Centre d'Economie de la Sorbonne
UMR 8174

UNIVERSITÉ PARIS 1 – PANTHÉON - SORBONNE –

Cahiers de la MSE

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Marius Valentin BOLDEA

2006.54

CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

Maison des Sciences Économiques, 106-112 boulevard de L'Hôpital, 75647 Paris Cedex 13
http://mse.univ-paris1.fr/Publicat.htm
ISSN : 1624-0340

ON THE EQUILIBRIUM IN A DISCRETE-TIME LUCAS MODEL WITH ENDOGENOUS LEISURE

MARIUS VALENTIN BOLDEA

Abstract. In this paper I study a discrete-time version of the Lucas model with the endogenous leisure but without physical capital. Under standard conditions I prove that the optimal human capital sequence is increasing. If the instantaneous utility function and the production function are Cobb-Douglas, I prove that the human capital sequence grow at a constant rate. I finish by studying the existence and the unicity of the equilibrium in the sense of Lucas or Romer.

AMS Classification Numbers: 91B62, 91B50, 37N4, 74H20, 74H25

Key words and phrases. Lucas Model, Human Capital, Externalities, Optimal Growth, Equilibrium.

I am grateful to Cuong Le Van and Pascal Gourdel for valuable comments and suggestions. I acknowledges too, the financial support of the Eiffel Doctorate Scholarship, program launched by the French Ministry of Foreign Affairs.

1. INTRODUCTION

Intertemporal models with elastic labor supply continue to be the standard setting used to model many issues in applied macroeconomics. Examples include the theories of business cycles fluctuation and the analysis of economic environments that are distorted due to the presence of taxation and monetary policies. The attempts to assess qualitatively the role of leisure in the process of growth include the studies of Chase [4] and Ryder et al. [11]. Introducing leisure choice in Lucas models, Lardon-De-Guevara et al. [6] and Ortigueira [9] show that elastic labor has important qualitative implications for the behavior of equilibrium paths in the long-run (multiple balanced growth paths), as well as, during the transitional period.

The role of knowledge or human capital has been proved to be crucial for the endogenous growth theory. In Romer [10], the author proves that the knowledge accumulated by the agents is the basic form of the capital. In Lucas [7], physical capital and human capital are used as inputs in the production process, but the crucial role is the human capital accumulation. In both papers continuous-time models are used. In Gourdel et al. [5] the authors show that, if the quality of the human capital accumulation is high, the economy will take off and there exists a unique equilibrium. Their model is a discrete-time version of the Lucas model. In each period, every individual consumes a quantity c_t.

Our model is a discrete-time Lucas model too, but in each period, every individual consumes a quantity c_t and spends a quantity of leisure l_t.

We first consider the social planner problem. We show that if the quality of the human capital accumulation is high, then the economy will take off, i.e. the optimal human capital sequence will grow over time. If we assume that the form of instantaneous utility function is $u(c,l) = c^{\mu} l^{1-\mu}$ and the production function $f(x) = x^{\alpha}$, then there exists a unique equilibrium. This equilibrium must be understood in the sense of Lucas [7] or Romer [10]. That is a human capital path such that, when it is used as externality, it will coincide with the solution to the optimal problem taking it as exogenously determined.

It is known that Lucas [7] and Romer [10] use continuous-time models. Xie [15] shows that a continuum of equilibria may exist in the Lucas model with physical and human capital. In this paper, we use discrete-time framework as in Gourdel and al. [5] or Le Van and Morhaim [13]. It reduces the complexity of the proofs of the existence of optimal solutions to the social planner problem and of equilibria as well.

The paper is organized as follows: in the second section, we present the model and prove the existence of the optimal solution to the social planner problem. In the third section, we prove the existence and uniqueness of equilibrium.

2. THE SOCIAL PLANNER PROBLEM

2.1. **The Model.** The following model is a discrete-time horizon version of the Lucas model without physical capital. We consider an intertemporal model where the social planner maximizes the utility of an infinitely lived representative consumer. In each period, every individual consumes a quantity c_t and spends a quantity of leisure l_t. The consumption good is produced through a production function using only labour as input. Effective labour is the sum of working hours combined with the human capital, which are devoted to the production process. More explicitly, we assume there exists a representative worker who has $h \in [0, +\infty)$ as human capital and devotes a fraction θ of his non-leisure time for working, the remaining fraction $(1-\theta)$ to human capital accumulation. Effective labour is $N^e = \theta h$. Given h, θ the production level is $G(h) f(\theta h)$. The term $G(h)$ represents the external effect of the human capital. The rate of growth of the human capital depends on the time devoted to human capital accumulation through a function ϕ. The model can be written as follows:

$$\max \sum_{t=0}^{+\infty} \beta^t u(c_t, l_t), \qquad (P)$$

under the constraints:

$$(1) \qquad \forall t \geq 0, \qquad 0 \leq c_t \leq G(h_t) f(\theta_t h_t),$$

$$(2) \qquad h_{t+1} = h_t (1 + \lambda \phi (1 - \theta_t)),$$

$$(3) \qquad 0 \leq \theta_t \leq 1, \quad h_0 > 0 \quad \textit{is} \quad \textit{given}$$

and:

$$u(c, l) = c^{\mu} l^{1-\mu},$$

with:

$$0 < \beta < 1, \quad l = 1 - \theta, \quad 0 < \mu < 1.$$

In the equation describing the dynamics of h_t, the parameter λ measures the quality of the human capital technology function ϕ.

We make the following assumptions:

H1: The utility function u, $u : \mathbb{R}^2 \to \mathbb{R}$, is a Cobb-Douglas: $u(c, l) = c^{\mu} l^{1-\mu}$, $0 < \mu < 1$.

H2: The production function f, $f : \mathbb{R} \to \mathbb{R}$, is a Cobb-Douglas: $f(x) = x^{\alpha}$, $0 < \alpha < 1$.

H3: The function G, $G : \mathbb{R} \to \mathbb{R}$, $G(x) = x^{\gamma}, 0 < \gamma < 1$.

H4: The function ϕ is strictly increasing and twice continuously differentiable, $\phi(0) = 0$, $\phi(1) = 1$, $\lambda > 0$.

H5: $0 < \beta (1 + \lambda)^{(\alpha+\gamma)\mu} < 1$.

2.2. **Optimal Solutions.** Let $\psi : [1, 1 + \lambda] \to \mathbb{R}$ be defined by:

$$(4) \qquad \psi(x) = 1 - \phi^{-1}\left(\frac{1}{\lambda}(x - 1)\right),$$

where ϕ^{-1} denotes the inverse map of ϕ. The function ψ gives the working time fraction when the human capital growth factor is x.

We list the properties of ψ:
(a) ψ is continuously differentiable, decreasing, $\psi(1) = 1$, $\psi(1+\lambda) = 0$, $\psi^{'}(1) = -\frac{1}{\lambda\phi^{'}(0)}$, $\psi^{'}(1+\lambda) = -\frac{1}{\lambda\phi^{'}(1)}$.
(b) If ϕ is (strictly) concave, then ψ is also (strictly) concave.

Observe that the problem is now equivalent to:

$$\max \sum_{t=0}^{+\infty} \beta^t h_t^{(\alpha+\gamma)\mu} \left[\psi\left(\frac{h_{t+1}}{h_t}\right)\right]^{\alpha\mu} \left[1 - \psi\left(\frac{h_{t+1}}{h_t}\right)\right]^{1-\mu},$$

under the constraints:
$\forall t \geq 0, \quad h_t \leq h_{t+1} \leq (1+\lambda) h_t$ and $h_0 > 0$ is given.

PROPOSITION 1. *Under assumptions $H1 - H5$, there exists a solution for problem (P).*

Proof. It is standard (see [12]) . □

The following proposition states that the optimal sequence of human capital is strictly increasing. In other words, the economy will take off.

PROPOSITION 2. *Any optimal human capital sequence $\boldsymbol{h} = (h_0, h_1, ..., h_t, ...)$ satisfies $h_0 < h_1 < ... < h_t < ...$.*

Proof. Since the problem is stationary, it suffices to show that for any $h_0 > 0$, the stationary sequence $(h_0, h_0, ..., h_0, ...)$ is not optimal.

Let $\varepsilon > 0$ sufficiently small such that $(1 + \lambda\phi(\varepsilon)) \leq (1+\lambda)$. Define the sequence $\boldsymbol{h} = (h_0, h_1, ..., h_t, ...)$ by:

$$\forall t \geq 1, \quad h_t = h_0 (1 + \lambda\phi(\varepsilon)).$$

The associated sequence of consumptions $c_\varepsilon = (c_{0\varepsilon}, c_{1\varepsilon}, ..., c_{t\varepsilon}, ...)$ is:

$$c_{0\varepsilon} = G(h_0) f(h_0(1-\varepsilon))$$

and:

$$c_{t\varepsilon} = G(h_0(1+\lambda\phi(\varepsilon))) f(h_0(1+\lambda\phi(\varepsilon))),$$

for any $t \geq 1$.

We also have $\theta_\varepsilon = (\theta_{0\varepsilon}, \theta_{1\varepsilon}, ..., \theta_{t\varepsilon}, ...)$, with:

$$\theta_{0\varepsilon} = 1 - \varepsilon, \quad \theta_{t\varepsilon} = 1, \quad \forall t \geq 1,$$

so:

$$l_{0\varepsilon} = \varepsilon, \quad l_{t\varepsilon} = 0, \quad \forall t \geq 1.$$

The sequences of consumptions $\boldsymbol{c}^*$ and $\boldsymbol{\theta}^*$ associated with $(h_0, h_0, ..., h_0, ...)$ are:

$$c_t^* = G(h_0) f(h_0), \quad \forall t \geq 0$$

and:

$$\theta_t^* = 1, \quad \forall t \geq 0,$$

so:

$$l_t^* = 0, \quad \forall t \geq 0.$$

We compare the utilities associated with these sequences. Let:

$$\Delta_\varepsilon = \sum_{t=0}^{+\infty} \beta^t u\left(c_t, l_t\right) - \sum_{t=0}^{+\infty} \beta^t u\left(c_t^*, l_t^*\right).$$

From the concavity of u, one gets:

$$\Delta_\varepsilon \geq u\left(c_{0\varepsilon}, l_{0\varepsilon}\right) - u\left(c_0^*, l_0^*\right) + \frac{\beta}{1-\beta}\left[u\left(c_{1\varepsilon}, l_{1\varepsilon}\right) - u\left(c_1^*, l_1^*\right)\right] \geq$$
$$\geq u_c\left(c_{0\varepsilon}, l_{0\varepsilon}\right)\left(c_{0\varepsilon} - c_0^*\right) + u_l\left(c_0^*, l_0^*\right)\left(l_{0\varepsilon} - l_0^*\right) +$$
$$+\frac{\beta}{1-\beta} u_c\left(c_{1\varepsilon}, l_{1\varepsilon}\right)\left(c_{1\varepsilon} - c_1^*\right) + \frac{\beta}{1-\beta} u_l\left(c_1^*, l_1^*\right)\left(l_{1\varepsilon} - l_1^*\right),$$

so, we have:

$$\Delta_\varepsilon \geq u_c\left(c_{0\varepsilon}, \varepsilon\right)\left[G\left(h_0\right) f\left(h_0\left(1-\varepsilon\right)\right) - G\left(h_0\right) f\left(h_0\right)\right] + u_l\left(c_0^*, 0\right)\varepsilon +$$
$$+\frac{\beta}{1-\beta} u_c\left(c_{1\varepsilon}, 0\right)\left[G\left(h_0\left(1+\lambda\phi\left(\varepsilon\right)\right)\right) f\left(h_0\left(1+\lambda\phi\left(\varepsilon\right)\right)\right) - G\left(h_0\right) f\left(h_0\right)\right].$$

Because $u_c\left(c_{1\varepsilon}, 0\right) = 0$, then:

$$\Delta_\varepsilon \geq u_c\left(c_{0\varepsilon}, \varepsilon\right)\left[G\left(h_0\right) f\left(h_0\left(1-\varepsilon\right)\right) - G\left(h_0\right) f\left(h_0\right)\right] + u_l\left(c_0^*, 0\right)\varepsilon.$$

We will show that:

$$\lim_{\varepsilon\to 0} \frac{\Delta_\varepsilon}{\varepsilon} = +\infty.$$

But:

$$\lim_{\varepsilon\to 0} \frac{\Delta_\varepsilon}{\varepsilon} \geq -u_c\left(c_0^*, 0\right) G\left(h_0\right) h_0 \lim_{\varepsilon\to 0} \frac{f\left(h_0\left(1-\varepsilon\right)\right) - f\left(h_0\right)}{-h_0\varepsilon} + \lim_{\varepsilon\to 0} u_l\left(c_0^*, \varepsilon\right) =$$
$$= -h_0 u_c\left(c_0^*, 0\right) G\left(h_0\right) f'\left(h_0\right) + \lim_{\varepsilon\to 0} u_l\left(c_0^*, \varepsilon\right) = \lim_{\varepsilon\to 0} u_l\left(c_0^*, \varepsilon\right) = +\infty.$$

We know that:

$$u_c\left(c_0^*, 0\right) = 0$$

and:

$$\lim_{\varepsilon\to 0} u_l\left(c_0^*, \varepsilon\right) = +\infty.$$

Then, $\Delta_\varepsilon > 0$, for ε small enough. In other words, the stationary sequence $\left(h_0, h_0, ..., h_0, ...\right)$ is not optimal. □

We define the value function of h_0 by:

$$\forall h_0 \geq 0, \quad V\left(h_0\right) = \max \sum_{t=0}^{+\infty} \beta^t h_t^{(\alpha+\gamma)\mu} \left[\psi\left(\frac{h_{t+1}}{h_t}\right)\right]^{\alpha\mu} \left[1 - \psi\left(\frac{h_{t+1}}{h_t}\right)\right]^{1-\mu},$$

under the constrains: $\forall t \geq 0, \quad h_t \leq h_{t+1} \leq (1+\lambda) h_t$ and h_0 given. Now, if we define the function:

$$F\left(x, y\right) = x^{(\alpha+\gamma)\mu} \left[\psi\left(\frac{y}{x}\right)\right]^{\alpha\mu} \left[1 - \psi\left(\frac{y}{x}\right)\right]^{1-\mu},$$

with $F : \mathbb{R}^2 \to \mathbb{R}$, then the value function become:

$$\forall h_0 \geq 0, \quad V\left(h_0\right) = \max \sum_{t=0}^{+\infty} \beta^t F\left(h_t, h_{t+1}\right),$$

under the constrains: $\forall t \geq 0, \quad h_t \leq h_{t+1} \leq (1+\lambda) h_t$ and h_0 given.

We now add assumptions in order to obtain uniqueness of optimal human capital paths which grow at constant rate:

H6: The function ϕ is concave.

H7:

$$\left[\psi'(\xi)\right]^2 + \psi(\xi)\,\psi''(\xi) > 0.$$

Examples:

I. $\phi(x) = x$.

$\psi(\xi) = -\frac{1}{\lambda}\xi + \frac{1+\lambda}{\lambda}$ *and so:* $\left[\psi'(\xi)\right]^2 + \psi(\xi)\,\psi''(\xi) > 0.$

II. $\phi(x) = \frac{\ln(1+x)}{\ln 2}$.

$\psi(\xi) = 2 - 2^{\frac{1}{\lambda}(\xi-1)}, \quad \psi'(\xi) = -\frac{\ln 2}{\lambda} \cdot 2^{\frac{1}{\lambda}(\xi-1)}, \quad \psi''(\xi) = -\frac{\ln^2 2}{\lambda^2} \cdot 2^{\frac{1}{\lambda}(\xi-1)},$

so:

$\left[\psi'(\xi)\right]^2 + \psi(\xi)\,\psi''(\xi) = 2 \cdot \frac{\ln^2 2}{\lambda^2} \cdot 2^{\frac{1}{\lambda}(\xi-1)} \cdot \left[2^{\frac{1}{\lambda}(\xi-1)} - 1\right] \geq 0.$

H8:

$$\mu > \frac{1}{1+\alpha}.$$

We therefore weaken *H5* in:

H5b: $\beta(1+\lambda)^{(\alpha+\gamma)\mu} < 1 - (\alpha+\gamma)\mu$.

Lemma 1. *(i) The value function satisfies the Bellman equation:*

$$V(h_0) = \max_{y \in [h_0, (1+\lambda)h_0]} \{F(h_0, y) + \beta V(y)\}, \quad \forall h_0 \geq 0.$$

(ii) We have $V = \lim_{n\to+\infty} T^n f$ *for any continuous function* f *on* $\mathbb{R}_+$, *where* T *is the following linear operator:*

$$\forall h \geq 0, \quad Tf(h) = \max_{y \in [h, (1+\lambda)h]} \{F(h, y) + \beta f(y)\}.$$

Proof. It is standard (see [12]). □

So, we can define a correspondence φ which associates any h_0 with its maximizer. Let:

$$\varphi(h_0) = argmax\{F(h_0, y) + \beta V(y) : y \in [h_0, (1+\lambda)h_0]\}.$$

PROPOSITION 3. *Assume H1-H2-H3-H4-H5-H6-H7-H8. Then, the function:*

$$F(x, y) = x^{(\alpha+\gamma)\mu}\left[\psi\left(\frac{y}{x}\right)\right]^{\alpha\mu}\left[1 - \psi\left(\frac{y}{x}\right)\right]^{1-\mu},$$

with $F : \mathbb{R}^2 \to \mathbb{R}$ *is a concave function in the second variable.*

Proof. It easy to see that:

$$F(x,y) = \left\{ x \left[\psi\left(\frac{y}{x}\right) \left[1 - \psi\left(\frac{y}{x}\right)\right]^{\eta} \right]^{\frac{\alpha}{\alpha+\gamma}} \right\}^{(\alpha+\gamma)\mu},$$

where:

$$\eta = \frac{1-\mu}{\alpha\mu}.$$

From H8, $\eta < 1$. If $f(\xi) = \psi(\xi)[1 - \psi(\xi)]^{\eta}$, then:

$$f'(\xi) = \psi'(\xi)[1-\psi(\xi)]^{\eta} - \eta\psi'(\xi)\psi(\xi)[1-\psi(\xi)]^{\eta-1},$$

so:

$$f''(\xi) = \psi''(\xi)[1-\psi(\xi)]^{\eta} - \eta\left[\psi'(\xi)\right]^2[1-\psi(\xi)]^{\eta-1} +$$
$$+\eta(\eta-1)\left[\psi'(\xi)\right]^2\psi(\xi)[1-\psi(\xi)]^{\eta-1} - \eta[1-\psi(\xi)]^{\eta-1}\left\{\left[\psi'(\xi)\right]^2 + \psi(\xi)\psi''(\xi)\right\}.$$

From H6 and H7, f is strict concave and so F is a strict concave function. □

Lemma 2. *Assume H1-H2-H3-H4-H5-H6-H7-H8.*
i) The correspondence φ *is upper semi-continuous.*
ii) If the sequence $\mathbf{h}$ *is optimal, then the mapping* φ *satisfies:* $\forall t \geq 0$, *then* $h^*_{t+1} = \varphi(h^*_t)$.

Proof. i) The statement is a consequence of the Maximum Theorem.
ii)We know that a feasible sequence $\mathbf{h}$ is optimal if, and only if,

$$V(h^*_t) = F\left(h^*_{t+1}, h^*_t\right) + \beta V\left(h^*_{t+1}\right).$$

This shows that $\forall t \geq 0, \quad h^*_{t+1} \in \varphi(h^*_t)$. But F is a concave function in y, so:

$$\forall t \geq 0, \quad h^*_{t+1} = \varphi(h^*_t).$$

□

In the following proposition, we show the uniqueness of optimal human capital paths with constant growth rate.

PROPOSITION 4. *Assume H1-H2-H3-H4-H5b-H6-H7-H8. Then, the optimal human capital sequence is unique and grows at constant rate* $\delta \in (1, 1+\lambda)$.

Proof. The proof is done in two steps.

Step 1. We claim that:
(i) The value function has the form: $V(h_0) = Ah_0^{(\alpha+\gamma)\mu}$ for some constant A.
(ii) Given h_0 the optimal value of the human capital of period **1** is $h^*_1 = \delta h_0$ with δ solution to:

$$\max_{v\in[1,1+\lambda]} \{[\psi(v)]^{\alpha\mu}[1-\psi(v)]^{1-\mu} + \beta A v^{(\alpha+\gamma)\mu}\}.$$

Proof claim (i): Let T the operator which associates any continuous function f on $\mathbb{R}_+$ with the function:

$$Tf(h) = \max_{y\in[h,(1+\lambda)h]} \{F(h,y) + \beta f(y)\}.$$

Take $h>0$. We have successively:

$$T0\left(h\right)=\max_{y\in[h,(1+\lambda)h]}\left\{F\left(h,y\right)\right\}=$$
$$=\max_{y\in[h,(1+\lambda)h]}\left\{h^{(\alpha+\gamma)\mu}\left[\psi\left(\frac{y}{h}\right)\right]^{\alpha\mu}\left[1-\psi\left(\frac{y}{h}\right)\right]^{1-\mu}\right\}=$$
$$=h^{(\alpha+\gamma)\mu}\max_{v\in[1,1+\lambda]}\left\{\left[\psi\left(v\right)\right]^{\alpha\mu}\left[1-\psi\left(v\right)\right]^{1-\mu}\right\}=A_1h^{(\alpha+\gamma)\mu}.$$

$$T^2 0\left(h\right)=\max_{y\in[h,(1+\lambda)h]}\left\{F\left(h,y\right)+\beta A_1 y^{(\alpha+\gamma)\mu}\right\}=$$
$$=\max_{y\in[h,(1+\lambda)h]}\left\{h^{(\alpha+\gamma)\mu}\left[\psi\left(\frac{y}{h}\right)\right]^{\alpha\mu}\left[1-\psi\left(\frac{y}{h}\right)\right]^{1-\mu}+\beta A_1 y^{(\alpha+\gamma)\mu}\right\}=$$
$$=h^{(\alpha+\gamma)\mu}\max_{v\in[1,1+\lambda]}\left\{\left[\psi\left(v\right)\right]^{\alpha\mu}\left[1-\psi\left(v\right)\right]^{1-\mu}+\beta A_1 v^{(\alpha+\gamma)\mu}\right\}=A_2h^{(\alpha+\gamma)\mu}.$$

By induction, we have:

$$T^n 0\left(h\right)=A_n h^{(\alpha+\gamma)\mu},$$

where $v=\frac{y}{h}$.

From Lemma 1 (ii) we know that $\lim_{n\to+\infty}T^n 0=V$ and from $A_n\to A$ when $n\to+\infty$ we obtain $V\left(h_0\right)=Ah^{(\alpha+\gamma)\mu}$, so we proved the first part of the claim.

Proof claim (ii): From Lemma 1 (i), we know that V satisfies Bellman equation:

$$V(h)=\max_{y\in[h,(1+\lambda)h]}\left\{F\left(h,y\right)+\beta V\left(y\right)\right\}=$$
$$=\max_{y\in[h,(1+\lambda)h]}\left\{h^{(\alpha+\gamma)\mu}\left[\psi\left(\frac{y}{h}\right)\right]^{\alpha\mu}\left[1-\psi\left(\frac{y}{h}\right)\right]^{1-\mu}+\beta A y^{(\alpha+\gamma)\mu}\right\}=$$
$$=h^{(\alpha+\gamma)\mu}\max_{v\in[1,1+\lambda]}\left\{\left[\psi\left(v\right)\right]^{\alpha\mu}\left[1-\psi\left(v\right)\right]^{1-\mu}+\beta A v^{(\alpha+\gamma)\mu}\right\}=$$
$$=h^{(\alpha+\gamma)\mu}\left\{\left[\psi\left(\delta\right)\right]^{\alpha\mu}\left[1-\psi\left(\delta\right)\right]^{1-\mu}+\beta A\delta^{(\alpha+\gamma)\mu}\right\},$$

where $\delta\in[1,1+\lambda]$.

From Lemma 2 (ii), the optimal policy φ is defined by $\varphi\left(h\right)=\delta h$ and $\mathbf{h}$ is optimal if, and only if, $h_t=\delta^t h_0$, $\forall t\geq 0$. Since the problem is stationary, if $\{h_t\}$ is an optimal sequence, then we have $h_t=\delta^t h_0$ for every t.

Step 2. From Proposition 2, an optimal sequence of human capital must satisfy $h_{t+1}>h_t$, $\forall t\geq 0$. Since u satisfies Inada condition, optimal consumption must be positive. Thus Euler equation holds for all $t\geq 0$. First, we put:

$$\mathcal{F}\left(\frac{h_{t+1}}{h_t}\right)=\left[\psi\left(\frac{h_{t+1}}{h_t}\right)\right]^{\alpha\mu}\left[1-\psi\left(\frac{h_{t+1}}{h_t}\right)\right]^{1-\mu}.$$

So:

$$(\mathcal{E}):\quad h_t^{(\alpha+\gamma)\mu}\mathcal{F}'\left(\frac{h_{t+1}}{h_t}\right)\frac{1}{h_t}+\beta\left(\alpha+\gamma\right)\mu h_{t+1}^{(\alpha+\gamma)\mu-1}\mathcal{F}\left(\frac{h_{t+2}}{h_{t+1}}\right)-$$
$$-\beta h_{t+1}^{(\alpha+\gamma)\mu}\frac{h_{t+2}}{h_{t+1}}\mathcal{F}'\left(\frac{h_{t+2}}{h_{t+1}}\right)\frac{1}{h_{t+1}}=0,$$

$$(\mathcal{E}): \quad h_t^{(\alpha+\gamma)\mu-1}\mathcal{F}'\left(\frac{h_{t+1}}{h_t}\right)+\beta(\alpha+\gamma)\mu h_{t+1}^{(\alpha+\gamma)\mu-1}\mathcal{F}\left(\frac{h_{t+2}}{h_{t+1}}\right)-$$

$$-\beta h_{t+1}^{(\alpha+\gamma)\mu-1}\frac{h_{t+2}}{h_{t+1}}\mathcal{F}'\left(\frac{h_{t+2}}{h_{t+1}}\right)=0.$$

From step 1 we have:

$$(\mathcal{E}): \quad \mathcal{F}'(\delta)+\beta(\alpha+\gamma)\mu\delta^{(\alpha+\gamma)\mu-1}\mathcal{F}(\delta)-\beta\delta^{(\alpha+\gamma)\mu-1}\delta\mathcal{F}'(\delta)=0,$$

so:

$$(\mathcal{E}): \quad x^{1-(\alpha+\gamma)\mu}-x\beta=-\beta(\alpha+\gamma)\mu\frac{\mathcal{F}(x)}{\mathcal{F}'(x)}.$$

Let:

$$L(x)=x^{1-(\alpha+\gamma)\mu}-x\beta$$

and:

$$K(x)=-\beta(\alpha+\gamma)\mu\frac{\mathcal{F}(x)}{\mathcal{F}'(x)}.$$

We know that $L(1)=1-\beta>0$, $L(1+\lambda)=(1+\lambda)^{1-(\alpha+\gamma)\mu}-\beta(1+\lambda)$ and from H5b:

$$L'(x)=[1-(\alpha+\gamma)\mu]x^{-(\alpha+\gamma)\mu}-\beta>0,$$

so L is a increasing function.

For K we know that $K(1)=0$, $K(1+\lambda)=0$ and from H7 and H8:

$$K'(x)=-\beta(\alpha+\gamma)\mu\left[1-\frac{\mathcal{F}(x)\mathcal{F}''(x)}{[\mathcal{F}'(x)]^2}\right]<0,$$

so K is a decreasing function.

But:

$$K(x)=-\beta(\alpha+\gamma)\mu\frac{[\psi(x)][1-\psi(x)]}{\psi'(x)[\alpha\mu+(\mu-\alpha\mu-1)\psi(x)]}.$$

Because $\alpha\mu+(\mu-\alpha\mu-1)\psi(x)\neq 0$ must hold, if $x\in(\xi,1+\lambda]$, then K is a positive decreasing function; if $x=\xi$, then K is not well defined and if $x\in[1,\xi)$, then K is a negative decreasing function, where:

$$\xi=1+\lambda\phi\left(\frac{1-\mu}{\alpha\mu-\mu+1}\right)\in(1,1+\lambda),$$

solution for $\mathcal{F}'(x)=0$ and:

$$\lim_{x\to\xi,x<\xi}K(x)=-\beta(\alpha+\gamma)\mu\lim_{x\to\xi,x<\xi}\frac{[\psi(x)][1-\psi(x)]}{\psi'(x)[\alpha\mu+(\mu-\alpha\mu-1)\psi(x)]}=-\infty,$$

$$\lim_{x\to\xi,x>\xi}K(x)=-\beta(\alpha+\gamma)\mu\lim_{x\to\xi,x>\xi}\frac{[\psi(x)][1-\psi(x)]}{\psi'(x)[\alpha\mu+(\mu-\alpha\mu-1)\psi(x)]}=+\infty.$$

Therefore, there exists a unique solution $x\in(1,1+\lambda)$ for equation $L(x)=K(x)$.

□

3. EQUILIBRIUM

We first define the concept of equilibrium (in the sense of Lucas or Romer).

Suppose we are given a sequence of human capital $\overline{\boldsymbol{h}} = \left(h_0, \overline{h}_1, ..., \overline{h}_t, ...\right)$. Consider the following model:

$$\max \sum_{t=0}^{+\infty} \beta^t u\left(c_t, l_t\right),$$

under the constraints:

$$\forall t \geq 0, \qquad 0 \leq c_t \leq G\left(\overline{h}_t\right) f\left(\theta_t h_t\right), \tag{5}$$

$$h_{t+1} = h_t\left(1 + \lambda\phi\left(1 - \theta_t\right)\right), \tag{6}$$

$$0 \leq \theta_t \leq 1, \quad h_0 > 0 \quad \textit{is} \quad \textit{given}. \tag{7}$$

The solution $\boldsymbol{h} = (h_0, h_1, ..., h_t, ...)$ to this problem depends on $\overline{\boldsymbol{h}}$. We write $\boldsymbol{h} = \Phi\left(\overline{\boldsymbol{h}}\right)$. An *equilibrium* is a sequence of human capital $\boldsymbol{h^*} = (h_0, h_1^*, ..., h_t^*, ...)$ such that $\boldsymbol{h^*} = \Phi\left(\boldsymbol{h^*}\right)$.

3.1. Existence and Uniqueness of Equilibrium.

We give below conditions for which an equilibrium $\boldsymbol{h^*}$ is strictly increasing.

PROPOSITION 5. *Assume H1-H2-H3-H4-H5b-H6. Then, any equilibrium $\boldsymbol{h^*}$ is strictly increasing.*

Proof. Assume the contrary. We have two cases.

Case 1. The optimal sequence $\boldsymbol{h^*}$ satisfies $h_t^* = h_T^*$ for any $t \geq T$. Define a sequence $\boldsymbol{h}$ by $h_t = h_t^*$, $\forall t \leq T$ and $h_t = h_T^* + \varepsilon$, $\forall t \geq T+1$ and the sequence $\boldsymbol{\theta}$ by $\theta_t = \theta_t^*$, $\forall t \leq T-1$, $\theta_T = 1-\varepsilon$ and $\theta_t = 1$, $\forall t \geq T+1$. We will show that, with $\boldsymbol{h^*}$ as externality, the intertemporal utility generated by $\boldsymbol{h}$ and $\boldsymbol{\theta}$ is greater than the one generated by $\boldsymbol{h^*}$ and $\boldsymbol{\theta^*}$, which contradicts the optimality of $\boldsymbol{h^*}$.

Let:

$$\Delta_\varepsilon = \sum_{t=0}^{+\infty} \beta^t u\left(c_t, l_t\right) - \sum_{t=0}^{+\infty} \beta^t u\left(c_t^*, l_t^*\right),$$

so:

$$\begin{aligned}
\Delta_\varepsilon = & \sum_{t=0}^{+\infty} \beta^t u\left(G\left(h_t^*\right) f\left(h_t \psi\left(\frac{h_{t+1}}{h_t}\right)\right), l_t\right) - \sum_{t=0}^{+\infty} \beta^t u\left(G\left(h_t^*\right) f\left(h_t^* \psi\left(\frac{h_{t+1}^*}{h_t^*}\right)\right), l_t^*\right) = \\
= & \beta^T\left[u\left(G\left(h_T^*\right) f\left(h_T \psi\left(\frac{h_{T+1}}{h_T}\right)\right), l_T\right) - u\left(G\left(h_T^*\right) f\left(h_T^* \psi\left(\frac{h_{T+1}^*}{h_T^*}\right)\right), l_T^*\right)\right] + \\
& + \sum_{t=T+1}^{+\infty} \beta^t\left[u\left(G\left(h_t^*\right) f\left(h_t \psi\left(\frac{h_{t+1}}{h_t}\right)\right), l_t\right) - u\left(G\left(h_t^*\right) f\left(h_t^* \psi\left(\frac{h_{t+1}^*}{h_t^*}\right)\right), l_t^*\right)\right].
\end{aligned}$$

From the definition of $\boldsymbol{h}^*$ and $\boldsymbol{h}$ we have:

$$\Delta_\varepsilon = \beta^T \left[u\left(G\left(h_T^*\right) f\left(h_T^* \psi\left(\frac{h_T^* + \varepsilon}{h_T^*} \right)\right), \varepsilon \right) - u\left(G\left(h_T^*\right) f\left(h_T^* \psi(1)\right), 0\right) \right] +$$
$$+ \left[u\left(G\left(h_T^*\right) f\left(\left(h_T^* + \varepsilon\right)\psi(1)\right), 0\right) - u\left(G\left(h_T^*\right) f\left(h_T^*\psi(1)\right), 0\right)\right] \sum_{t \geq T+1} \beta^t.$$

Because $\psi(1) = 1$ and from the concavity of u, f and ψ one gets:

$$\Delta_\varepsilon \geq \beta^T u_c\left(G\left(h_T^*\right) f\left(h_T^* \psi\left(\frac{h_T^* + \varepsilon}{h_T^*}\right)\right), \varepsilon\right) G\left(h_T^*\right) f^{'}\left(h_T^* \psi\left(\frac{h_T^* + \varepsilon}{h_T^*}\right)\right) \psi^{'}\left(\frac{h_T^* + \varepsilon}{h_T^*}\right) \epsilon +$$

$$+\beta^T u_l\left(G\left(h_T^*\right) f\left(h_T^*\right), l_T^*\right)\varepsilon +$$

$$+u_c\left(G\left(h_T^*\right) f\left(h_T^* + \varepsilon\right), 0\right) G\left(h_T^*\right) f^{'}\left(h_T^* + \varepsilon\right)\varepsilon \sum_{t\geq T+1} \beta^t + u_l\left(G\left(h_T^*\right) f\left(h_T^*\right), l_{T+1}^*\right)\cdot 0 \cdot \sum_{t \geq T+1} \beta^t,$$

so:

$$\Delta_\varepsilon \geq \beta^T u_c\left(G\left(h_T^*\right) f\left(h_T^* \psi\left(\frac{h_T^* + \varepsilon}{h_T^*}\right)\right), \varepsilon\right) G\left(h_T^*\right) f^{'}\left(h_T^* \psi\left(\frac{h_T^* + \varepsilon}{h_T^*}\right)\right) \psi^{'}\left(\frac{h_T^* + \varepsilon}{h_T^*}\right) \epsilon +$$

$$+\beta^T u_l\left(G\left(h_T^*\right) f\left(h_T^*\right), l_T^*\right)\varepsilon + u_c\left(G\left(h_T^*\right) f\left(h_T^* + \varepsilon\right), 0\right) G\left(h_T^*\right) f^{'}\left(h_T^* + \varepsilon\right)\varepsilon \sum_{t\geq T+1} \beta^t.$$

We will show that:

$$\lim_{\varepsilon \to 0} \frac{\Delta_\varepsilon}{\varepsilon} = +\infty.$$

But:

$$\lim_{\varepsilon \to 0} \frac{\Delta_\varepsilon}{\varepsilon} \geq \beta^T u_c\left(G\left(h_T^*\right) f\left(h_T^*\right), 0\right) G\left(h_T^*\right) f^{'}\left(h_T^*\right) \psi^{'}(1) +$$
$$+\beta^T \lim_{\varepsilon \to 0} u_l\left(G\left(h_T^*\right) f\left(h_T^*\right), \varepsilon\right) + u_c\left(G\left(h_T^*\right) f\left(h_T^*\right), 0\right) G\left(h_T^*\right) f^{'}\left(h_T^*\right) \sum_{t \geq T+1} \beta^t.$$

We know that:

$$u_c\left(G\left(h_T^*\right) f\left(h_T^*\right), 0\right) = 0,$$

and:

$$\lim_{\varepsilon \to 0} u_l\left(G\left(h_T^*\right) f\left(h_T^*\right), \varepsilon\right) = +\infty.$$

Hence $\Delta_\varepsilon > 0$ for $\varepsilon > 0$ sufficiently small.

Case 2. The optimal sequence $\boldsymbol{h}^*$ satisfies $h_t^* = h_T^*$ for any $T \leq t \leq T + \tau$ and $h_t^* < h_{t+1}^*$, $\forall t < T$ and $t \geq T + \tau$. Define a sequence $\boldsymbol{h}$ by $h_t = h_t^*$, $\forall t \neq T + \tau$ and $h_{T+\tau}^* < h_{T+\tau} = h_{T+\tau}^* + \varepsilon < h_{T+\tau+1}^*$ and a sequence $\boldsymbol{\theta}$ by $\theta_t = \theta_t^*$, $\forall t \neq T + \tau - 1$

and $\theta_{T+\tau-1} = 1-\varepsilon$. As previously, we will show that, with $\boldsymbol{h}^*$ as externality, the intertemporal utility generated by $\boldsymbol{h}$ and $\boldsymbol{\theta}$ is greater than the one generated by $\boldsymbol{h}^*$ and $\boldsymbol{\theta}^*$.

Let:

$$\Delta_\varepsilon = \sum_{t=0}^{+\infty} \beta^t u\left(G\left(h_t^*\right) f\left(h_t \psi\left(\frac{h_{t+1}}{h_t}\right)\right), l_t\right) - \sum_{t=0}^{+\infty} \beta^t u\left(G\left(h_t^*\right) f\left(h_t^* \psi\left(\frac{h_{t+1}^*}{h_t^*}\right)\right), l_t^*\right),$$

so:

$$\begin{aligned}\Delta_\varepsilon = \beta^{T+\tau-1}\Bigg[u\left(G\left(h_{T+\tau-1}^*\right) f\left(h_{T+\tau-1}\psi\left(\frac{h_{T+\tau}}{h_{T+\tau-1}}\right)\right), l_{T+\tau-1}\right) - \\ -u\left(G\left(h_{T+\tau-1}^*\right) f\left(h_{T+\tau-1}^*\psi\left(\frac{h_{T+\tau}^*}{h_{T+\tau-1}^*}\right)\right), l_{T+\tau-1}^*\right)\Bigg] + \\ +\beta^{T+\tau}\Bigg[u\left(G\left(h_{T+\tau}^*\right) f\left(h_{T+\tau}\psi\left(\frac{h_{T+\tau+1}}{h_{T+\tau}}\right)\right), l_{T+\tau}\right) - \\ -u\left(G\left(h_{T+\tau}^*\right) f\left(h_{T+\tau}^*\psi\left(\frac{h_{T+\tau+1}^*}{h_{T+\tau}^*}\right)\right), l_{T+\tau}^*\right)\Bigg].\end{aligned}$$

From definition of $\boldsymbol{h}^*$ and $\boldsymbol{h}$ we have:

$$\begin{aligned}\Delta_\varepsilon = \beta^{T+\tau-1}\left[u\left(G\left(h_T^*\right) f\left(h_T^*\psi\left(\frac{h_T^*+\varepsilon}{h_T^*}\right)\right), \varepsilon\right) - u\left(G\left(h_T^*\right) f\left(h_T^*\psi(1)\right), 0\right)\right] + \\ +\beta^{T+\tau}\left[u\left(G\left(h_T^*\right) f\left(\left(h_T^*+\varepsilon\right)\psi\left(\frac{h_{T+\tau+1}^*}{h_T+\varepsilon}\right)\right), l_{T+\tau}^*\right) - u\left(G\left(h_T^*\right) f\left(h_T^*\psi\left(\frac{h_{T+\tau+1}^*}{h_T^*}\right)\right), l_{T+\tau}^*\right)\right]\end{aligned}$$

and from the concavity of u, f and ψ and because $\psi(1) = 1$ one gets:

$$\begin{aligned}\Delta_\varepsilon \geq \beta^{T+\tau-1}u_c\left(G\left(h_T^*\right) f\left(h_T^*\psi\left(\frac{h_T^*+\varepsilon}{h_T^*}\right)\right), \varepsilon\right) G\left(h_T^*\right) f'\left(h_T^*\psi\left(\frac{h_T^*+\varepsilon}{h_T^*}\right)\right)\psi'\left(\frac{h_T^*+\varepsilon}{h_T^*}\right)\varepsilon + \\ +\beta^{T+\tau-1}u_l\left(G\left(h_T^*\right) f\left(h_T^*\right), l_{T+\tau}^*\right)\varepsilon + \\ +\beta^{T+\tau}u_c\left(G\left(h_T^*\right) f\left(\left(h_T^*+\varepsilon\right)\psi\left(\frac{h_{T+\tau+1}^*}{h_T^*+\varepsilon}\right)\right), l_{T+\tau}^*\right) G\left(h_T^*\right) f'\left(\left(h_T^*+\varepsilon\right)\psi\left(\frac{h_{T+\tau+1}^*}{h_T^*+\varepsilon}\right)\right)\cdot \\ \cdot\left[\psi\left(\frac{h_{T+\tau+1}^*}{h_T^*+\varepsilon}\right) - \frac{h_{T+\tau+1}^*}{h_T^*+\varepsilon}\psi'\left(\frac{h_{T+\tau+1}^*}{h_T^*+\varepsilon}\right)\right]\varepsilon + \\ +\beta^{T+\tau}u_l\left(G\left(h_T^*\right) f\left(h_T^*\psi\left(\frac{h_{T+\tau+1}^*}{h_T^*}\right)\right), l_{T+\tau}^*\right)\cdot 0,\end{aligned}$$

so:

$$\begin{aligned}\Delta_\varepsilon \geq \beta^{T+\tau-1}u_c\left(G\left(h_T^*\right) f\left(h_T^*\psi\left(\frac{h_T^*+\varepsilon}{h_T^*}\right)\right), \varepsilon\right) G\left(h_T^*\right) f'\left(h_T^*\psi\left(\frac{h_T^*+\varepsilon}{h_T^*}\right)\right)\psi'\left(\frac{h_T^*+\varepsilon}{h_T^*}\right)\varepsilon + \\ +\beta^{T+\tau-1}u_l\left(G\left(h_T^*\right) f\left(h_T^*\right), l_{T+\tau}^*\right)\varepsilon + \\ +\beta^{T+\tau}u_c\left(G\left(h_T^*\right) f\left(\left(h_T^*+\varepsilon\right)\psi\left(\frac{h_{T+\tau+1}^*}{h_T^*+\varepsilon}\right)\right), l_{T+\tau}^*\right) G\left(h_T^*\right) f'\left(\left(h_T^*+\varepsilon\right)\psi\left(\frac{h_{T+\tau+1}^*}{h_T^*+\varepsilon}\right)\right)\cdot \\ \cdot\left[\psi\left(\frac{h_{T+\tau+1}^*}{h_T^*+\varepsilon}\right) - \frac{h_{T+\tau+1}^*}{h_T^*+\varepsilon}\psi'\left(\frac{h_{T+\tau+1}^*}{h_T^*+\varepsilon}\right)\right]\varepsilon.\end{aligned}$$

But:

$$\lim_{\varepsilon\to 0}\frac{\Delta_\varepsilon}{\varepsilon} \geq \beta^{T+\tau-1}u_c\left(G\left(h_T^*\right) f\left(h_T^*\right), 0\right) G\left(h_T^*\right) f'\left(h_T^*\right)\psi'(1) +$$

$$+\beta^{T+\tau-1}\lim_{\varepsilon\to 0} u_l\left(G\left(h_T^*\right)f\left(h_T^*\right),\varepsilon\right)+$$

$$+\beta^{T+\tau}u_c\left(G\left(h_T^*\right)f\left(h_T^*\psi\left(\frac{h_{T+\tau+1}^*}{h_T^*}\right)\right),l_{T+\tau}^*\right)G\left(h_T^*\right)f'\left(h_T^*\psi\left(\frac{h_{T+\tau+1}^*}{h_T^*}\right)\right)\cdot$$

$$\cdot\left[\psi\left(\frac{h_{T+\tau+1}^*}{h_T^*}\right)-\frac{h_{T+\tau+1}^*}{h_T^*}\psi'\left(\frac{h_{T+\tau+1}^*}{h_T^*}\right)\right].$$

We know that:

$$u_c\left(G\left(h_T^*\right)f\left(h_T^*\right),0\right)=0,\ \lim_{\varepsilon\to 0}u_l\left(G\left(h_T^*\right)f\left(h_T^*\right),\varepsilon\right)=+\infty.$$

So:

$$\lim_{\varepsilon\to 0}\frac{\Delta_\varepsilon}{\varepsilon}=+\infty.$$

Hence $\Delta_\varepsilon > 0$ for $\varepsilon > 0$ sufficiently small. □

The following proposition gives necessary and sufficient conditions for a sequence $\boldsymbol{h^*}$ to be an equilibrium.

PROPOSITION 6. *Assume H1-H2-H3-H4-H5b-H6-H7-H8. A sequence* $\boldsymbol{h^*}$ *is an equilibrium starting from* $h_0 > 0$ *if, and only if, it satisfies the following conditions:*

(1) Interiority:

$$\forall t\geq 0,\quad h_t^* < h_{t+1}^* < (1+\lambda)\,h_t^*,\quad h_0^*=h_0>0,$$

(2) Euler equation: $\forall t\geq 0,$

$$(\mathcal{E}):\quad \left(h_t^*\right)^{(\alpha+\gamma)\mu-1}\mathcal{F}'\left(\frac{h_{t+1}^*}{h_t^*}\right)+\beta\alpha\mu\left(h_{t+1}^*\right)^{(\alpha+\gamma)\mu-1}\mathcal{F}\left(\frac{h_{t+2}^*}{h_{t+1}^*}\right)-$$

$$-\beta\left(h_{t+1}^*\right)^{(\alpha+\gamma)\mu-1}\frac{h_{t+2}^*}{h_{t+1}^*}\mathcal{F}'\left(\frac{h_{t+2}^*}{h_{t+1}^*}\right)=0.$$

(3) Transversality condition:

$$\lim_{t\to+\infty}\beta^t\left(h_t^*\right)^{(\alpha+\gamma)\mu}\left[\alpha\mu\mathcal{F}\left(\frac{h_{t+1}^*}{h_t^*}\right)-\frac{h_{t+1}^*}{h_t^*}\mathcal{F}'\left(\frac{h_{t+1}^*}{h_t^*}\right)\right]=0$$

where:

$$\mathcal{F}\left(\frac{h_{t+1}^*}{h_t^*}\right)=\left[\psi\left(\frac{h_{t+1}^*}{h_t^*}\right)\right]^{\alpha\mu}\left[1-\psi\left(\frac{h_{t+1}^*}{h_t^*}\right)\right]^{1-\mu}.$$

Proof. 1. Let $\boldsymbol{h^*}$ be an equilibrium.

(1) From the previous proposition, we have $h_{t+1}^* > h_t^*$, $\forall t\geq 0$. Since u satisfies Inada condition, optimal consumptions must be positive at each period. Hence, $h_{t+1}^* < (1+\lambda)\,h_t^*$, for every t.

If not:

$$h_{t+1}^*=(1+\lambda)\,h_t^*,$$

but:

$$h_{t+1}^*=h_t^*\left(1+\lambda\phi\left(1-\theta_t\right)\right),$$

so $\theta_t = 0$ and $c_t = G\left(\overline{h}_t\right) f\left(\theta_t h_t\right) = 0$. Contradiction.

(2) Since the optimal path $\boldsymbol{h}^{*}$ is interior, Euler equation must hold.

$$\forall t \geq 0, \quad h_t^* < h_{t+1}^* < (1+\lambda)\, h_t^*.$$

Let y, with $y \in V$ and:

$$h_t^* < y < (1+\lambda)\, h_t^*,$$
$$y < h_{t+2}^* < (1+\lambda)\, y,$$

where V is a open neighborhood of h_{t+1}^*. Consider the following sequence $\boldsymbol{h}$ defined as follows:

$$\forall \tau \neq t+1, h_\tau = h_t^*, \quad h_{t+1} = y.$$

The sequence $\boldsymbol{h}$ is feasible, so we have:

$$F\left(h_t^*, h_{t+1}^*\right) + \beta F\left(h_{t+1}^*, h_{t+2}^*\right) \geq F\left(h_t^*, y\right) + \beta F\left(y, h_{t+2}^*\right),$$

where:

$$F\left(h_t^*, h_{t+1}^*\right) = \left(h_t^*\right)^{(\alpha+\gamma)\mu} \mathcal{F}\left(\frac{h_{t+1}^*}{h_t^*}\right).$$

Thus, the function:

$$\Phi\left(y\right) = F\left(h_t^*, y\right) + \beta F\left(y, h_{t+2}^*\right),$$

has a local maximum at h_{t+1}^*. By writing $\Phi'\left(h_{t+1}^*\right) = 0$, we obtain the Euler equation.

(3) We now prove that the transversality condition also holds. Let:

$$V_{\boldsymbol{h}^*}\left(h_0\right) = \max \sum_{t=0}^{+\infty} \beta^t \left(h_t^*\right)^{\gamma\mu} \left[\psi\left(\frac{h_{t+1}}{h_t}\right) h_t\right]^{\alpha\mu} \left[1 - \psi\left(\frac{h_{t+1}}{h_t}\right)\right]^{1-\mu},$$

under the constraints:

$$\forall t \geq 0, \quad h_t \leq h_{t+1} \leq (1+\lambda)\, h_t, \quad h_0 > 0$$

and the sequence $\boldsymbol{h}^*$ is an externality. Let:

$$H\left(x, y\right) = \left\{x \cdot \psi\left(\frac{y}{x}\right) \left[1 - \psi\left(\frac{y}{x}\right)\right]^{\eta}\right\}^{\alpha\mu},$$

where:

$$\frac{1-\mu}{\alpha\mu} = \eta.$$

From H8, $\eta < 1$.

But if:

$$h\left(\xi\right) = \psi\left(\xi\right) \left[1 - \psi\left(\xi\right)\right]^{\eta},$$

then:

$$h'\left(\xi\right) = \psi'\left(\xi\right) \left[1 - \psi\left(\xi\right)\right]^{\eta} - \eta \psi'\left(\xi\right) \psi\left(\xi\right) \left[1 - \psi\left(\xi\right)\right]^{\eta-1},$$

so:

$$h''\left(\xi\right) = \psi''\left(\xi\right) \left[1 - \psi\left(\xi\right)\right]^{\eta} - \eta \left[\psi'\left(\xi\right)\right]^2 \left[1 - \psi\left(\xi\right)\right]^{\eta-1} + \eta\left(\eta - 1\right) \left[\psi'\left(\xi\right)\right]^2 \psi\left(\xi\right) \left[1 - \psi\left(\xi\right)\right]^{\eta-1} -$$
$$-\eta \left[1 - \psi\left(\xi\right)\right]^{\eta-1} \left\{\left[\psi'\left(\xi\right)\right]^2 + \psi\left(\xi\right) \psi''\left(\xi\right)\right\}.$$

From H6 and H7, h is strict concave and then, H and V_{h^*} are a strict concave functions.

One can easily prove that:

$$0 \leq V_{\boldsymbol{h}^*}(h_0) \leq (h_0^*)^{\gamma\mu} h_0^{\alpha\mu} \sum_{t=0}^{+\infty} \left[\beta(1+\lambda)^{(\alpha+\gamma)\mu}\right]^t < \infty, \tag{8}$$

with:

$$\psi\left(\frac{h_{t+1}}{h_t}\right) \leq 1$$

and:

$$1-\psi\left(\frac{h_{t+1}}{h_t}\right) \leq 1.$$

Moreover, $V'_{\boldsymbol{h}^*}(h_0)$ exists and: (see [12], Benveniste-Scheinkman)

$$V'_{\boldsymbol{h}^*}(h_0) = F_1(h_0, h_1),$$

with:

$$F(h_t, h_{t+1}) = (h_t^*)^{\gamma\mu} h_t^{\alpha\mu} \mathcal{F}\left(\frac{h_{t+1}}{h_t}\right).$$

So:

$$V'_{\boldsymbol{h}^*}(h_0) = (h_0^*)^{\gamma\mu} h_0^{\alpha\mu-1}\left[\alpha\mu\mathcal{F}\left(\frac{h_1}{h_0}\right) - \frac{h_1}{h_0}\mathcal{F}'\left(\frac{h_1}{h_0}\right)\right]. \tag{9}$$

From the concavity of $V_{\boldsymbol{h}^*}$, we have:

$$V_{\boldsymbol{h}^*}(h_t^*) = V_{\boldsymbol{h}^*}(h_t^*) - V_{\boldsymbol{h}^*}(0) \geq V'_{\boldsymbol{h}^*}(h_t^*)\, h_t^*.$$

From relation (9), we get:

$$V_{\boldsymbol{h}^*}(h_t^*) \geq (h_t^*)^{(\alpha+\gamma)\mu-1}\left[\alpha\mu\mathcal{F}\left(\frac{h_{t+1}^*}{h_t^*}\right) - \frac{h_{t+1}^*}{h_t^*}\mathcal{F}'\left(\frac{h_{t+1}^*}{h_t^*}\right)\right] h_t^* \geq 0,$$

because:

$$-\frac{h_{t+1}^*}{h_t^*}\mathcal{F}'\left(\frac{h_{t+1}^*}{h_t^*}\right) = -\frac{h_{t+1}^*}{h_t^*}\psi'\left(\frac{h_{t+1}^*}{h_t^*}\right)\left[\psi\left(\frac{h_{t+1}^*}{h_t^*}\right)\right]^{\alpha\mu-1}\left[1-\psi\left(\frac{h_{t+1}^*}{h_t^*}\right)\right]^{-\mu}\cdot$$

$$\cdot\left\{\alpha\mu\left[1-\psi\left(\frac{h_{t+1}^*}{h_t^*}\right)\right] - (1-\mu)\left[\psi\left(\frac{h_{t+1}^*}{h_t^*}\right)\right]\right\} \geq 0.$$

From relation (8) and assumption H5b, we have:

$$\lim_{t\to+\infty} \beta^t V_{\boldsymbol{h}^*}(h_t^*) = 0,$$

so the previous inequality yields:

$$\lim_{t\to\infty} \beta^t (h_t^*)^{(\alpha+\gamma)\mu}\left[\alpha\mu\mathcal{F}\left(\frac{h_{t+1}^*}{h_t^*}\right) - \frac{h_{t+1}^*}{h_t^*}\mathcal{F}'\left(\frac{h_{t+1}^*}{h_t^*}\right)\right] = 0.$$

2. The proof that these conditions are sufficient are standard since, given the externality $\boldsymbol{h}^*$, we have a concave optimal growth problem.(see [12]) □

PROPOSITION 7. *Assume H1-H2-H3-H4-H5b-H6-H7-H8. Then, there exists a unique equilibrium* $\boldsymbol{h}^*$. *It grows at constant rate* ν^*. *This rate is smaller than the one in the social planner problem.*

Proof. From the previous proposition, $\boldsymbol{h}^*$ satisfies the three conditions of the previous proposition.

Our strategy of proof is to show that:

(1) the Euler equation admits a solution $\boldsymbol{h}^*$ which grows at the constant rate $\nu^* \in (1, 1+\lambda)$. Moreover, this solution satisfies the three conditions of the previous proposition and thus, is optimal,

(2) any other solution $\boldsymbol{h}$ to Euler equation does not satisfy the transversality condition. Hence, again from the previous proposition, it is not optimal. From that, one concludes that there exists a unique equilibrium.

We will show that there exists a solution $\boldsymbol{h}^*$ to Euler equation which grows at constant rate ν^*. Indeed, from Euler equation, ν^* must solve the following equation:

$$\nu^{1-(\alpha+\gamma)\mu} - \nu\beta = -\beta\alpha\mu \frac{\mathcal{F}(\nu)}{\mathcal{F}'(\nu)},$$

where:

$$\mathcal{F}(\nu) = [\psi(\nu)]^{\alpha\mu} [1-\psi(\nu)]^{1-\mu}.$$

Let:

$$L(x) = x^{1-(\alpha+\gamma)\mu} - x\beta$$

and:

$$H(x) = -\beta\alpha\mu \frac{\mathcal{F}(x)}{\mathcal{F}'(x)}.$$

We know that $L(1) = 1-\beta > 0$, $L(1+\lambda) = (1+\lambda)^{1-(\alpha+\gamma)\mu} - \beta(1+\lambda)$ and from H5b:

$$L^{'}(x) = [1-(\alpha+\gamma)\mu]\, x^{-(\alpha+\gamma)\mu} - \beta > 0,$$

so L is a increasing function.

For H we know that $H(1) = 0$, $H(1+\lambda) = 0$ and from H7 and H8:

$$H^{'}(x) = -\beta\alpha\mu \left[1 - \frac{\mathcal{F}(x)\mathcal{F}^{''}(x)}{[\mathcal{F}'(x)]^2}\right] < 0,$$

so H is a decreasing function.

But:

$$H(x) = -\beta\alpha\mu \frac{[\psi(x)][1-\psi(x)]}{\psi'(x)[\alpha\mu + (\mu-\alpha\mu-1)\psi(x)]}.$$

Because $\alpha\mu + (\mu-\alpha\mu-1)\psi(x) \neq 0$ must hold, if $x \in (\xi, 1+\lambda]$, then H is a positive decreasing function; if $x = \xi$, then H is not well defined and if $x \in [1, \xi)$, then H is a negative decreasing function, where:

$$\xi = 1 + \lambda\phi\left(\frac{1-\mu}{\alpha\mu-\mu+1}\right) \in (1, 1+\lambda),$$

ξ solution for $\mathcal{F}^{'}(x) = 0$ and:

$$\lim_{x\to\xi, x<\xi} H(x) = -\beta\alpha\mu \lim_{x\to\xi, x<\xi} \frac{[\psi(x)][1-\psi(x)]}{\psi'(x)[\alpha\mu + (\mu-\alpha\mu-1)\psi(x)]} = -\infty,$$

$$\lim_{x\to\xi, x>\xi} H(x) = -\beta\alpha\mu \lim_{x\to\xi, x>\xi} \frac{[\psi(x)][1-\psi(x)]}{\psi'(x)[\alpha\mu + (\mu-\alpha\mu-1)\psi(x)]} = +\infty.$$

Therefore, there exists a unique solution $\nu^* \in (1, 1+\lambda)$ for equation $L(x) = H(x)$.

It is easy to show that this rate is smaller than the one in the social planner problem which solves $L(\nu) = K(\nu)$, with:

$$K(\nu) = -\beta\mu(\alpha+\gamma)\frac{\mathcal{F}(\nu)}{\mathcal{F}'(\nu)} = -\beta\alpha\mu\frac{\mathcal{F}(\nu)}{\mathcal{F}'(\nu)} - \beta\gamma\mu\frac{\mathcal{F}(\nu)}{\mathcal{F}'(\nu)} > H(\nu).$$

Let $\boldsymbol{h}^*$ be defined by $h_0^* = h_0$, $h_{t+1}^* = \nu^* h_t^*$, $\forall t$. Obviously, it satisfies interiority condition and Euler equation. It remains to show that $\boldsymbol{h}^*$ also satisfies the transversality condition.

Since:

$$\beta^t (h_t^*)^{(\alpha+\gamma)\mu}\left[\alpha\mu\mathcal{F}\left(\frac{h_{t+1}^*}{h_t^*}\right) - \frac{h_{t+1}^*}{h_t^*}\mathcal{F}'\left(\frac{h_{t+1}^*}{h_t^*}\right)\right] \leq$$
$$\leq (h_0)^{(\alpha+\gamma)\mu}\left[\alpha\mu\mathcal{F}(\nu^*) - \nu^*\mathcal{F}'(\nu^*)\right]\left[\beta(1+\lambda)^{(\alpha+\gamma)\mu}\right]^t,$$

then from H5:

$$\lim_{t\to+\infty}\beta^t (h_t^*)^{(\alpha+\gamma)\mu}\left[\alpha\mu\mathcal{F}\left(\frac{h_{t+1}^*}{h_t^*}\right) - \frac{h_{t+1}^*}{h_t^*}\mathcal{F}'\left(\frac{h_{t+1}^*}{h_t^*}\right)\right] = 0,$$

which is the transversality condition.

The proof of the uniqueness is rather long. It will be done in three steps. The idea is to prove that for any solution to Euler equation different from the one which grows at rate ν^*, the rate growth converges to $1+\lambda$. This property is crucial to prove that this solution does not satisfy the transversality condition and, from the previous proposition, is not optimal. One obviously concludes that the equilibrium is unique and grows at rate ν^*.

Step 1. Let $\nu_t = \frac{h_{t+1}}{h_t}$ and $\delta = \frac{1}{1-\alpha\mu}$. Euler equation can be written as:

$$\nu_t^{1-(\alpha+\gamma)\mu}\left[h(\nu_t)\right]^{\alpha\mu-1}h'(\nu_t) = \beta\left\{\nu_{t+1}\left[h(\nu_{t+1})\right]^{\alpha\mu-1}h'(\nu_{t+1}) - \left[h(\nu_{t+1})\right]^{\alpha\mu}\right\}, \tag{10}$$

with:

$$h(x) = \psi(x)\left[1-\psi(x)\right]^{\eta}$$

and:

$$\frac{1-\mu}{\alpha\mu} = \eta.$$

If:

$$\delta = \frac{1}{1-\alpha\mu},$$

then:

$$\frac{h(\nu_{t+1})}{\left[h(\nu_{t+1}) - \nu_{t+1}h'(\nu_{t+1})\right]^{\delta}} = \beta^{\delta}\frac{h(\nu_t)}{\left[-\nu_{t+1}^{1-(\alpha+\gamma)\mu}h'(\nu_t)\right]^{\delta}}, \tag{11}$$

or:

$$\Phi(\nu_{t+1}) = \Psi(\nu_t), \tag{12}$$

with:

$$\Phi(x) = \frac{h(x)}{\left[h(x) - xh'(x)\right]^{\delta}}$$

and:

$$\Psi(x) = \beta^{\delta} \frac{h(x)}{\left[-x^{1-(\alpha+\gamma)\mu} h'(x)\right]^{\delta}}.$$

We will show that $\nu_{t+1} = I(\nu_t)$ with $I' > 0$. Indeed, tedious computations give:

$$\Phi'(x) = \frac{h'(x)\left[h(x) - xh'(x)\right] + \delta x h(x) h''(x)}{\left[h(x) - xh'(x)\right]^{\delta+1}} < 0$$

and:

$$\Psi'(x) = \beta^{\delta} \frac{h'(x)\left[-x^{1-(\alpha+\gamma)\mu} h'(x)\right] + \delta h(x)\left\{\left[1-(\alpha+\gamma)\mu\right] x^{-(\alpha+\gamma)\mu} h'(x) + x^{1-(\alpha+\gamma)\mu} h''(x)\right\}}{\left[-x^{1-(\alpha+\gamma)\mu} h'(x)\right]^{\delta+1}} < 0.$$

Hence, one can write $\nu_{t+1} = I(\nu_t)$ with $I' > 0$.

Observe that Euler equation (11) has only three fixed points which are 1, ν^* and $1+\lambda$. We have shown that the sequence $\boldsymbol{h^*}$ with $h_t^* = (\nu^*)^t h_0$, $\forall t$ is an equilibrium. The sequence $\boldsymbol{h^*}$ with $h_t^* = h_0$ is obviously not optimal and $\forall t$ and $\boldsymbol{h^*}$ with $h_t^* = (1+\lambda)^t h_0$, $\forall t$, is not optimal since the associated consumptions equal zero at every date.

Step 2. Consider a non-stationary sequence ν which satisfies Euler equation (11) and $\forall t$, $1 \leq \nu_t \leq 1+\lambda$. We will show that such a sequence converges to $1+\lambda$. In view of the monotonicity of I, since $\nu_1 \leq \nu_0$ implies that $\nu_2 = I(\nu_1) \leq I(\nu_0) = \nu_1$ (respectively $\nu_1 \geq \nu_0$ $\nu_2 \geq \nu_1$), by an easy induction, the sequence ν is weakly monotonous. Hence it is converging to a fixed-point of I: either ν^*, 1 or $1+\lambda$.

We will show that to assume that ν converges to ν^* leads to a contradiction. Its convergence to 1 is obviously not possible. Indeed, let $\varepsilon_t = \nu^* - \nu_t$. First, observe that $\nu_0 \neq \nu^*$ implies that for all t, $\varepsilon_t \neq 0$. When $t \to +\infty$, then $\varepsilon_{t+1} \sim I'(\nu^*)\varepsilon_t$. Let us compute $I'(\nu^*)$. We obtain, after tedious computations:

$$I'(\nu^*) = \lim_{t\to\infty} \frac{\nu^* - \nu_{t+1}}{\nu^* - \nu_t} = \lim_{t\to\infty} \frac{\Psi_1'(\nu_t)}{\Phi_1'(\nu_{t+1})} = \frac{\Psi_1'(\nu^*)}{\Phi_1'(\nu^*)} =$$

$$= \frac{(1-\alpha\mu)\nu^*\left[h'(\nu^*)\right]^2 - h(\nu^*)\left\{\left[1-(\alpha+\beta)\mu\right]h'(\nu^*) + \nu^* h''(\nu^*)\right\}}{(1-\alpha\mu)\nu^*\left[h'(\nu^*)\right]^2 - \beta(\nu^*)^{1+(\alpha+\gamma)\mu} h(\nu^*) h''(\nu^*)},$$

with:

$$\Psi_1(x) = \frac{x^{1-(\alpha+\gamma)\mu} h'(\nu^*)}{h^{1-\alpha\mu}(\nu^*)}, \quad \Phi_1(x) = \frac{xh'(\nu^*) - h(\nu^*)}{h^{1-\alpha\mu}(\nu^*)}$$

and:

$$\Psi_1'(x) = -x^{-(\alpha+\gamma)\mu} \frac{(1-\alpha\mu)x\left[h'(x)\right]^2 - h(x)\left\{\left[1-(\alpha+\beta)\mu\right]h'(x) + xh''(x)\right\}}{h^{2-\alpha\mu}(x)},$$

$$\Phi_1'(x) = -\beta \frac{(1-\alpha\mu)x\left[h'(x)\right]^2 - (1-\alpha\mu)h(x)h'(x) - xh(x)h''(x)}{h^{2-\alpha\mu}(x)},$$

$$(\nu^*)^{1-(\alpha+\gamma)\mu} h'(\nu^*) = \beta\left[\nu^* h'(\nu^*) - h(\nu^*)\right].$$

Since $\beta(1+\lambda)^{(\alpha+\gamma)\mu} < 1$ and $1 < \nu^* < 1+\lambda$, $h''(\nu^*) < 0$, we have $-\nu^* h''(\nu^*) > -\beta(\nu^*)^{1+(\alpha+\gamma)\mu} h''(\nu^*)$. Hence, $I'(\nu^*) > 1$. In particular, for t large enough, the sequence $(|\varepsilon_t|)$ is increasing, which contradicts $\nu_t \to \nu^*$.

Step 3. Because:

$$\mathcal{F}\left(\frac{h_{t+1}}{h_t}\right) = \left[h\left(\frac{h_{t+1}}{h_t}\right)\right]^{\alpha\mu},$$

with:

$$h\left(\frac{h_{t+1}}{h_t}\right) = \psi\left(\frac{h_{t+1}}{h_t}\right)\left[1-\psi\left(\frac{h_{t+1}}{h_t}\right)\right]^{\frac{1-\mu}{\alpha\mu}},$$

then:

$$\mathcal{F}'\left(\frac{h_{t+1}}{h_t}\right) = \alpha\mu\left[h\left(\frac{h_{t+1}}{h_t}\right)\right]^{\alpha\mu-1} h'\left(\frac{h_{t+1}}{h_t}\right)$$

and the transversality condition becomes:

$$\lim_{t\to+\infty} \alpha\mu\beta^t (h_t)^{(\alpha+\gamma)\mu}\left[h\left(\frac{h_{t+1}}{h_t}\right)\right]^{\alpha\mu-1}\left[h\left(\frac{h_{t+1}}{h_t}\right) - \frac{h_{t+1}}{h_t}h'\left(\frac{h_{t+1}}{h_t}\right)\right] = 0.$$

Since $\frac{h_{t+1}}{h_t} \to 1+\lambda$ when $t\to\infty$ and:

$$h'(1+\lambda) = \psi'(1+\lambda) > -\infty,$$

the transversality condition is equivalent to:

$$\lim_{t\to+\infty} \beta^t (h_t)^{(\alpha+\gamma)\mu}\left[h\left(\frac{h_{t+1}}{h_t}\right)\right]^{\alpha\mu-1} = 0.$$

Let us denote by $\nu_t = \frac{h_{t+1}}{h_t}$, $\varepsilon_t = 1+\lambda-\nu_t$ and $S_t = \beta^t (h_t)^{(\alpha+\gamma)\mu}[h(\nu_t)]^{\alpha\mu-1}$. When $t\to\infty$, then $\nu_t \to 1+\lambda$ and $\varepsilon_t \to 0$. Consequently:

$$h(\nu_t) = h(\nu_t) - h(1+\lambda) \sim -h'(1+\lambda)(1+\lambda-\nu_t) = -h'(1+\lambda)\varepsilon_t,$$

so:

$$[h(\nu_t)]^{\alpha\mu-1} \sim \left[-h'(1+\lambda)\varepsilon_t\right]^{\alpha\mu-1}.$$

It follows that $S_t \sim \hat{S}_t\left[-h'(1+\lambda)\varepsilon_t\right]^{\alpha\mu-1}$ with $\hat{S}_t = \beta^t (h_t)^{(\alpha+\gamma)\mu}(\varepsilon_t)^{\alpha\mu-1}$. Hence, in order to prove that the transversality does not hold, we will prove that $\lim_{t\to+\infty}\hat{S}_t > 0$.

For this we have:

$$\varepsilon_{t+1} = (1+\lambda) - \nu_{t+1} = I(1+\lambda) - I(\nu_t) \sim I'(1+\lambda)(1+\lambda-\nu_t) = I'(1+\lambda)\varepsilon_t.$$

Let us now remark that $I'(1+\lambda) < 1$ and this implies in particular the summability of (ε_t). Indeed, we obtain, after tedious computations:

$$I'(1+\lambda) = \left[\beta(1+\lambda)^{(\alpha+\gamma)\mu}\right]^{\frac{1}{1-\alpha\mu}} < 1.$$

$$I'(1+\lambda) = \lim_{t\to\infty}\frac{1+\lambda-\nu_{t+1}}{1+\lambda-\nu_t} = \frac{\Psi'(1+\lambda)}{\Phi'(1+\lambda)},$$

with:

$$\Psi^{'}(1+\lambda)=\beta^{\delta}\frac{-(1+\lambda)^{1-(\alpha+\gamma)\mu}\left[h^{'}(1+\lambda)\right]^{2}}{\left[-(1+\lambda)^{1-(\alpha+\gamma)\mu}h'(1+\lambda)\right]^{\delta+1}}$$

and:

$$\Phi^{'}(x)=\frac{-(1+\lambda)\left[h^{'}(1+\lambda)\right]^{2}}{\left[-(1+\lambda)h'(1+\lambda)\right]^{\delta+1}}.$$

Letting $\pi_t=\frac{\hat{S}_{t+1}}{\hat{S}_t}$, with classical notations, we can write:

$$\pi_t=\beta(\nu_t)^{(\alpha+\gamma)\mu}\left(\frac{\varepsilon_{t+1}}{\varepsilon_t}\right)^{\alpha\mu-1}=\beta(1+\lambda-\varepsilon_t)^{(\alpha+\gamma)\mu}\left[I^{'}(1+\lambda)+\frac{1}{2}I^{''}(1+\lambda)\varepsilon_t+o(\varepsilon_t)\right]^{\alpha\mu-1}.$$

In view of the computation of $I^{'}(1+\lambda)$,

$$\pi_t=\left(\frac{1+\lambda-\varepsilon_t}{1+\lambda}\right)^{(\alpha+\gamma)\mu}\left[1+\frac{I^{''}(1+\lambda)\varepsilon_t}{2I^{'}(1+\lambda)}+o(\varepsilon_t)\right]^{\alpha\mu-1},$$

where $\frac{o(\varepsilon_t)}{\varepsilon_t}\to 0$, when $t\to\infty$.

Therefore, the sequence $\left(\frac{\ln(\pi_t)}{\varepsilon_t}\right)$ converges. The summability of (ε_t) implies the summability of $(\ln\pi_t)$ which is equivalent to the convergence of the infinite product $(\pi_0\pi_1\ldots\pi_t)$ to a positive limit. Since $\hat{S}_{t+1}=(\pi_0\pi_1\ldots\pi_t)\hat{S}_0$, we proved that $\hat{S}_t\nrightarrow 0$. □

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Marius Valentin Boldea, CERMSEM, Maison des Sciences Economiques, 106-112 BD de l'Hopital, 75647 Paris Cedex 13, France.

E-mail address: mvboldea@univ-paris1.fr