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Consensus, Communication and Knowledge: an Extension with Bayesian Agents

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Abstract

Parikh and Krasucki [1990] showed that pairwise communication of the value of a function f leads to a consensus about the communicated value if the function f is *convex*. They showed that *union consistency* of f may not be sufficient to guarantee consensus in any communication protocol. Krasucki [1996] proved that consensus occurs for any *union consistent* function if the protocol contains no cycle. We show that if agents communicate their optimal action, namely the action that maximizes their expected utility, then consensus obtains in any fair protocol for any action space.

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1 Introduction

Aumann [1976] proved that if two individuals have the same prior beliefs, then common knowledge of their posterior beliefs for an event implies the equality of these posteriors. Geanakoplos and Polemarchakis [1982] extended Aumann's result to a dynamic framework, and showed that communication of posterior beliefs leads to a situation of common knowledge

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of these posteriors. Cave [1983] and Bacharach [1985] proved these agreement results considering *union consistent*[1] functions more general than posterior beliefs. In all of these settings, communication is public, as achieved *e.g.* by auctions. Parikh and Krasucki [1990] investigated the case where communication is not public but in pairs. They defined an updating process along which agents communicate with each other, according to a protocol upon which they have agreed beforehand. At each stage one of the agents transmits to another agent the value of a certain function f, which depends on the set of states of the world she conceives as possible at that stage. Parikh and Krasucki [1990] showed that two conditions guarantee that eventually, all agents will communicate the same value (a situation we will refer to as a consensus): 1) a *fairness* condition on the communication protocol, which imposes that every agent has to be sender and receiver of the communication infinitely many times; 2) a *convexity* condition on the function whose value is communicated. Let Ω be the set of states of the world. A function $f : 2^\Omega \to \mathbb{R}$ is convex if $\forall\ X, Y \in 2^\Omega$ such that $X \cap Y = \emptyset$, there exists $\alpha \in]0, 1[$ such that $f(X \cup Y) = \alpha f(X) + (1 - \alpha)f(Y)$. This condition is satisfied by conditional probabilities for instance, and is more restrictive than Cave's union consistency.

Parikh and Krasucki's convexity condition may not apply in some contexts, as shown in the following example. An individual contemplates buying a car. The set of available decisions is $\{$*buy, not buy*$\}$. Suppose that we re-label the decisions in $\mathbb{R}$, with for instance 1 standing for *buy* and 0 standing for *not buy*. The convexity condition implies that if $f(X) = 0$ and $f(Y) = 1$ for some X, Y such that $X \cap Y = \emptyset$, then $f(X \cup Y) \in\ ]0, 1[$, which does not correspond to any decision in $\{$*buy,not buy*$\}$. Hence there are some decision spaces for which, even after a re-labelling in $\mathbb{R}$, we may not be able to apply the convexity condition.

Parikh and Krasucki [1990] showed by a counter-example that *weak convexity*[2] and union consistency are not sufficient to guarantee that consensus occurs in any fair protocol. Krasucki [1996] investigated what restrictions on the communication protocol should be imposed to guarantee the consensus with any union consistent function. He showed that if the protocol is fair and contains no cycle, then communication of the value of any union consistent function leads to consensus.

[1] Let Ω be the set of states of the world. $f : 2^\Omega \to \mathcal{D}$ is union consistent if $\forall\ X, Y \in 2^\Omega$ such that $X \cap Y = \emptyset$, $f(X) = f(Y) \Rightarrow f(X \cup Y) = f(X) = f(Y)$.

[2] Let Ω be the set of states of the world. $f : 2^\Omega \to \mathbb{R}$ is weakly convex if $\forall\ X, Y \in 2^\Omega$ such that $X \cap Y = \emptyset$, there exists $\alpha \in [0, 1]$ such that $f(X \cup Y) = \alpha f(X) + (1 - \alpha)f(Y)$.

In this note, we give a new condition on f for consensus to emerge in any fair communication protocol. This condition is that the function whose values are communicated is the maximizer of a conditional expected utility. Contrary to Parikh and Krasucki's convexity condition, this condition applies to any action space.

Even after an appropriate re-labelling of the image of f in $\mathbb{R}$, the functions we consider may not be representable by weakly convex functions. Furthermore, there exist weakly convex functions that do not obey our condition. Hence the class of functions we look at have a non-empty intersection with the class of weakly convex functions, but there is no inclusion relation among them. On the other hand, for any decision space, the functions we consider are union consistent.

2 Reaching a consensus

Let Ω be a finite set of states of the world. We consider a group of N agents, each of them endowed with a partition Π_i of Ω. All agents share some prior belief P on Ω. We note $\Pi_i(\omega)$ the cell of Π_i that contains ω. $\Pi_i(\omega)$ is the set of states that i judges possible when state ω occurs. As in Parikh and Krasucki [1990], agents communicate the value of a function $f : 2^\Omega \to \mathcal{D}$, according to a $fair$ protocol Pr. A protocol is a pair of functions $(s(.), r(.)) : \mathbb{N} \to \{1, \dots, N\}^2$ where $s(t)$ stands for the sender and $r(t)$ the receiver of the communication which takes place at time t. A protocol is *fair*[3] if no participant is blocked from the communication, that is if every agent is a sender and a receiver infinitely many times, and everyone receives information from every other, possibly indirectly, infinitely many times. Except fairness, we do not make any assumption on the protocol. We assume that $\mathcal{D}$ can be any compact subset of a topological space.

Agents share a common payoff function $U : \mathcal{D} \times \Omega \to \mathbb{R}$, which depends on the chosen action $d \in \mathcal{D}$ and on the realized state of the world. We assume that $U(., \omega)$ is continuous on $\mathcal{D}$ for all ω. What is communicated by an agent is the action that maximizes her expected utility, computed with respect to the common belief P. In order to avoid indifference cases, we make the assumption that given any event, all actions have different expected utility conditional on

[3]Given a protocol $(s(t), r(t))$ consider the directed graph whose vertices are the participants $\{1, \dots, N\}$ and such that there is an edge from i to j iff there are infinitely many t such that $s(t) = i$ and $r(t) = j$. The protocol is fair if the graph above is strongly connected.

this event. That is to say given an event $F \subseteq \Omega$, $\forall\, d, d' \in \mathcal{D},\ E(U(d,.) \mid F) \neq E(U(d',.) \mid F)$. Without this assumption, the set of maximizing actions of an agent may not be a singleton, and we would have to specify the way agents choose between indifferent actions.

The function $f : 2^{\Omega} \rightarrow \mathcal{D}$ is then defined by:

$$\forall\, E \subseteq \Omega,\ f(E) = \text{argmax}_{d \in \mathcal{D}} \mathrm{E}(U(d,.) \mid E)$$

Suppose now that Pr is some given protocol. The set of possible states for an agent i at time t if the state of the world is ω is denoted $C_i(\omega, t)$ and is defined by the following recursive process:

$$C_i(\omega, 0) = \Pi_i(\omega)$$
$$C_i(\omega, t+1) = C_i(\omega, t) \cap \{\omega' \in \Omega \mid f(C_{s(t)}(\omega', t)) = f(C_{s(t)}(\omega, t))\}\ if\ i = r(t),$$
$$C_i(\omega, t+1) = C_i(\omega, t)\ otherwise.$$

The next result states that for all ω, $f(C_i(\omega, t))$ has a limiting value which does not depend on i.

Theorem 1 *There is a $T \in \mathbb{N}$ such that for all ω, i, and all t, $t' \geq T$, $C_i(\omega, t) = C_i(\omega, t')$. Moreover, if the protocol is fair, then for all i, j, for all ω, $f(C_i(\omega, T)) = f(C_j(\omega, T))$.*

We now discuss the properties of the function f defined as the argmax of an expected utility. First, f is clearly *union consistent* for any action space. Second, f may not be representable by a weakly convex function, namely a one to one function $g : \mathcal{D} \rightarrow \mathbb{R}$ may fail to exist such that $g \circ f$ is weakly convex. If such a function g were to exist, learning and consensus properties of f and $g \circ f$ would be the same. Therefore, the functions f we consider would be particular weakly convex functions, for which consensus obtains in any fair protocol. We show that it is not the case with the following counter example.

Consider the case where $\Omega = \{1, 2, 3, 4\}$, $\mathcal{D} = \{a, b, c\}$, P is uniform ($P(\omega) = 1/4\ \forall\, \omega$) and the utility function U is defined by:

$$U(a,1) = 1, U(a,2) = 0, U(a,3) = 1, U(a,4) = 0$$
$$U(b,1) = 0, U(b,2) = 1, U(b,3) = 2/3, U(b,4) = 2/3$$
$$U(c,1) = 2/3, U(c,2) = 2/3, U(c,3) = 0, U(c,4) = 1$$

We have in particular:

$$f(\{1\}) = a,\ f(\{2\}) = b,\ f(\{3\}) = a,\ f(\{4\}) = c,\ f(\{1,2\}) = c,\ f(\{3,4\}) = b$$

For any one to one function $g : \mathcal{D} \to \mathbb{R}$, six cases are possible. We show that in each case, $g \circ f$ is not weakly convex.

1. If $g(a) < g(b) < g(c)$, then $g \circ f(\{1\}) < g \circ f(\{2\}) < g \circ f(\{1,2\})$.
2. If $g(a) < g(c) < g(b)$, then $g \circ f(\{3\}) < g \circ f(\{4\}) < g \circ f(\{3,4\})$.
3. If $g(b) < g(a) < g(c)$, then $g \circ f(\{3,4\}) < g \circ f(\{3\}) < g \circ f(\{4\})$.
4. If $g(b) < g(c) < g(a)$, then $g \circ f(\{3,4\}) < g \circ f(\{4\}) < g \circ f(\{3\})$.
5. If $g(c) < g(a) < g(b)$, then $g \circ f(\{1,2\}) < g \circ f(\{1\}) < g \circ f(\{2\})$.
6. If $g(c) < g(b) < g(a)$, then $g \circ f(\{1,2\}) < g \circ f(\{2\}) < g \circ f(\{1\})$.

Finally, there exist weakly convex functions that cannot be defined as the argmax of an expected utility. Such an example can be found in Parikh and Krasucki [1990, p 185]: they exhibit a weakly convex function f such that consensus may fail to occur in some protocols. It can be shown easily that it is not possible to find a utility function U and a probability P such that this function f is the argmax of the conditional expectation of U.

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Proof: [Theorem 1]

1) As Ω is finite, the first part of the theorem is evident. In the sequel, we will note $C_i(\omega)$ the limiting value of $C_i(\omega, t)$, and C_i the information partition of agent t at equilibrium.

2) As in Parikh and Krasucki [1990], we prove the second part of the theorem for $N = 3$ and for a "round-robin protocol", namely such that for all t, $s(t) = t$ mod 3 and $r(t) = (t+1)$ mod 3. Note that this is sufficient to prove the theorem for any fair protocol. Our argument only uses the fact that we are able to find a chain $t_1 < t_2 < \cdots < t_p$, with $T \leq t_1$, such that: (a) $s(t_1) = 1$, (b) the receiver at t_j is the sender at t_{j+1}, (c) the chain passes through all participants, finally returning to 1. This is implied by the fact that the protocol is fair.

Let M_{ij} be the partition of common knowledge among agents i and j at equilibrium, that is M_{ij} is the finest partition of Ω such that $\forall\ \omega,\ C_i(\omega) \subseteq M_{ij}(\omega)$ and $C_j(\omega) \subseteq M_{ij}(\omega)$. By consequence, $\forall\ \omega$, $M_{ij}(\omega)$ is a disjoint union of cells of C_i and a disjoint union of cells of C_j. $\sum_{C_i(k)\subseteq M_{ij}(\omega)}$ will denote the sum on all cells of C_i composing $M_{ij}(\omega)$.

At equilibrium, agent 1 communicates her optimal action to agent 2, agent 2 communicates her optimal action to agent 3 and agent 3 communicates her optimal action to agent 1. By consequence, the action taken by agent 1 is common knowledge among 1 and 2. Hence we have for all ω:

$$M_{12}(\omega) \subseteq \{\omega' \in \Omega \mid f(C_1(\omega')) = f(C_1(\omega))\}$$

As $M_{12}(\omega)$ is a disjoint union of cells of C_1, union consistency of f implies that $f(M_{12}(\omega)) = f(C_1(k))$ $\forall\ k \in M_{12}(\omega)$.

- **Result 1** $E(U(f(M_{12}(\omega)),.) \mid M_{12}(\omega)) = E[E(U(f(C_1(\centerdot)),.) \mid C_1(\centerdot)) \mid M_{12}(\omega)]$

Proof: For all $\omega' \in M_{12}(\omega)$, $f(C_1(\omega')) = f(M_{12}(\omega))$. Then $E[E(U(f(C_1(\centerdot)),.) \mid C_1(\centerdot)) \mid M_{12}(\omega)] = E[E(U(f(M_{12}(\omega)),.) \mid C_1(\centerdot)) \mid M_{12}(\omega)]$. As M_{12} is coarser than C_1, the law of iterated expectations implies that $E[E(U(f(M_{12}(\omega)),.) \mid C_1(\centerdot)) \mid M_{12}(\omega)] = E(U(f(M_{12}(\omega)),.) \mid M_{12}(\omega)]$.

- **Result 2** $E(U(f(M_{12}(\omega)),.) \mid M_{12}(\omega)) \leq \displaystyle\sum_{C_2(k)\subseteq M_{12}(\omega)} \frac{P(C_2(k))}{P(M_{12}(\omega))} E(U(f(C_2(k)),.) \mid C_2(k))$

Proof: By definition, $\forall\, k \in M_{12}(\omega)$ we have:

$$E(U(f(M_{12}(\omega)),.) \mid C_2(k)) \leq E(U(f(C_2(k)),.) \mid C_2(k))$$

It implies that:

$$\sum_{C_2(k)\subseteq M_{12}(\omega)} P(C_2(k))E(U(f(M_{12}(\omega)),.) \mid C_2(k)) \leq \sum_{C_2(k)\subseteq M_{12}(\omega)} P(C_2(k))E(U(f(C_2(k)),.) \mid C_2(k))$$

that is:

$$P(M_{12}(\omega))E(U(f(M_{12}(\omega)),.) \mid M_{12}(\omega)) \leq \sum_{C_2(k)\subseteq M_{12}(\omega)} P(C_2(k))E(U(f(C_2(k)),.) \mid C_2(k))\Box$$

- **Result 3** $\forall\, i, j,\ E[E(U(f(C_i(\centerdot)),.) \mid C_i(\centerdot))] = E[E(U(f(C_j(\centerdot)),.) \mid C_j(\centerdot))]$

Proof:

$$E[E(U(f(C_1(\centerdot)),.) \mid C_1(\centerdot))] = \sum_{M_{12}(\omega)\subseteq\Omega} P(M_{12}(\omega)E[E(U(f(C_1(\centerdot)),.) \mid C_1(\centerdot)) \mid M_{12}(\omega)]$$

Yet by results **1** and **2**, we have

$$P(M_{12}(\omega))E[E(U(f(C_1(\centerdot)),.) \mid C_1(\centerdot)) \mid M_{12}(\omega)] \leq \sum_{C_2(k)\subseteq M_{12}(\omega)} P(C_2(k))E(U(f(C_2(k)),.) \mid C_2(k))$$

Then

$$\begin{aligned} E[E(U(f(C_1(\centerdot)),.) \mid C_1(\centerdot))] &\leq \sum_{M_{12}(\omega)\subseteq\Omega}\ \sum_{C_2(k)\subseteq M_{12}(\omega)} P(C_2(k))E(U(f(C_2(k)),.) \mid C_2(k)) \\ &= \sum_{C_2(k)\subseteq\Omega} P(C_2(k))E(U(f(C_2(k)),.) \mid C_2(k)) \\ &= E[E(U(f(C_2(\centerdot)),.) \mid C_2(\centerdot))] \end{aligned}$$

Applying the same reasoning, we get

$$E[E(U(f(C_2(\centerdot)),.) \mid C_2(\centerdot))] \leq E[E(U(f(C_3(\centerdot)),.) \mid C_3(\centerdot))]$$

and

$$E[E(U(f(C_3(\centerdot)),.) \mid C_3(\centerdot))] \leq E[E(U(f(C_1(\centerdot)),.) \mid C_1(\centerdot))]$$

Hence $E[E(U(f(C_i(\centerdot)),.) \mid C_i(\centerdot))] = E[E(U(f(C_j(\centerdot)),.) \mid C_j(\centerdot))]$ for all i, j. $\Box$

• **Result 4** *For all $\omega \in \Omega$, we have*

$E(U(f(C_1(\omega)),.) \mid C_2(\omega)) = E(U(f(C_2(\omega)),.) \mid C_2(\omega))$

$E(U(f(C_2(\omega)),.) \mid C_3(\omega)) = E(U(f(C_3(\omega)),.) \mid C_3(\omega))$

$E(U(f(C_3(\omega)),.) \mid C_1(\omega)) = E(U(f(C_1(\omega)),.) \mid C_1(\omega))$

Proof:

By **Result 3**, the inequality can not be strict in **Result 2**. Then we have:

$$P(M_{12}(\omega))E(U(f(M_{12}(\omega)),.) \mid M_{12}(\omega)) = \sum_{C_2(k)\subseteq M_{12}(\omega)} P(C_2(k))E(U(f(C_2(k)),.) \mid C_2(k))$$

By definition, $E(U(f(C_1(k)),.) \mid C_2(k)) \leq E(U(f(C_2(k)),.) \mid C_2(k))$ for all $k \in M_{12}(\omega)$.

If $\exists\, k$ such that $E(U(f(C_1(k)),.) \mid C_2(k)) < E(U(f(C_2(k)),.) \mid C_2(k))$, then

$$\sum_{C_2(k)\subseteq M_{12}(\omega)} P(C_2(k))E(U(f(C_1(k)),.) \mid C_2(k)) < \sum_{C_2(k)\subseteq M_{12}(\omega)} P(C_2(k)E(U(f(C_2(k)),.) \mid C_2(k))$$

that is:

$$P(M_{12}(\omega))E(U(f(M_{12}(\omega)),.) \mid M_{12}(\omega)) < \sum_{C_2(k)\subseteq M_{12}(\omega)} P(C_2(k)E(U(f(C_2(k)),.) \mid C_2(k))$$

which is a contradiction.

Hence we have $E(U(f(C_1(k)),.) \mid C_2(k)) = E(U(f(C_2(k)),.) \mid C_2(k))$ for all $k \in M_{12}(\omega)$. As it is true for all ω, we have $E(U(f(C_1(k)),.) \mid C_2(k)) = E(U(f(C_2(k)),.) \mid C_2(k))$ for all $k \in \Omega$. The same reasoning applies for 2, 3 and 3, 1. □

From **Result 4** and the assumption that all actions bring different expected utilities, we have

$$f(C_1(\omega)) = f(C_2(\omega)) = f(C_3(\omega))\ \forall\ \omega \in \Omega$$