

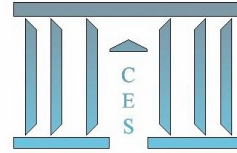


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Wavelet Method for Locally Stationary Seasonal Long Memory Processes

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Abstract

Long memory processes have been extensively studied over the past decades. When dealing with the financial and economic data, seasonality and time-varying long-range dependence can often be observed and thus some kind of non-stationarity can exist inside financial data sets. To take into account this kind of phenomena, we propose a new class of stochastic process: the locally stationary k -factor Gegenbauer process. We describe a procedure of estimating consistently the time-varying parameters by applying the discrete wavelet packet transform (DWPT). The robustness of the algorithm is investigated through simulation study. An application based on the error correction term of fractional cointegration analysis of the Nikkei Stock Average 225 index is proposed.

Keywords: Discrete wavelet packet transform, Gegenbauer process, Nikkei Stock Average 225 index, Non-stationarity, Ordinary least square estimation.

JEL Classification: C13, C14, C15, C22, C63, G15.

1 Introduction

In the last several decades, the long memory models have been widely investigated. The fractional integrated model (FARIMA) was introduced by

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Granger and Joyeux (1980) and Hosking (1981) using the differencing operator $(I - B)^d$, where $-1/2 < d < 1/2$: it permits to model singularity or a pole at the zero frequency in the spectrum. An extension of FARIMA model has been developed permitting singularities at non-zero (seasonal and/or cyclical) frequencies. It is the class of the k -factor Gegenbauer autoregressive moving average model introduced by Gray et al. (1989) and Giraitis et Leipus (1995). A stochastic process $(y_t)_t$ is defined as a k -factor GARMA process if it satisfies the relationship

$$\phi(B) \prod_{i=1}^k (I - 2\nu_i B + B^2)^{d_i} y_t = \theta(B) \varepsilon_t, \quad (1)$$

where $(\varepsilon_t)_t$ is a white noise, B is the backshift operator, and $\phi(B)$ and $\theta(B)$ are polynomials of order p and q . Most of the studies on model (1) have been done in a stationary setting. This means that if the ν_i are distinct frequencies in equation (1) then one assumes that $|d_i| < \frac{1}{2}$ when $|\nu_i| < 1$, and $|d_i| < \frac{1}{4}$ when $|\nu_i| = 1$. The model (1) includes most of the well known long memory processes: the fractional integrated process (when $k = 1$, $\nu_i = 1$, $\phi(B) = I$ and $\theta(B) = I$); the FARIMA process (when $k = 1$ and $\nu_i = 1$), and the k -factor Gegenbauer process (when $\phi(B) = I$ and $\theta(B) = I$), for a review we refer to Guégan (2005).

This previous model has been extensively used in economics and finance to model existence of seasonals inside data sets assuming stationarity. We can cite for instance the works of Diebold and Rudebush (1989), Sowell (1992), Gil-Alana and Robinson (2001), Porter Hudak (1990), Franses and Ooms (1997), Arteche and Robinson (2000), Ferrara and Guégan (2001), Diongue and Guégan (2004), Gil-Alana and Hualde (2008) and Diongue et al. (2009). But, in practice, series cannot always make stationary even by transformation or sometimes it has no sense to make the data sets stationary. Working with the existence of non-stationarity does not mean that we observe explosions. Recent works in ecology illustrate this fact, Whitcher and Jensen (2000) and Cavanaugh et al. (2002).

It exist some works where the authors investigate long memory models in a non-stationarity setting. Beran and Terrin (1996) permit the long memory parameter of FARMA models to evolve between $1/2$ and 1 , see also Velasco and Robinson (2000), Shimotsu and Phillips (2005), and Moulines et al. (2008). Other works deal with a time-varying long memory parameter. A locally stationary FARIMA model is introduced in Jensen (1999a, b) and

Whitcher and Jensen (2000) using the operator $(I - B)^{d(t)}$. Cavanaugh et al (2002) explored the time-varying fractional Brownian motion. For these last both works, the authors developed an estimation procedure based on wavelets techniques, succeeding in capturing the local changes in the series. However, although these previous models take into account the local changes, they do not permit at the same time the existence of seasonalities characterized with time-varying long memory parameter, feature which is common for a lot of data sets, in meteorology, finance and economics.

In this paper, we introduce an estimate for the fractional difference parameters of a non-stationary k -factor Gegenbauer process that is allowed to vary smoothly over time, i.e., $d(t)$. The non-stationary k -factor Gegenbauer process that we consider is a member of the non-stationary class of processes known as locally stationary process introduced by Dalhaus (1996). The model introduced in this paper, we call it a locally stationary k -factor Gegenbauer model.

In the stationary case, estimation methods for long memory models with seasonalities, have been developed using Whittle methods, by Diongue et al (2004), and using semi-parametric methods by Robinson (1995) and Arteche and Robinson (2000). In this paper, we proceed in a different way to estimate $d(t)$ and develop a procedure based on wavelet method. Indeed, the strength of the wavelet method lies on its capability to simultaneously localize a process in time and scale. At high scales, the wavelet has a small centralized time support enabling it to focus on short-lived time phenomena such as singularity point. At low scales, the wavelet has a large support allowing it to identify long periodic behavior. By moving from low to high scales, the wavelet zooms in a process's behavior, identifying singularities, jumps and cups (Mallat and Zhong 1992; Mallat and Hwang 1992; Mallat 1999). Thus, this approach appears pertinent to solve the estimation problem in presence on seasonalities and non-stationarity. In this paper we focus on the discrete wavelet packet transform (DWPT) which permits to decorrelate the spectrum of the process. We provide an approximate log-linear relationship between the time-varying variance of the DWPT coefficients and the time-varying long memory parameter $d(t)$, then applying locally the OLS regression method, we obtain local estimates for the time-varying parameters. Our method may be regarded as an extension of the work of Cavanaugh et al. (2002): here we use a wavelet packet transform which needs to be adapted to the existence of seasonalities. In another hand, the main difference between our work and Whitcher's (2004) approach lies in that we dealt with non-stationary processes while Whitcher (2004) considered only stationary

processes with seasonalities.

The paper is organized as follows. Section 2 introduces the methodology: model and estimation procedure. Section 3 studies the robustness of the time-varying long memory estimates using simulation experiments for finite samples. Section 4 proposes an application based on the Nikkei Stock Average 225 index. Section 5 concludes.

2 Methodology

To get an asymptotic estimation theory for $d_i(t)$, $i = 1, \dots, k$, we introduce an estimate that is allowed to vary smoothly over time and we use a method permitting to determine the location in time of this long memory parameter.

We begin to define the locally stationary k -factor Gegenbauer model and to describe its time-varying spectral density. The DWPT of a finite length vector is then defined in terms of filtering. We look at the local wavelet variance based on the DWPT method and describe a procedure for estimating the local fractional parameters.

2.1 Local stationary k -factor Gegenbauer model

In the first step, we restrict to a 1-factor Gegenbauer model and we assume that we observe $(y_t)_t$, $t = 1, \dots, N$, such that:

$$(I - 2\nu B + B^2)^{d(t)} y_t = \varepsilon_t, \quad (2)$$

$(\varepsilon_t)_t$ being a Gaussian white noise and $\cos^{-1} \nu$ being the Gegenbauer frequency, and B is the backshift operator, $y_{t-j} = B^j y_t$. We assume that the time-varying fractional parameter is such that $d(t) < 1/2$. In order to estimate $d(t)$, providing an asymptotic theory, we need to make N tends to infinity. To avoid instability for $d(t)$, we suppose that we observe $d(t)$ on a finer grid (making $d(t)$ rescaled on $[0, 1]$), that we observe $(y_t)_t$ such that:

$$(I - 2\nu B + B^2)^{d(t/N)} y_{t,N} = \varepsilon_t. \quad (3)$$

Letting N tends to infinity means that we have, in the sample $y_{1,N}, \dots, y_{N,N}$, more and more observations for each value of $d(t)$.

Now, we are going to characterize this local stationary process through its spectral density: this tool being privilege in presence of seasonalities, to the

autocovariance function.

The stochastic process defined in (2) is a Gegenbauer process that is locally stationary in the sense of Dalhaus (1996), with realizations of length N . Its spectral density is such that:

$$f_N(\lambda) = \frac{\sigma_{\varepsilon_t}^2}{2\pi} \frac{1}{2|\cos\lambda - \nu|^{2d(t/N)}}, \quad -1/2 < \lambda < 1/2. \quad (4)$$

Because the process $(y_{t,N})_t$ is non-stationary, increasing the number of observations by measuring new realizations of the process tell us nothing about the process' behavior at the beginning of the period. As a result, we fix the time period and as N increases we liken it to measuring the process at higher and higher levels of resolution on a fixed time interval.

The spectral density for the process defined in (3) is an even, 2π -periodic function that is uniformly Lipschitz continuous in $t/N \in [0, 1]$. The time-varying spectral density function has the specific behavior:

$$f_N(\lambda) \sim |\lambda - \cos^{-1} \nu|^{-2d(t/N)}, \quad \text{as } \lambda \rightarrow \cos^{-1} \nu.$$

Then, if $d(t/N) > 0$, $f_N(\lambda)$ is smooth for frequencies around $\cos^{-1} \nu$, but is unbounded when $\lambda \rightarrow \cos^{-1} \nu$. In other words the behavior of $(y_{t,N})_t$ is concentrated over the frequency associated with seasonality. This behavior can be extended in the case that we have several explosions inside the spectral density. This means that, on the interval $[0, N]$, we observe the locally stationary k -factor Gegenbauer process:

$$\prod_{i=1}^k (I - 2\nu_i B + B^2)^{d_i(t/N)} y_{t,N} = \varepsilon_t, \quad (5)$$

where $(\varepsilon_t)_t$ is a Gaussian white noise, and $\cos^{-1} \nu_i$ are Gegenbauer frequencies. Then, the time-varying spectral density is:

$$f_N(\lambda) = \frac{\sigma_{\varepsilon}^2}{2\pi} \prod_{i=1}^k |\cos(\lambda) - \nu_i|^{-2d_i(t/N)}. \quad (6)$$

2.2 Discrete Wavelet Packet Transforms

In this Section we introduce the wavelet technique that we use for the estimation of the parameters $d_i(t)$ introduced in (5). In order to decorrelate long memory time series, Jensen (1999a, b), and Cavanaugh et al. (2002) used the

orthonormal discrete wavelet transform (DWT). In presence of seasonalities, we need a more general wavelet transform to decorrelate the process $(y_{t,N})_t$ given in (5). Using the discrete wavelet packet transform (DWPT) we approximately decorrelate the spectrum of the process $(y_{t,N})_t$ introduced in (5).

We assume that we observe a single realization $(y_{1,N}, \dots, y_{N,N})$ of the locally stationary long memory model given in (5), and that N is an integer multiple of 2^J , where J is any positive integer. To realize this approximate decorrelation, we use the minimum-bandwidth discrete-time (MBDT) wavelets with length L , where $L < N$ (denoted by MB(L)), introduced by Morris and Peravali (1999): it permits to approximately decorrelate the spectrum and to choose the adaptive orthonormal basis (Whitcher 2004).

Let h_0, \dots, h_{L-1} be the unit scale wavelet (high-pass) filter. Thus, the scaling (low-pass) coefficients may be computed via the "quadrature mirror relationship"

$$g_l = (-1)^{l+1} h_{L-l-1}, \quad l = 0, 1, \dots, L-1.$$

We define

$$u_{n,l} = \begin{cases} g_l, & \text{if } n \bmod 4 = 0 \text{ or } 3 \\ h_l, & \text{if } n \bmod 4 = 1 \text{ or } 2, \end{cases}$$

as the approximate filter at a given node of the wavelet packet tree. Instead of one particular filtering sequence, the DWPT executes all possible filtering combinations to construct a wavelet packet tree, denoted by $\mathcal{T} = \{(j, n) | j = 0, \dots, J-1; n = 0, \dots, 2^j - 1\}$.

The DWPT coefficients are then calculated using the pyramid algorithm of filtering and downsampling (Mallat 1999). Denote $W_{j,n,K}$ the K -th element of length $N_j (= N/2^j)$, corresponding to the wavelet coefficient vector $\mathbf{W}_{j,n}$, $(j, n) \in \mathcal{T}$ with $\mathbf{W}_{0,0} = y_{1,N}$. Given the DWPT coefficients $W_{j-1, [\frac{n}{2}], K}$, where $[\cdot]$ represents the "integer part" operator, then the coefficient $W_{j,n,K}$ is calculated by

$$W_{j,n,K} \equiv \sum_{l=0}^{L-1} u_{n,l} W_{j-1, [\frac{n}{2}], 2K+1-l \bmod N_{j-1}}, \quad K = 0, 1, \dots, N_j - 1. \quad (7)$$

An adaptive orthonormal basis $\mathcal{B} \subset \mathcal{T}$ is obtained when a collection of DWPT coefficients is retained such that band-pass frequencies are disjoint and cover the frequency interval $[0, \frac{1}{2}]$ (Percival and Walden 2000; Gençay et al. 2001).

For the locally stationary 1-factor Gegenbauer process defined in (3), its

time-varying spectral density is expressed in equation (4). Thus by applying the logarithmic transform to both sides of equation (4), we get

$$\log f_N(\lambda) = C - 2d\left(\frac{t}{N}\right) \log 2|\cos \lambda - \nu|, \quad -\frac{1}{2} < \lambda < \frac{1}{2}. \quad (8)$$

In order to take into account the local behavior of the process $(y_{t,N})$, we partition the time interval $[0, 1)$ into 2^l ($0 < l < J - 1$) non-overlapping subintervals I_h :

$$I_h = [h2^{-2l}, (h+1)2^{-2l}), \quad h = 0, \dots, 2^l - 1,$$

on which we assume that the process $(y_{t,N})_t$ is locally stationary. This means that the time-varying parameter $d\left(\frac{t}{N}\right)$ is constant on these intervals.

Now, since the time-varying wavelet variance provides an estimate of the spectral density function, we are going to use the following logarithmic transformation of this variance to estimate locally the long memory parameter. Let be the following relationship:

$$\log \sigma^2(\lambda_{j,n}, t) = \alpha(t) + \beta(t) \log 2|\cos \mu_{j,n} - \nu| + u(t), \quad (9)$$

where $\sigma^2(\lambda_{j,n}, t)$ is the variance of the DWPT coefficients $\mathbf{W}_{j,n}$ associated, at time t , with the frequency interval $\lambda_{j,n} = \left(\frac{n}{2^{j+1}}, \frac{n+1}{2^{j+1}}\right]$ (where $n = 0, \dots, 2^j - 1$; $j = 0, \dots, J - 1$); $\mu_{j,n}$ is the midpoint of the interval $\lambda_{j,n}$; $\beta(t)$ is the slope of the log-linear relationship at time t , and $d(t) = -\beta(t)/2$. We assume that the previous relationship is verified for the k explosions observed on the periodogram, thus, in the following we use the following approximation to estimate $d_i(t), i = 1, \dots, k$:

$$\log \sigma_i^2(\lambda_{j,n}, t) = \alpha_i(t) + \beta_i(t) \log 2|\mu_{j,n} - \cos^{-1}(\nu_i)| + u_i(t), \quad (10)$$

where $u_i(t)$ is a sequence of correlated random variables, Arteche and Robinson (2000).

2.3 Procedure for Estimating $d_i(t)$ ($i = 1, \dots, k$)

Using the previous approach, we present now a general procedure for estimating the time-varying parameters $d_i(t)$ ($i = 1, \dots, k$). We assume that the sample size is dyadic ($N = 2^J$), otherwise we repeat the last data value several times to achieve such a sample size. We detail the different steps:

1. We first detect the Gegenbauer frequency $\cos^{-1} \nu_1$ corresponding to the highest explosion in the periodogram. This frequency is fixed all along the procedure.
2. We compute the DWPT coefficient vectors $\mathbf{W}_{j,n}$ of length N_j using the formula (7), where $j = 0, \dots, J-1$; $n = 0, \dots, 2^j - 1$.
3. We associate to the vector $\mathbf{W}_{j,n}$ an adaptive orthonormal basis \mathcal{B} , such that the squared gain function of the wavelet filter associated with $\mathbf{W}_{j,n}$ is sufficiently small at the Gegenbauer frequency. Practically, we define $\mathcal{U}_{j,n}(f) = |U_{j,n}(f)|^2$ to be the squared gain function for the wavelet packet filter $u_{j,n,l}$, where $U_{j,n}(f)$ is the discrete Fourier transform (DFT) of

$$u_{j,n,l} = \sum_{k=0}^{L_j-1} u_{n,k} u_{j-1, [\frac{n}{2}], l-2^{j-1}k}, \quad l = 0, \dots, L_j - 1,$$

with $u_{1,0,l} = g_l$, $u_{1,1,l} = h_l$ and $L_j = (2^j - 1)(L - 1) + 1$, g_l and h_l being the scaling filter and the wavelet filter defined before.

4. The basis selection procedure involves selecting the combination of wavelet basis functions such that $\mathcal{U}_{j,n}(f_G) < \epsilon$ for some $\epsilon > 0$ at the minimum level j . However, the method of basis selection is not unique and the basis is not unique either. We apply the white noise tests like the portmanteau test to determine the best adaptive orthonormal basis that decorrelates the observed time series.
5. We partition the sampling interval $[0, 1)$ into 2^l non-overlapping subintervals of equal length, where l is an integer chosen such that $0 < l < (J - 1)$. " l " depends on the length of the data and the required precision. The 2^l subintervals are as follows

$$I_h = [h2^{-l}, (h + 1)2^{-l}), \quad \text{where } h = 0, \dots, 2^l - 1.$$

6. We locate the DWPT coefficients $W_{j,n,K}$ on each subinterval I_h . In order to construct the local estimates for the time-varying long memory parameter $d_1(t)$, we proceed according to the Heisenberg uncertainty principle: every DWPT coefficient vector is mapped to a rectangle (Heisenberg box) defined in the time-frequency plane with the boxes covering completely the plane.

7. Since the DWPT coefficient vector $\mathbf{W}_{j,n} = (W_{j,n,K})$ corresponds to the frequency interval $\lambda_{j,n} = (\frac{n}{2^{j+1}}, \frac{n+1}{2^{j+1}}]$, we obtain the corresponding time interval on the time-frequency plane with width $2^j/N$, whereas, the length of the vector $\mathbf{W}_{j,n}$ is N_j . Therefore, we partition the elements of the vectors $\mathbf{W}_{j,n}$ with equal length $N_j/2^l = 2^{J-j-l}$ and attach them sequentially.
8. On each subinterval I_h ($h = 0, \dots, 2^l - 1$), we consider the bivariate collection of data

$$\{(\log 2|\mu_{j,n} - f_1|, \log \sigma_1^2(\lambda_{j,n})) \mid 0 \leq n \leq 2^j - 1; 0 \leq j \leq J - 1\},$$

and we use the approximate log-relationship (10). On each subinterval I_h , we carry out the ordinary least squares (OLS) regression to get the local estimates for the slope $\beta_1(t)$. Thus we obtain 2^l local estimates for the parameter $\beta_1(t)$. Since $d_1(t) = -\frac{\beta_1(t)}{2}$, we get 2^l local estimates for the parameter $d_1(t)$.

9. We omit the first and the last estimates to avoid the boundary effects. We smooth the estimated 2^l points by two local polynomial methods: spline method and loess (locally weighted scatter plot smoothing) method. Thus, we obtain two smoothed curves for $d_1(t)$ which approximate locally the true parameter curve. Finally, we denote these estimates, $\hat{d}_1(t)$.
10. Knowing ν_1 , the above steps (2-9) permit to get the estimates $\hat{d}_1(t)$.
11. Now, we proceed in the same way to estimate the other time-varying long memory parameters, corresponding to each Gegenbauer frequency. First we calculate $y_{t,N}^1 := (1 - 2\nu_1 B + B^2)^{\hat{d}_1(t)} y_{t,N}$, where $\hat{d}_1(t)$ is obtained in the previous steps. We need to interpolate some points such that the vector $\hat{d}_1(t)$ is of length N , due to the fact that the number of points on the smoothed curves is less than N if we adopt the loess smoothing method, for instance.
12. Assuming that we know the frequency $\cos^{-1} \nu_2$, we repeat the above steps 1 to 9 on the vector $y_{t,N}^1$ to get the estimate $\hat{d}_2(t)$.
13. We proceed in the same way for other Gegenbauer frequencies until the $(k+1)$ -th step providing the white noise $(\varepsilon_t)_t$.

At the end, there is no more peak in the periodogram, and we have k pairs of estimations for the Gegenbauer frequencies and time-varying long memory parameters.

2.4 Consistency for estimates $\hat{d}_i(t)$ ($i = 1, \dots, k$)

In this subsection, we study the properties of the estimates $\hat{d}_i(t)$ ($i = 1, \dots, k$). To get $\hat{d}_i(t)$, we have previously established a linear regression between the variance of the DWPT wavelet coefficients $W_{j,n,t}$ and the long memory parameters $d_i(t)$. Some similar approaches have been developed, in a stationary setting, by Geweke and Porter-Hudak (1983), Robinson (1995), Hurvich and Beltrao (1981) and Arteché (1998). They obtain the consistency of the constant long memory parameter d_i . We extend their results.

Here, we assume that the assumptions $\mathbf{A}_1 - \mathbf{A}_2$ and $\mathbf{A}_4 - \mathbf{A}_5$ introduced in Arteché (1998) are verified for $f_N(\lambda)$ defined in (6). The assumptions \mathbf{A}_1 and \mathbf{A}_2 specify the local behavior of the spectrum. The assumption \mathbf{A}_4 corresponds to the "trimming" condition introduced first in Robinson (1995). Now, under these assumptions and in the case of a 1-factor stationary Gegenbauer model, the asymptotic normality of the long memory parameter is known, and has been proved by Arteché (1998).

Lemma 2.1 *Consider the Gegenbauer model $(I - 2\nu_1 B + B^2)^{d_1} y_t = \varepsilon_t$ with the previous assumptions $\mathbf{A}_1 - \mathbf{A}_2$ and $\mathbf{A}_4 - \mathbf{A}_5$. Let \hat{d}_1 be the least squares estimate of d_1 obtained from the following regression:*

$$\log I(\omega + \lambda_j) = c + d_1(-2 \log \lambda_j) + u_j, \quad j = l + 1, \dots, m, \quad (11)$$

where $I(\lambda)$ denotes the periodogram $c = \log C - \eta$, η is the Euler's constant, $u_j = \log\left(\frac{I(\omega + \lambda_j)}{C \lambda_j^{-2d_1}}\right) + \eta$, and $\lambda_j = \frac{2\pi j}{n}$ are Fourier frequencies. Then

$$2\sqrt{m}(\hat{d}_1 - d_1) \rightarrow^d N\left(0, \frac{\pi}{6}\right).$$

Now we consider the locally stationary 1-factor Gegenbauer process $(I - 2\nu B + B^2)^{d_1(t)} y_t = \varepsilon_t$. The parameter $d_1(t)$ has been locally estimated on a sequence of intervals I_h ($h = 0, \dots, 2^l - 1$) using the locally stationary process $y_{t,N}$, then using the previous lemma and the relationship (10) we get

$$\forall h, \sqrt{l}(\hat{d}_1(h) - d_1(h)) \rightarrow^d N\left(0, \frac{\pi}{24}\right), \quad h = 0, 1, \dots, 2^l - 1.$$

In order to get a smoothed curve for $\hat{d}_1(t)$, we have smoothed the 2^l independent estimates $(\hat{d}_1(0), \dots, \hat{d}_1(2^l - 1))$ using two local polynomial methods: spline method and loess method. This means, that for each method there exists a set of basis function $\hat{\omega}_h(t)$ such that $\sum_{h=0}^{2^l-1} \omega_h(t)^2 = C < \infty$, where C is a constant. Thus we get:

$$\tilde{d}_1(t) = \sum_{h=0}^{2^l-1} \hat{\omega}_h(t) \hat{d}_1(h).$$

We can remark that $E[\tilde{d}_1(t)] = \sum_{h=0}^{2^l-1} \hat{\omega}_h(t) E[\hat{d}_1(h)] = 0$, and $Var[\tilde{d}_1(t)] = Var[\sum_{h=0}^{2^l-1} \hat{\omega}_h(t) \hat{d}_1(h)] = C \frac{\pi}{24} = C_1$. Thus, assuming that N tends to infinity, we obtain:

$$\sqrt{l}(\tilde{d}_1(t) - d_1(t)) \rightarrow^d N(0, C_1).$$

If we observe k explosions on the periodogram, we apply the same approach for each couple $(\nu_i, d_i(t), i = 1, \dots, k)$.

3 Simulation experiments

In this section, we carry out some Monte Carlo simulations to establish the robustness of the estimation of the parameter function $d_i(t)$ using wavelet approach, for finite samples. We focus on model (2) assuming that $(\varepsilon_t)_t$ is a Gaussian noise:

$$(I - 2\nu B + B^2)^{d(t)} y(t) = \varepsilon(t). \quad (12)$$

We use linear and quadratic functions for $d(t)$:

1. $d(t)$ is linear: $d_1(t) = 0.2t + 0.1$
2. $d(t)$ is quadratic: $d_2(t) = 0.3(t - 0.5)^2 + 0.1$.

For convenience, we assume that the data points that we observe $[y_1, \dots, y_N]'$ are equally spaced on the time interval and are scaled on the time interval $[0, 1)$, using the transformation $t_i = \frac{i-1}{N}$ (where $i = 1, \dots, N = 2^J$). Here, $N = 4096 = 2^{12}$ ($J = 12$) and $\cos^{-1}\nu = 1/3$.

For the estimation procedure, we use the MB(16) wavelet filter ($L = 16$), and we choose the adaptive orthonormal basis using portmanteau test with $p = 0.01$. We partition the sampling interval $[0, 1)$ into $2^6 = 64$ subintervals ($l = 6$), and we get 64 local estimates for $d(t)$. Finally, we smooth the estimates using two local polynomial methods, spline method and loess method. We replicate the simulations 100 times for each locally stationary 1-factor Gegenbauer process (12) using the two previous functions $d(t)$. We carry out the code on the computer Mac OS X 10.5.1 Léopard, written in language R with the help of the package "waveslim".

We denote $(y_{1,t})_t$ the process (12) with linear parameter function $d_1(t) = 0.2t + 0.1$ and $(y_{2,t})_t$ the process (12) with quadratic parameter function $d_2(t) = 0.3(t - 0.5)^2 + 0.1$. On Figures 1 and 5, we provide the trajectories for the processes $(y_{1,t})_t$ and $(y_{2,t})_t$ respectively. On Figures 3 and 7 we provide their autocorrelation functions. Finally on Figures 4 and 8 we exhibit the true linear and true quadratic function $d(t)$, with the estimated curves smoothed by spline method and loess method. In Table 1, we provide the mean of the estimated Gegenbauer frequencies, and the bias and of the RMSE for $\hat{d}(t)$ using 100 simulations.

	Gegenbauer frequency $\hat{\lambda}_1$	bias of $\hat{d}(t)$	RMSE of $\hat{d}(t)$
Y_1	0.33	spline: -0.115436 loess: -0.089052	spline: 0.1039803 loess: 0.0632354
Y_2	0.33	spline: -0.006068 loess: -0.001749	spline: 0.01021453 loess: 0.009949698

Table 1: Estimation of Gegenbauer frequencies, bias and RMSE of $(y_{1,t})_t$, $(y_{2,t})_t$.

In summary, we observe that:

1. each estimated curve approximates the general shape of the time-varying parameter function. The rebuilding of the curve smoothed using the loess method appears better than that using the spline method;
2. the small values for the bias and the RMSE of the estimated parameter suggest that our algorithm is robust. Comparing the two smoothing methods, we find that in most cases, the loess method performs a little better than the spline method.

Figure 5.1: Sample path of the LSGP Y_1

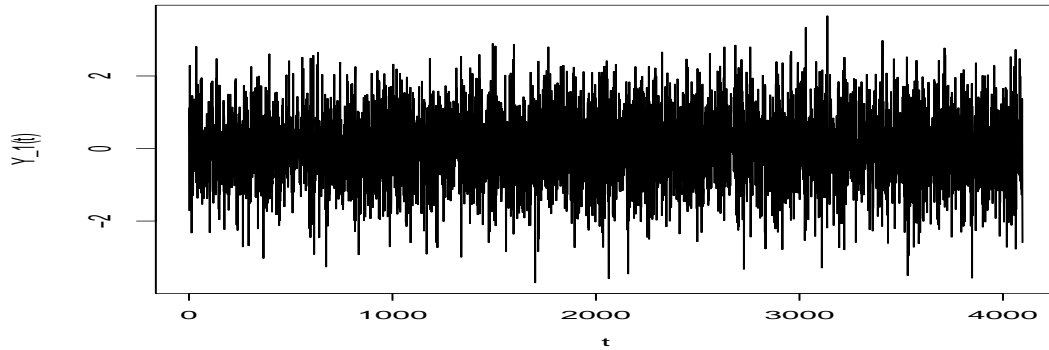


Figure 1: Sample path of $(y_{1,t})_t$

Figure 5.3: ACF of the LSGP Y_1

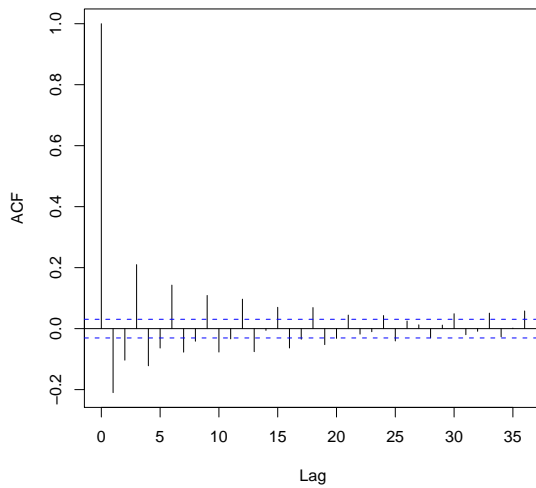


Figure 2: ACF of $(y_{1,t})_t$

Figure 5.2: Spectrum of the LSGP Y_1

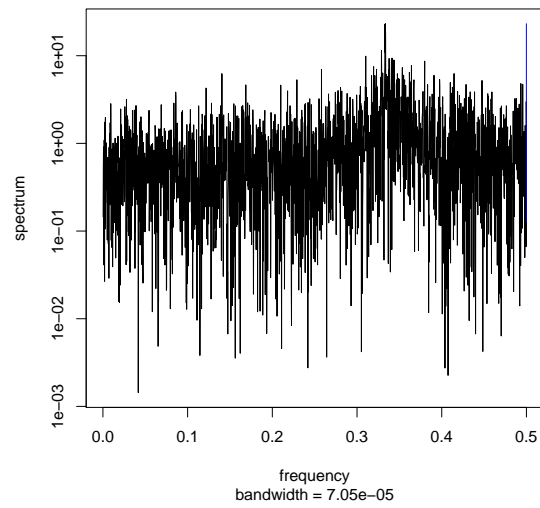


Figure 3: Spectrum of $(y_{1,t})_t$

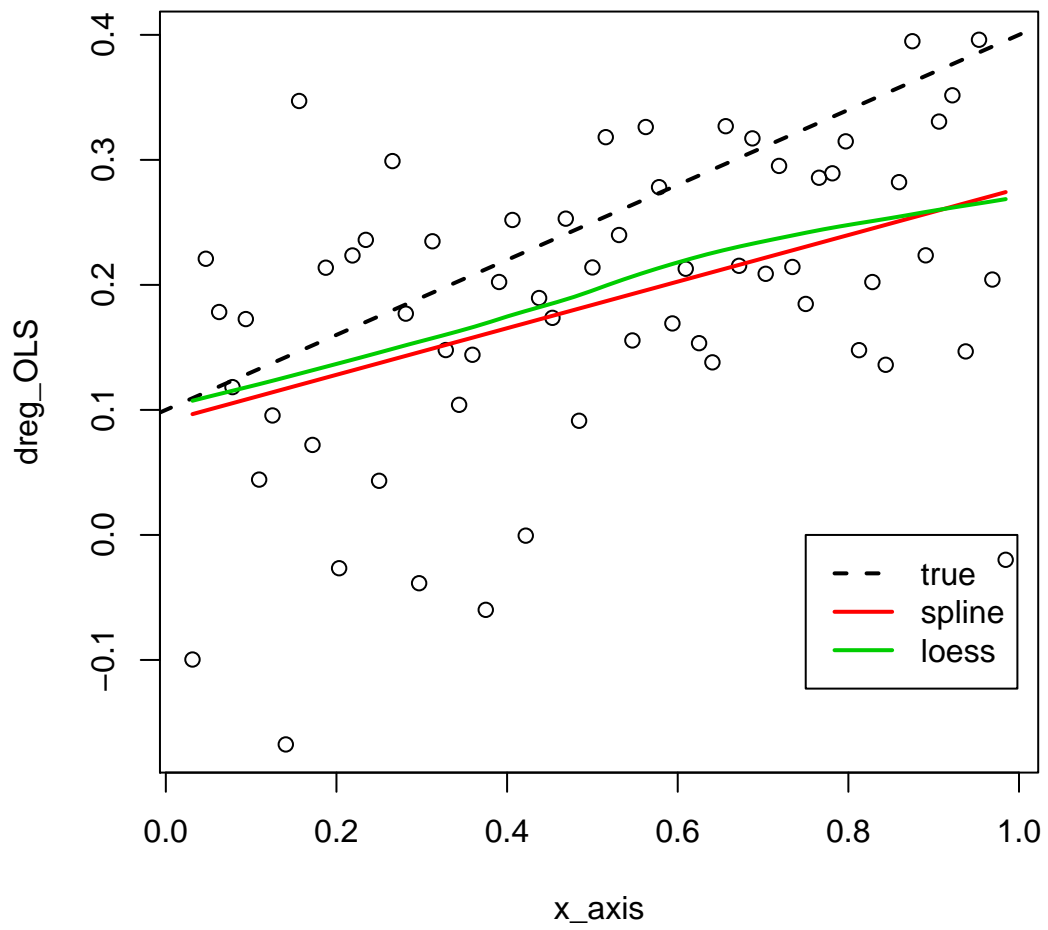


Figure 4: $\tilde{d}_1(t)$ (smoothed by spline and loess method) for $(y_{1,t})_t$

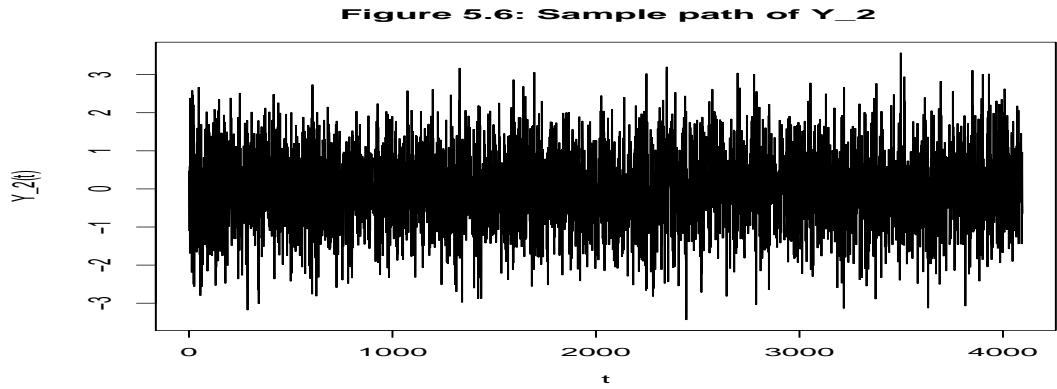


Figure 5: Sample path of $(y_{2,t})_t$

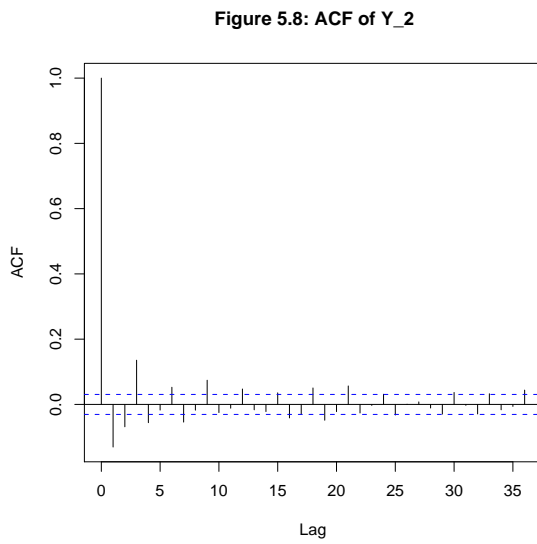


Figure 6: ACF of $(y_{2,t})_t$

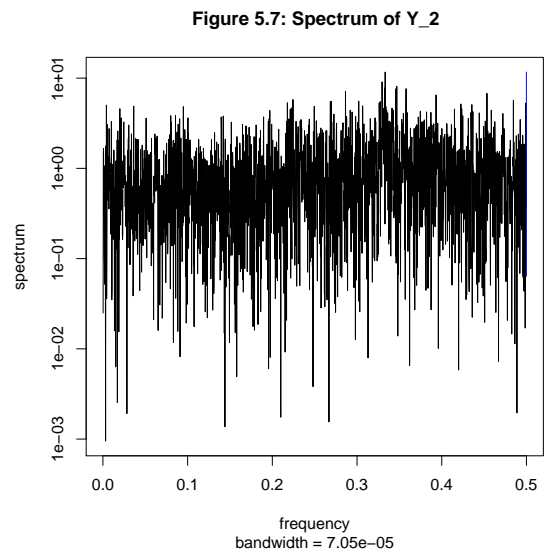


Figure 7: Spectrum of $(y_{2,t})_t$

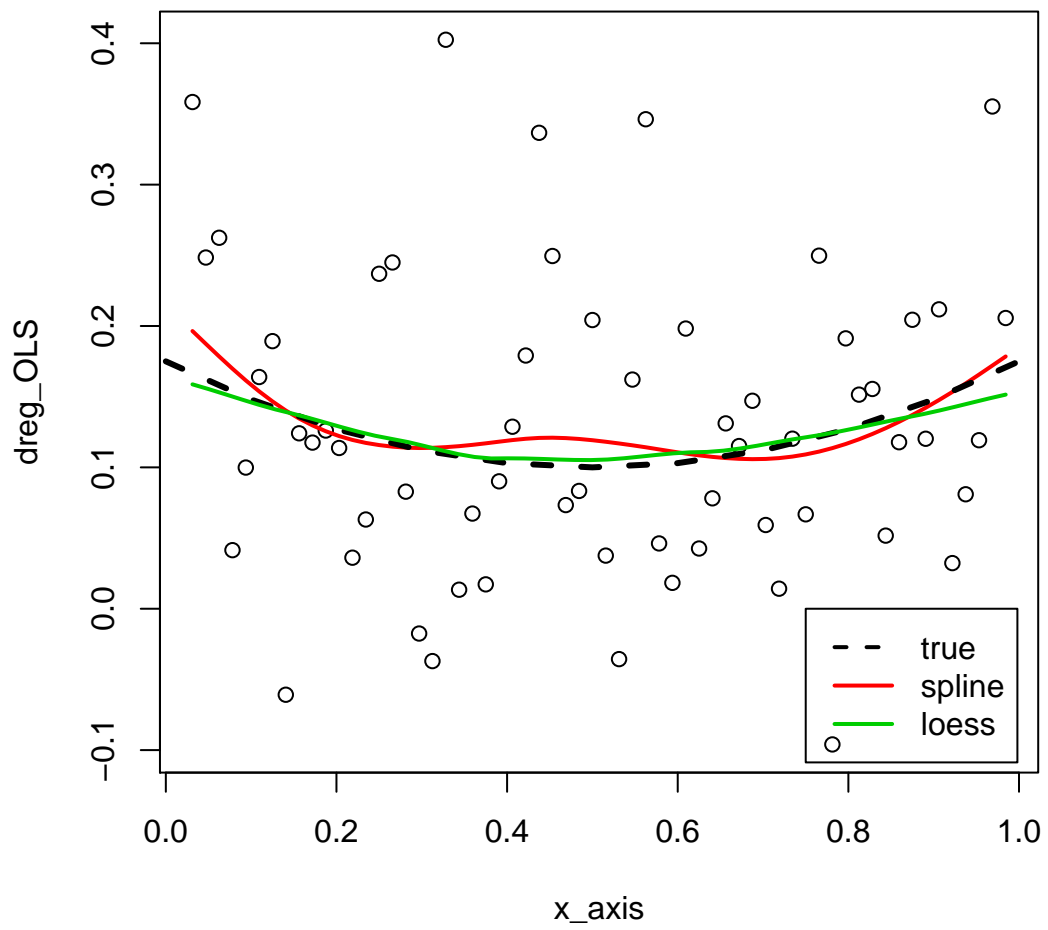


Figure 8: $\tilde{d}_1(t)$ (smoothed by spline and loess method) for $(y_{2,t})_t$

4 Application to the Nikkei Stock Average 225 index data

4.1 The data set

In this section we consider the Nikkei Stock Average 225 (NSA 225) spot index and futures price which correspond to 4096 daily observations of the spot index and the futures price of the NSA 225, covering the period from January 2nd, 1989 through September 13rd, 2004. Daily closing values of the spot index and the settlement prices of the futures contracts are used. The regular futures contracts mature in March, June, September and December. For further details on the futures price series, we refer to Lien and Tse (1999). Data sets are available from Thomson data stream.

4.2 Modeling

Figure 9 represents the spot index and futures prices from January 2nd 1989 to September 13rd 2004. $(S_t)_t$ denotes the logarithm of the spot price and $(F_t)_t$ the logarithm of the futures price. Lien and Tse (1999) assumed that $(S_t)_t$ and $(F_t)_t$ are both integrated of order one and they modeled the relationship between $(S_t)_t$ and $(F_t)_t$ using an error correction model (ECM), proposed by Engle and Granger (1987). Current prices are affected by the past prices and error correction term and the authors used the following relationship:

$$\Delta S_t = \phi_0 + \sum_{i=1}^p \phi_i \Delta S_{t-i} + \sum_{j=1}^q \psi_j \Delta F_{t-j} + \gamma Z_{t-1} + \varepsilon_{S_t}, \quad (13)$$

where $\phi_i, i = 0, \dots, p$, $\phi_j, j = 1, \dots, q$ and γ are real numbers, and $(Z_t)_t$ is such that $Z_t = F_t - S_t$, for $t = 1, \dots, T$. Figure 10 represents the error correction term $(Z_t)_t$, which is the difference between the log futures prices and the log spot prices.

Our aim is to estimate the error correction term $(Z_t)_t$ using the method that we have developed previously. Indeed, Lien and Tse (1999) and Ferrara and Guégan (2001) have already considered this problem using stationary models on a shorter period (from January 1989 to August 1997 and from May 1992 to August 1996 respectively). In the error correction model (ECM) of Lien and Tse (1999), the spot prices and futures prices are integrated of order one but the bias (the difference between the futures price and the spot indexes) is fractionally integrated. Whereas, Ferrara and Guégan (2001) modeled the

bias in the ECM using stationary Gegenbauer process, which has been proved to be more efficient than the modeling used by Lien and Tse (1999).

However, we would like to consider an even longer data set, which is not necessarily globally stationary. In another way, since we observe the existence of volatility in $(Z_t)_t$, it seems appropriate to model the series $(Z_t)_t$ by time-varying models. Thus, we propose to model the series using a locally stationary 1-factor gegenbauer process.

In the first step, we use the wavelet multiresolution analysis to remove the time-varying mean. For this purpose, we apply the Maximal Overlap Discrete Wavelet Transform (MODWT) (J=6) with a Daubechies least asymmetric (LA(8)) wavelet filter and perform the multiresolution analysis that we provide on Figure 11. The wavelet details, D_1, \dots, D_6 , exhibit zero mean, while the wavelet smooth S_6 , associated with the low frequency $[0, 1/64]$, captures the trend of the series. To remove the time-dependent mean, we ignore the wavelet smooth and sum up the six wavelet details. Thus we get the residuals: $Z_t - S_{6,t}$ which still keep the periodicity in the data set.

In the second step, we obtain the estimate of Gegenbauer frequency $\cos^{-1} \nu = 0.015$ which corresponds to the highest explosion in the periodogram. Thus, we apply the DWPT on the residuals, choosing the orthonormal basis, locating the DWPT coefficients on the partitioned 64 subintervals, calculating locally the variance, and carrying out the OLS regression on each subinterval. Then, we get the estimated curves smoothed by spline and loess method.

Thus, the estimated model for the series $(Z_t)_t$ is:

$$(I - 2 \times 0.995B + B^2)^{\hat{d}(t)}(Z_t - S_{6,t}) = \varepsilon_t,$$

where $\hat{d}(t)$ is the estimated curve provided on Figure 12. The thin real curve is the estimated parameter function smoothed by spline method, and the thick real curve is the one smoothed by loess method. In Figure 12, we also provide the estimation results in dashed line and dotted line using two semi-parametric methods say, Robinson method (1995) and Whittle method (1951), regarding the parameter function as a constant in the stationary Gegenbauer model. $S_{6,t}$ is the wavelet smooth obtained using the multiresolution analysis given in Figure 11.

Comparing our result with the result proposed by Ferrara and Guégan (2001) on a shorter time period using the Whittle approach, we get close behavior.

Thus, the method we developed here permits to extend the previous result. Indeed, on a shorter time period, it seems reasonable to assume the stationarity for the series. However on a longer time period, it seems more robust to work locally. Thus, this approach permits to work with non-stationary data sets without making them stationary.

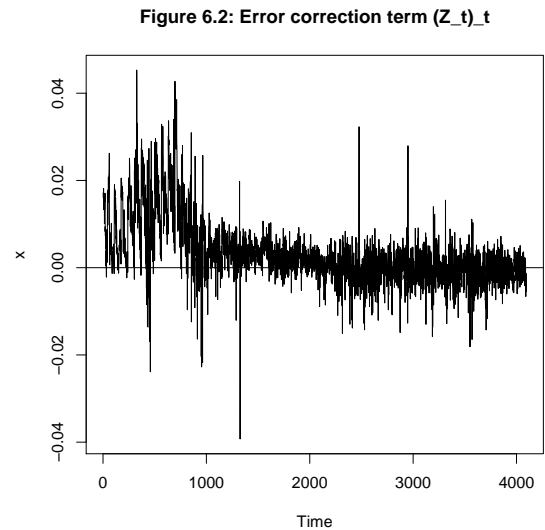
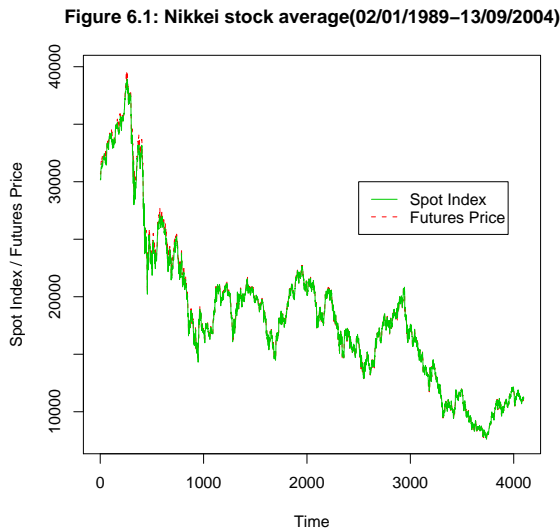


Figure 9: NSA (02/01/1989-13/09/2004)

Figure 10: Error Correction Term $(Z_t)_t$

5 Conclusion

In this paper, we have proposed a new class of model: the locally stationary k -factor Gegenbauer process with time-varying parameters to adequately model non-stationary time series. It can be regarded as an extension of the stationary k -factor Gegenbauer process by taking into account the time-varying parameter functions. We have proposed, discussed and investigated an algorithm for estimating the time-varying parameters of this new model. We investigate the consistency of the estimated parameters through simulations experiments. The estimation algorithm proposed in this paper does not restrict the type of parameter function: other time varying functions can be considered. We retain for estimation procedure the ordinary least squares method.

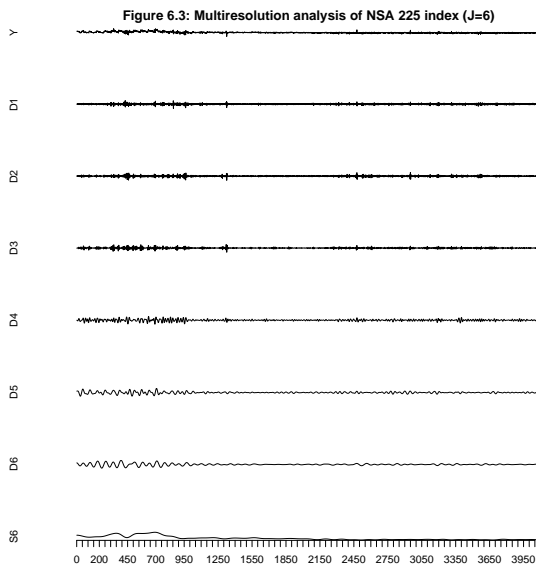


Figure 11: MRA of $(Z_t)_t$ ($J = 6$)

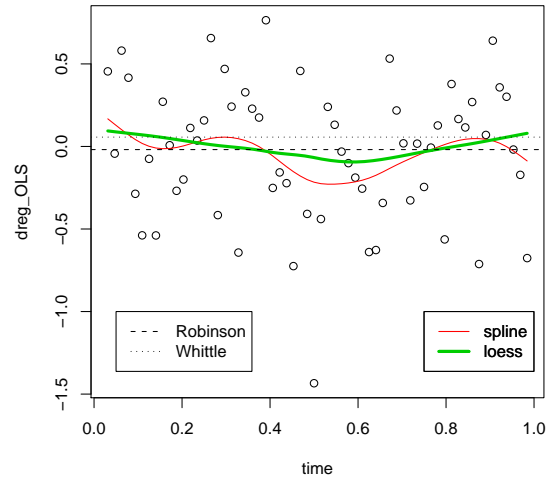


Figure 12: $\tilde{d}(t)$ by loess method

Therefore, we apply our algorithm to the error correction model for the data of NSA 225 spot index and futures price for a long time period (4096 data points). For the same time period considered by other authors, we get a similar result. However, we obtain an overall estimation for the parameter function, which grasps the local characteristics much more precisely. This example shows the interest of the methodology developed in that paper permitting to be free of the stationary assumption in the modeling of long memory behavior in presence of seasonalities.

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