No-Arbitrage Condition and Existence of Equilibrium with Dividends*

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Abstract

In this paper we first give an elementary proof of existence of equilibrium with dividends in an economy with possibly satiated consumers. We then introduce a no-arbitrage condition and show that it is equivalent to the existence of equilibrium with dividends.

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1 Introduction

In the Arrow-Debreu model (1954), the authors impose a nonsatiation assumption which states that for every consumer,whatever the commodity bundle may be, there exists another consumption bundle she/he strictly prefers. It is well-known, that in presence of satiation, a Walras equilibrium may not exist since for every price, there could be a consumer who maximizes her/his preference in the interior of her/his budget set. In presence of financial assets, satiation is rather a rule than an exception. Both the mean-variance CAPM and the expected-utility model with negative returns exhibit satiation (see e.g. Nielsen (1989), Dana, Le Van and Magnien (1997), Section 5).

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The absence of the nonsatiation condition with fixed prices was studied by Drèze and Muller (1980) by introducing the notion of coupons equilibrium, Aumann and Drèze (1986) with the concept of dividends, Mas-Colell (1992) who used the term of slack equilibrium. In Debreu (1959, *Theory of Value*), the notion of an *equilibrium relative to the price system* can be viewed as an equilibrium with possibly negative dividends. We can cite other authors who worked on nonsatiation: e.g. Makarov (1981), Kajii (1996), Florig and Yildiz (2002),Konovalov (2005), and for a continuum of consumers, Cornet, Topuzu and Yildiz (2003).

In this paper we first give an easy proof of existence of equilibria with dividends. For Aumann and Drèze, a dividend is a "cash allowance added to the budget by each trader. Its function is to distribute among the nonsatiated agents the surplus created by the failure of the satiated agents to use their entire budget". Here, we introduce an additional good (e.g. financial asset, or paper money) that the satiated agents will want to have in order to fill up their budget sets. For that, they will buy this additional good from the nonsatiated agents. More precisely, we will introduce an intermediary economy by adding another good that any agent would like to have if she/he meets satiation. In this economy, the nonsatiation condition is satisfied. There thus exists a Walras equilibrium. We show that this equilibrium actually corresponds to an equilibrium with dividends for the initial economy. It is interesting to notice that we show that, at this equilibrium, the satiated agents will buy the additional good from the nonsatiated agents and if an agent is not satiated then the value of the additional good will be zero for that agent. It is important to note that the idea to introduce an additional good is not new when one considers the *equilibrium with paper money* of Kajii (1996). What is new in this paper is the mechanism of exchange: it is defined clearly with well-defined partial extended preferences that the satiated consumers who meet satiation points will buy additional good from the consumers who do not meet satiation.

Second, we allow our model to have financial assets. If we assume that the production sets satisfy in particular the inaction and irreversibility conditions (see Debreu, 1959) and the utility functions satisfy the No-Half Line Condition (see e.g. Werner, 1986, Page and Wooders 1996, Dana, Le Van and Magnien, 1999, Allouch, Le Van, Page, 2002), then there exists an equilibrium with dividends iff there exists a no-arbitrage price. Usually, no-arbitrage conditions are introduced in an exchange economy with financial markets. Here, we introduce a no-arbitrage condition in an economy with production. We think of two-period models where firms produce consumption goods using capital goods and the consumers buy, in the first period, consumption goods and assets. An opportunity of arbitrage is a system of prices of commodities (consumption goods or assets) for which, either at least one consumer, without cost, can increase without bound her/his consumption, or one firm produces more and more because her/his profit increases without bound.

The paper is organized as follows. The model is presented in Section 2. The main result is given in Section 3. In Section 4, we introduce the no-arbitrage price condition and prove that existence of equilibrium is equivalent to existence of no-arbitrage prices. In Section 5, Appendix 1 gives a proof of Theorem 2 of Section 3. In Section 6, Appendix 2 presents an example of economies with production where the no-arbitrage condition is satisfied.

2 The Model

We consider an economy having l goods, J producers, and I consumers. We suppose that the numbers of the producers and the consumers are finite. For each $i \in I$, let $X_i \subset \mathbb{R}^l$ denote the set of consumption goods, let $u_i : X_i \longrightarrow \mathbb{R}$ denote the utility and let $e_i \in \mathbb{R}^l$ be the initial endowment. Furthermore for each $j \in J$, let $Y_j \subset \mathbb{R}^l$ denote the producing set of the producer j.

Let θ_{ij} be the ratio of the profit that consumer i can get from the producer j. We suppose that $0 \leq \theta_{ij} \leq 1, \sum_{i \in I} \theta_{ij} = 1$. Let $p \in \mathbb{R}^l$ denote the price of the goods.

In the sequel we will denote this economy by

$$\mathcal{E} = \left\{(X_i, u_i, e_i)_{i \in I}, (Y_j)_{j \in J}, (\theta_{ij})_{i \in I, j \in J}\right\}.$$

2.1 Preliminaries

We recall that a function u_i is said to be *quasiconcave* if its level-set

$$L^\alpha = \{x_i \in X_i : u_i(x_i) \geq \alpha\}$$

is convex for each $\alpha \in R$.

The function u_i is *strictly quasiconcave* if and only if $x_i, x'_i \in X_i$, $u_i(x'_i) > u_i(x_i)$ and $\lambda \in [0,1)$, then

$$u_i(\lambda x_i + (1-\lambda)x'_i) > u_i(x_i).$$

It means that

$$u_i(\lambda x_i + (1-\lambda)x'_i) > \min(u_i(x_i), u_i(x'_i)).$$

The function u_i is *upper semicontinuous* if and only if L^α is closed for each α.

Let S_i denote the set of satiation points of u_i. Then

$$S_i = \{x'_i \in X_i : u_i(x'_i) \geq u_i(x_i), \text{ for any } x_i \in X_i\}.$$

By this definition, the function u_i has no satiation point if for all $x_i \in X_i$ there exists $x'_i \in X_i$ such that $u_i(x'_i) > u_i(x_i)$. It is easy to check that S_i is convex and closed.

2.2 Definition 1

A *Walras equilibrium* of $\mathcal{E}$ is a list $((x_i^*)_{i\in I}, (y_j^*)_{j\in J}, p^*) \in (\mathbb{R}^l)^{|I|} \times (\mathbb{R}^l)^{|J|} \times (\mathbb{R}^l \setminus \{0\})$ which satisfies
(a) $\sum_{i\in I} x_i^* = \sum_{i\in I} e_i + \sum_{j\in J} y_j^*$ (Market clearing);
(b) for each i one has

$$p^*.x_i^* = p^*.e_i + \sum_{j\in J} \theta_{ij}.\sup p^*.Y_j$$

(butget constraint), and for each $x_i \in X_i$,with $u_i(x_i) > u_i(x_i^*)$, it holds

$$p^*.x_i > p^*.e_i + \sum_{j\in J} \theta_{ij}.\sup p^*.Y_j.$$

(c) For each $j \in J, y_j^* \in Y_j$ and $p^*.y_j^* = \sup p^*.Y_j$, where $\sup p.Y_j = \sup_{y_j\in Y_j} p.y_j$.

A *Walras quasi-equilibrium* is a list $((x_i^*)_{i\in I}, (y_j^*)_{j\in J}, p^*) \in (\mathbb{R}^l)^{|I|} \times (\mathbb{R}^l)^{|J|} \times (\mathbb{R}^l \setminus \{0\})$ which satisfies (a), (c), and (b) with the following change:

$$u_i(x_i) > u_i(x_i^*) \Rightarrow p^*.x_i \geq p^*.e_i + \sum_{j\in J} \theta_{ij}.\sup p^*.Y_j.$$

2.3 Definition 2

An *equilibrium with dividends* $(d_i^*)_{i\in I} \in \mathbb{R}_+^{|I|}$of $\mathcal{E}$ is a list $((x_i^*)_{i\in I}, (y_j^*)_{j\in J}, p^*) \in (\mathbb{R}^l)^{|I|} \times (\mathbb{R}^l)^{|J|} \times \mathbb{R}^l$ which satisfies:
(a) $\sum_{i\in I} x_i^* = \sum_{i\in I} e_i + \sum_{j\in J} y_j^*$ (Market clearing);
(b) for each i one has

$$p^*.x_i^* \leq p^*.e_i + \sum_{j\in J} \theta_{ij}. \sup p^*.Y_j + d_i^*$$

(butget constraint), and for each $x_i \in X_i$,with $u_i(x_i) > u_i(x_i^*)$, it holds

$$p^*.x_i > p^*.e_i + \sum_{j\in J} \theta_{ij}. \sup p^*.Y_j \ + d_i^*$$

(c) For each $j \in J, y_j^* \in Y_j$ and $p^*.y_j^* = \sup p^*.Y_j$, where $\sup p.Y_j = \sup_{y_j\in Y_j} p.y_j$.

2.4 Definition 3

A *feasible allocation* is the list $((x_i)_{i\in I}, (y_j)_{j\in J}) \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j$ which satisfies $\sum_{i\in I} x_i = \sum_{i\in I} e_i + \sum_{j\in J} y_j$. We denote by A the set of feasible allocations and by A_i the projection of A on the i^{th} component.

The main purpose of this paper is to give an easy proof of existence of equilibrium with dividends of economy $\mathcal{E}$ when satiation points occur in the preferences of the consumers. Our idea is to introduce an intermediary economy with an additional good (think of financial asset or money paper) that the consumers want to possess when they meet satiation. In this new economy, there is no satiation point. Hence, an equilibrium exists under appropriate assumptions. We show that this equilibrium is an equilibrium with dividends for the initial economy. It is worth to point out that at this equilibrium point, the consumers who meet satiation points will buy the additional good from the consumers who do not meet satiation.

2.5 The Assumptions

We now list our assumptions.
(H_1) For each $i \in I$, the set X_i is nonempty closed convex;

(H_2) For each $i \in I$, the function u_i is strictly quasiconcave and upper semicontinuous;
(H_3) For each $j \in J$, the set Y_j is nonempty closed convex and $Y = \sum_{j\in J} Y_j$ is closed.
(H_4) The feasible set A is compact.
(H_5) For every i, $e_i \in int(X_i - \sum_{j\in J} \theta_{ij} Y_j)$. Moreover, for every $i \in I, x_i \in A_i$ the set $\{x'_i : u_i(x'_i) > u_i(x_i)\}$ is relatively open in X_i.

Remark 1 *(1) Assumptions (H_1), (H_2) are standard.*
(2) Assumption (H_3) can be relaxed as follows: for each $j \in J$, the set Y_j is nonempty and the total production set $Y = \sum_j Y_j$ is closed and convex (see Remark 5 (1) below).
(3) Assumption (H_4) is satisfied when the consumption sets are the positive orthant $\mathbb{R}^l_+$, the production sets satisfy $0 \in Y_j, \forall j$, the total production set satisfies $Y \cap (-Y) = \{0\}$ (irreversibility) and $Y \cap \mathbb{R}^l_+ = \{0\}$ (one cannot produce without using input). It is also satisfied in a financial exchange economy with strictly concave utility functions and a no-arbitrage condition (see e.g. Page (1987) or Page and Wooders (1996)). We give in Appendix 2 two examples of economies with production and assets where the no-arbitrage condition is satisfied.
(4) Assumption (H_5) ensures that any quasi-equilibrium is actually an equilibrium.

3 The Results

We first give an existence of Walras equilibrium theorem when there exists no satiation.

Theorem 2 *Assume $(H_1) - (H_4)$ and*
(i)

$$\forall i, \;\; e_i \in (X_i - \sum_{j\in J} \theta_{ij} Y_j)$$

$$\forall i, \forall x_i \in X_i, \exists x'_i \in X_i \text{ such that } u_i(x'_i) > u_i(x_i).$$

then there exists a quasi-equilibrium.
(ii) If we add H_5 and

$$\forall i, \forall x_i \in X_i, \exists x_i' \in X_i \text{ such that } u_i(x_i') > u_i(x_i),$$

then there exists an equilibrium.

Proof. We adapt the proof given in Dana, Le Van and Magnien (1999) for an exchange exconomy. A detailed proof is given in Appendix 1. ■

We now come to our main result which is a corollary of the previous theorem.

Theorem 3 *Assume $(H_1)-(H_5)$. Then there exists an equilibrium with dividends.*

Proof. Let us introduce the intermediary economy

$$\widehat{\mathcal{E}} = \left\{(\widehat{X}_i, \widehat{u}_i, \widehat{e}_i)_{i\in I}, (\widehat{Y}_j)_{j\in J}, (\theta_{ij})_{i\in I, j\in J}\right\}$$

where: $\hat{X}_i = X_i \times \mathbb{R}_+,\ \hat{e}_i = (e_i, \delta_i)$ with $\delta_i > 0$ for any $i \in I$ and $\hat{Y}_j = (Y_j, 0)$ for any $j \in J$, and the utilities $\widehat{u}_i$ are defined as follows (recall that S_i is the set of satiation points for agent i): let $\mu > 0$, $M_i = \max\{u_i(x) : x \in X_i\}$.
- If $x_i \notin S_i$, then $\widehat{u}_i(x_i, d_i) = u_i(x_i)$ for any $d_i \geq 0$.
- If $x_i \in S_i$, then $\widehat{u}_i(x_i, d_i) = u_i(x_i) + \mu d_i = M_i + \mu d_i$ for any $d_i \geq 0$.
We will check that Assumption (H_2) is satisfied for every $\widehat{u}_i$.
To prove that $\widehat{u}_i$ is quasi-concave and upper semi-continuous, it suffices to prove that the set $\hat{L}_i^\alpha = \{(x_i, d_i) \in X_i \times \mathbb{R}_+ : \hat{u}_i(x_i, d_i) \geq \alpha\}$ is closed and convex for every α. We have two cases:
Case 1: $\alpha < M_i$. We claim that $\hat{L}_i^\alpha = L_i^\alpha \times \mathbb{R}_+$. Indeed, let $(x_i, d_i) \in \hat{L}_i^\alpha$. It follows $\hat{u}_i(x_i, d_i) \geq \alpha$ and there are two possibilities for x_i:
+ If $x_i \notin S_i$, then $\hat{u}_i(x_i, d_i) = u_i(x_i)$. It implies $u_i(x_i) \geq \alpha$ or $x_i \in L_i^\alpha$ and hence $(x_i, d_i) \in L_i^\alpha \times \mathbb{R}_+$.
+ If $x_i \in S_i$, then $u_i(x_i) = M_i > \alpha$. This follows $x_i \in L_i^\alpha$ and $(x_i, d_i) \in L_i^\alpha \times \mathbb{R}_+$.
So, $\hat{L}_i^\alpha \subset L_i^\alpha \times \mathbb{R}_+$. It is obvious $L_i^\alpha \times \mathbb{R}_+ \subset \hat{L}_i^\alpha$.
Case 2: $\alpha \geq M_i$. We claim that $\hat{L}_i^\alpha = S_i \times \left\{d_i : d_i \geq \frac{\alpha - M_i}{\mu}\right\}$. Indeed, if $\hat{u}_i(x_i, d_i) \geq \alpha$, then $x_i \in S_i$. In this case, $\widehat{u}_i(x_i, d_i) = M_i + \mu d_i \geq \alpha$, and hence $d_i \geq \frac{\alpha - M_i}{\mu}$. The converse is obvious.

It is also obvious that S_i is closed and convex. We have proved that $\widehat{u}_i$ is upper semicontinuous and quasi-concave for every i.

We now prove that $\widehat{u}_i$ is strictly quasi-concave.
Indeed, take $M_i = u_i(x)$ with $x \in S_i$ and $(x_i, d_i), (x'_i, d'_i) \in X_i \times R_+$ such that $\hat{u}_i(x'_i, d'_i) > \hat{u}_i(x_i, d_i)$. For any $\lambda \in]0, 1[$, we verify that

$$\hat{u}_i(\lambda x_i + (1-\lambda)x'_i, \lambda d_i + (1-\lambda)d'_i) > \hat{u}_i(x_i, d_i).$$

Since $\hat{u}_i(x'_i, d'_i) > \hat{u}_i(x_i, d_i)$, we can consider the following cases:
Case 1: $x'_i \in S_i, x_i \in S_i$. We have

$$\hat{u}_i(x_i, d_i) = M_i + \mu d_i,\ \hat{u}_i(x'_i, d'_i) = M_i + \mu d'_i.$$

It follows that $d'_i > d_i$. Hence

$$\lambda d_i + (1-\lambda)d'_i > \lambda d_i + (1-\lambda)d_i = d_i.$$

Since $\lambda x_i + (1-\lambda)x'_i \in S_i$, we deduce

$$\hat{u}_i(\lambda x_i + (1-\lambda)x'_i, \lambda d_i + (1-\lambda)d'_i) =$$

$$M_i + \mu(\lambda d_i + (1-\lambda)d'_i) > M_i + \mu d_i = \hat{u}_i(x_i, d_i).$$

Case 2: $x'_i \in S_i, x_i \notin S_i$. It implies $u_i(x'_i) > u_i(x_i)$. Since u_i is a strictly quasi-concave function, we obtain

$$u_i(\lambda x_i + (1-\lambda)x'_i) > u_i(x_i).$$

2a: If $\lambda x_i + (1-\lambda)x'_i \in S_i$, then

$$\hat{u}_i(\lambda x_i + (1-\lambda)x'_i, \lambda d_i + (1-\lambda)d'_i) = M_i + \mu(\lambda d_i + (1-\lambda)d'_i) > u_i(x_i) = \hat{u}_i(x_i, d_i).$$

2b: If $\lambda x_i + (1-\lambda)x'_i \notin S_i$, then

$$\hat{u}_i(\lambda x_i + (1-\lambda)x'_i, \lambda d_i + (1-\lambda)d'_i) = u_i(\lambda x_i + (1-\lambda)x'_i) > u_i(x_i) = \hat{u}_i(x_i, d_i).$$

Case 3 $x'_i \notin S_i, x_i \notin S_i$. We have

$$\hat{u}_i(x_i, d_i) = u_i(x_i),\ \ \hat{u}_i(x'_i, d'_i) = u_i(x'_i).$$

This follows $u_i(x'_i) > u_i(x_i)$. Similarly as above we consider

3a: If $\lambda x_i + (1-\lambda)x'_i \in S_i$, then

$$\hat{u}_i(\lambda x_i + (1-\lambda)x'_i, \lambda d_i + (1-\lambda)d'_i) = M_i + \mu(\lambda d_i + (1-\lambda)d'_i)$$

$$> u_i(x_i) = \hat{u}_i(x_i, d_i).$$

3b: If $\lambda x_i + (1-\lambda)x'_i \notin S_i$, then

$$\hat{u}_i(\lambda x_i + (1-\lambda)x'_i, \lambda d_i + (1-\lambda)d'_i) = u_i(\lambda x_i + (1-\lambda)x'_i) > u_i(x_i) = \hat{u}_i(x_i, d_i).$$

We have proved that the function $\hat{u}_i$ is strictly quasi-concave.
It remains to prove that the $\hat{u}_i$ has no satiation point.
Indeed, let $(x_i, d_i) \in X_i \times \mathbb{R}_+$. We consider the following cases
Case 1: $x_i \notin S_i$. Take $x'_i \in X_i$ such that $u_i(x'_i) > u_i(x_i)$ and $d'_i = d_i$.We have $\hat{u}_i(x'_i, d_i) \geq u_i(x'_i) > u_i(x_i) = \hat{u}_i(x_i, d_i)$.
Case 2: $x_i \in S_i$. Take $x'_i = x_i$ and $d'_i > d_i$. We have

$$\hat{u}_i(x'_i, d'_i) = \hat{u}_i(x'_i) + \mu d'_i > u_i(x_i) + \mu d_i = \hat{u}_i(x_i, d_i).$$

We have proved that the $\hat{u}_i$ has no satiation point.
Let us consider the feasible set $\widehat{A}$ of $\widehat{\mathcal{E}}$. We have:

$$\widehat{A} = \{((x_i, d_i)_{i\in I}, (y_j, 0)_{j\in J}) : \forall i, x_i \in X_i, d_i \in \mathbb{R}_+, \forall j, y_j \in Y_j \text{ and}$$

$$\sum_{i\in I} x_i = \sum_{i\in I} e_i + \sum_{j\in J} y_j, \sum_{i\in I} d_i = \sum_{i\in I} \delta_i.\}$$

It is obvious that $\widehat{A}$ is compact.
It is also obvious that Assumptions $(H_1), (H_2), (H_3)$ are fulfilled in economy $\widehat{\mathcal{E}}$.
Apply Theorem 2, part (i).
There exists a quasi-equilibrium $\big((x^*_i, d^*_i)_{i\in I}, (y^*_j, 0)_{j\in J}, (p^*, q^*)\big)$ with $(p^*, q^*) \neq (0, 0)$. It satisfies:

$$(i) \sum_{i\in I}(x^*_i, d^*_i) = \sum_{i\in I}(e_i, \delta_i) + \sum_{j\in J}(y^*_j, 0),$$

$$(ii) \text{ for any } i \in I,\ p^*.x^*_i + q^* d^*_i = p^*.e_i + \sum_{j\in J} \theta_{ij} \sup(p^* \cdot Y_j + q^* \times 0) + q^*\delta_i,$$

and

$$(iii) \text{ for any } j \in J,\ p^* \cdot y_j^* = \sup(p^* \cdot Y_j).$$

Observe that since $\mu > 0$, the price q^* must be nonnegative.
We claim that $\big((x_i^*)_{i\in I}, (y_j^*)_{j\in J}, p^*)\big)$ is an equilibrium with dividends $(q^*\delta_i)_{i\in I}$. Indeed, first, we have

$$\forall i \in I,\ p^*.x_i^* \leq p^*.e_i + \sum_{j\in J} \theta_{ij} p^* \cdot y_j^* + q^*\delta_i.$$

Now, let $x_i \in X_i, u_i(x_i) > u_i(x_i^*)$. That implies $x_i^* \notin S_i$ and hence $\hat{u}_i(x_i^*, d_i^*) = u_i(x_i^*)$. We also have $\hat{u}_i(x_i, 0) = u_i(x_i)$. That means $\hat{u}(x_i, 0) > \hat{u}_i(x_i^*, d_i^*)$. This implies

$$p^*.x_i = p^*.x_i + q^* \times 0 \geq p^*.e_i + \sum_{j\in J} \theta_{ij} \sup p^*.Y_j + (q^*\delta_i).$$

We claim that

$$p^*.x_i > p^*.e_i + \sum_{j\in J} \theta_{ij} \sup p^*.Y_j + (q^*\delta_i).$$

Assume the contrary, i.e.

$$p^*.x_i = p^*.e_i + \sum_{j\in J} \theta_{ij} \sup p^*.Y_j + (q^*\delta_i). \tag{1}$$

Then, since

$$e_i \in int(X_i - \sum_{j\in J} \theta_{ij} Y_j),$$

we have

$$\inf p^*.(X_i - \sum_{j\in J} \theta_{ij} Y_j) < p^*.e_i.$$

This means that there exists $x_i' \in X_i, y_j' \in Y_j$ such that

$$p^*.(x_i' - \sum_{j\in J} \theta_{ij} y_j') < p^*.e_i$$

which implies

$$p^*.x_i' < \sum_{j\in J} \theta_{ij} p^*.y_j' + p^*.e_i \leq \sum_{j\in J} \theta_{ij} p^*.y_j^* + p^* e_i + q^*\delta_i. \tag{2}$$

Let $x_i^\lambda = \lambda x_i' + (1-\lambda)x_i$ with $\lambda > 0$. Since $\{x_i : u_i(x_i) > u_i(x_i^*)\}$, by assumption, is relatively open, we have

$$u_i(x_i^\lambda) > u_i(x_i^*). \tag{3}$$

for every λ sufficiently small. On the other hand, from (1) and (2) we have

$$p^*.(\lambda x_i' + (1-\lambda)x_i) = \lambda p^*.x_i' + (1-\lambda)p^*.x_i$$

$$< \lambda(\sum_{j\in J}\theta_{ij}p^*.y_j^* + p^*.e_i + q^*\delta_i) + (1-\lambda)(\sum_{j\in J}\theta_{ij}p^*.y_j^* + p^*.e_i + q^*\delta_i)$$

or

$$p^*.x_i^\lambda + q^* \times 0 < p^*.e_i + \sum_{j\in J}\theta_{ij}p^*.y_j^* + q^*\delta_i. \tag{4}$$

Since $\hat{u}_i(x_i^\lambda, 0) = u_i(x_i^\lambda)$ and $\hat{u}_i(x_i^*, d_i^*) = u_i(x_i^*)$, relations (3) and (4) contradict the fact that $\big((x_i^*, d_i^*)_{i\in I}, (y_j^*, 0)_{j\in J}, (p^*, q^*)\big)$ is a quasi-equilibrium of the intermediary economy. ■

Corollary 4 *Assume $(H_1)-(H_4)$. Let $((x_i^*)_{i\in I}, (y_j^*)_{j\in J}, p^*)$ be an equilibrium with dividends (d_i^*). If consumer i is non-satiated, then*

$$p^*.x_i^* = p^*.e_i + \sum_{j\in J}\theta_{ij}.\sup p^*.Y_j + q^*\delta_i,$$

and $p^* \neq 0$.
Suppose that every consumer is non-satiated. Then an equilibrium with dividends will be reduced to a Walras equilibrium. That is the dividend is zero and the equilibrium price is non-zero.

Proof. First, we prove that, if x_i^* is not a satiation point, then $q^*d_i^* = 0$. Indeed, let $u_i(x_i) = \hat{u}_i(x_i, 0) > u_i(x_i^*) = \hat{u}_i(x_i^*, d_i^*)$. We then have

$$p^*.x_i \geq p^*.e_i + \sum_{j\in J}\theta_{ij}\sup p^*.Y_j + q^*\delta_i = p^*.x_i^* + q^*d_i^*.$$

For any $\lambda \in \,]0,1[\,$, from the strict quasi-concavity of u_i, we have $u_i(\lambda x_i + (1-\lambda)x_i^*) > u_i(x_i^*)$ and hence $p^*.(\lambda x_i + (1-\lambda)x_i^*) \geq p^*.x_i^* + q^*d_i^*$. Letting λ converge to zero, we obtain $q^*d_i^* \leq 0$. Thus $q^*d_i^* = 0$. That means that a consumer who does not meet satiation point will sell her/his endowment

of the additional good if $q^* > 0$. Observe also that $p^* \neq 0$ (if not we have $0 = q^*\delta_i$; this implies $q^* = 0$: a contradiction with $(p^*, q^*) \neq 0$).
One deduces from that, if x_i^* is not a satiation point for every $i \in I$, then $q^* = 0$, since $\sum_{i\in I} d_i = \sum_{i\in I} \delta_i > 0$. In this case, $p^* \neq 0$, and $((x_i^*)_{i\in I}, (y_j^*)_{j\in J}, p^*)$ is a Walras equilibrium. ■

Remark 5 *(1) We can replace (H_3) by $(H_3$ bis): "The total production set $\sum_{j\in J} Y_j$ is closed, non-empty and convex" as in Florig and Yildiz (2002), i.e., we do not require every Y_j be convex. Indeed, we replace the sets Y_j by their closed convex hulls $\overline{co}Y_j$. Let $((x_i^*), (y_j^*), p^*)$ be an equilibrium with dividends (d_i^*) of this new economy. This implies that every y_j^* is in $\overline{co}Y_j$. It is obvious that for any j*

$$p^* \cdot y_j^* = \max_{y\in\overline{co}Y_j} p^* \cdot y = \sup_{y\in Y_j} p^* \cdot y.$$

By assumption, $\sum_j Y_j$ is closed and convex. We then have $\sum_j Y_j = \sum_j \overline{co}Y_j$ [1]*. Hence there exist $(\zeta_j^*) \in \Pi_j Y_j$ such that $\sum_j \zeta_j^* = \sum_j y_j^*$. Since $\sum_i x_i^* = \sum_i e_i + \sum_j y_j^*$, and since $p^* \cdot \zeta_j^* \leq p^* \cdot y_j^*, \forall j$, we must have $p^* \cdot \zeta_j^* = p^* \cdot y_j^* = \max p^* \cdot Y_j$ for every j. That means that $((x_i^*), (\zeta_j^*), p^*)$ is an equilibrium with dividends for the initial economy.*
(2) Let $I_1 = \{i \in I : x_i^*$ is not a satiation point$\}$, and $I_2 = I \setminus I_1$. From Corollary 4, $q^* d_i^* = 0$, for any $i \in I_1$. Thus $\sum_{i\in I_1} q^*\delta_i = \sum_{i\in I_2} q^*.d_i^* - \sum_{i\in I_2} q^*\delta_i$. This shows that the group of agents who meet satiation buy the additional good from the group of agents who do not meet satiation.

4 No-arbitrage condition and existence of equilibrium with dividends

If we assume that $0 \in Y_j$ for every j, and if $((x_i^*)_{i\in I}, (y_j^*)_{j\in J}, p^*)$ is an equilibrium with dividends, we will have

$$p^*.e_i = p^*.e_i + \sum_j \theta_{ij}\, p^*.0 \leq p^*.e_i + \sum_{j\in J} \theta_{ij} p^*.y_j^* + q^*\delta_i.$$

[1] It comes from three facts. (i)We always have $\sum_j coY_j = co \sum_j Y_j$ (see e.g. Florenzano, Le Van and Gourdel, 2001, p. 16), (ii) $\sum_j \overline{co}Y_j \subset \overline{\sum_j coY_j}$ and $\sum_j Y_j$ is closed and convex.

Hence, for every i, we have $u_i(x_i^*) \geq u_i(e_i)$. We therefore define the set of *individually rational feasible allocations* $\widetilde{A}$. More precisely:

$$\widetilde{A} = \left\{ ((x_i),(y_j)) \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j : \sum_{i\in I} x_i = \sum_{i\in I} e_i + \sum_{j\in J} y_j, \forall i, u_i(x_i) \geq u_i(e_i) \right\}.$$

We will replace (H_4) by

$(H_4 bis)$ The set $\widetilde{A}$ is compact.

We have the following result:

Theorem 6 *(i) Assume* $(H_1),(H_2),(H_3),(H_4bis),(H_5)$ *, for every* j, $0 \in Y_j$ *and*

$$\forall i, \forall x_i \in X_i, \exists x_i' \in X_i \text{ such that } u_i(x_i') > u_i(x_i).$$

Then there exists a Walras equilibrium.

(ii)Assume $(H_1),(H_2),(H_3),(H_4bis),(H_5)$ *and for every* j, $0 \in Y_j$*. Then there exists an equilibrium with dividends.*

Proof. The proof is similar to the one of Theorem 2. One just replaces the feasible set A by the set of individually rational feasible allocations $\widetilde{A}$. ■

Let $P_i = \{x_i \in X_i : u_i(x_i) \geq u_i(e_i)\}$, and W_i be the recession cone of P_i. Elements in W_i which are different from zero will be called useful vectors for agent i (see Werner,1987). Let Z_j denote the recession cone of Y_j. Take some $\gamma_j \in Y_j$. Then $\gamma_j + \lambda z_j \in Y_j, \forall \lambda \geq 0$, $\forall z_j \in Z_j$. We call useful production vector for firm j any vector $z_j \in Z_j \setminus \{0\}$ (the producer can produce an infinitely large quantity $\gamma_j + \lambda z_j, \lambda \geq 0$).

Let $p \in \mathbb{R}^l$. We say that there exists an *opportunity of arbitrage* associated with p if either there exists $i \in I$, $w_i \in W_i \setminus \{0\}$, such that $p.w_i \leq 0$, or there exists $j \in J$, $z_j \in Z_j$, such that $p.z_j > 0$. In other words, with such a price p, either the consumer i will increase without bounds her/his consumption or firm j will produce an infinite quantity.

A price vector $p \in \mathbb{R}^l$ is a *no-arbitrage price* for the economy if $\forall i \in I$, $w_i \in W_i \setminus \{0\} \Longrightarrow p.w_i > 0$, and $\forall j \in J$, $z_j \in Z_j \Longrightarrow p.z_j \leq 0$.

We introduce the following **No-Arbitrage Condition**:

(NA) There exists a no-arbitrage price for the economy.

Remark 7 *Our No-Arbitrage Condition coincides with the one for an exchange economy, i.e. when* $Y_j = \{0\}$, $\forall j$.

Let us replace (H_3) by
(H_3ter) For each $j \in J$, the set Y_j is nonempty closed convex and $Y = \sum_{j\in J} Y_j$ is closed. Moreover, for every j, $0 \in Y_j$ and $Y \cap -Y = \{0\}$.
We have the following result

Theorem 8 *(i) Assume* $(H_1), (H_2), (H_3ter), (H_5)$ *and* (NA). *Then there exists an equilibrium with dividends.*
(ii) Assume the following No-Halfline Condition :
(NHL) *For* $i \in I$, *if* $w_i \in W_i \setminus \{0\}$, *then for any* $x \in P_i$, *there exists* $\lambda > 0$, *such that* $u_i(x + \lambda w_i) > u_i(x)$.
Then:
$((x_i^*)_{i\in I}, (y_j^*)_{j\in J}, p^*)$ *is an equilibrium with dividends* $\Rightarrow$ p^**is a no-arbitrage price.*

Proof. (i) It suffices to prove that $\widetilde{A}$ is compact. Assume the contrary. Then there is a sequence $\left((x_i^n)_i, (y_j^n)_j\right)_{n=1,..,\infty} \in \widetilde{A}$ such that $\sigma_n = \sum_i \|x_i^n\| + \sum_j \left\|y_j^n\right\| \to +\infty$ when $n \to \infty$. Since

$$\frac{\sum_i x_i^n}{\sigma_n} = \frac{\sum_i e_i}{\sigma_n} + \frac{\sum_j y_j^n}{\sigma_n}$$

We can assume, without loss of generality, that

$$\left(\left(\frac{x_i^n}{\sigma_n}\right)_i, \left(\frac{y_j^n}{\sigma_n}\right)_j\right) \to \left((w_i)_i, (z_j)_j\right) \in \left(\left(\prod_i W_i\right) \times \left(\prod_j Z_j\right)\right) \setminus \{0\}.$$

Moreover, we have

$$\sum_i w_i = \sum_j z_j.$$

Let p be a no-arbitrage price. If $(w_i)_i \neq 0$, we have a contradiction: $0 < p.\sum_i w_i = p.\sum_j z_j \leq 0$. If $(w_i)_i = 0$, then $\sum_j z_j = 0$. We have:$\sum_{k\neq j} z_k = -z_j$. From (H_3ter), $\sum_{k\neq j} z_k \in Y$ and $z_j \in Y$. Hence $z_j \in Y \cap -Y$. This implies $z_j = 0$. We have shown that, in this case, we have $(z_j)_j = 0$ and a contradiction with $\left((w_i)_i, (z_j)_j\right) \neq 0$.
We have proved that $\widetilde{A}$ is compact.
(ii) Let $((x_i^*)_{i\in I}, (y_j^*)_{j\in J}, p^*)$ be an equilibrium with dividends. It is obvious that $p^*.z_j \leq 0$, for every $z_j \in Z_j$ since $y_j^* + z_j \in Y_j$ and $p^*.y_j^* = \max p^*.Y_j$.

We have two cases.
Case 1. There exists some $i \in I$ such that x_i^* is not a satiation point. From Corollary 4, $p^* \neq 0$. If $w_i \in W_i \setminus \{0\}$, then Condition (NHL) implies $u_i(x_i^* + \lambda w_i) > u_i(x_i^*)$, for some $\lambda > 0$. Since $((x_i^*)_{i\in I}, (y_j^*)_{j\in J}, p^*)$ is an equilibrium, we have $p^*.w_i > 0$.
Case 2. For any $i \in I$, x_i^* is a satiation point. Condition (NHL) implies that $W_i = \{0\}$, for every i. No-arbitrage Condition is satisfied in this case with p^*. ■

Remark 9 *The No Halfline Condition is satisfied with strictly concave functions.*

5 Appendix 1: Proof of Theorem 2

5.1 Gale-Nikaido-Debreu Lemma

We will make use of the following lemma the proof of which can be found in Florenzano and Le Van (1986):

Lemma 10 (Gale-Nikaido-Debreu) *Let P be a closed nonempty convex cone in the linear space $\mathbb{R}^l$. Let P^0 be the polar cone of P and S be the unit sphere in $\mathbb{R}^l$. Suppose that the multivalued mapping Z from $S \cap P$ to $\mathbb{R}^l$ is upper semicontinuous and $Z(p)$ is nonempty convex compact. Suppose further that for every $p \in S \cap P, \exists z \in Z(p)$ such thar $p.z \leq 0$. Then there exists $\overline{p} \in S \cap P$ satisfying*

$$Z(\overline{p}) \cap P^0 \neq \emptyset,$$

where $P^0 = \{q \in \mathbb{R}^l : q.p \leq 0,\ \forall p \in P\}$.

5.2 Proof of Theorem 2

We consider a sequence of truncated economies.
Let $B(0,n)$ denote the ball centered at 0 with radius n. Let

$$X_i^n = X_i \cap B(0,n) \quad , \quad Y_j^n = Y_j \cap B(0,n)$$

where $i \in I, j \in J$. Since $e_i \in X_i$, we have $e_i \in X_i^n$ for all n is large enough. For every $(p,q) \in S \cap (\mathbb{R}^l \times \mathbb{R}_+)$, where S is the unit sphere of $\mathbb{R}^{l+1}$, define the multivalued mapping

$$\xi_i^n, Q_i^n : \mathbb{R}^l \times \mathbb{R}_+ \longrightarrow X_i$$

by setting
$\xi_i^n(p,q) = \Big\{ x_i \in X_i^n : p.x_i \leq p.e_i + \sum_{j\in J} \theta_{ij} \prod_j^n(p) + q \Big\}$,
$Q_i^n(p,q) = \Big\{ x_i \in \xi_i^n(p,q) : \text{if} \quad x_i' \in X_i^n \quad \text{with} \quad u_i(x_i') > u_i(x_i) \quad \text{then} \quad p.x_i' \geq p.e_i + \sum_{j\in J} \theta_{ij} \prod_j^n(p) + q \Big\}$, where $\prod_j^n(p) = \max p.Y_j^n$.

Under the assumptions mentioned in Theorem 2 we have the following lemma:

Lemma 11 *For each $i \in I$ the mapping Q_i^n is upper semicontinuous having nonempty compact convex values.*

Proof. From the definition it is easy to see that ξ_i^n is upper semicontinuous having nonempty convex compact values. From the definition of the mapping Q_i^n we have the following properties:
Let $x \in \xi_i^n(p,q)$ and $u_i(x) = \max u_i(x_i)$, with $x_i \in \xi_i^n(p,q)$ then $x \in Q_i^n(p,q)$. Indeed, let $x_i' \in X_i^n$ and $u_i(x_i') > u_i(x)$, then $x_i' \notin \xi_i^n(p,q)$. Hence by the definition of this set we have $p.x_i' \geq p.e_i + \sum_{j\in J} \theta_{ij} \prod_j^n(p) + q$, and therefore $x \in Q_i^n(p,q)$. This implies that $Q_i^n(p,q)$ nonempty for every $(p,q) \in S \cap (\mathbb{R}^l \times \mathbb{R}_+)$.
For every $x_i, y_i \in Q_i^n(p,q)$ and $\lambda \in [0,1]$, since $\xi_i^n(p,q)$ is convex we have $\lambda x_i + (1-\lambda) y_i \in \xi_i^n(p,q)$. On the other hand, since u_i is strictly quasiconcave, $u_i(\lambda x_i + (1-\lambda) y_i) > \min(u_i(x_i), u_i(y_i))$. Hence, for each $x_i' \in X_i^n$ and $u_i(x_i') > u_i(\lambda x_i + (1-\lambda) y_i)$, it follows that $u_i(x_i') > \min(u_i(x_i), u_i(y_i))$. Thus $p.x_i' \geq p.e_i + \sum_{j\in J} \theta_{ij} \prod_j^n(p) + q$. Hence $\lambda x_i + (1-\lambda) y_i \in Q_i^n(p,q)$ which means that $Q_i^n(p,q)$ is convex.
The mapping Q_i^n is closed. Indeed, let

$$(p^k, q^k, x_i^k) \in graph Q_i^n$$

and assume that $(p^k, q^k) \to (p, q), x_i^k \to x_i$. Since $x_i^k \in Q_i^n(p^k, q^k) \subset \xi_i^n(p^k, q^k)$ and ξ_i^n is closed, we have $x_i \in \xi_i^n(p, q)$. On the other hand, let $x_i' \in X_i{}^n$ with $u_i(x_i') > u_i(x_i)$, by the upper semicontinuity of u_i we see that $u_i(x_i') > u_i(x^k{}_i)$ for all k large enough. Since $x_i^k \in Q_i^n(p^k, q^k)$, we have

$$p^k.x_i' \geq p^k.e_i + \sum_{j\in J} \theta_{ij} \prod\nolimits_j^n (p^k) + q^k.$$

Letting $k \to +\infty$ we obtain

$$p.x_i' \geq p.e_i + \sum_{j\in J} \theta_{ij} \prod\nolimits_j^n (p) + q.$$

This implies that $x_i \in Q_i^n(p, q)$. Hence Q_i^n is closed. But, since

$$Q_i^n(p, q) \subset \xi_i^n(p, q) \subset X_i^n$$

for all $(p, q) \in S \cap (\mathbb{R}^l \times \mathbb{R}_+), n \geq 1$ and X_i^n is compact, we see that Q_i^n is a compact mapping. Hence Q_i^n is upper semicontinuous. ■

a) Under assumptions $(H_1) - (H_4)$ we now show that there exists quasi-equilibrium. Let $\Phi_j^n(p)$ denote the solution-set of $\prod_j^n(p)$, that means $y_j \in \Phi_j^n(p)$ if and only if $p.y_j = \max p.Y_j{}^n$. Define the mapping z^n by setting, for each $(p, q) \in S \cap (\mathbb{R}^l \times \mathbb{R}_+)$,

$$z^n(p, q) = (\sum_{i\in I} Q_i^n(p, q) - \sum_{i\in I} e_i - \sum_{j\in J} \phi_j^n(p)) \times \{-|I|\}$$

where S stands for the unit sphere in $\mathbb{R}^{l+1}$. By virtue of Lemma 11, from the assumptions of the theorem it is easy to see that z^n is upper semicontinuous having nonempty convex compact values. Note that for any x in $z^n(p, q)$ we can write

$$x = (\sum_{i\in I} x_i^n - \sum_{i\in I} e_i - \sum_{j\in J} y_j^n) \times (-|I|)$$

where $x_i^n \in Q_i^n(p, q)$ and $y_j^n \in \Phi_j{}^n(p)$. Since $x_i^n \in Q_i^n(p, q)$, that implies

$$p.x_i^n \leq p.e_i + \sum_{j\in J} \theta_{ij} \prod\nolimits_j^n (p) + q = p.e_i + \sum_{j\in J} \theta_{ij} p.y_j^n + q$$

or

$$p.\sum_{i\in I} x_i^n \leq p.\sum_{i\in I} e_i + \sum_{i\in I}\sum_{j\in J} \theta_{ij} p.y_j^n + |I|q = p.\sum_{i\in I} e_i + p.\sum_{j\in J} y_j^n + |I|q.$$

Thus

$$p.(\sum_{i\in I} x_i^n - \sum_{i\in I} e_i - \sum_{j\in J} y_j^n) - |I|q \leq 0.$$

Hence $(p,q).x \leq 0$ for every $(p,q) \in S \cap \mathbb{R}^l \times \mathbb{R}_+$, and $x \in z^n(p,q)$. Applying the Gale-Nikaido-Debreu Lemma , we can conclude that there exists $(p^n,q^n) \in S \cap (\mathbb{R}^l \times \mathbb{R}_+)$ such that

$$z^n(p^n,q^n) \cap (\mathbb{R}^l \times \mathbb{R}_+)^0 \neq \emptyset.$$

Since $(\mathbb{R}^l \times \mathbb{R}_+)^0 = (O_{\mathbb{R}^l} \times \mathbb{R}_-)$, it follows that for every $i \in I, j \in J$ there exists $x_i^n \in Q_i^n(p^n,q^n), y_j^n \in \Phi_j^n(p^n)$ satisfying

$$\sum_{i\in I} x_i^n - \sum_{i\in I} e_i - \sum_{j\in J} y_j^n = 0, \tag{5}$$

$$p^n.x_i^n \leq p^n.e_i + \sum_{j\in J} \theta_{ij} \prod\nolimits_j^n (p^n) + q^n$$

for every $i \in I$, and

$$p^n.x_i' \geq p^n.e_i + \sum_{j\in J} \theta_{ij} \prod\nolimits_j^n (p^n) + q^n. \tag{6}$$

for every $x_i' \in X_i^n$ which satisfies $u_i(x_i') > u_i(x_i^n)$.
From (5) we have $(x_i^n, y_j^n) \in A$. Since A is compact, without loss of generality, we may assume that

$$(x_i^n, y_j^n) \longrightarrow (x_i^*, y_j^*).$$

Since $(p^n,q^n) \in S \cap (\mathbb{R}^l \times \mathbb{R}_+)$ and $S \cap (\mathbb{R}^l \times \mathbb{R}_+)$ is compact, we can also assume $(p^n,q^n) \longrightarrow (p^*,q^*)$. From (5) and (6) it implies

$$\sum_{i\in I} x_i^* - \sum_{i\in I} e_i - \sum_{j\in J} y_j^* = 0, \tag{7}$$

$$p^*.x_i^* \leq p^*.e_i + \sum_{j\in J} \theta_{ij} \prod\nolimits_j (p^*) + q^* \quad \text{for every } i \in I, \tag{8}$$

where $\prod_j(p^*) = \max\{p^*.Y_j\}$.
Let $x_i \in X_i$ satisfy $u_i(x_i) > u_i(x^*)$. Define

$$x_i^\lambda = \lambda x_i + (1-\lambda)x_i^*,$$

where $\lambda \in (0,1]$. Since u_i is strictly quasiconcave, it implies $u_i(x_i^\lambda) > u_i(x_i^*)$. Moreover, since u_i is upper semicontinous and $x_i^n \to x_i^*$, for every n large enough, we have $u_i(x_i^\lambda) > u_i(x_i^n)$. Thus by (6) we obtain

$$p^n.x_i^\lambda \geq p^n.e_i + \sum_{j\in J} \theta_{ij} \prod\nolimits_j^n (p^n) + q^n$$

or

$$p^n.(\lambda x_i + (1-\lambda)x_i^*) \geq p^n.e_i + \sum_{j\in J} \theta_{ij} \prod\nolimits_j^n (p^n) + q^n.$$

Let $n \to +\infty$ we obtain

$$\lambda p^*.x_i + (1-\lambda)p^*.x_i^* \geq p^*.e_i + \sum_{j\in J} \theta_{ij} \prod_j (p^*) + q^*.$$

Let $\lambda \to 0$ we get

$$p^*.x_i^* \geq p^*.e_i + \sum_{j\in J} \theta_{ij} \prod\nolimits_j (p^*) + q^*. \tag{9}$$

Then from (8) and (9) follows

$$p^*.x_i^* = p^*.e_i + \sum_{j\in J} \theta_{ij} \prod\nolimits_j (p^*) + q^* \text{ for every } i \in I,$$

and hence

$$p^*.\sum_{i\in I} x_i^* = p^*.\sum_{i\in I} e_i + \sum_{i\in I}\sum_{j\in J} \theta_{ij} \prod\nolimits_j (p^*) + |I|q^*.$$

or

$$p^*.(\sum_{i\in I} x_i^* - \sum_{i\in I} e_i - \sum_{j\in J} y_j^*) = |I|q^*$$

But, from $\sum_{i\in I} x_i^* - \sum_{i\in I} e_i - \sum_{j\in J} y_j^* = 0$ follows $|I|q^* = 0$. Hence $q^* = 0$ and $p^* \neq 0$. Thus $((x_i^*)_{i\in I}, (y_j^*)_{j\in J}, p^*)$ is a quasi-equilibrium.
b) Now we show that if, in addtion, (H_5) is satisfied, then this quasi-equilibrium is in fact an equilibrium. Take $x_i \in X_i$ such that $u_i(x_i) > u_i(x_i^*)$. By the just proved preceeding part we have

$$p^*.x_i \geq p^*.e_i + \sum_{j\in J} \theta_{ij} \sup p^*.Y_j = p^*.e_i + \sum_{j\in J} \theta_{ij} p^*.y_j^*.$$

In contrary we suppose that

$$p^*.x_i = p^*.e_i + \sum_{j\in J} \theta_{ij} p^*.y_j^*. \tag{10}$$

Then, since

$$e_i \in int(X_i - \sum_{j\in J} \theta_{ij} Y_j),$$

we have

$$\inf p^*.(X_i - \sum_{j\in J} \theta_{ij} Y_j) < p^*.e_i.$$

This means that there exists $x_i' \in X_i, y_j' \in Y_j$ such that

$$p^*.(x_i' - \sum_{j\in J} \theta_{ij} y_j') < p^*.e_i$$

which implies

$$p^*.x_i' < \sum_{j\in J} \theta_{ij} p^*.y_j' + p^*.e_i \leq \sum_{j\in J} \theta_{ij} p^*.y_j^* + p^* e_i. \tag{11}$$

Let $x_i^\lambda = \lambda x_i' + (1-\lambda)x_i$ with $\lambda > 0$. Since $\{x_i : u_i(x_i) > u_i(x_i^*)\}$, by assumption, is open, we have

$$u_i(x_i^\lambda) > u_i(x_i^*). \tag{12}$$

for every λ sufficiently small. On the other hand, from (10) and (11) we have

$$p^*.(\lambda x_i' + (1-\lambda)x_i) = \lambda p^*.x_i' + (1-\lambda)p^*.x_i$$

$$< \lambda(\sum_{j\in J} \theta_{ij} p^*.y_j^* + p^*.e_i) + (1-\lambda)(\sum_{j\in J} \theta_{ij} p^*.y_j^* + p^*.e_i)$$

or

$$p^*.x_i^\lambda < p^*.e_i + \sum_{j\in J} \theta_{ij} p^*.y_j^*. \tag{13}$$

From (12) and (13) we arrive at a contradiction to the assumption that $((x_i^*)_{i\in I}, (y_j^*)_{j\in J}, p^*)$ is a quasi-equilibrium. The theorem is proved.

6 Appendix 2: An example of economies where the no-arbitrage condition is satisfied

Consider a two-period economy with two consumers and one firm. There exists one consumption good, one capital good, two assets. In the second period, there are two states of nature. Firm produces in period 1. Consumer i consumes c_0^i in period 1, c_s^i in period 2 if state s occurs. She/he owns $\alpha_i k_0$ capital stock (k_0 is the initial capital stock, α_i is the share between the two consumers of this capital stock). She/he buys in period 1, $\theta_1^i,\ \theta_2^i$ assets which yield in period 2, $v_s^{i,1}\theta_1^i + v_s^{i,2}\theta_2^i$ consumption goods if state s occurs. The preference of consumer i is represented by a concave, increasing function u^i. Consumer i solves the problem ($\mathcal{P}$):

$$\max u^i(c_0^i, c_1^i, c_2^i)$$

under the constraints

$$p_0 c_0^i + q.\theta^i \leq \alpha_i r k_0 + \beta_i \pi^*$$

and

$$0 \leq c_s^i \leq e_s^i + v_s^{i,1}\theta_1^i + v_s^{i,2}\theta_2^i$$

where p_0 is the price of consumption good in period 1, q is the price of assets, π^* is the profit of firm, β_i is the share of profit, r is the price of the capital good and e_s^i is the initial endowment in state s.
Firm solves the problem ($\mathcal{Q}$):

$$\pi^* = \max_k \{p_0 F(k) - rk\}$$

where F is a concave production function, increasing and $F(0) = 0$.
An equilibrium is a list $(p_0^*, q^*, r^*, c_0^{*i}, c_1^{*i}, c_2^{*i}, k^*)$ such that
(i) $(c_0^{*i}, c_1^{*i}, c_2^{*i})$ solve problem ($\mathcal{P}$) with $p_0 = p_0^*, q = q^*, r = r^*$,
(ii) k^* solves ($\mathcal{Q}$) with $p_0 = p_0^*, r = r^*$,
and (iii)

$$c_0^{*1} + c_0^{*2} = F(k^*)$$

$$c_s^{*i} = e_s^i + v_s^{i,1}\theta_1^{*i} + v_s^{i,2}\theta_2^{*i},\ \forall s = 1,2$$

$$\sum_{i=1}^{2} \theta_1^{*i} = 0, \sum_{i=1}^{2} \theta_2^{*i} = 0$$

and finally

$$k^* = k_0.$$

Since the functions u^i are increasing, the equilibrium problem is equivalent to the following.
$(c_0^{*i}, \theta_1^{*i}, \theta_2^{*i})$ solve:

$$\max u^i(c_0^i, e_1^i + v_1^{i,1}\theta_1^i + v_1^{i,2}\theta_2^i, e_2^i + v_2^{i,1}\theta_1^i + v_2^{i,2}\theta_2^i)$$

under the constraints

$$p_0^* c_0^i + q^*.\theta^i \leq \alpha_i r^* k_0 + \beta_i \pi^*$$

where $\pi^* = \max_k p_0^* F(k) - r^* k = p_0^* F(k^*) - r^* k^*$ and

$$c_0^{*1} + c_0^{*2} = F(k^*),$$

$$\sum_{i=1}^{2} \theta_1^{*i} = 0, \sum_{i=1}^{2} \theta_2^{*i} = 0$$

$$k^* = k_0.$$

Let $A_i = \{(\theta_1, \theta_2) : e_s^i + v_s^{i,1}\theta_1 + v_s^{i,2}\theta_2 \geq 0, \text{ for } s = 1, 2\}$.
The consumption set for consumer i is $X_i = \mathbb{R}_+ \times \mathbb{R}_+ \times A_i$ (the second factor corresponds to the capital good). Let $O^+ A_i$ denote the recession cone of A_i. Then the recession cone of X_i is $W_i = \mathbb{R}_+ \times \mathbb{R}_+ \times O^+ A_i$.
The production set for firm is

$$Y = \{(y, -k, \theta_1, \theta_2) \in \mathbb{R}_+ \times \mathbb{R}_- \times \{0\} \times \{0\} : y \leq F(k)\}.$$

Its recession cone is $Z = \mathbb{R}_- \times \mathbb{R}_- \times \{0\} \times \{0\}$.
Let $S_i = \{(q_1, q_2) : q_1 w_1 + q_2 w_2 > 0, \ \forall (w_1, w_2) \in O^+ A_i \setminus \{(0,0)\}\}$.
Assume $S = S_1 \cap S_2 \neq \emptyset$. Then the No-arbitrage Condition holds. Indeed, let $p = (1, 1, s)$ with $s \in S$. Then we have $p \cdot w > 0$ for all $w \in W_i \setminus \{0\}$ and $p \cdot z < 0, \forall z \in Z \setminus \{0\}$.

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