivalently, as some further thought will reveal, each of X and Z is a non-decreasing function of some third function from Ω to $\mathbb{R}$, and this other function can always be taken to be $X + Z$.

That is, all the interquantile intervals are shorter for X than for Y. The definition of dispersion doesn't require F nor G to possess finite first moments, and even if they do, these moments need not be equal. In fact, $Y \sim G$ is more dispersed than $X \sim F$ if and only if $Y + c$ is more dispersed than X, for arbitrary $c \in \mathbb{R}$. Gathering different properties obtained in the literature on this subject (see [8, 19, 21]), it is possible to obtain the following connection between the Bickel & Lehmann dispersive order and M-MPIR.

Proposition 1 *When two random variables X and Y in V have the same expected value, Y is a M-MPIR of X if and only if F_Y is more dispersed than F_X in the sense of Bickel & Lehmann.*

>From now on, we concentrate on M-MPIR, the mean-preserving version of comparative dispersion and on the corresponding notion of attitude to risk, monotone risk aversion, to be viewed as aversion to M-MPIR. More formally:

Definition 4 A DM is monotone risk averse (respectively, monotone risk seeking) if for any X and Y in V with equal means such that Y is a M-MPIR of X, the DM weakly prefers X to Y (respectively, Y to X).

2.3 Motivation for the study of monotone mean-preserving increase in risk and the dispersive order: insurance and portfolio management under the EU model

To gain some intuition on M-MPIR, note that, as a consequence of Definition 3, the distributions F and G must single cross. Thus (see Diamond & Stiglitz [14]), G is a MPIR (in the sense of Rothschild & Stiglitz) with respect to F. But Definition 3 (i.e. dispersion) not only makes F and G single cross - it makes G single cross every horizontal translation of F. This property is very meaningful for Insurance: a horizontal translation of F is the distribution of a random variable $X - c$ obtained from an F-distributed X by the subtraction of an arbitrary constant c. This constant may play the role of insurance premium for an otherwise fair contract that replaces the distribution G of the uninsured position Y by the distribution F of the random variable X. Since the distribution of $X - c$ single crosses the distribution of Y and the utility function u is non-decreasing, the distributions of $u(X - c)$ and $u(Y)$ single cross as well. Choosing the premium c so that $E(u(X - c)) = E(u(Y))$, $u(Y)$ becomes a MPIR with respect to $u(X - c)$, so $E(\phi(u(X - c))) \geq E(\phi(u(Y)))$ for all non-decreasing and concave ϕ. Since for EU DM's this characterizes $v(\cdot) = \phi(u(\cdot))$ as displaying more risk aversion than u, this proves that a Arrow-Pratt more risk averse EU DM will always be ready to pay higher premiums for monotone mean-preserving reductions in risk. This property does not necessarily hold for standard mean-preserving reductions in risk, as shown by Ross [23] and others. This apparent drawback of the usual (Arrow-Pratt) notion of comparative risk aversion is in fact a drawback of the automatic use of second degree stochastic dominance as the reference for reduction of risk by insurance. Landsberger & Meilijson [18] proved that M-MPIR is the weakest order compatible with the Arrow-Pratt index of risk aversion, in the class of all non-decreasing utility functions. Jewitt [15] introduced a weaker order, still stronger than Rothschild & Stiglitz's, that characterizes risk reductions compatible with the Arrow-Pratt index in the class of non-decreasing concave utility functions. We respectfully credit Jewitt with the technical idea for connecting insurance premia with horizontal shifts of distributions. Landsberger & Meilijson [19] have presented other applications of this stochastic order to insurance.

In the following sections we study a role played by monotone mean-preserving risk in the RDEU model. A forthcoming paper [9] analyzes other roles played by monotone risk and its weaker version Location-independent Risk by Jewitt, in the same model. We finish this section on monotone risk by illustrating an application to Portfolio management.

In the standard problem of designing an optimal portfolio based on one safe and one risky asset, a natural prediction is that if the risky asset becomes riskier (in the usual Rothschild & Stiglitz (1970) sense, see [24]), the EU strongly risk averse investors will want less of it. This prediction was shown to be wrong by Rothschild & Stiglitz in their 1971 paper (see [25]), where they present the following necessary and sufficient condition on the utility function u under which a MPIR will always lead to a reduction in the allocation to the risky asset: $u'(\cdot)$ must be convex[2] and the relative index of risk aversion $-xu''(x)/u'(x)$ must be bounded from above by 1. This somewhat counter-intuitive result, that has puzzled many economists, has lead to various attempts to restrict not only the type of risk aversion postulated, but also the notion of increase in risk itself. In one of these contributions, Quiggin [21] showed that an EU DM with DARA (Decreasing Absolute Risk Aversion) utility function will reduce the allocation to the risky asset if it is subjected to a M-MPIR. It should be noted that the CARA utility function $-\exp\{-Kx\}$, a special case of DARA, has unbounded relative index of risk aversion, and as such, it does not meet the Rothschild & Stiglitz (1971) necessary conditions. Hence, a EU DM with CARA utility function will

[2] u' convex is termed "prudence" by Kimball [16].

coherently reduce the allocation on a monotonely riskier asset while failing to do so under the standard notion of riskier asset.

3 The RDEU model

Variants of the Rank Dependent Expected Utility theory due to Quiggin [20] have been treated by Yaari [32], Chew, Karni & Safra [11], Segal [28, 29] and Allais [2]. General axiomatization can be found in Wakker [30], Quiggin & Wakker [22], Chateauneuf [6].

Definition 5 A DM behaves in accordance with the Rank Dependent Expected Utility model (RDEU) if the DM is characterized by a continuous, strictly increasing utility function u in conjunction with a probability-perception function $f : [0;1] \to [0;1]$ that is strictly increasing and satisfies $f(0) = 0; f(1) = 1$. Such a DM (weakly) prefers the random variable X to the random variable Y if and only if $V(X) \geq V(Y)$, where the RDEU functional V is given by

$$V(X) = V_{u;f}(X) = -\int_{-\infty}^{\infty} u(x)\,df(P(X > x))$$

$$(4) \qquad = \int_{-\infty}^{0} [f(P(u(X) > t)) - 1]\,dt + \int_{0}^{\infty} f(P(u(X) > t))\,dt \, .$$

If the support of the random variable X is a finite set, $V(X)$ can be written as

$$V(X) = \sum_{i=1} u(x_i) \left[f(\sum_{j=i} p_j) - f(\sum_{j=i+1} p_j) \right]$$

$$= u(x_1) + \sum_{i=2} [u(x_i) - u(x_{i-1})] f(\sum_{j=i} p_j)$$

$$(5) \qquad = u(x_1) + \sum_{i=2} [u(x_i) - u(x_{i-1})] f(v_{i-1}) \, ;$$

where $v_i = \sum_{j=i+1}^{n} p_j = P(X > x_i)$.

If the perception function f is the identity function $f(v) \equiv v$, then $V(X) = V_{u;I}(X)$ is the expected utility $E[u(X)]$ of the random variable. The Yaari functional (see [32]) is the special case $V(X) = V_{I;f}(X)$, in which the utility on outcomes is the identity function $u(x) \equiv x$. If both perception and utility are identity functions, then $V(X) = V_{I;I}(X)$ is simply the expected value $E[X]$ of the random variable.

In some sharp sense, M-MPIR plays for Dual Theory (in fact, for the RDEU model with concave utility) the role played by MPIR for the EU model: for equal–mean X and Y in

V, all RDEU DM's with linear u and pessimistic f ($f(p) \le p$) prefer X to Y, if and only if Y is a M-MPIR with respect to X (see [5, 9] and implicitly [32]).

Example 1. Spreading out a two-point distribution — a necessary condition for monotone risk aversion.

Let $0 < v < 1$ and $-\infty < x_1 < x_2 < x_3 < \infty$, let $x_4 = x_3 + (x_2 - x_1)(1 - v)/v$. Consider the M-MPIR from the two-point distribution $L(X) = (x_2; 1 - v; x_3; v)$ (see (3)) to the two-point distribution $L(Y) = (x_1; 1 - v; x_4; v)$. Since $E(X) = E(Y)$; monotone risk aversion implies that $V(X) \ge V(Y)$ for all choices of v and x_i as above. Using the representation $V(X) = u(x_2) + f(v)[u(x_3) - u(x_2)]$ (see (5)), this inequality is readily seen to be equivalent to

$$(6) \qquad \frac{u(x_4) - u(x_3)}{x_4 - x_3} \Big/ \frac{u(x_2) - u(x_1)}{x_2 - x_1} \le \frac{1 - f(v)}{1 - v} \Big/ \frac{f(v)}{v} :$$

Hence, a necessary condition for monotone risk aversion is that the supremum of the left-hand side of (6), related to the index of greediness G_u (see (2)) of the utility function u, be less than or equal to the infimum of the right-hand side of (6), the index of pessimism P_f (see (1)) of the probability perception function f. However, the supremum of the LHS of (6) should be taken over vectors $(x_1; x_2; x_3; x_4)$ satisfying $x_1 < x_2 < x_3 < x_4$ and also $x_4 = x_3 + (x_2 - x_1)(1 - v)/v$, so this supremum could in principle depend on v. By the following Lemma 1, this supremum, independent of v, is equal to the index of greediness G_u of the utility function u, to be compared with the index of pessimism P_f of the probability perception function f.

Lemma 1 Let $u : \mathbb{R} \to \mathbb{R}$ be continuous and strictly increasing and let $\lambda > 0$. Denote

$$(7) \qquad E_\lambda = \left\{ \overline{x} \mid \overline{x} = (x_1; x_2; x_3; x_4) \in \mathbb{R}^4 \ ; \ x_1 < x_2 \le x_3 < x_4 \ ; \ \frac{x_4 - x_3}{x_2 - x_1} = \lambda \right\}$$

$$(8) \qquad G_u(\lambda) = \sup_{\overline{x} \in E_\lambda} \left[\frac{u(x_4) - u(x_3)}{x_4 - x_3} \Big/ \frac{u(x_2) - u(x_1)}{x_2 - x_1} \right] :$$

Then $G_u(\lambda) = G_u$:

The lemma is proved in the Appendix. As noted following the definition of the index of non-concavity, if u is differentiable, then G_u is expressible as $G_u = \sup_{y<x} \frac{u'(x)}{u'(y)}$ and Lemma 1 is superfluous.

The next subsections introduce more formally these indices of pessimism and greediness as well as their dual of optimism and thriftiness (or non-convexity), and analyze some of their main properties.

3.1 Index of pessimism

A RDEU DM is pessimistic under risk if the DM assesses lotteries as a EU DM who, for each $x \in \mathbb{R}$, undervalues the probability to win at least x; this is the effect of a probability perception function f such that $f(v) \leq v$ for all $v \in [0;1]$. For any probability perception function f lying under the identity function, $1 - f(v) \geq 1 - v$ and $f(v) \leq v$. Thus, by (1), the index of pessimism $P_f = \inf_{0<v<1}[\frac{1-f(v)}{1-v}/\frac{f(v)}{v}]$ of such an f satisfies $P_f \geq 1$. For a probability v of winning, $(1-v)/v$ is the odds-ratio against winning. Pessimists exaggerate this odds ratio by amplifying it to $(1-f(v))/f(v)$. The index of pessimism can be intuitively understood as the minimal such amplification factor.

Example 2. Kink perception function. Let a kink function be the pointwise maximum of two increasing linear functions defined on $[0;1]$: one with slope less than 1 going through $(0;0)$, the other with slope exceeding 1 going through $(1;1)$. If f is a kink, then

$$P_f = \min(1/f'(0); f'(1)) > 1 : \tag{9}$$

Example 3. Hyperbolic perception function. For $P > 1$, let the function

$$f_P(v) = \frac{v}{v + (1-v)P} \tag{10}$$

be called the hyperbolic perception function with index of pessimism P. It is an iso-pessimistic perception function in the sense that the expression $[(1-f_P(v))/(1-v)]/[f_P(v)/v]$, whose infimum over v generally defines the index of pessimism, is identically equal to the index of pessimism P of f_P.

The following proposition summarizes some basic properties of the index of pessimism P_f. Properties (i)-(v) are of intrinsic interest. Property (vi) is a technical lemma needed in the sequel.

Proposition 2

(i) For perception functions f and g, if $f(v) \leq g(v)$ for all $v \in (0;1)$, then $P_f \geq P_g$.

(ii) If f is a non-linear, convex perception function, then $P_f > 1$.

The following property characterizes strict pessimism.

(iii) For a perception function f, $P_f > 1$ if and only if f is majorized by some non-linear, convex perception function. Furthermore, every perception function f with $P_f > 1$

is majorized by the hyperbolic perception function with the same index of pessimism P_f (see (9)).

(iv) Fix $P > 1$ and let $L_P(v) = v/P$ and $R_P(v) = 1 + (v - 1)P$ be the linear support functions of the hyperbolic perception function f_P at 0 and 1 respectively. Then $P_f = P$ for every perception function f satisfying $L_P \le f \le f_P$ or $R_P \le f \le f_P$.

The following property gives an alternative definition of P_f.

(v) For a pessimistic perception function f, $P_f = Q$, where

$$\begin{aligned} Q &= \inf\{P > 1 \mid f(v) > f_P(v) \text{ for some } v \in (0,1)\} \\ &= \sup\{P \ge 1 \mid f(v) \le f_P(v) \text{ for all } v \in (0,1)\}\,. \end{aligned} \tag{11}$$

(vi) The index of pessimism of a pessimistic DM satisfies

$$P_f = \inf_{0<v_2 \le v_1<1} \frac{1 - f(v_1)}{1 - v_1} \Big/ \frac{f(v_2)}{v_2}\,. \tag{12}$$

The proof is in the Appendix. This proposition further justifies P_f as an index of pessimism — the more f "hangs down" below the diagonal, the more pessimistic the DM is, and the larger P_f is. The index of pessimism is the same for all perception functions that are "sandwiched" between an hyperbolic perception function and the kink that supports it from below at 0 and at 1.

Example 4. Power-type perception functions. For $\alpha > 1$, the perception functions $f(v) = v^\alpha$ and $g(v) = 1 - (1 - v)^{1/\alpha}$ have index of pessimism α.

Sketch of proof: By property (iv), f is sandwiched between R_α and f_α, while g is sandwiched between L_α and f_α. The details are skipped.

Optimism and index of optimism

By duality, a RDEU DM with a probability perception function f is optimistic under risk if $f(v) \ge v$ for all $v \in [0,1]$ with an index of optimism O_f

$$O_f = \inf_{0<v<1}\left[\frac{f(v)}{1 - f(v)} \Big/ \frac{v}{1 - v}\right] = \inf_{0<v<1}\left[\frac{f(v)}{v} \Big/ \frac{1 - f(v)}{1 - v}\right] \tag{13}$$

where $O_f \ge 1$ for an optimist.

3.2 Index of non-concavity (or greediness)

For a strictly increasing utility function u, (2) defines its index of non-concavity. Since non-decreasing functions are differentiable almost everywhere, the ratio in (2) can be made arbitrarily close to 1 by concentrating all four points x_i close to a point of differentiability of u. Hence, $G_u \geq 1$. The value 1 corresponds exclusively to concavity. For differentiable u the index of non-concavity can be expressed in the simpler form

$$G_u = \sup_{y<x} \frac{u'(x)}{u'(y)} \tag{14}$$

Intuitively, as the maximal possible amplification factor of marginal utility from a low level of wealth to a higher level of wealth, it measures "greed" - valuing an additional cent more when rich than when poor. We propose to call it index of greediness.

A function u^* is said to be more concave than a function u if there is a concave, strictly increasing function ϕ such that for all $x \in R$, $u^*(x) = \phi(u(x))$. In this definition, neither of the two utility functions is required to be concave[3]. Since

$$\begin{aligned} \frac{u^*(x_4) - u^*(x_3)}{x_4 - x_3} / \frac{u^*(x_2) - u^*(x_1)}{x_2 - x_1} &= \frac{\phi(u(x_4)) - \phi(u(x_3))}{x_4 - x_3} / \frac{\phi(u(x_2)) - \phi(u(x_1))}{x_2 - x_1} \\ &= \left[\frac{\phi(u(x_4)) - \phi(u(x_3))}{u(x_4) - u(x_3)} / \frac{\phi(u(x_2)) - \phi(u(x_1))}{u(x_2) - u(x_1)}\right]\left[\frac{u(x_4) - u(x_3)}{x_4 - x_3} / \frac{u(x_2) - u(x_1)}{x_2 - x_1}\right] \\ &\leq \frac{u(x_4) - u(x_3)}{x_4 - x_3} / \frac{u(x_2) - u(x_1)}{x_2 - x_1}\,; \end{aligned} \tag{15}$$

it follows directly from the definition (see (2)) of the index of greediness that $G_{u^*} \leq G_u$.

Proposition 3 summarizes the above properties of this index:

Proposition 3 (i) $G_u \geq 1$.

(ii) $G_u = 1$ if and only if u is concave.

(iii) If u is differentiable,

$$G_u = \sup_{y<x} \frac{u'(x)}{u'(y)}\,. \tag{16}$$

(iv) If the utility function u^* is more concave than the utility function u, then $G_{u^*} \leq G_u$.

Index of non-convexity (or thriftiness)

By duality, we can define, for a strictly increasing utility function u, an index of non-convexity (or thriftiness):

$$T_u = \sup_{x_1<x_2 \leq x_3<x_4} \left[\frac{u(x_2) - u(x_1)}{x_2 - x_1} / \frac{u(x_4) - u(x_3)}{x_4 - x_3}\right]. \tag{17}$$

$T_u \geq 1$ and the value 1 corresponds exclusively to convexity.

[3] In EU theory, the utility function u^* is then said to display (Arrow-Pratt) more risk aversion than the utility function u.

Moreover, for a differentiable u the index of non-convexity can be expressed in the simpler form

$$T_u = \sup_{x<y} \frac{u'(x)}{u'(y)} \tag{18}$$

4 Characterization of monotone risk aversion

We can now state the main result:

4.1 Main result

Theorem 1 A RDEU DM with probability perception function f and utility function u is monotone risk averse if and only if the DM's index of pessimism exceeds the DM's index of non-concavity: $P_f \geq G_u$.

The theorem will be proved in Section 5. Let us first emphasize some implications of this result.

1. Since $G_u \geq 1$, the fact that $P_f \geq G_u$ for monotone risk averse DM's, implies that $P_f \geq 1$, or $f(v) \leq v$. In other words, pessimism is a necessary condition for monotone risk aversion. This fact also follows from (i) monotone risk aversion implies weak risk aversion and (ii) weak risk aversion implies pessimism: $f(v) \leq v$ for $v \in [0;1]$ (see for instance Chateauneuf & Cohen [7]).

2. If f is the identity function then $P_f = 1$. However, there are pessimistic f other than the identity function itself, for which $P_f = 1$: it suffices that $f(p) = p$ for some $p \in (0;1)$ or that $f'(p) = 1$ at either $p = 0$ or $p = 1$. Thus, concavity of u is necessary for monotone risk aversion only for EU maximizers and for RDEU DM "locally close" to being EU maximizers, in this precise sense.

3. Quiggin (see [21]) uses the concept of monotone risk aversion only for DM with concave u. Theorem 1 proves that concavity of u is not a necessary condition for monotone risk aversion. Similarly, convexity is not necessary for monotone risk-seeking attitudes (see the following corollary).

4. If f crosses the diagonal, the DM is neither monotone risk averse nor monotone risk seeking. In fact, not even weakly so.

The following theorem is presented as a corollary since its proof is analogous to that of Theorem 1.

Corollary 1 A RDEU DM with probability perception function f and utility function u is monotone risk seeking if and only if the DM's index of optimism exceeds the DM's index of non-convexity: $O_f \geq T_u$.

4.2 Examples and discussion

4.2.1 Risk-seeking attitude with diminishing marginal utility

Restrict attention to random variables with values in $[0;1]$ and assume accordingly that the utility function is defined in the unit interval.

Example 5. CARA[4] utility and power-type perception functions. Let a RDEU DM's choices among lotteries on $[0;1]$ be characterized by the concave CARA utility function $u(x) = -ae^{-bx}$ and the optimistic power-type perception function $f(v) = 1 - (1-v)^h$, with $a > 0$, $b > 0$ and $h > e^b$. As is easy to ascertain, $O_f = h$ and $T_u = e^b$. Hence, in spite of the concavity of the utility function, the DM is monotone risk seeking.

Note, however, that a RDEU DM with the same utility function $u(x) = -ae^{-bx}$ as above but pessimistic power-type probability perception function $f(v) = v^k$ with $k \geq 1$, is monotone risk averse, since $P_f = k$ and $G_u = 1$.

In summary, a DM with a given concave utility function can be monotone risk averse or monotone risk seeking or in fact neither, depending on the probability perception function.

4.2.2 Mean-preserving increase in risk does not imply monotone mean-preserving increase in risk.

In the following example, Y is a mean-preserving spread but not a monotone mean-preserving spread of X, because X and Z are not comonotone.

Example 6. A non-monotone mean-preserving spread. Consider four equally likely points and two random variables Y and X with respective values

Y	0	1000	3000	4000
X	0	2000	2000	4000

and difference $Z = Y - X$

Z	0	-1000	1000	0

Any RDEU DM with linear u ($G_u = 1$) and pessimistic f ($P_f \geq 1$) is monotone risk averse because $P_f \geq G_u$. For any f satisfying moreover $f(1/2) > \frac{f(1/4)+f(3/4)}{2}$, it is easy to see that $V(Y) > V(X)$ and thus this monotone risk averse DM strictly prefers the more

[4] borrowing EU terminology that may be inappropriate beyond EU

risky random variable Y. By Chew, Karni & Safra [11], this could not have been possible with a convex f.

4.2.3 Comparison of monotone risk aversion with weak and strong risk aversion

As expressed in the Introduction, these three kinds of risk aversion are logically related so that strong r.a. implies monotone r.a., which in turn implies weak r.a. The three are equivalent in EU theory. In RDEU theory, pessimism is necessary for each of the three and concavity of u is necessary only for strong r.a. Furthermore, the three hold simultaneously only in the class of DM's with concave u and convex f (Chew, Karni & Safra [11]). In the class of DM's with concave u, weak and monotone r.a. are equivalent and hold as long as f displays pessimism (Quiggin [20, 21]), but do not imply strong r.a. However, if u is allowed not to be concave, weak r.a. does not imply monotone r.a., as the following example shows, which we present without proof: a DM with $u(x) = x^2$ on $[0; \infty)$ and $f(p) = p^2$, easily seen not to be monotone risk averse because $P_f = 2$ and $G_u = \infty$, is weakly risk averse ([10], in preparation).

There is, thus, an essential difference in the attitude of RDEU DM's to the Quiggin - Bickel & Lehmann and Rothschild & Stiglitz notions of increase in risk: for the latter, there is no degree of pessimism that can compensate for departures from concavity of u and still make the DM averse to risk. The main theorem of this paper shows that for the stronger monotone notion of risk (and thus, weaker notion of risk aversion), such a compensation exists and is completely separable in the sense that no joint condition in terms of both u and f is needed. Rather, the compensation is based on a comparison of intrinsic indices of pure type, one for greediness displayed by u and the other for pessimism displayed by f.

5 Proof of Theorem 1

5.1 Necessity of the condition $P_f \geq G_u$

Consider the setup introduced in Example 1, for some given $v \in (0; 1)$. By Lemma 1, it is enough to prove that for every $(y_1; y_2; y_3; y_4) \in E_\lambda$, where $\lambda = (1 - v)/v$, a monotone risk averse DM characterized by $(u; f)$ verifies

$$\frac{u(y_4) - u(y_3)}{y_4 - y_3} \Big/ \frac{u(y_2) - u(y_1)}{y_2 - y_1} \leq \frac{1 - f(v)}{1 - v} \Big/ \frac{f(v)}{v} : \qquad (19)$$

Let a random variable X have values $y_2 < y_3$ and respective probabilities $1 - v$ and v.

The RDEU functional for a DM characterized by a pair $(u; f)$ is (see (5))

$$V(X) = u(y_2) + f(v)[u(y_3) - u(y_2)] \, . \tag{20}$$

Let us apply a monotone mean-preserving spread to this distribution in the following way: Since $(y_1; y_2; y_3; y_4) \in E_\lambda$, there exists $\gamma > 0$ such that $y_1 = y_2 - \gamma v$ and $y_4 = y_3 + \gamma(1 - v)$. The random variable Y taking values $y_1 < y_4$ with respective probabilities $1 - v$ and v is then riskier than X in the sense of a M-MPIR. Since

$$V(Y) = u(y_1) + f(v)[u(y_4) - u(y_1)] \, ; \tag{21}$$

then $V(Y) - V(X) \leq 0$ if and only if $(1 - f(v))(u(y_1) - u(y_2)) + f(v)(u(y_4) - u(y_3)) \leq 0$. But $(y_4 - y_3)/(y_2 - y_1) = (1 - v)/v$, so the required inequality (19) follows.

5.2 Sufficiency of the condition $P_f \geq G_u$

We must compare $V(X)$ and $V(Y)$ for an arbitrary pair $(X; Y)$ of (equal-mean) random variables for which Y is a M-MPIR with respect to X. The proof will proceed in two steps: (i) sufficiency of the condition for random variables with finitely many values, and (ii) a continuity argument to extend sufficiency of the condition to the set $\mathcal{V}$, composed of bounded random variables in general.

5.2.1 The discrete case

As in Landsberger & Meilijson [18] and Chateauneuf, Cohen & Meilijson [9], let the result of an out-stretch (v,w;a,b) (where $0 < v \leq w < 1$ and $a, b > 0$) of a distribution function F be the distribution G obtained from F by shifting to the right by an amount b the section of F above height $1 - v$ and by a to the left the section of F below height $1 - w$. (If F is an income distribution, then the poorest $(1 - w)$-quantile of the population becomes poorer by the constant amount a and the richest v-quantile becomes richer by the constant amount b). Clearly, if $vb = (1 - w)a$, then $(v; w; a; b)$ is a mean-preserving out-stretch. Notice that these out-stretches may entail splitting an atom in each side in two, one part of which gets shifted and the other stays in place. It is intuitively true and easy to prove (see [18]) that (i) if G is obtained from F by a mean-preserving out-stretch then G is a M-MPIR with respect to F, and (ii) if F and G are supported by finite sets and G is a M-MPIR with respect to F, then there is a finite sequence $F = F_1; F_2; \ldots; F_k = G$ of distributions such that for each $1 \leq i < k$, F_{i+1} is a mean-preserving out-stretch of F_i. Due to these facts, it is enough to prove that $V(Y) \leq V(X)$ whenever the distribution of Y is a mean-preserving out-stretch of that of X. Let $(v; w; a; b)$ be this stretch and let

$$L(X) = (x_1, 1 - v_1; x_2, v_1 - v_2; \ldots; x_i, v_{i-1} - v_i; \ldots; x_j, v_{j-1} - v_j; \ldots; x_{n-1}, v_{n-2} - v_{n-1}; x_n, v_{n-1}) \tag{22}$$

be the distribution of X, with atoms $x_1 < x_2 < \cdots < x_n$ and probabilities $P(X > x_k) = v_k$. Let $i < j$ be chosen so that $v_{i-1} \geq w \geq v_i \geq v_{j-1} \geq v \geq v_j$. Then $V(X) - V(Y)$ is

$$\begin{aligned} \Delta V = {} & [u(x_1) - u(x_1 - a)]\{1 - f(v_1)\} + [u(x_2) - u(x_2 - a)]\{f(v_1) - f(v_2)\} + \cdots \\ & + [u(x_i) - u(x_i - a)]\{f(v_{i-1}) - f(w)\} - [u(x_j + b) - u(x_j)]\{f(v) - f(v_j)\} \\ & - [u(x_{j+1} + b) - u(x_{j+1})]\{f(v_j) - f(v_{j+1})\} - \cdots - [u(x_n + b) - u(x_n)]\{f(v_{n-1})\}. \end{aligned} \tag{23}$$

To prove non-negativity of (23), it is enough to prove its non-negativity when each term in square brackets with a + sign in front is replaced by the minimum of these terms, and each term in square brackets with a − sign is replaced by the maximum of these. Let the minimum of $u(x_k) - u(x_k - a)$ over $k \leq i$ be attained at $k = l$ and the maximum of $u(x_k + b) - u(x_k)$ over $k \geq j - 1$ be attained at $k = m$. Then, since $(1 - w)a = vb$,

$$\begin{aligned} \Delta V & \geq [u(x_l) - u(x_l - a)]\{1 - f(w)\} - [u(x_m + b) - u(x_m)]\{f(v)\} \\ & = \left\{ \frac{u(x_l) - u(x_l - a)}{a} \frac{1 - f(w)}{1 - w} - \frac{u(x_m + b) - u(x_m)}{b} \frac{f(v)}{v} \right\} vb \\ & \geq \frac{u(x_l) - u(x_l - a)}{a} \frac{f(v)}{v} \left\{ \left[\frac{1 - f(w)}{1 - w} \Big/ \frac{f(v)}{v}\right] - G_u \right\} vb. \end{aligned} \tag{24}$$

Since by assumption $P_f \geq G_u$, the last term is non-negative by Proposition 2 (vi). This completes the proof of sufficiency of the condition $P_f \geq G_u$, dealing with random variables supported by a finite set. It remains to present a continuity argument that will extend sufficiency to general bounded random variables.

5.2.2 Continuity arguments for the general case

Let us now consider two elements X and Y of V, such that Y is a M-MPIR of X. We can assume without loss of generality that $Y = X + Z$ with Z and X comonotone and $E(Z) = 0$.

We use the standard uniform approximations of bounded random variables by non-decreasing step functions, which preserve comonotonicity.

Thus, let X_n and Z_n be such approximations of X and Z. Explicitly,

$$X_n = \sum_{i=0}^{n2^n - 1} \left[\frac{i}{2^n} 1_{\{\frac{i}{2^n} \leq X < \frac{i+1}{2^n}\}} - \frac{i+1}{2^n} 1_{\{-\frac{i+1}{2^n} \leq X < -\frac{i}{2^n}\}}\right] \tag{25}$$

where 1_A denotes the characteristic function of the event A. A similar expression defines Z_n. X_n and Z_n are comonotone, since they are non-decreasing functions of comonotone random variables. (One of the equivalent definitions of comonotonicity of X and Z is the existence of a random variable U of which each of X and Z is a non-decreasing function. Obviously, so are X_n and Z_n, automatically).

For n large enough so that X, Y and Z are supported by the interval $(-n; n)$,

$$X - \frac{1}{2^n} \leq X_n \leq X \text{ and } Z - \frac{1}{2^n} \leq Z_n \leq Z . \tag{26}$$

Hence, with $Y_n = X_n + Z_n$,

$$Y - 1/2^{n-1} \leq Y_n \leq Y . \tag{27}$$

The second statement in (26) and the fact that $E(Z) = 0$ entail $E(Z_n) \uparrow 0$. Define then $\alpha_n \downarrow 0$ by $E(Z_n + \alpha_n) = 0$ and set $Z'_n = Z_n + \alpha_n; Y'_n = X_n + Z'_n$.

Clearly, X_n and Y'_n are finite-support random variables and Y'_n has been obtained from X_n by a M-MPIR. It follows from part (i) of the proof that for each n,

$$V(X_n) \geq V(Y'_n) . \tag{28}$$

For a random variable T, $V(T)$ is the Choquet integral of $u(T)$ with respect to the capacity $c = f \circ P$, i.e., $V(T) = \int u(T) d f \circ P$. Since the Choquet integral is monotone and comonotonically additive (see [5, 13, 26, 27]), (28) will entail the desired result $V(X) \geq V(Y)$. For the sake of completeness, we present a direct argument.

$Y'_n \geq Y_n$ implies $u(Y'_n) \geq u(Y_n)$ and, a-fortiori, $V(Y'_n) \geq V(Y_n)$. By (28),

$$V(X_n) \geq V(Y_n); \forall n \in N . \tag{29}$$

Since the utility function u is continuous, it is uniformly continuous on any compact subset of R. Hence, from (26) and (27), $\forall \epsilon \exists N(\epsilon)$ s.t. $n \geq N(\epsilon)$ implies that

$$u(X) - \epsilon \leq u(X_n) \leq u(X) ; u(Y) - \epsilon \leq u(Y_n) \leq u(Y) . \tag{30}$$

Monotonicity, comonotonic additivity and the normalization property $\int 1_\Omega df \circ P = 1$ lead to the two inequalities: $\forall n \geq N(\epsilon)$,

$$V(X) - \epsilon \leq V(X_n) \leq V(X) ; V(Y) - \epsilon \leq V(Y_n) \leq V(Y) . \tag{31}$$

Therefore, $\lim_{n \to \infty} V(X_n) = V(X)$ and $\lim_{n \to \infty} V(Y_n) = V(Y)$. Hence, by (29), $V(X) \geq V(Y)$.

Remark A RDEU DM was postulated in Definition 5 to have a continuous, strictly increasing utility function u and a strictly increasing probability perception function f. The preceding proof could be somewhat simplified if we added differentiability assumptions on u.

6 Appendix

Proof of Lemma 1

Obviously, $G_u(\lambda) \leq G_u$. We have to prove the opposite inequality $G_u(\lambda) \geq G_u$, i.e., for any $(x_1; x_2; x_3; x_4) \in R^4$ such that $x_1 < x_2 \leq x_3 < x_4$ and any $\varepsilon > 0$, there exists a $(y_1; y_2; y_3; y_4) \in E_\lambda$ such that

$$\frac{u(y_4) - u(y_3)}{y_4 - y_3} = \frac{u(y_2) - u(y_1)}{y_2 - y_1} \geq \frac{u(x_4) - u(x_3)}{x_4 - x_3} = \frac{u(x_2) - u(x_1)}{x_2 - x_1} - \varepsilon :$$

By continuity of u, there exists some $x_0 \in (x_3; x_4)$ such that for every $x \in (x_0; x_4)$,

$$\frac{u(x) - u(x_3)}{x - x_3} = \frac{u(x_2) - u(x_1)}{x_2 - x_1} \geq \frac{u(x_4) - u(x_3)}{x_4 - x_3} = \frac{u(x_2) - u(x_1)}{x_2 - x_1} - \varepsilon :$$

Divide the interval $(x_1; x_2)$ into K sub-intervals of equal length $\Delta = (x_2 - x_1)/K$ such that $\lambda\Delta < x_4 - x_0$. This guarantees that the sequence $x_3; x_3 + \lambda\Delta; x_3 + 2\lambda\Delta; x_3 + 3\lambda\Delta; \ldots$ has some element $x_3 + k\lambda\Delta$ (call it x) in the interval $(x_0; x_4)$.

Since $(u(x) - u(x_3))/(x - x_3) = (1/k) \sum_{i=0}^{k-1} [u(x_3 + (i+1)\lambda\Delta) - u(x_3 + i\lambda\Delta)]/\lambda\Delta$, there is a sub-interval $(y_3; y_4) = (x_3 + i\lambda\Delta; x_3 + (i+1)\lambda\Delta)$ of $(x_3; x_4)$ such that $(u(y_4) - u(y_3))/(y_4 - y_3) \geq (u(x) - u(x_3))/(x - x_3)$. Similarly, there exists a sub-interval $(y_1; y_2) = (x_1 + j\Delta; x_1 + (j+1)\Delta)$ of $(x_1; x_2)$ along which $(u(y_2) - u(y_1))/(y_2 - y_1) \leq (u(x_2) - u(x_1))/(x_2 - x_1)$. This completes the proof.

Proof of Proposition 2

Property (i) follows easily from the definition of P_f (see (1)) as the infimum of $(v/(1 - v))(1/f(v) - 1))$, that majorizes the corresponding expression involving g.

Property (ii) follows from Property (i) by observing that every non-linear, convex perception function is separated from the identity function by some kink function (see (9).

One direction of the proof of Property (iii) is clear: If $f \leq g$ and g is a non-linear, convex perception function, then by Properties (i) and (ii), $P_f \geq P_g > 1$. As for the opposite direction, assume that $P_f > 1$. Let $g(v) = f_{P_f}(v) = v/(v + (1 - v)P_f)$ be the hyperbolic perception function with index of pessimism P_f and re-write the inequality

$$P_f = \inf_w \{[(1 - f(w))/(1 - w)]/[f(w)/w]\} \leq [(1 - f(v))/(1 - v)]/[f(v)/v]$$

simply as $f \leq g$.

Property (iv) is proved by first applying Property (i) to obtain that $P_f \geq P_{f_P} = P$, and then observing that $P_f = \inf_{v \in (0;1)}\{\cdots\} \leq \inf_{v \in I}\{\cdots\}$ for every sub-interval $I \subset (0;1)$. The inequality $P_f \leq P$ follows by taking $I = (0;\varepsilon)$ and minorizing f by L_P on I, or by taking $I = (1-\varepsilon;1)$ and minorizing f by R_P on I.

Proof of Property (v): To see that $P_f \geq Q$, observe that this is trivial if $Q = 1$ and concentrate on the case $Q > 1$. Take any $\varepsilon \in (0;Q - 1)$. By definition of Q, $f_{Q-\varepsilon}$ majorizes f. Hence, by Property (i), $P_f \geq Q - \varepsilon$. Since this is true for all sufficiently small ε, $P_f \geq Q$. To prove the opposite inequality, take any $P > Q$ and any $v \in (0;1)$ at which $f(v) > f_P(v)$. Then,

$$P_f = \inf_w \frac{1 - f(w)}{1 - w} \Big/ \frac{f(w)}{w} \leq \frac{1 - f(v)}{1 - v} \Big/ \frac{f(v)}{v} < \frac{1 - f_P(v)}{1 - v} \Big/ \frac{f_P(v)}{v} \equiv P\,.$$

Since $P_f < P$ for all $P > Q$, the inequality $P_f \leq Q$ follows.

Proof of Property (vi): It is clear that the right-hand side of (12) is less than or equal P_f, since the infimum over $v_2 \leq v_1$ is less than or equal to the infimum over $v_2 = v_1$, that is P_f by definition. To see that the infimand in the right-hand side of (12) is greater or equal than P_f, it is enough to show that for arbitrary $0 < v_2 < v_1 < 1$,

$$\frac{1 - f(v_1)}{1 - v_1} \Big/ \frac{f(v_2)}{v_2} \geq \min\left[\frac{1 - f(v_1)}{1 - v_1} \Big/ \frac{f(v_1)}{v_1}\,;\; \frac{1 - f(v_2)}{1 - v_2} \Big/ \frac{f(v_2)}{v_2}\right]. \tag{32}$$

Otherwise, $f(v_2)/v_2 > f(v_1)/v_1$ and $(1 - f(v_2))/(1 - v_2) > (1 - f(v_1))/(1 - v_1)$, so $f(v_1) > 1 - (1 - f(v_2))(1 - v_1)/(1 - v_2) > 1 - (1 - f(v_1)v_2/v_1)(1 - v_1)/(1 - v_2)$. Extracting $f(v_1)$ from the first and third terms implies that $f(v_1) > v_1$, a contradiction.

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