General equilibrium of financial markets:
An introduction *

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Abstract

The purpose of this paper is to explain the role of financial assets in allowing individual agents of an economy to make at time 0 some limited commitments into the future which, at some extent, redistribute their revenue among several time periods and different states of the world. It is done studying in different contexts the general equilibrium of a simple two-period exchange model, under weaker assumptions and in a more general setting than the ones usually described in the literature. Several equilibrium existence theorems are stated and proved. Even in this simple framework, they often require a rather sophisticated mathematical background and are of deep economic significance. Moreover,they are a necessary step towards further developments (including infinite horizon, continuous time, continuum of states of the world, default and collateral securities, . . .).

Keywords and phrases: general equilibrium, incomplete financial markets, arbitrage, numeraire assets, nominal assets, real assets, pseudo-équilibre.

Summary

1   Introduction

In order to understand the role of financial assets for the allocation of resources in a world in which time and uncertainty enter in an essential way, we introduce in this paper the simplest possible intertemporal model with two time periods and an a priori uncertainty at the first period about which of a finite number of possible states of the world (equivalently states of nature) will prevail at the second period.

We introduce also the pure exchange economy with finitely many consumers and a same finite set of goods at the first period and in each state of nature at the second period which will be extensively studied in this course. As usually, consumers are defined by their consumption set, their preferences and their initial endowment. To describe their tastes, rather than to assume complete preference preorderings on each consumption set, we use preference correspondences. In this way, we posit and will maintain in the following a minimal set of weak assumptions which will be used to get for different contexts equilibrium existence proofs in the spirit of the simultaneous approach to general equilibrium.

In addition to spot commodity markets at each time and in each state of nature, there is, at the first period, a financial market for a finite number of assets which yield, at the second period, contingent returns in each state of the world. A complete description of the financial structure of the model requires to specify first how are denominated the returns of the different assets. As we will see later, one can distinguish nominal, numeraire and real assets according as returns are denominated in units of account, in units of a same bundle of goods chosen as "numéraire" or in units of a given list of commodity bundles associated to the asset and to each state of nature. For simplicity, one generally assumes that the assets of the given financial structure are of the same kind ; obviously, in the "real" world, agents can use a richer and more complex array of financial instruments for which the general description given below is also relevant.

To define financial equilibrium, we assume that each agent is given with a portfolio set which describes what portfolios are available for him, i.e., we introduce the possibility of (institutional) bounds on short-selling of assets. As it was first remarked by Radner (1972) in a similar context, under the assumption that the individual portfolio sets of agents are bounded from below, a financial equilibrium is easily proved to exist, whatever be the kind of assets available on the financial market.

On the contrary, if there is no bound on short-selling of assets, the answer which can be given to the equilibrium existence problem depends heavily on the kind of assets considered in the financial side of the model. An objective of this paper is to state and to prove equilibrium existence theorems in each usual setting for the financial structure: numeraire, nominal or real assets.

The paper is organized as follows. In the next section, we describe the basic two-period exchange model, originated from Arrow (1953), and define financial equilibrium. In Section 3, we describe some typical examples of assets. In Section 4, we prove existence of financial equilibrium when the individual portfolio sets are assumed to be bounded from below. The proof is an adaptation of Radner's proof to our more general setting. Then, relaxing this constraint by letting the inferior bound for portfolios tend to infinity, we prove in Section 5 that equilibrium exists with numeraire assets and no bound on short-selling of assets.

As we will see, the so-called no-arbitrage condition is, in every context, a necessary condition for existence of equilibrium. Roughly speaking, it states that it is impossible at equilibrium to get positive financial returns without spending at time 0 some amount of money on the asset market. If there is no bound on short-selling, this condition determines at equilibrium a relation between asset prices and their financial returns. This relation is the basis of the asset pricing theory developed in Finance. Section 6 is devoted to the proof of this relation and to its consequences. In Section 7, we use this condition to give an equilibrium existence proof for the nominal asset case. Finally, in Section 8, we prove with an example that, with real assets and without bounds on short-selling of assets, equilibrium may fail to exist. This negative result, firstly noticed by Hart (1975), contrasts with the equilibrium existence theorems for nominal and numeraire assets given in the previous sections and motivates the definition and the existence theorem for a weaker equilibrium concept (pseudo-equilibrium) which coincides with equilibrium when the rank of the (pseudo-) equilibrium return matrix $V(p)$ is equal to the number of assets. This coincidence at the pseudo-equilibrium commodity prices is proved to hold "generically", i.e., roughly speaking, for randomly chosen endowments or asset structures, so that the equilibrium existence result stated in Section 8 is, for the real asset case, a result of generic existence. Here, we will prove only the existence of pseudoequilibrium and will stop before any differential topology considerations.

As it will be made clear at the end of Section 8, the rank of $V(p)$ is constant at (pseudo-) equilibrium with nominal assets, or with numeraire assets as long as desirability assumptions guarantee a strictly positive value of numeraire in every state of the world. Thus, as they are formulated in this course, the equilibrium existence results obtained in Section 5 for numeraire assets and in Section 7 for nominal assets can be viewed as consequences of the pseudo-equilibrium existence result. However, beginning with the abstract concept of pseudo-equilibrium would not have facilitated understanding the equilibrium existence problem in financial markets.

It is also worth noticing that the didactic order followed in this paper departs somewhat from the historical order in which appeared concepts and results. Arrow (1953) is the pioneering article for asset economies formulated in pure theory but the Arrow complete securities model is equivalent to the Arrow-Debreu

model. The archetype model, with multiple commodities and multiple budget constraints, was first formulated by Radner (1972). Applied to a financial economy, the equilibrium obtained by Radner was criticized, as contingent on the particular amount (and the particular shape) of a priori bounds on portfolios. After Hart (1975)'s counterexample, the equilibrium existence puzzle could seem intractable and stayed unsolved until a celebrated (but unpublished) paper by Cass (1984) who showed that existence of equilibrium could be guaranteed if the assets promise delivery in fiat money, i.e., in units of account. Almost simultaneously, Werner (1985) gave also a proof of existence of equilibrium with nominal assets, Geanakoplos and Polemarchakis (1986) showed the same for numeraire assets (the existence proof given in Section 5 is inspired by Chae (1988)), while Duffie and Shafer (1985) established generic existence in the real asset case. General equilibrium theory of financial markets was then in place.

2 The basic two period exchange economy

Let us consider two time periods $t = 0$ and $t = 1$, an a priori uncertainty about which of a positive finite number S of possible states of the world, $s = 1, \ldots, S$, will occur at time $t = 1$, and a positive finite number L of divisible goods, $\ell = 1, \ldots, L$, available at $t = 0$ and in each state of nature at period 1. For convenience, $s = 0$ denotes the state of the world (known with certainty) at period 0. The commodity space of the model is $\mathbb{R}^{L(1+S)}$. On this point, it should be noticed that all the hereafter definitions and results could easily be extended to a T-period framework but that infinite time horizon and/or more than finitely many states of the world in the model are out of the scope of this course.

On such a stochastic structure, we consider a pure exchange economy with a positive finite number I of consumers, $i = 1, \ldots, I$, each one characterized by a consumption set $X^i \subset \mathbb{R}^{L(1+S)}$, a preference correspondence $P^i : X \to X^i$ where $X = \prod_{i=1}^I X^i$ and an endowment vector $\omega^i = (\omega_0^i, \omega_1^i) \in \mathbb{R}^{L(1+S)}$. Since consumer i does not know which state of nature will occur at period 1, $\omega_1^i = (\omega^i(s))_{s=1}^S$ can be thought of as a random variable. For $x \in X$, $P^i(x)$ is interpreted as the set of consumption plans in X^i which are strictly preferred to x^i by the consumer i, given the consumption plans $(x^j)_{j \neq i}$ of the other agents. Since correspondence P^i describes possible rankings between elements of $X^i \subset \mathbb{R}^{L(1+S)}$, it should be emphasized that it expresses, as much as a comparison between different goods, the time preference of consumer i and his/her attitude toward risk. This general framework obviously encompasses the case where each consumer i is assumed to have a complete preorder on his/her consumption set X^i ; a fortiori, it encompasses the case where preferences of consumer i are represented by a von Neumann - Morgenstern expected utility function $U^i(x_0^i, x_1^i) = \sum_{s=1}^S \rho_s^i u^i(x^i(0), x^i(s))$ (here $\rho_s^i > 0$ denotes the (subjective) probability of state s and $\sum_{s=1}^S \rho_s^i = 1$).

An equilibrium definition depends on the definition adopted for the consumer budget sets. If, given a price system $p = (p(s)) \in \mathbb{R}^{L(1+S)}$, the budget set of i is

$$B^i_{A-D}(p) = \{x^i \in X^i \mid p \cdot x^i \leq p \cdot \omega^i\},$$

an *Arrow-Debreu equilibrium* is standardly defined as a pair $(\bar{p}, \bar{x})$ of a price and an allocation such that

(i) $\sum_{i=1}^I \bar{x}^i = \sum_{i=1}^I \omega^i$

(ii) for every i, $\bar{x}^i \in B^i_{A-D}(\bar{p})$ and $B^i_{A-D}(\bar{p}) \cap P^i(\bar{x}) = \emptyset$.

Following Arrow (1953), one can interpret $x^i \in B^i_{A-D}(p)$ as a contingent good, that is, $x^i(s)$ is bought at time 0 contingent to the fact that state s will occur. Then the just defined equilibrium is, in a nontransitive context, the same as the one described in Debreu's Theory of Value, Chapter 7. Such a definition provides the consumers with all potentially desirable credit arrangements.

At the opposite side, for the same pure exchange model on the same stochastic structure, a *pure spot market equilibrium* is characterized by budget sets of the form

$$B^i_{SM}(p) = \{x^i \in X^i \mid \forall s = 0, 1, \ldots, S, \ p(s) \cdot x^i(s) \leq p(s) \cdot \omega^i(s)\}$$

and the corresponding conditions (i) and (ii) for an equilibrium pair $(\bar{p}, \bar{x})$. With such budget sets, no financial transfer is allowed to the consumers from a period to another or from one state of the world to another.

In the following, we model the fact that financial instruments enable the agents to make at time 0 some limited commitments into the future. We assume that, in addition to the different commodity spot markets, there exists at time 0 a financial market for a positive finite number J of assets, $j = 1, \ldots, J$, bought (or sold) by the agents at time 0 and which deliver a random return across the states of the world at $t = 1$.

An *asset j* is a contract which promises to deliver in each state s of period $t = 1$ and for a given price system of commodities $p = (p(s))_{s=0}^S \in \mathbb{R}^{L(1+S)}$ the financial return $v^j(p, s)$, so that asset j is described by the vector map $p \to (v^j(p, s))_{s=1}^S$. The matrix map V

$$p \to V(p) = (v^j(p, s))_{\substack{s=1,\ldots,S \\ j=1,\ldots,J}},$$

which gives for each p the $S \times J$–matrix of financial returns, summarizes the financial asset structure.

Let us call *portfolio* an asset bundle $z \in \mathbb{R}^J$ with the convention :

$z_j > 0$ represents a quantity of asset j bought at period 0,

$z_j < 0$ represents a quantity of asset j sold at period 0.

If we assume that portfolios are constrained, that is, each agent i is given with a portfolio set $Z^i \subset \mathbb{R}^J$ which describes the portfolios available for him, then the definition of a financial economy is the following:

Definition 2.1 *A financial economy $\mathcal{E}$ is a collection*

$$((X^i, P^i, \omega^i, Z^i)_{i=1}^I, V).$$

Given commodity and asset prices (p, q) measured in units of account, the budget set of i is now :

$$B_{FM}^i(p, q) = \left\{ x^i \in X^i \left| \begin{array}{l} \exists z^i \in Z^i, \\ p(0) \cdot x^i(0) + q \cdot z^i \leq p(0) \cdot \omega^i(0) \\ p(s) \cdot x^i(s) \leq p(s) \cdot \omega^i(s) + v(p, s) \cdot z^i, \ \ \forall s = 1, \ldots, S \end{array} \right. \right\}$$

where $v(p, s)$ denotes the sth row of matrix $V(p)$.

If we adopt the compact notations :

- $p \, \square \, x^i$ denotes the vector $(p(s) \cdot x^i(s))_{s=0}^S$.

- $W(p, q)$ denotes the $(1 + S) \times J$-matrix $\begin{pmatrix} -q \\ V(p) \end{pmatrix}$,

the budget set can be written

$$B_{FM}^i(p, q) = \{x^i \in X^i \mid \exists z^i \in Z^i, p \, \square \, (x^i - \omega^i) \leq W(p, q) z^i\}.$$

With this definition, as in the pure spot market equilibrium case, each consumer faces a system of $(S + 1)$ budget constraints, one at each state of the world. $W(p, q)$, sometimes called *full matrix of returns*, summarizes the possible financial transfers between period 0 and the different states of the world at period 1 which enable each consumer to redistribute (at some extent) revenue across the different states of the world.

Definition 2.2 *A financial equilibrium is a pair of actions and admissible prices $((\bar{x}^i, \bar{z}^i)_{i=1}^I, (\bar{p}, \bar{q}))$ such that*

(i) for each i, $\bar{x}^i \in X^i$, $\bar{z}^i \in Z^i$, $\bar{p} \, \square \, (\bar{x}^i - \omega^i) = W(\bar{p}, \bar{q})\bar{z}^i$ and

$$P^i(\bar{x}) \bigcap B_{FM}^i(\bar{p}, \bar{q}) = \emptyset$$

(ii) $\sum_{i=1}^I (\bar{x}^i - \omega^i) = 0$ and $\sum_{i=1}^I \bar{z}^i = 0$.

Classically, (i) means that each $(\bar{x}^i, \bar{z}^i)$ is an optimal budget feasible plan for agent i, given $(\bar{p}, \bar{q})$. Note that with (i) we require, in coherence with (ii), each budget constraint to be binded at equilibrium. (ii) is a couple of market clearing

conditions under the implicit hypothesis that no production or intertemporal storage is possible and assets are in zero net supply.

If we note that for every i,

$$\bar{p} \,\square\, (\bar{x}^i - \omega^i) = \bar{p} \,\square\, \Big(-(\sum_{j \neq i}(\bar{x}^j - \omega^j))\Big) = W(\bar{p}, \bar{q})(-\sum_{j \neq i} \bar{z}^i),$$

we have the following :

Remark 2.1 *Assume that $\forall i$, Z^i equals the same vector subspace Z of $\mathbb{R}^J$. Then, in the previous definition, the condition $\sum_{i=1}^{I} \bar{z}^i = 0$ is redundant in the following sense: by changing the portfolio of any one agent, it is easy to associate a financial equilibrium with any $((\bar{x}^i, \bar{z}^i)_{i=1}^I, \bar{p}, \bar{q})$ satisfying all the other conditions of Definition 2.2 but not necessarily $\sum_{i=1}^{I} \bar{z}^i = 0$.*

Let us denote by $\widehat{X}$ the set of all attainable consumption allocations :

$$\widehat{X} := \{(x^i) \in \prod_{i=1}^{I} X^i \,|\, \sum_{i=1}^{I}(x^i - \omega^i) = 0\}$$

and by $\widehat{X}^i$ the projection of $\widehat{X}$ on X^i. We will maintain in this course the following assumptions.

On the consumption side, we set:

C.1 For every i, X^i is a closed, convex and bounded below subset of $\mathbb{R}^{L(1+S)}$

C.2 The correspondences $P^i : X \to X^i$ are lower semicontinuous on X and have convex open values in X^i. Moreover, $x^i \notin P^i(x)$ and the preference correspondences satisfy an additional convexity property: $[y^i \in P^i(x)$ and $0 < \lambda \leq 1]$ imply $[x^i + \lambda(y^i - x^i) \in P^i(x)]$

C.3 (Survival assumption) For every i, $\omega^i \in \mathrm{int}X^i$ (the interior of X^i)

C.4 (nonsatiation at every date-event pair and at every component of an attainable consumption allocation) For every $x \in \widehat{X}$, for every i, for every $s = 0, 1, \ldots, S$, there exists an $x'^i \in X^i$, differing from x^i only at s, such that $x'^i \in P^i(x)$.

Assumptions C.1 – C.3 are standard in a nontransitive context. As it is well known, the existence of an Arrow-Debreu equilibrium requires a weaker nonsatiation assumption than C.4. The strong form of nonsatiation in C.4 is specific of equilibrium models with multiple budget constraints. The existence of a pure spot market equilibrium will be proved later under Assumptions C.1 – C.4.

On the financial side, we assume :

F.1 For every i, Z^i is closed, convex, with $0 \in \mathrm{int}Z^i$

F.2 The map $p \to V(p)$ is continuous.

3   Main kinds of assets

We now describe three usual settings for a financial structure.

Definition 3.1 *A real asset j is a contract which promises to deliver (i.e. to pay the value of) in each state s at time $t = 1$ a vector $a^j(s)$ of quantities of the L goods.*

A real asset is thus characterized by an element $a^j = (a^j(s)) \in \mathbb{R}^{LS}$. Given a commodity price system $p = (p(s))_{s=0}^{S}$, the vector $(p(s) \cdot a^j(s))_{s=1}^{S}$ expresses the financial return of asset j across states of nature at period $t = 1$, denominated in units of account. Thus if all assets are real (real case), $A = (a^j(s)) \in \mathbb{R}^{LSJ}$ summarizes the financial structure and the $(S \times J)$-matrix

$$V(p) = (p(s) \cdot a^j(s))_{\substack{s=1,\dots,S \\ j=1,\dots,J}}$$

completely describes, given commodity prices p, the financial returns at time $t = 1$ allowed by the *real asset structure.*

An example of real asset is that of contingent commodity. A *contingent commodity* is a contract which promises to deliver one unit of good ℓ in state s and nothing otherwise. If there is available at date 0 a complete set of such contingent contracts, then $J = SL$. The asset $a^{s\ell}$ is defined by $a^{s\ell}(s) = e^{\ell}$, the ℓ th vector of the natural basis, $a^{s\ell}(s') = 0$ if $s' \neq s$. Given a commodity price system p, $v^{s\ell}(p,s) = p_{\ell}(s)$, $v^{s\ell}(p,s') = 0$ if $s' \neq s$, so that

$$V(p) = \begin{pmatrix} p_1(1) & \dots & p_L(1) & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & \dots & 0 & p_1(2) & \dots & p_L(2) & \dots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots & \ddots & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & & p_1(S) & \dots & p_L(S) \end{pmatrix}.$$

It should be noticed that if p satisfies $p(s) \neq 0 \ \forall s = 1, \dots S$, the rank of this matrix is equal to S. It will be seen later that at equilibrium, $B_{FM}(\bar{p}, \bar{q})$ coincides then with $B_{A-D}(\bar{p})$.

An other example which involves a smaller number of real assets is that of futures contracts. A *futures contract* for good ℓ is a contract which promises to deliver one unit of good ℓ in each state of nature s at date $t = 1$. In this case, $a^{\ell}(s) = e^{\ell}$, $\forall s = 1, \dots, S$ and $v^{\ell}(p,s) = p_{\ell}(s)$. If there is a futures contract for each good, then $J = L$ and

$$V(p) = \begin{pmatrix} p_1(1) & p_2(1) & \dots & p_L(1) \\ p_1(2) & p_2(2) & \dots & p_L(2) \\ \vdots & \vdots & \dots & \vdots \\ p_1(S) & p_2(S) & \dots & p_L(S) \end{pmatrix}.$$

Assume now that each agent holds initial ownership shares of firms. Let there be J firms and suppose that the production decision y^j of firm j has already been made. The *equity* of firm j is a real asset $(y^j(s))_{s=1}^S$ and we can think of a portfolio as a net trade of shares. $v^j(p,s) = p(s) \cdot y^j(s)$ and

$$V(p) = (p(s) \cdot y^j(s))_{\substack{s=1,\ldots,S \\ j=1,\ldots,J}}$$

Definition 3.2 *Let a consumption bundle $e \in \mathbb{R}^L$ be chosen as a unit of "numéraire". Numeraire assets are a particular case of real assets where $a^j(s) = r^j(s)e$ (with $r^j(s) \in \mathbb{R}$) denotes the random return of asset j across the states of the world at $t = 1$.*

If all assets are numeraire assets, e and the $(S \times J)$-matrix $R = (r^j(s))_{\substack{s=1,\ldots,S \\ j=1,\ldots,J}}$ summarize the *numeraire asset structure*.

$$V(p) = ((p(s) \cdot e)r^j(s))_{\substack{s=1,\ldots,S \\ j=1,\ldots,J}} = \begin{pmatrix} p(1) \cdot e & 0 & \ldots & 0 \\ 0 & p(2) \cdot e & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & p(S) \cdot e \end{pmatrix} R.$$

Definition 3.3 *A nominal (or purely financial) asset structure is described by a $(S \times J)$-matrix $R = (r^j(s))_{\substack{s=1,\ldots,S \\ j=1,\ldots,J}}$, where the vector $(r^j(s))_{s=1}^S \in \mathbb{R}^S$ describes the random return of asset j, directly denominated in units of account.*

In this case, the return matrix $V(p)$ does not depend on p. More precisely, $V(p) \equiv R$ and R summarizes the *nominal asset structure*.

Bonds and financial futures are typical examples of nominal assets. Same examples hold for numeraire assets, excepted for that in the numeraire case, financial contracts are " indexed" (to the value of the numeraire).

4   Bounded portfolios. Existence of equilibrium

We now assume that the portfolio sets of the agents are bounded below. Such an assumption may correspond to (possibly individual) institutional restrictions on asset trading, independent of current prices. More precisely, we replace in this section Assumption F.1 by:

F'.1 For every i, Z^i is a closed, convex, bounded below subset of $\mathbb{R}^J$, with $0 \in \text{int} Z^i$.

The purpose of this section is to prove under Assumptions C.1 – C.4, F'.1, F.2 on a financial economy $\mathcal{E}$ the existence of a financial equilibrium. The strategy

of the proof is close to the one used in a simultaneous optimization approach for a standard Arrow-Debreu economy.

4.1 Truncating the economy

Recall that $\widehat{X} = \{(x^i) \in \prod_{i=1}^I X^i \,|\, \sum_{i=1}^I (x^i - \omega^i) = 0\}$. Let us define $\widehat{Z} := \{(z^i) \in \prod_{i=1}^I Z^i \,|\, \sum_{i=1}^I z^i = 0\}$. It follows from the previous assumptions that $\widehat{X}$ and $\widehat{Z}$ are compact. The same is true for each $\widehat{X}^i$, the projection of $\widehat{X}$ on X^i, for $\widehat{Z}^i$, the projection of $\widehat{Z}$ on Z^i, and also for each $\widehat{X}^i(s) = \{x^i(s) \in \mathbb{R}^L \,|\, x^i \in \widehat{X}^i\}$. We can choose a real number r such that $\forall i$, $\forall s$, $\widehat{X}^i(s) \subset B_o(0, r)$ (where $B_o(0, r)$ is the open ball in $\mathbb{R}^L$ with center 0 and radius r) and $\widehat{Z}^i \subset B_o(0, r)$ (where $B_o(0, r)$ is now an open ball in $\mathbb{R}^J$).

Let, in each case, $\overline{B_o(0, r)}$ denote the closure of $B_o(0, r)$ and let us set: $\widetilde{X}^i = X^i \cap \prod_{s=0}^S \overline{B_o(0, r)}$, $\widetilde{Z}^i = Z^i \cap \overline{B_o(0, r)}$.

To $\mathcal{E}$, we associate the economy

$$\widetilde{\mathcal{E}} = ((\widetilde{X}^i, \widetilde{P}^i, \omega^i, \widetilde{Z}^i)_{i=1}^I, V)$$

where each $\widetilde{P}^i$ is deduced from P^i in an obvious manner.

4.2 Existence of financial equilibrium in the compact economy $\widetilde{\mathcal{E}}$

Proposition 4.1 *Under the assumptions C.1 – C.4, F'1, F.2 on $\mathcal{E}$, $\widetilde{\mathcal{E}}$ has a financial equilibrium $((\bar{x}^i, \bar{z}^i)_{i=1}^I, (\bar{p}, \bar{q}))$ with for every s, $\bar{p}(s) \neq 0$.*

Proof. The proof is done in several steps. Let $\Pi = \{(p, q) \in \mathbb{R}^{L(1+S)} \times \mathbb{R}^J \,|\, \forall s \, \|p(s)\| \leq 1, \|q\| \leq 1\}$ denote a set of admissible prices for commodities and assets. Given $(p, q) \in \Pi$, $(x, z) \in \widetilde{X} \times \widetilde{Z}$, following ideas originating from Bergstrom (1976), we define the "modified" budget sets of i

$$B'^i(p, q) = \{(x^i, z^i) \in \widetilde{X}^i \times \widetilde{Z}^i \,|\, p \,\square\, (x^i - \omega^i) \leq W(p, q, x)z^i + \gamma(p, q)\}$$

$$B''^i(p, q) = \{(x^i, z^i) \in \widetilde{X}^i \times \widetilde{Z}^i \,|\, p \,\square\, (x^i - \omega^i) \ll W(p, q, x)z^i + \gamma(p, q)\}$$

where $\gamma(p, q) \in \mathbb{R}^{1+S}$ is defined by

$$\gamma_0(p, q) = 1 - \min\{1, \|p(0)\| + \|q\|\}$$

$$\gamma_s(p, q) = 1 - \|p(s)\|, \quad s = 1, \ldots, S.$$

Claim 4.1 $\forall (p, q) \in \Pi$, $B''^i(p, q) \neq 0$.

Indeed, if $(p, q) \in \Pi$, let x^i be such that $p \,\square\, (x^i - \omega^i) \leq 0$ with a strict inequality at state $s \in \{0, \dots, S\}$ when $p(s) \neq 0$ (recall that $\omega^i \in \text{int } X^i$). Now, if either $p(0) \neq 0$ or $[p(0) = 0$ and $q = 0]$, then $(x^i, 0) \in B''^i(p, q)$. If $p(0) = 0$ and $q \neq 0$, recalling that $0 \in \text{int } Z^i$, we can choose $z \in Z^i$ such that $q \cdot z < 0$, $v(p, s) \cdot z > p(s) \cdot (x^i(s) - \omega^i(s)) - \gamma_s(p, q)$, $s = 1, \dots, S$, and $(x^i, z) \in B''^i(p, q)$. $\square$

Claim 4.2 $\forall i$, $\forall (p, q) \in \Pi$, B''^i is lower semicontinuous on Π.

Indeed, it follows from the convexity and the nonemptiness of $B''^i(p, q)$ that $\forall (p, q) \in \Pi$, $B'^i(p, q) = \overline{B''^i(p, q)}$. Then the claim follows from the fact that B''^i has obviously an open graph. $\square$

Claim 4.3 $\forall i$, B'^i is upper semicontinuous with closed convex values.

Indeed, B'^i has a closed graph with convex values in the compact convex set $\widetilde{X}^i \times \widetilde{Z}^i$. $\square$

We now introduce an additional agent and, as in Gale and Mas-Colell (1975-1979), we set the following reaction correspondences defined on $(\Pi \times \prod_{i=1}^{I}(\widetilde{X}^i \times \widetilde{Z}^i))$.

$$\psi^i(p, q, x, z) = \begin{cases} B'^i(p, q) & \text{if } (x^i, z^i) \notin B'^i(p, q) \\ B''^i(p, q) \cap (\widetilde{P}^i(x) \times \widetilde{Z}^i) & \text{if } (x^i, z^i) \in B'^i(p, q) \end{cases}$$

$$\psi^0(p, q, x, z) = \{(p', q') \in \Pi \mid (p' - p) \cdot (\sum_{i=1}^{I}(x^i - \omega^i)) + (q' - q) \cdot \sum_{i=1}^{I} z^i > 0\}.$$

Claim 4.4 $\forall i = 0, 1, \dots, I$, ψ^i is lower semicontinuous.

Indeed, ψ^0 has an open graph. If $i \neq 0$, it follows from the lower semicontinuity of B'^i together with the fact that B'^i is upper semicontinuous with nonempty closed values, from the lower semicontinuity of $\widetilde{P}^i$, hence the lower semicontinuity of $(p, q, x, z) \to B''^i(p, q) \cap (\widetilde{P}^i(x) \times \widetilde{Z}^i)$ and from the remark that $B''^i(p, q) \cap (\widetilde{P}^i(x) \times \widetilde{Z}^i) \subset B'^i(p, q)$. $\square$

Remark that, by construction, $(p, q) \notin \psi^0(p, q, x, z)$ and that, since $x^i \notin P^i(x)$, $(x^i, z^i) \notin \psi^i(p, q, x, z)$, $i = 1, \dots, I$. It then follows from the Gale and Mas-Colell fixed point theorem: there exists $(\bar{p}, \bar{q}, \bar{x}, \bar{z}) \in (\Pi \times \prod_{i=1}^{I}(\widetilde{X}^i \times \widetilde{Z}^i))$ such that

$$(\bar{x}^i, \bar{z}^i) \in B'^i(\bar{p}, \bar{q}) \text{ and } B''^i(\bar{p}, \bar{q}) \cap (\widetilde{P}^i(\bar{x}) \times \widetilde{Z}^i) \neq \emptyset, \; i = 1, \dots, I \qquad (1)$$

$$p \cdot \sum_{i=1}^{I}(\bar{x}^i - \omega^i) + q \cdot \sum_{i=1}^{I} \bar{z}^i \leq \bar{p} \cdot \sum_{i=1}^{I}(\bar{x}^i - \omega^i) + \bar{q} \cdot \sum_{i=1}^{I} \bar{z}^i, \; \forall (p, q) \in \Pi. \qquad (2)$$

Claim 4.5 $\sum_{i=1}^{I} \bar{z}^i = 0$.

Indeed if not, from $q \cdot \sum_{i=1}^{I} \bar{z}^i \leq \bar{q} \cdot \sum_{i=1}^{I} \bar{z}^i$, $\forall q, \|q\| \leq 1$, it follows $\|\bar{q}\| = 1$, $\gamma_0(\bar{p}, \bar{q}) = 0$, $\bar{q} \cdot \sum_{i=1}^{I} \bar{z}^i > 0$. From budget constraints, $\bar{p}(0) \cdot (\bar{x}^i(0) - \omega^i(0)) + \bar{q} \cdot \bar{z}^i \leq 0$, $i = 1, \ldots, I$. Summing on i,

$$\bar{p}(0) \cdot \sum_{i=1}^{I} (\bar{x}^i(0) - \omega^i(0)) + \bar{q} \cdot \sum_{i=1}^{I} \bar{z}^i \leq 0.$$

But, taking $p(0) = 0$ and $p(s) = \bar{p}(s)$, $s = 1, \ldots, S$, we have also from (2)

$$\bar{q} \cdot \sum_{i=1}^{I} \bar{z}^i \leq \bar{p}(0) \cdot \sum_{i=1}^{I} (\bar{x}^i(0) - \omega^i(0)) + \bar{q} \cdot \sum_{i=1}^{I} \bar{z}^i$$

which, with $\bar{q} \cdot \sum_{i=1}^{I} \bar{z}^i > 0$, implies:

$$0 < \bar{p}(0) \cdot \sum_{i=1}^{I} (\bar{x}^i(0) - \omega^i(0)) + \bar{q} \cdot \sum_{i=1}^{I} \bar{z}^i,$$

a contradiction. $\qquad\qquad\square$

Claim 4.6 $\sum_{i=1}^{I} \bar{x}^i = \sum_{i=1}^{I} \omega^i$.

Indeed, if not, for some $s = 0, 1, \cdots, S$, we deduce from (2): $\|\bar{p}(s)\| = 1$, $\gamma_s(\bar{p}, \bar{q}) = 0$, $\bar{p}(s) \cdot \sum_{i=1}^{I} (\bar{x}^i(s) - \omega^i(s)) > 0$. From budget constraints at s, we have: $\bar{p}(s) \cdot (\bar{x}^i(s) - \omega^i(s)) \leq W(\bar{p}, \bar{q})_s \cdot \bar{z}^i$, $i = 1, \ldots I$, where $W(\bar{p}, \bar{q})_s$ denotes the row s of the matrix $W(\bar{p}, \bar{q})$. Summing on i and due to $\sum_{i=1}^{I} \bar{z}^i = 0$, we get $\bar{p}(s) \cdot \sum_{i=1}^{I} (\bar{x}^i(s) - \omega^i(s)) \leq 0$, a contradiction. $\qquad\square$

Claim 4.7 *Each $\bar{x}^i$ with $\bar{z}^i$ is optimal in $B'^i(\bar{p}, \bar{q})$.*

This follows from the openness in $\widetilde{X}^i$ of values of $\widetilde{P}^i$ and from the nonemptiness of $B''^i(\bar{p}, \bar{q})$. $\qquad\qquad\square$

Claim 4.8 $\gamma(\bar{p}, \bar{q}) = 0$ *and* $\forall s \neq 0$, $\|\bar{p}(s)\| = 1$;

Indeed, since $\bar{x} \in \widehat{X}$, by Assumption C.4 and since $[y^i \in P^i(x)$ and $0 < \lambda \leq 1]$ imply $[x^i + \lambda(y^i - x^i) \in P^i(x)]$ (Assumption C.2), we have local nonsatiation at each $\bar{x}^i$ for each date-event pair. From this, it follows that

$$\bar{p} \,\square\, (\bar{x}^i - \omega^i) = W(\bar{p}, \bar{q})\bar{z}^i + \gamma(\bar{p}, \bar{q}), i = 1, \ldots, I.$$

Summing on i, we get $I\gamma(\bar{p}, \bar{q}) = 0$, i.e., $\gamma(\bar{p}, \bar{q}) = 0$. $\qquad\qquad\square$

Claim 4.9 $(\bar{p}, \bar{q}, \bar{x}, \bar{z})$ *is an equilibrium of $\mathcal{E}$ and $\bar{p}(0) \neq 0$.*

The first assertion was proved in Claims 4.5 – 4.8. The last assertion follows from Assumption C.4, i.e., from the local nonsatiation of preferences for each $\bar{x}^i$ at state $s = 0$. $\square$

Remark 4.1 *At this stage, it is important to emphasize that the equilibrium asset price vector $\bar{q}$ may be equal to 0.*

4.3   Existence of financial equilibrium in the initial economy

Proposition 4.2 *Under the assumptions C.1 – C.4, F'.1, F.2, the economy $\mathcal{E}$ has a financial equilibrium with commodity prices satisfying for every s, $\bar{p}(s) \neq 0$.*

Proof. Let $(\bar{p}, \bar{q}, (\bar{x}^i, \bar{z}^i)_{i=1}^I)$ be a financial equilibrium of $\widetilde{\mathcal{E}}$ obtained from Proposition 4.1. The allocation $(\bar{x}^i, \bar{z}^i)_{i=1}^I$ satisfies Condition *(ii)* of Definition 2.2. On the other hand, $\bar{p} \,\square\, (\bar{x}^i - \omega^i) = W(\bar{p}, \bar{q})\bar{z}^i$, $i = 1, \dots, I$. Let us prove that each $\bar{x}^i$ is optimal with $\bar{z}^i$ in the budget set

$$B_{FM}^i(\bar{p}, \bar{q}) = \{x^i \in X^i \mid \exists z^i \in Z^i, \bar{p} \,\square\, (x^i - \omega^i) \leq W(\bar{p}, \bar{q})z^i\}.$$

If not, let for some i, $(x^i, z^i) \in X^i \times Z^i$ be such that $\bar{p} \,\square\, (x^i - \omega^i) \leq W(\bar{p}, \bar{q})z^i\}$ and $x^i \in P^i(\bar{x})$. Recall that $\forall i, \forall s, \bar{x}^i(s) \in$ belongs to $B_o(0, r)$, the open ball in $\mathbb{R}^L$, while $\bar{z}^i$ belongs to $B_o(0, r)$, the open ball in $\mathbb{R}^J$. Then, it is easy to find $\lambda : 0 < \lambda \leq 1$ such that $(\bar{x}^i, \bar{z}^i) + \lambda((x^i - \bar{x}^i), (z^i - \bar{z}^i)) \in \widetilde{X}^i \times \widetilde{Z}^i$ and satisfies the same budget constraints. As we have also: $\bar{x}^i + \lambda(x^i - \bar{x}^i) \in P^i(\bar{x})$, we have got a contradiction with the optimality of $\bar{x}^i$ in the budget set of i in the economy $\widetilde{\mathcal{E}}$. $\square$

4.4   Existence of a pure spot market equilibrium

Proposition 4.3 *Under Conditions C.1 – C.4, a pure spot market exchange economy $\mathcal{E} = ((X^i, P^i, \omega^i)_{i=1}^I$, as defined in Section 2, has an equilibrium.*

Proof. Recall the definition of the budget sets:

$$B_{SM}^i(p) = \{x^i \in X^i \mid \forall s = 0, 1, \dots, S, \ p(s) \cdot x^i(s) \leq p(s) \cdot \omega^i(s)\}.$$

It is easy to see that the same proof as previously is working. The only difference is in defining $\gamma(0) = 1 - \|p(0)\|$, for getting the modified budget sets. Let $\Pi = \{p \in \mathbb{R}^{L(1+S)} \mid \forall s \ \|p(s)\| \leq 1\}$ be the set of admissible commodity prices. The proof of the nonemptiness of $B^{''i}(p)$, $\forall p \in \Pi$ is even simpler. The rest of the proof is the same. $\square$

5 Existence of equilibrium with numeraire assets

We begin now to study the financial equilibrium of an economy without bounds on short-selling of assets. In this section, we assume that $e \in \mathbb{R}^L$ is a "numéraire" and that a $(S \times J)$-matrix $R = (r^j(s))_{\substack{s=1,\dots,S \\ j=1,\dots,J}}$ describes the random returns of assets, denominated in units of the "numéraire", across the states of the world at time $t = 1$, so that the complete description of the economy is

$$\mathcal{E} = ((X^i, P^i, \omega^i, Z^i)_{i=1}^I, R).$$

Then, for every admissible price,

$$V(p) = ((p(s) \cdot e) r^j(s))_{\substack{s=1,\dots,S \\ j=1,\dots,J}} = \begin{pmatrix} p(1) \cdot e & 0 & \dots & 0 \\ 0 & p(2) \cdot e & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & p(S) \cdot e \end{pmatrix} R.$$

If $p(s) \cdot e > 0$, $\forall s = 1, \dots, S$, then rank $V(p) = \text{rank } R$. In particular, if rank $R = J$ ($J \leq S$), then rank $V(p) = J$. In the following, we will assume that rank $R = J$ (a costless assumption, since it is enough to remove redundant assets) and we will look for equilibrium commodity prices $\bar{p}$ such that rank $V(\bar{p}) = \text{rank } R = J$. With this remark in mind, we set on $\mathcal{E}$ the following assumptions:

On the consumption side,

C.1 For every i, X^i is a closed, convex and bounded below subset of $\mathbb{R}^{L(1+S)}$

C'.2 The correspondences $P^i : X \to X^i$ have an open graph with convex values. Moreover, $x^i \notin P^i(x)$ and the preference correspondences satisfy an additional convexity property: $[y^i \in P^i(x)$ and $0 < \lambda \leq 1]$ imply $[x^i + \lambda(y^i - x^i) \in P^i(x)]$

C.3 (Survival assumption) For every i, $\omega^i \in \text{int} X^i$ (the interior of X^i)

C'.4 (desirability of numeraire at every date-event pair and at every component of an attainable consumption allocation) For every $x \in \widehat{X}$, for every i, for every $s = 0, 1, \dots, S$, there exists $\lambda > 0$ such that $x^i + \lambda e^s \in P^i(x)$, where e^s is defined by $e^s(s) = e$, $e^s(s') = 0$, $s' \neq s$.

On the financial side,

F".1 For every i, $Z^i = \mathbb{R}^J$

F'.2 rank $R = J$.

Remark 5.1 *Assumption C'2 reinforces Assumption C. 2 set in Section 2 and used in Section 4 for the model with bounded portfolios. This strengthening is dependent on the technique of proof that we use below. The desirability of numeraire*

(Assumption C'.4), together with C'.2, implies local nonsatiation at every component of an attainable consumption allocation and for each date-event pair, as postulated in C.4. Without changing the result to be stated and proved below, it would be possible to make separately this last assumption (i.e. Assumption C. 4) and to assume the desirability of numeraire only for one consumer in each state of the world.

We will prove the following theorem:

Theorem 5.1 *Under C.1, C'.2, C.3, C'.4, F".1, F'.2, the economy $\mathcal{E}$ has a financial equilibrium $(\bar{p}, \bar{q}, \bar{x}, \bar{z})$ with $\bar{p}(s) \cdot e > 0$, $s = 0, 1, \ldots, S$.*

Proof. To each $\nu \in \mathbb{N}$, let us associate the financial economy

$$\mathcal{E}^\nu = ((X^i, P^i, \omega^i, Z^{i\nu})_{i=1}^I, R)$$

differing from $\mathcal{E} = ((X^i, P^i, \omega^i, Z^i)_{i=1}^I, R)$ by the definition of the portfolio sets $Z^{i\nu} = \left\{ z \in \mathbb{R}^J \mid z_j \geq -\nu, j = 1, \ldots, J \right\}$. It follows from Proposition 4.2 in Section 4 that $\mathcal{E}^\nu$ has an equilibrium $(\bar{p}^\nu, \bar{q}^\nu, (\bar{x}^{i\nu}, z^{i\nu})_{i=1}^I)$ with $\|\bar{p}^\nu(s)\| = 1$, $0 \neq \|\bar{p}^\nu(0)\| \leq 1$, $\|\bar{q}^\nu\| \leq 1$. It satisfies:

$$\sum_{i=1}^I \bar{x}^{i\nu} = \sum_{i=1}^I \omega^i, \ \ \sum_{i=1}^I \bar{z}^{i\nu} = 0 \tag{3}$$

$$\forall i = 1, \ldots I, \ \bar{p}^\nu \square (\bar{x}^{i\nu} - \omega^i) = W(\bar{p}^\nu, \bar{q}^\nu)\bar{z}^{i\nu} \text{ and } P^i(\bar{x}^\nu) \cap B_{FM}^i(\bar{p}^\nu, \bar{q}^\nu) = \emptyset. \tag{4}$$

It follows from C'4 that $\forall s = 0, 1, \ldots S$, $\bar{p}^\nu(s) \cdot e > 0$. In the following lemma, we prove a stronger result.

Lemma 5.1 *$\forall s = 1, \ldots, S$, there exists $\varepsilon > 0$ such that $\forall \nu \in \mathbb{N}$, $p^\nu(s) \cdot e \geq \varepsilon$.*

Proof. Let $S(0, 1)$ denote the closed sphere in $\mathbb{R}^L$ with center 0 and radius 1. For each $s = 1, \ldots, S$, let us define

$$P_s = \left\{ p(s) \in S(0, 1) \ \middle| \ \begin{array}{l} \text{for some } i_s, \text{ some } x \in \widehat{X}, \ x'^{i_s} \in P^{i_s}(x) \text{ with} \\ x'^{i_s}(s') = x^{i_s}(s') \ \forall s' \neq s \ \Rightarrow \\ p(s) \cdot x'^{i_s}(s) \geq p(s) \cdot x^{i_s}(s) \geq p(s) \cdot \omega^i(s) \end{array} \right\}$$

Claim 5.1 *$\forall \nu$, $\bar{p}^\nu(s) \in P_s$.*

To see this, choose i_s such that $v(\bar{p}^\nu, s) \cdot \bar{z}^{i\nu} \geq 0$ together with $\bar{x}^\nu = (\bar{x}^{i\nu})_{i=1}^I$.
$\square$

Claim 5.2 *P_s is a closed (hence compact) subset of $S(0, 1)$.*

Indeed, consider a sequence $(p^k(s))_k$ of elements of P_s converging to $p(s)$. Without loss of generality, one can assume that for some i_s and for each k, there exists x^k such that $x'^{i_s} \in P^{i_s}(x^k)$, $x'^{i_s}(s') = x^{i_s k}(s')$ $\forall s' \neq s$ imply $p^k(s) \cdot x'^{i_s}(s) \geq p^k(s) \cdot x^{i_s k}(s) \geq p^k(s) \cdot \omega^i(s)$. Without loss of generality, one can also assume that $x^k \to x \in \widehat{X}$ which implies $p(s) \cdot x^{i_s}(s) \geq p(s) \cdot \omega^i(s)$. Assume now $x'^{i_s} \in P^{i_s}(x)$ with $x'^{i_s}(s') = x^{i_s}(s')$ $\forall s' \neq s$. Let us define $x'^{i_s k}$ by $x'^{i_s k}(s) = x'^{i_s}(s)$ and $x'^{i_s k}(s') = x^{i_s k}(s')$, $s' \neq s$. Obviously, $x'^{i_s k} \to x'^{i_s}$ and from Assumption C'.2, we deduce successively $x'^{i_s k} \in P^{i_s}(x^k)$ for k large enough, and $p^k(s) \cdot x'^{i_s k}(s) \geq p^k(s) \cdot x^{i_s k}(s)$. Passing to limit, we get: $p(s) \cdot x'^{i_s}(s) \geq p(s) \cdot x^{i_s}(s)$.

$\square$

Claim 5.3 $\forall p(s) \in P_s$, $p(s) \cdot e > 0$.

Let i_s and $x \in \widehat{X}$ as in the definition of P_s. In view of Assumption C. 3 (Survival Assumption), there exists $a^{i_s} \in X^{i_s}$ such that $p(s) \cdot a^{i_s}(s) < p(s) \cdot \omega^{i_s} \leq p(s) \cdot x^{i_s}(s)$. Since $x^{i_s} + \lambda e^s \in P^{i_s}(x)$, in view of Assumption C'.2, it is possible to find x'^{i_s} satisfying simultaneously: $x'^{i_s}(s') = x^{i_s}(s')$ $\forall s' \neq s$, $p(s) \cdot x'^{i_s}(s) < p(s) \cdot x^{i_s}(s)$, $x'^{i_s} + \lambda e^s \in P^{i_s} x)$. It then follows that $p(s) \cdot \lambda e^s(s) \geq p(s) \cdot (x^{i_s}(s) - x'^{i_s}(s)) > 0$.

$\square$

Claim 5.4 *There exists $\varepsilon > 0$ such that $\forall p(s) \in P_s$, $p(s) \cdot e \geq \varepsilon$.*

For all $p(s) \in P_s$, let us define ε_p such that $p(s) \cdot e > \varepsilon_p > 0$. A compactness argument

ends the proof. Since $P_s \subset \cup_p \{p(s) \in B(0,1) \mid p(s) \cdot e > \varepsilon_p\}$, there exist $\varepsilon_{p_1}, \ldots, \varepsilon_{p_r}$ such that $P_s \subset \cup_{k=1}^r \{p(s) \in B(0,1) \mid p(s) \cdot e > \varepsilon_{p_k}\}$. If $\varepsilon := \min\{\varepsilon_{p_1}, \ldots, \varepsilon_{p_r}\}$, $\varepsilon > 0$ and $\forall p(s) \in P_s$, $p(s) \cdot e > \varepsilon$.

$\square$

Now, claims 5.1 and 5.4 prove the lemma.

$\square$

End of the proof of Theorem 5.1. Without loss of generality, we can assume $\bar{p}^\nu \to \bar{p}$, $\bar{q}^\nu \to \bar{q}$, $\bar{x}^{i\nu} \to \bar{x}^i$, $i = 1, \ldots, I$, $W(\bar{p}^\nu, \bar{q}^\nu) \to W(\bar{p}, \bar{q})$. From (3), we deduce $\sum_{i=1}^I \bar{x}^i = \sum_{i=1}^I \omega^i$. From (4), we deduce $\bar{p} \square (\bar{x}^i - \omega^i) = \lim_\nu W(\bar{p}^\nu, \bar{q}^\nu)$. Note that we have for each i:

$$\left(\bar{p}^\nu(s) \cdot (\bar{x}^{i\nu} - \omega^i(s))\right)_{s=1}^S = V(\bar{p}^\nu)\bar{z}^{i\nu} = \begin{pmatrix} \bar{p}^\nu(1) \cdot e & & 0 \\ & \ddots & \\ 0 & & \bar{p}^\nu(s) \cdot e \end{pmatrix} R\bar{z}^{i\nu}$$

with rank $V(\bar{p}^\nu) = \text{rank } R = J$. It follows from Lemma 5.1 that $\bar{p}(s) \cdot e > 0$ $\forall s$. Hence we have also: rank $V(\bar{p}) = \text{rank } R = J$, so that $\bar{z}^{i\nu} \to \bar{z}^i$. It then follows from (3) that $\sum_{i=1}^I \bar{z}^i = 0$.

It remains to check the optimality of each $\bar{x}^i$ in $B_{FM}^i(\bar{p}, \bar{q}) = \{x^i \in X^i \mid \exists z^i \in \mathbb{R}^J, \ \bar{p} \,\square\, (x^i - \omega^i) \leq W(\bar{p}, \bar{q}) z^i\}$. Assume on the contrary that $x^i \in P^i(\bar{x}) \cap B_{FM}^i(\bar{p}, \bar{q})$. In view of C.3, without loss of generality, one can assume:

$$\bar{p} \,\square\, (x^i - \omega^i) \ll W(\bar{p}, \bar{q}) z^i.$$

For ν large enough, $z^i \geq -\nu$, $x^i \in P^i(\bar{x}^{i\nu})$ and

$$\bar{p}^\nu \,\square\, (x^i - \omega^i) \ll W(\bar{p}^\nu, \bar{q}^\nu) z^i$$

which contradicts the optimality of $\bar{x}^{i\nu}$ in $B_{FM}^i(\bar{p}^\nu, \bar{q}^\nu)$. $\qquad\square$

6 No-arbitrage condition and completeness of the markets

Let us introduce this section with the very simple following proposition.

Proposition 6.1 *Let $\mathcal{E} = ((X^i, P^i, \omega^i, Z^i)_{i=1}^I, V)$ be an economy with financial markets satisfying C.2 and C.4 and let $((\bar{x}^i, \bar{z}^i)_{i=1}^I, (\bar{p}, \bar{q}))$ be an equilibrium of $\mathcal{E}$. Then it does not exist $(z, \lambda) \in \mathbb{R}^J \times \mathbb{R}_{++}$ such that $W(\bar{p}, \bar{q}) \, z > 0$ with for some i, $z'^i = \bar{z}^i + \lambda \, z \in Z^i$.*

Proof. If not, since $\bar{p} \,\square\, \bar{x}^i = \bar{p} \,\square\, \omega^i + W(\bar{p}, \bar{q}) \bar{z}^i$, $\bar{x}^i$ (together with $\bar{z}^i + \lambda \, z$) belongs to $B_{FM}^i(\bar{p}, \bar{q})$ with, for some s, $\bar{p}(s) \cdot (\bar{x}^i(s) - \omega^i(s)) < (W(\bar{p}, \bar{q}))_s \cdot z'^i$. In view of C.2 and C.4, this strict inequality in the budget constraint at s contradicts the optimality of $\bar{x}^i$ in $B_{FM}^i(\bar{p}, \bar{q})$. $\qquad\square$

The economic meaning of this result is that, at equilibrium, the financial market must not offer arbitrage opportunities at any agent. In view of studying the consequences of absence of arbitrage for the financial market at equilibrium, we now assume, as in the traditional approach in Finance Theory, that there is no restriction on the portfolios available to the agents, i.e. that $\forall i = 1, \ldots I$, $Z^i = \mathbb{R}^J$.

Definition 6.1 *A financial market (p, q, V) is said arbitrage-free if there is no portfolio $z \in \mathbb{R}^J$ such that $W(p, q) \, z > 0$.*

Note that in the nominal case, this notion does not depend on the commodity prices vector p.

As it is usual, we can invoke the strict separation theorem to characterize the no-arbitrage condition as a linear relationship between asset prices and returns.

Lemma 6.1 *If the market system (p, q, V) does not permit any arbitrage, there exists $\lambda = (\lambda_s)_{s=0,\ldots,S} \gg 0$ such that $\lambda_0 = 1$ and $\lambda_0 q = \sum_{s=1}^{S} \lambda_s v(p, s)$.*

Proof. Let us denote by Δ_S the unit simplex of $\mathbb{R}^{1+S}$

$$\Delta_S = \{w = (w_s)_{s=0,\ldots,S} \mid \forall s, w_s \geq 0 \text{ and } \|w\|_1 = 1\}.$$

It follows from the no-arbitrage condition that $ImW(p, q) \cap \Delta_S = \emptyset$. Since Δ_S is compact, we can apply the strict separation theorem to get $\beta \in \mathbb{R}^{1+S} \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that

$$\sup_{x \in ImW(p,q)} \beta \cdot x \leq \alpha < \inf_{y \in \Delta_S} \beta \cdot y.$$

Since the linear functional is majorized on the linear space $ImW(p, q)$, it is identically equal to 0 on $ImW(p, q)$. Then $\alpha \geq 0$ and it follows from the right inequality that $\beta_s > 0 \ \forall s \in S$. If we denote by λ the vector (β_s/β_0), it satisfies $\forall z \in \mathbb{R}^J$,

$$\lambda \cdot W(p, q) \, z = -\lambda_0 q \cdot z + \sum_{s=1}^{S} \lambda_s v(p, s) \cdot z = \left(-\lambda_0 q + \sum_{s=1}^{S} \lambda_s v(p, s)\right) \cdot z$$

Since this quantity is identically equal to 0 for all $z \in \mathbb{R}^J$, one deduces the conclusion. $\qquad\square$

Moreover, if we multiply by λ_s each inequality defining the budget set $B^i(p, q)$ and if we sum, we get for any $x^i \in B^i_{FM}(p, q)$, $\pi \cdot x^i \leq \pi \cdot \omega^i$, with $\pi = (\lambda_s p(s))_{s=0}^{S}$, that is,

$$B^i_{FM}(p, q) \subset B^i(p, q, \lambda) \stackrel{\text{def}}{=} B^i_{A-D}(\pi).$$

Definition 6.2 *The arbitrage-free market system (p, q, V) is said complete when the rank of the matrix $V(p)$ is equal to S.*

Once again, it is useful to emphasize that except for the case of nominal assets, this notion depends strongly on the commodity prices vector p. The meaning of this definition is that any vector of financial returns $t = (t_s)_{s=1}^{S} \in \mathbb{R}^S$ at time 1 can then be got using at time 0 a convenient portfolio $z \in \mathbb{R}^J$ (obviously it implies $J \geq S$).

Proposition 6.2 *Let $\lambda = (\lambda_s)_{s=0,\ldots,S} \gg 0$ associated with the arbitrage-free financial market (p, q, V). If (p, q, V) is complete, then the inequality $\pi \cdot x^i \leq \pi \cdot \omega^i$ (with $\pi = (\lambda_s p(s))_{s \in S}$) implies the existence of some portfolio z^i such that $(x^i, z^i) \in B^i_{FM}(p, q)$. In this case (and only in this case), λ is unique (up to a normalization) and it is equivalent for each consumer to optimize in the budget set $B^i_{FM}(p, q)$ previously defined or in the Arrow-Debreu budget set $B^i(p, q, \lambda) = B^i_{A-D}(\pi) = \{x^i \in X^i \mid \pi \cdot x^i \leq \pi \cdot \omega^i\}$.*

Proof. If $x^i \in B^i(p, q, \lambda)$, there exists some vector $x'^i \in \mathbb{R}^{L(1+S)}$ satisfying $\pi \cdot (x'^i - \omega^i) = 0$ and for every good ℓ, for each state s, $x'^i_\ell(s) \geq x^i_\ell(s)$ if $p_\ell(s) \geq 0$ and $x'^i_\ell(s) \leq x^i_\ell(s)$ if $p_\ell(s) \leq 0$. This can be equivalently rewritten as $\lambda \cdot (p \, \square \, (x'^i - \omega^i)) = 0$. Since $\text{rank } W(p, q) = \text{rank } V(p, q) = S$, one has the existence of some portfolio z^i such that $p \, \square \, (x'^i - \omega^i) = W(p, q)(z^i)$, and consequently x'^i associated with z^i satisfies all the budget constraints. Finally, by positivity of $p(s) \cdot (x'^i(s) - x^i(s))$, one deduces that x^i associated with z^i satisfies all the budget constraints, which means $x^i \in B^i_{FM}(p, q)$. Recalling that $Im W(p, q) \subset \lambda^\perp$, it follows from the fact that λ is a non zero-vector that $\text{rank } W(p, q) = \text{rank } V(p) = S$ if and only if $Im W(p, q) = \lambda^\perp$ (and if and only if λ is unique up to a constant factor). $\qquad \square$

From the previous proposition, it follows that if the no-arbitrage market system (p, q, V) is complete, the coefficient λ_s may be unambiguously interpreted as the present value of a unit of account available at node s. The vector $\lambda = (\lambda_s)$ is called a vector of strictly positive node (present value) prices.

In view of Lemma 6.1, λ_s is also the price of a portfolio with financial returns equal to one unit of account in state s and zero elsewhere. Note that, in view of completeness of the market system, such a portfolio z^s (called "Arrow-Debreu security") exists for every s and is unambiguously priced by no-arbitrage.

To end with the interpretation of node prices when the no-arbitrage market system is complete, let us call, as in Duffie (1992), *actualization factor* the quantity $\widetilde{\lambda}_0 \overset{\text{def}}{=} \lambda_1 + \ldots + \lambda_S$. In view of completeness of the market system, let us consider the non-risky portfolio with financial returns equal to one unit of account in each state of nature ; $\widetilde{\lambda}_0$ is the value of this portfolio. In other words, $\widetilde{\lambda}_0$ is the price to pay today to get with certainty one unit of account tomorrow. Then the vector $\widetilde{\lambda} \overset{\text{def}}{=} (\lambda_s/\widetilde{\lambda}_0)_{1 \leq s \leq S}$ can be interpreted as a vector of probability on the future. Since

$$\frac{q}{\widetilde{\lambda}_0} = \sum_{\sigma=1}^{S} \widetilde{\lambda}_\sigma v(p, \sigma) = E_{\widetilde{\lambda}}(v(p, \cdot)),$$

it appears that the normalized price of a portfolio can be thought of as the expected value of the future income it will give, calculated under a special "risk-neutral" probability. This property is very used in Finance since the article of Harrison and Kreps (1979).

When the no-arbitrage market system is incomplete, one has the following proposition due to Cass (1984).

Proposition 6.3 *Let $(\overline{x}, \overline{z}) = ((\overline{x}^i, \overline{z}^i))_{i=1}^I$ be an attainable allocation of the economy $\mathcal{E}$ and $(\overline{p}, \overline{q})$ be a commodity/asset price system. Let us assume that $(\overline{p}, \overline{q}, V)$ is arbitrage-free and that $\lambda = (\lambda_s)_{s=0}^S$ is some associated system of node prices. Under the assumption C.4, the assertion (i) implies the assertion (ii) :*

(i) $\bar{x}^1$ is optimal for agent 1 in the budget set $B^1(\bar{p}, \bar{q}, \lambda)$, and for $i = 2, \ldots, I$, $\bar{x}^i$ (with $\bar{z}^i$) is optimal for agent i in the budget set $B_{FM}^i(\bar{p}, \bar{q})$.

(ii) $((\bar{x}^i, \bar{z}^i)_{i=1}^I, (\bar{p}, \bar{q}))$ is a financial equilibrium of the economy $\mathcal{E}$.

Proof. In view of Assumption C.4, if $\bar{z}^i$ is the portfolio associated with $\bar{x}^i$, one has for every $i \neq 1$, $\bar{p} \,\square\, (\bar{x}^i - \omega^i) = W(\bar{q})\bar{z}^i$. We can use Remark 2.1 to construct the portfolio $\bar{z}^1$ of agent 1. It follows of this construction that $\bar{p} \,\square\, (\bar{x}^1 - \omega^1) = W(\bar{q})\bar{z}^1$. Recalling that $B_{FM}^1(\bar{p}, \bar{q}) \subset B^1(\bar{p}, \bar{q}, \lambda)$, one deduces that $P^1(\bar{x}) \bigcap B_{FM}^1(\bar{p}, \bar{q}) = \emptyset$.

$\square$

This proposition will be useful in equilibrium existence proofs. Its main meaning is that there exist several (possibly personalized) systems of node prices to calculate at equilibrium the present value of a consumption bundle, equivalently to evaluate the present value of an unit of account available in the different states of nature at time $t = 1$. As to the probabilistic interpretation of node prices, let us first remark that, given an incomplete arbitrage-free market structure, a non-risky portfolio may not exist. If such a portfolio exists, there is an essential ambiguity on the node price system to be used in the computation of the actualization factor.

7 Existence of equilibrium with nominal assets

We now come back to Assumptions C.1, C.2, C.3, C.4, as set in Section 2, on the consumption side of an economy $\mathcal{E} = ((X^i, P^i, \omega^i, Z^i)_{i=1}^I, V)$. On the financial side, we assume:

F".1 For every i, $Z^i = \mathbb{R}^J$.

F".2 For every p, $V(p) = R$, where R is a $S \times J$-real matrix such that rank $R = J$.

The theorem to be proved is the following:

Theorem 7.1 *Under the previous assumptions, given $\lambda = (\lambda_s)_{s=0,\ldots,S} \in \mathbb{R}^{(1+S)}$ such that $\lambda_0 = 1$ and $\lambda \gg 0$, $\mathcal{E}$ has an equilibrium $(\bar{p}, \bar{q}, (\bar{x}^i)_{i=1}^I, (\bar{z}^i)_{i=1}^I)$ with $\bar{q} = \sum_{s=1}^S \lambda_s\, r(s)$ (where $r(s)$ denotes the sth row of matrix R).*

Proof. The proof is a variant of the existence proof given in Section 4. As in Section 4, the proof will be done in several steps.

7.1   Using the Cass trick

Let $\lambda \gg 0$ with $\lambda_0 = 1$. We will prove the existence of a *Cass equilibrium* defined as follows:

(i) $\bar{x}^1$ is optimal in $B_{A-D}^1(\bar{\pi}) = \{x^1 \in X^1 \mid \bar{\pi} \cdot x^1 \leq \bar{\pi} \cdot \omega^1\}$, where $\bar{\pi} = (\lambda_s \bar{p}(s))_{s=0}^S$

(ii) $\forall i = 2, \ldots, I$, $\bar{x}^i$ is optimal with $\bar{z}^i$ in $B_{FM}^i(\bar{p}, \bar{q}) = \{x^i \in X^i \mid \exists z^i \in Z^i,\ \bar{p} \square (x^i - \omega^i) \leq \begin{pmatrix} -q \\ R \end{pmatrix} z^i\}$, with $\bar{p} \square (\bar{x}^i - \omega^i) = \begin{pmatrix} -q \\ R \end{pmatrix} \bar{z}^i$

(iii) $\sum_{i=1}^I \bar{x}^i = \sum_{i=1}^I \omega^i$

(iv) $\bar{q} = \sum_{s=1}^S \lambda_s\ r(s)$.

As it was seen in Proposition 6.3, it is then possible to define $\bar{z}^1$ such that $\bar{z}^1 + \sum_{i=2}^I \bar{z}^i = 0$ and $\bar{x}^1$ is optimal with $\bar{z}^1$ in $B_{FM}^1(\bar{p}, \bar{q}) = \{x^1 \in X^1 \mid \exists z^1 \in Z^1,\ \bar{p} \square (x^1 - \omega^1) \leq \begin{pmatrix} -q \\ R \end{pmatrix} z^1\}$.

7.2   Truncating the economy

Recall that $\forall i = 1, \ldots I$, $\forall s = 0, 1, \ldots S$,

$$\widehat{X}^i = \{x^i \in X^i \mid x^i + \sum_{j \neq i} x^j = \sum_{i=1}^I \omega^i \text{ for some } (x^j)_{j \neq i} \in \prod_{j \neq i} X^j\}$$

$$\widehat{X}^i(s) = \{x^i(s) \in \mathbb{R}^L \mid x^i \in \widehat{X}^i\}.$$

Exactly as in Section 4, we can choose a real number r such that $\forall i$, $\forall s$, $\widehat{X}^i(s) \subset B_o(0, r)$ (where $B_o(0, r)$ is the open ball in $\mathbb{R}^L$ with center 0 and radius r). We now define

$$\widetilde{\mathcal{E}} = ((\widetilde{X}^i, \widetilde{P}^i, \omega^i, Z^i)_{i=1}^I, V)$$

where each $\widetilde{X}^i = X^i \cap \prod_{s=0}^S \overline{B_o(0, r)}$ and each $\widetilde{P}^i$ is deduced from P^i in an obvious manner.

7.3   Modifying the budget sets in $\widetilde{\mathcal{E}}$

Let us first restrict ourselves to prices p such that if $\pi = (\lambda_s p(s))_{s=0}^S$, then $\|\pi\| \leq 1$. Let

$$\Pi = \{\pi \in \mathbb{R}^{L(1+S)} \mid \|\pi\| \leq 1\}$$

denote the set of admissible prices. We set

$$\gamma_s(\pi) = \frac{1}{I\lambda_s(1+S)}(1 - \|\pi\|),\ s = 0, 1, \ldots, S$$

$$\gamma = (\gamma_s(\pi))_{s=0}^{S}.$$

With in mind the obvious remark that the function $p \to \pi$ is bijective and bi-continuous, for given $\pi \in \Pi$, we consider for $i = 2, \ldots, S$, the following modified budget sets:

$$B_{\gamma}^{i}(\pi) = \{x^i \in \widetilde{X}^i \mid \exists z^i \in \mathbb{R}^J, \ p \,\square\, (x^i - \omega^i) \leq W(p, \bar{q})z^i + \gamma(\pi)\}$$

$$B_{\gamma}^{'i}(\pi) = \{x^i \in \widetilde{X}^i \mid \exists z^i \in \mathbb{R}^J, \ p \,\square\, (x^i - \omega^i) \ll W(p, \bar{q})z^i + \gamma(\pi)\}$$

$$B_{\gamma}^{''i}(\pi) = \begin{cases} \omega^i & \text{if } B_{\gamma}^{'i}(\pi) = \emptyset \\ B^i(\pi) & \text{if } B_{\gamma}^{'i}(\pi) \neq \emptyset \end{cases}.$$

For $i = 1$, according to what is usually called "the Cass trick", we consider :

$$B_{\gamma}^{1}(\pi) = \{x^1 \in \widetilde{X}^1 \mid \pi \cdot (x^1 - \omega^1) \leq \sum_{s=0}^{S} \lambda_s \gamma_s(\pi)\}$$

$$B_{\gamma}^{'1}(\pi) = \{x^1 \in \widetilde{X}^1 \mid \pi \cdot (x^1 - \omega^1) < \sum_{s=0}^{S} \lambda_s \gamma_s(p)\}.$$

Remark that, in view of Assumption C.3, $\forall \pi \in \Pi$, $B_{\gamma}^{'1}(\pi) \neq \emptyset$. It is not the case for $B_{\gamma}^{'i}(\pi)$, $i = 2, \ldots I$. It is for this reason that we define $B_{\gamma}^{''i}$.

As in Gale and Mas-Colell (1975-1979), for $(\pi, x) \in \Pi \times \prod_{i=1}^{I} \widetilde{X}^i$, we set the following reaction correspondences

$$\psi^i(\pi, x) = \begin{cases} B_{\gamma}^{''i}(\pi) & \text{if } x^i \notin B_{\gamma}^{i}(\pi) \\ B_{\gamma}^{'i}(\pi) \cap P^i(x) & \text{if } x^i \in B_{\gamma}^{i}(\pi) \end{cases} \quad i = 2, \ldots I$$

$$\psi^1(\pi, x) = \begin{cases} B_{\gamma}^{1}(\pi) & \text{if } x^1 \notin B_{\gamma}^{1}(\pi) \\ B_{\gamma}^{'1}(\pi) \cap P^i(x) & \text{if } x^1 \in B_{\gamma}^{1}(\pi) \end{cases}$$

$$\psi^0(\pi, x) = \{\pi' \in \Pi \mid (\pi' \cdot \sum_{i=1}^{I}(x^i - \omega^i) > (\pi \cdot \sum_{i=1}^{I}(x^i - \omega^i))\}.$$

Claim 7.1 $\forall i = 1, \ldots, I$, B_{γ}^{i} *is a closed correspondence.*

Indeed, for $i = 1$, this follows from the definition of the budget set and the continuity of the function $\pi \to \gamma(\pi)$.

For $i \neq 1$, assume that $x^{i\nu}$ (with $z^{i\nu}$) belongs to $B_{\gamma}^{i}(\pi^\nu)$ with $x^{i\nu} \to x^i$ and $\pi^\nu \to \pi$. We first prove that $(z^{i\nu})$ is a bounded sequence. Indeed, if not, without loss of generality, we can assume: $\|z^{i\nu}\| \to \infty$ and from $\frac{1}{\|z^{i\nu}\|}(p^\nu \,\square\, (x^{i\nu} - \omega^i)) \leq \begin{pmatrix} -\bar{q} \\ R \end{pmatrix} \frac{z^{i\nu}}{\|z^{i\nu}\|} + \frac{1}{\|z^{i\nu}\|}\gamma(\pi^\nu)$, we deduce: $0 \leq \begin{pmatrix} -\bar{q} \\ R \end{pmatrix} z$ for some $z \in \mathbb{R}^J$, $z \neq 0$. If $\begin{pmatrix} -\bar{q} \\ R \end{pmatrix} z > 0$, we have a contradiction with the fact that $\bar{q}$ does not allow

arbitrage. If $\begin{pmatrix} -\bar{q} \\ R \end{pmatrix} z = 0$, in view of Assumption F".2, $Rz = 0 \Rightarrow z = 0$, which contradicts $z \neq 0$. We have thus proved that $(z^{i\nu})$ is a bounded sequence. Without loss of generality, $z^{i\nu} \rightarrow z^i \in \mathbb{R}^J$, and, by passing to limit in the inequalities defining $B_\gamma^i(\pi^\nu)$, we get $x^i \in B_\gamma^i(\pi)$.   □

Note that in view of the convexity and compactness of $\widetilde{X}^i$, the correspondence $\pi \rightarrow B_\gamma^i(\pi)$ is upper semicontinuous with closed, convex values.

Claim 7.2 $\forall i = 2, \ldots, I$, the correspondences $B_\gamma''^i$ and ψ^i are lower semicontinuous with convex values. So are ψ^1 and ψ^0.

Indeed, let us first consider the case $i \neq 1$.

Let V open in $\mathbb{R}^{L(1+S)}$ be such that $V \cap B_\gamma''^i(\pi) \neq \emptyset$. If $B_\gamma'^i(\pi) = \emptyset$, $\omega^i \in V$ so that $\forall \pi' \in \Pi$, $V \cap B_\gamma''^i(\pi) \neq \emptyset$. If $B_\gamma'^i(\pi) \neq \emptyset$, $V \cap B_\gamma^i(\pi) \neq \emptyset \Rightarrow V \cap B_\gamma'^i(\pi) \neq \emptyset$, which, implies the existence of a neighborhood U of π such that $\pi' \in U \Rightarrow V \cap B_\gamma'^i(\pi') \neq \emptyset \Rightarrow V \cap B_\gamma''^i(\pi') \neq \emptyset$.

Let V open in $\mathbb{R}^{L(1+S)}$ be such that $V \cap \psi^i(\pi, x) \neq \emptyset$. If $x^i \notin B_\gamma^i(\pi)$, it follows from the fact that $B_\gamma^i(\pi)$ is convex and closed and from the upper semicontinuity of B_γ^i that there exist neighborhoods U and W of π and x respectively such that $x' \in W$ and $\pi' \in U \Rightarrow x'^i \notin B_\gamma^i(\pi')$. Now, since $V \cap B_\gamma''^i(\pi) \neq \emptyset$, in view of the lower semicontinuity of B''^i, there exists a neighborhood U' of π such that $\pi' \in U' \Rightarrow V \cap B_\gamma''^i(\pi') \neq \emptyset$. Finally, $\pi' \in U \cap U'$ and $x'^i \in W$ imply $V \cap \psi^i(\pi, x) \neq \emptyset$. If $x^i \in B_\gamma^i(\pi)$, $V \cap \psi^i(\pi, x) = V \cap B'^i(\pi) \cap P^i(x) \neq \emptyset$ and there exist a neighborhood U of π, W of x such that for every π' of U, for every x' of W, one has $V \cap B_\gamma'^i(\pi') \cap P^{(}x') \neq \emptyset$, thus, in account of $B_\gamma'^i(\pi') \cap P^{(}x') \subset B_\gamma^i(\pi')$, $V \cap \psi^i(\pi', x) \neq \emptyset$.

The proof of the lower semicontinuity of ψ^1 is standard. The lower semicontinuity of ψ^0 is obvious. So is the convexity of values of all correspondences.   □

7.4   Existence of a Cass equilibrium in $\widetilde{\mathcal{E}}$

Remark that, by construction, one has for every $(\pi, x) \in \Pi \times \prod_{i=1}^I \widetilde{X}^i$, $\pi \notin \psi^0(\pi, x)$, $x^i \notin \psi^i(\pi, x)$, $i = 1, \ldots I$, and that correspondences $B_\gamma''^i$, $i = 1, \ldots, I$, have nonempty values. Also $B_\gamma'^1$ has nonempty values. It follows from the Gale and Mas-Colell theorem that $\exists (\bar{\pi}, \bar{x}) \in \Pi \times \prod_{i=1}^I \widetilde{X}^i$ such that:

$$\forall i = 1, \ldots, I, \ \bar{x}^i \in B_\gamma^i(\bar{\pi}) \text{ and } P^i(\bar{x}) \cap B'^i(\bar{\pi}) = \emptyset \tag{5}$$

$$\forall \pi \in \Pi, \ \pi \cdot (\sum_{i=1}^I \bar{x}^i - \sum_{i=1}^I \omega^i) \leq \bar{\pi} \cdot (\sum_{i=1}^I \bar{x}^i - \sum_{i=1}^I \omega^i). \tag{6}$$

Claim 7.3 $\sum_{i=1}^{I} \bar{x}^i = \sum_{i=1}^{I} \omega^i$

Indeed if not, $\|\bar{\pi}\| = 1$, $\gamma(\bar{\pi}) = 0$, $\bar{\pi} \cdot (\sum_{i=1}^{I} \bar{x}^i - \sum_{i=1}^{I} \omega^i) > 0$. But for $i = 2, \ldots, I$ and for some $\bar{z}^i \in \mathbb{R}^J$, $\bar{p} \,\square\, (\bar{x}^i - \omega^i) \leq W(\bar{p}, \bar{q})\bar{z}^i$. Premultiplying by λ_s each budget constraint and summing on s, we get: $\bar{\pi} \cdot (\bar{x}^i - \omega^i) \leq 0$. Also, $\bar{\pi} \cdot (\bar{x}^1 - \omega^1) \leq 0$. Summing on i, we get $\bar{\pi} \cdot (\sum_{i=1}^{I} \bar{x}^i - \sum_{i=1}^{I} \omega^i) \leq 0$ which yields a contradiction. $\qquad\square$

Claim 7.4 $\bar{\pi} \neq 0$ *and* $\forall s = 0, 1, \ldots, S,\ \bar{p}(s) \neq 0$.

Indeed, if $\bar{\pi} = 0$, then $\forall s$, $\gamma_s(\bar{\pi}) > 0$. One deduces that $\forall i$, $B_\gamma'^i(\bar{\pi}) = B_\gamma^i(\bar{\pi}) = \widetilde{X}^i$. In view of C.2, C.4, this contradicts the relation $P^i(\bar{x}) \cap B'^i(\bar{\pi}) = \emptyset$ in (5). We have thus proved $\bar{\pi} \neq 0$.

Now, from the survival assumption C.3, $B_\gamma'^1(\bar{\pi}) \neq \emptyset$, and for consumer 1, (5) and Assumption C.2 implies $P^1(\bar{x}) \cap B_\gamma^1(\bar{\pi}) = \emptyset$, which shows, using C.4 and recalling the definition of π, that $\bar{p}(s) \neq 0$, $s = 0, 1, \ldots, S$. $\qquad\square$

Claim 7.5 $\|\bar{\pi}\| = 1$.

Indeed, for $i = 2, \ldots, I$, the nonemptiness of $B'^i(\bar{\pi})$ follows from C.3 and the previous claim. Then, using C.4, we get, if $\bar{z}^i$ finances $\bar{x}^i$:

$$\bar{p} \,\square\, (\bar{x}^i - \omega^i) = W(\bar{p}, \bar{q})\bar{z}^i + \gamma(\bar{\pi}).$$

Premultiplying by λ_s and summing on s the budget constraints, we get:

$$\bar{\pi} \cdot (\bar{x}^i - \omega^i) = \sum_{s=0}^{S} \lambda_s \gamma_s(\bar{\pi}) = \frac{1}{I}(1 - \|\bar{\pi}\|).$$

For $i = 1$, we have also:

$$\bar{\pi} \cdot (\bar{x}^1 - \omega^1) = \sum_{s=0}^{S} \lambda_s \gamma_s(\bar{\pi}) = \frac{1}{I}(1 - \|\bar{\pi}\|).$$

Summing on i, we get $0 = 1 - \|\bar{\pi}\|$. $\qquad\square$

End of the proof of Theorem 7.1. By Claim 7.5, $\gamma(\bar{\pi}) = 0$. Then it follows from (5), Claims 3 and 4 that $(\bar{p}, \bar{q}, (\bar{x}^i)_{i=1}^{I}, (\bar{z}^i)_{i=2}^{I})$ is a Cass equilibrium of $\widetilde{\mathcal{E}}$. As already seen, if $\bar{z}^1$ is defined by $\bar{z}^1 + \sum_{i=2} \bar{z}^i = 0$, $(\bar{p}, \bar{q}, (\bar{x}^i)_{i=1}^{I}, (\bar{z}^i)_{i=1}^{I})$ is an equilibrium of $\widetilde{\mathcal{E}}$. It follows from C.2 and the definition of the truncated economy $\widetilde{\mathcal{E}}$ that it is also an equilibrium of $\mathcal{E}$. $\qquad\square$

8　An introduction to the real asset case

8.1　An example of non-existence of equilibrium with real assets

When there is no bound on short-selling, does a financial equilibrium exist for all kinds of assets ? It results from the counter-example of Hart (1975) that the answer is negative in the case of real assets, even with stronger assumptions than the assumptions used in this course. We give here an example taken from Magill and Shafer (1990).

As in many models, the consumption takes only place in the second period. We consider an economy with two agents, two commodities and two states of nature tomorrow. The financial markets is composed of two real assets. The utility functions, endowments, assets are as follows :

$$u^i(x_1^i, x_2^i) = \rho_1 U^i(x^i(1)) + \rho_2 U^i(x^i(2)),$$
$\rho_1 > 0,\ \rho_2 > 0,\ \rho_1 + \rho_2 = 1$ for $i = 1, 2$

$$U^i(\xi) = \alpha_1^i \ln(\xi_1) + \alpha_2^i \ln(\xi_2)$$
$\alpha_1^i > 0,\ \alpha_2^i > 0,\ \alpha_1^i + \alpha_2^i = 1$ for $i = 1, 2$.

$$\omega^1(1) = (1 - \varepsilon, 1 - \varepsilon), \qquad \omega^1(2) = (\varepsilon, \varepsilon)$$

$$\omega^2(1) = (\varepsilon, \varepsilon), \qquad \omega^2(2) = (1 - \varepsilon, 1 - \varepsilon)$$

$A = (a^1(s), (a^2(s))_{s=1,2}$, where $a^1(1) = a^1(2) = (1, 0)$ and $a^2(1) = a^2(2) = (0, 1)$. It follows that $V(p) = \begin{pmatrix} p_1(1) & p_2(1) \\ p_1(2) & p_2(2) \end{pmatrix}$.

Claim 8.1 *If $\varepsilon < 1/2$ and $\alpha_1^1 \neq \alpha_1^2$, then there does not exist any financial equilibrium.*

Proof of the claim. We will prove this by contradiction, with a discussion on the rank of the return matrix $V(\bar{p})$ of a financial equilibrium $(\bar{x}^i, \bar{z}^i, \bar{p})$. Let us first remark that since the utility function is strictly monotone in each state, we can replace all budget's inequalities by equalities. We can also remark that the rank of the return matrix $V(\bar{p})$ is equal to one if $\bar{p}(1)$ is collinear to $\bar{p}(2)$, and is equal to two if if $\bar{p}(1)$ is not collinear to $\bar{p}(2)$

First case, the rank is equal to two, in this case the financial market is complete and the allocation can be viewed as an Arrow Debreu equilibrium allocation. Since the allocation equilibrium is strictly positive, we can write the Kuhn and Tucker's conditions and we deduce that $\frac{x_\ell^i(1)}{x_\ell^i(2)} = \frac{\bar{p}_\ell(2)\rho_1}{\bar{p}_\ell(1)\rho_2}$ for all good ℓ, all consumer i, and each state $s = 1, 2$. Consequently $\frac{x_\ell^1(1)}{x_\ell^1(2)} = \frac{x_\ell^2(1)}{x_\ell^2(2)} = \frac{\bar{p}_\ell(2)\rho_1}{\bar{p}_\ell(1)\rho_2}$. Since at equilibrium, one has for all node $s = 1, 2$ and all good ℓ, $x_\ell^1(s) + x_\ell^2(s) = \omega_\ell^1(s) + \omega_\ell^2(s) =$

1, it follows from the previous equation that $\frac{\bar{p}_\ell(2)\rho_1}{\bar{p}_\ell(1)\rho_2} = 1$. Consequently $\bar{p}(1)$ is collinear to $\bar{p}(2)$, which contradicts the assumption that the rank is equal to two.

Second case, the rank is equal to 1, in this case at equilibrium the financial market is reduced to the trading of a unique financial instrument with positive returns. Hence, it follows from the no-arbitrage condition that the price q of this unique instrument is strictly positive. This is why there would not be any agent that will be able to buy this instrument since there is no endowment in the first period. Consequently the equilibrium is without financial transfer, and the financial equilibrium is a pure spot market equilibrium. Once again the allocation equilibrium is strictly positive and we deduce from the Kuhn and Tucker's conditions that

$$(x_1^i(s), x_2^i(s)) = \left(\frac{\alpha_1^i \bar{p}(s) \cdot \omega^i(s)}{\bar{p}_1(s)}, \frac{\alpha_2^i \bar{p}(s) \cdot \omega^i(s)}{\bar{p}_2(s)}\right)$$

for all consumer i, and each state $s = 1, 2$. Consequently
$$(x_1^1(s) + x_1^2(s), x_1^1 2(s) + x_2^2(s)) =$$

$((\alpha_1^1 \bar{p}(s) \cdot \omega^1(s) + \alpha_1^2 \bar{p}(s) \cdot \omega^2(s))/\bar{p}_1(s), ((\alpha_2^1 \bar{p}(s) \cdot \omega^1(s) + \alpha_2^2 \bar{p}(s) \cdot \omega^2(s))/\bar{p}_2(s))$. Since at equilibrium, one has for all node $s = 1, 2$ and all good ℓ, $x_\ell^1(s) + x_\ell^2(s) = \omega_\ell^1(s) + \omega_\ell^2(s) = 1$, it follows from the usual normalization of the price and from the previous equation that

$$(1, 1) = \left(\frac{\alpha_1^1(1-\varepsilon) + \alpha_1^2\varepsilon}{\bar{p}_1(1)}, \frac{\alpha_2^1(1-\varepsilon) + \alpha_2^2\varepsilon}{\bar{p}_2(1)}\right) = \left(\frac{\alpha_1^1\varepsilon + \alpha_1^2(1-\varepsilon)}{\bar{p}_1(2)}, \frac{\alpha_2^1\varepsilon + \alpha_2^2(1-\varepsilon)}{\bar{p}_2(2)}\right).$$

Since $\alpha^1 \neq \alpha^2$ and $\varepsilon \neq 1/2$, one computes that $\bar{p}(1)$ is not collinear to $\bar{p}(2)$, which contradicts the assumption that the rank is equal to one.   □

8.2   Definition and existence of pseudo-equilibrium

The previous example suggests that solving the equilibrium existence problem in the general setting described in Section 2 by Assumptions C.1 – C.4 and F.1, F.2 (i.e. without assuming that assets are numeraire or nominal) for a financial economy $\mathcal{E}$ requires a weakened definition of financial equilibrium.

Let $G^J(\mathbb{R}^{1+S})$ denote the set of all J-dimensional linear subspaces of $\mathbb{R}^{1+S}$. Such a collection is called $(1 + S) \times J$ Grassmanian manifold.

Definition 8.1   *A pseudo-equilibrium of $\mathcal{E}$ is an element $(\bar{x}, \bar{p}, \bar{q}, \bar{E})$ of $\prod_{i=1}^I X^i \times \mathbb{R}^{L(1+S)} \times \mathbb{R}^J \times G^J(\mathbb{R}^{1+S})$ such that*

(i) for each i, $\bar{x}^i$ is optimal in the budget set

$$B^i(\bar{p}, \bar{E}) = \{x^i \in X^i \mid \exists t^i \in \bar{E}, \bar{p} \,\square\, (x^i - \omega^i) \leq t^i\}$$

(ii) $\sum_{i=1}^{I} \bar{x}^i = \sum_{i=1}^{I} \omega^i$

(iii) $\{t \in \mathbb{R}^{L(1+S)} \mid t = W(\bar{p}, \bar{q}) \, z, \; z \in \mathbb{R}^J\} \subset \bar{E}$.

The relation between this equilibrium concept and the standard financial equilibrium concept is given in the following proposition.

Proposition 8.1 *Let* $(\bar{x}, \bar{p}, \bar{q}, \bar{E})$ *be a pseudo-equilibrium of* $\mathcal{E}$. *If* $\bar{E} = \{t \in \mathbb{R}^{L(1+S)} \mid t = W(\bar{p}, \bar{q}) \, z, \; z \in \mathbb{R}^J\}$ *and if* $(\bar{p}, \bar{q})$ *does not allow arbitrage, then there exists* $\bar{z} = (\bar{z}^i)_{i=1}^{I}$ *such that* $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ *is a financial equilibrium.*

Proof. Let $(\bar{t}^i)_{i=1}^{I}$ be the financial transfers associated to $\bar{x}^i$. Note that $\bar{p} \square (x^i - \omega^i) \leq \bar{t}^i \; \forall i$, this implies $\sum_{i=1}^{I} \bar{t}^i \geq 0$. Let $\bar{z}^i$ be such that $W(\bar{p}, \bar{q}) \, \bar{z}^i = \bar{t}^i$. Then $W(\bar{p}, \bar{q}) \sum_{i=1}^{I} \bar{z}^i \geq 0$ and no arbitrage imply $\sum_{i=1}^{I} \bar{t}^i = 0$ and $\forall i, \; \bar{p} \square (\bar{x}^i - \omega^i) = \bar{t}^i = W(\bar{p}, \bar{q}) \, \bar{z}^i$. Summing on i, we get $W(\bar{p}, \bar{q}) \sum_{i=1}^{I} \bar{z}^i = 0$ and thus $\sum_{i=1}^{I} \bar{z}^i = 0$ (the rank of $W(\bar{p}, \bar{q})$ is equal to J). As each equilibrium budget set $B^i(\bar{p}, \bar{E})$ can be identified with $B_{FM}^i(\bar{p}, \bar{q}) = \{x^i \in X^i \mid \exists z^i \in \mathbb{R}^J, \bar{p} \square (x^i - \omega^i) \leq W(\bar{p}, \bar{q}) \, z^i\}$, the proof is complete. $\square$

The theorem we intend to prove is the following:

Theorem 8.1 *Under Assumptions C.1 – C.4, F.1, F.2, given* $\lambda = (\lambda_s)_{s=0,\ldots,S} \in \mathbb{R}^{(1+S)}$ *such that* $\lambda_0 = 1$ *and* $\lambda \gg 0$, $\mathcal{E}$ *has a pseudo-equilibrium* $((\bar{x}^i)_{i=1}^{I}, \bar{p}, \bar{q}, \bar{E})$ *with* $\bar{q} = \sum_{s=1}^{S} \lambda_s \, v(s, \bar{p})$ *and* $\bar{E} \subset \lambda^{\perp}$.

8.2.1 A fixed point like theorem

Let us first endow $G^J(\mathbb{R}^{1+S})$ with the following metric:

$$\delta(E, E') = d_H(E \cap B(0,1), E' \cap B(0,1))$$

where $B(0,1)$ is the closed unit-ball in $\mathbb{R}^{1+S}$ and d_H the Hausdorff distance:

$$d_H(A, B) = \max\{\max\{d(x,B) \mid x \in A\}, \max\{d(y,A) \mid y \in B\}\}.$$

The following observation will be useful in the sequel:

Proposition 8.2 $G^J(\mathbb{R}^{1+S})$ *is a compact metric space and the correspondence* $\varphi : G^J(\mathbb{R}^{1+S}) \to \mathbb{R}^{1+S}$, *defined by* $\varphi(E) = E$, *is closed and lower semicontinuous.*

We will admit the following fixed point-like theorem:

Theorem 8.2 *Let V be an Euclidean space and $G^J(V)$ the set of all J-dimensional linear subspaces of V, endowed with the metric previously defined. Let X^i, $i = 1, \ldots, n$ be nonempty, convex and compact subsets of some Euclidean space and $X = \prod_{i=1}^n X^i \times G^J(V)$. On X, $\forall i = 1, \ldots n$, let us consider a lower semicontinuous and convex valued correspondences $\Phi^i : X \to X^i$, and $\forall j = 1, \ldots J$, continuous functions $\theta^j : X \to V$. Then there exists $\bar{x} = (\bar{x}^1, \ldots \bar{x}^n, \bar{E}) \in X$ with*

$$\forall i = 1, \ldots, n, \ \bar{x}^i \in \Phi^i(\bar{x}) \text{ or } \Phi^i(\bar{x}) = \emptyset \tag{7}$$

$$\forall j = 1, \ldots J, \ \theta^j(\bar{x}) \in \bar{E}. \tag{8}$$

The Gale and Mas-Colell lemma (see Theorem A.1) is obviously a particular case of this theorem.

8.2.2   Existence of pseudo-equilibrium

The proof of theorem 8.1 mimics the proof given in the nominal case. Given $\lambda = (\lambda_s)_{s=0,\ldots,S} \in \mathbb{R}^{(1+S)}$ such that $\lambda_0 = 1$ and $\lambda \gg 0$, one looks for a "Cass pseudo-equilibrium", i.e. $(\bar{x}, \bar{p}, \bar{q}, \bar{E})$ such that

(i) $\bar{x}^1$ is optimal in $B^1_{A-D}(\bar{\pi}) = \{x^1 \in X^1 \mid \bar{\pi} \cdot x^1 \leq \bar{\pi} \cdot \omega^1\}$, where $\bar{\pi} = (\lambda_s \bar{p}(s))_{s=0}^S$

(ii) $\forall i = 2, \ldots, I$, $\bar{x}^i$ is optimal in $B^i(\bar{p}, \bar{E}) = \{x^i \in X^i \mid \exists t^i \in \bar{E}, \ \bar{p} \,\square\, (x^i - \omega^i) \leq t^i\}$ (iii) $\sum_{i=1}^I \bar{x}^i = \sum_{i=1}^I \omega^i$

(iv) $\{W(\bar{p}, \bar{q}) \, z \mid z \in \mathbb{R}^J\} \subset \bar{E} \subset \lambda^\perp$.

Remark that $\forall i = 2, \ldots I$, $B^i(\bar{p}, \bar{E}) \subset B^i_{A-D}(\bar{\pi})$. Defining $\bar{t}^1$ by $\bar{t}^1 = \bar{p} \,\square\, (\bar{x}^1 - \omega^1)$, we get $\bar{t}^1 = \bar{p} \,\square\, (-\sum_{i=2}^I (\bar{x}^i - \omega^i)) = -\sum_{i=2}^I \bar{t}^i \in \bar{E}$, so that $\bar{x}^1 \in B^1(\bar{p}, \bar{E})$ and is obviously optimal in $B^1(\bar{p}, \bar{E})$. A Cass pseudo-equilibrium is thus a pseudo-equilibrium.

Let us truncate the consumption sets by closed balls containing in their interior all $\widehat{X}^i(s)$, $\forall i$, $\forall s$ and replaces $\mathcal{E}$ by $\widetilde{\mathcal{E}} = ((\widetilde{X}^i, \widetilde{P}^i, \omega^i, Z^i)_{i=1}^I, V)$ where each $\widetilde{X}^i$ is convex and compact. Π and $\gamma(\pi)$ are defined as in Section 7. The modified budget sets in $\widetilde{\mathcal{E}}$ are for $i = 2, \ldots I$

$$B^i_\gamma(\pi, E) = \{x^i \in \widetilde{X}^i \mid \exists t^i \in E, \ p \,\square\, (x^i - \omega^i) \leq t^i + \gamma(\pi)\}$$

$$B'^i_\gamma(\pi, E) = \{x^i \in \widetilde{X}^i \mid \exists t^i \in E, \ p \,\square\, (x^i - \omega^i) \ll t^i + \gamma(\pi)\}$$

$$B''^i_\gamma(\pi, E) = \begin{cases} \omega^i & \text{if } B'^i_\gamma(\pi, E) = \emptyset \\ B^i(\pi, E) & \text{if } B'^i_\gamma(\pi) \neq \emptyset \end{cases}$$

and for $i = 1$,

$$B^1_\gamma(\pi) = \{x^1 \in \widetilde{X}^1 \mid \pi \cdot (x^1 - \omega^1) \leq \sum_{s=0}^S \lambda_s \gamma_s(\pi)\}$$

$$B_\gamma^{'1}(\pi) = \{x^1 \in \widetilde{X}^1 \mid \pi \cdot (x^1 - \omega^1) < \sum_{s=0}^{S} \lambda_s \gamma_s(p)\}.$$

The reaction correspondences are defined on $\Pi \times \prod_{i=1}^{I} \widetilde{X}^i \times G^J(\lambda^\perp)$ by

$$\psi^i(\pi, x, E) = \begin{cases} B_\gamma^{''i}(\pi, E) & \text{if } x^i \notin B_\gamma^i(\pi, E) \\ B_\gamma^{'i}(\pi, E) \cap P^i(x) & \text{if } x^i \in B_\gamma^i(\pi, E) \end{cases} \quad i = 2, \ldots I$$

$$\psi^1(\pi, x, E) = \begin{cases} B_\gamma^1(\pi) & \text{if } x^1 \notin B_\gamma^1(\pi) \\ B_\gamma^{'1}(\pi) \cap P^1(x) & \text{if } x^1 \in B_\gamma^1(\pi) \end{cases}$$

$$\psi^0(\pi, x, E) = \{\pi' \in \Pi \mid (\pi' \cdot \sum_{i=1}^{I}(x^i - \omega^i) > (\pi \cdot \sum_{i=1}^{I}(x^i - \omega^i))\}.$$

One defines also the functions $\theta^j : \Pi \times \prod_{i=1}^{I} \widetilde{X}^i \times G^J(\lambda^\perp) \to \lambda^\perp$, $j = 1, \ldots, J$ by:

$$\theta^j(\pi, x, E) = \left(-\sum_{s=1}^{S} \lambda_s v^j(s, p), \ v^j(1, p), \ v^j(2, p), \ldots, \ v^j(S, p) \right).$$

One has to check the condition of the fixed-point like theorem. The lower semi-continuity of each ψ^i is routine, using the observation in Proposition 8.2. Using the fixed-point like theorem, one finds $(\bar{p}, \bar{x}, \bar{E})$ such that

$$\forall i = 1, \ldots, I, \ \bar{x}^i \in B_\gamma^i(\bar{\pi}, E) \text{ and } P^i(\bar{x}) \cap B^{'i}(\bar{\pi}, E) = \emptyset \tag{9}$$

$$\forall \pi \in \Pi, \ \pi \cdot (\sum_{i=1}^{I} \bar{x}^i - \sum_{i=1}^{I} \omega^i) \leq \bar{\pi} \cdot (\sum_{i=1}^{I} \bar{x}^i - \sum_{i=1}^{I} \omega^i) \tag{10}$$

$$\forall j = 1, \ldots, J, \ \theta^j(\bar{\pi}, \bar{x}, \bar{E}) \in \bar{E}, \tag{11}$$

i.e., defining $\bar{q} = \sum_{s=1}^{S} \lambda_s v(s, \bar{p})$,

$$\{W(\bar{p}, \bar{q}) \, z \mid z \in \mathbb{R}^J\} \subset \bar{E}.$$

As in the nominal case, one then proves successively that:

$\sum_{i=1}^{I} \bar{x}^i = \sum_{i=1}^{I} \omega^i$

$\bar{\pi} \neq 0$ and $\forall s = 0, 1, \ldots, S, \ \bar{p}(s) \neq$

$\|\pi\| = 1$

$(\bar{p}, \bar{x}, \bar{E}, \bar{q})$ is a Cass pseudo-equilibrium of $\widetilde{\mathcal{E}}$, hence a pseudo-equilibrium of $\mathcal{E}$ To end, two remarks are in order:

Remark 8.1 *This theorem contains the existence theorem in the nominal case. Indeed, in this case,* $\operatorname{rank} V(\bar{p}) = \operatorname{rank} R = J$ *and* $\bar{E} = \{W(\bar{p}, \bar{q})z \mid z \in \mathbb{R}^J\}$.

Remark 8.2 *This theorem contains also the existence theorem in the case of numeraire assets. Indeed, with the assumptions of the previous results, one has also* $\operatorname{rank} W(\bar{p}, \bar{q}) = J$ *and* $\bar{E} = \{W(\bar{p}, \bar{q})z \mid z \in \mathbb{R}^J\}$. *In this case, as the budget constraint at s is homogeneous in* $p(s)$, *the choice of* λ *appears as an artifact of the proof which does not influence equilibrium commodity prices ad allocations. One can say the same for the pseudo-equilibrium obtained in case of real assets (the case generally considered in the literature).*

A   Appendix

In this appendix, we summarize the most important definitions and properties dealing with correspondences. In order to keep the notions as simple as possible, we will restrict our attention to the case of Euclidean spaces, though for some definitions, the general topological formulation may be more practical. The proofs of stated results are not given.

In the Subsection A.1, we define the two notions of semicontinuity and give a characterization by sequences. Related concepts of continuity are introduced in Subsection A.2, and we establish the main relations with the concepts of previous subsections. In Subsection A.3, we study different operations for correspondences (composition, closure, intersection, product, etc.) which are very useful in Mathematical Economics. Finally, in the last subsection, we state the Maximum Theorem, the Gale and Mas-Colell lemma and Kakutani's theorem .

A.1   Upper and lower semicontinuity of correspondences

Let us consider a nonempty subset X of an Euclidean space $\mathbb{R}^{\ell}$ (respectively a nonempty subset Y of an Euclidean space $\mathbb{R}^{\ell'}$). We will denote by $\mathcal{P}(Y)$, the set of all subsets of Y, including the empty set.

Definition A.1 *A correspondence $\varphi : X \to Y$ is a mapping from X with values in $\mathcal{P}(Y)$; it associates with each element x in X, a subset $\varphi(x)$ of Y. Let $\varphi : X \to Y$ be a correspondence. The set $\{(x,y) \in X \times Y \mid y \in \varphi(x)\}$ is called the graph of φ and is denoted by $gr(\varphi)$. If $A \subset X$, we define $\varphi(A) = \cup_{x \in A}\varphi(x)$.*

Definition A.2 *(superior and inferior inverse of a correspondence) Let $\varphi : X \to Y$ be a correspondence, $B \subset Y$. We define :*
$\varphi^{+}(B) = \{x \in X \mid \varphi(x) \subset B\}$ (superior inverse of φ),
$\varphi^{-}(B) = \{x \in X \mid \varphi(x) \cap B \neq \emptyset\}$ (inferior inverse of φ).

Definition A.3 *(upper and lower semicontinuity of a correspondence):*
Let $\varphi : X \to Y$ be a correspondence. It is said to be lower semicontinuous at x_0, if for all open set $V \subset Y$ such that $V \cap \varphi(x_0)$ is non-empty, there exists a neighborhood U of x_0 in X such that : for all $x \in U$, $V \cap \varphi(x)$ is non-empty. The correspondence φ is said to be lower semicontinuous if it is lower semicontinuous at each point of X.
It is said to be upper semicontinuous at x_0 if for all open set $V \subset Y$ such that $V \supset \varphi(x_0)$, there exists a neighborhood U of x_0 in X such that : for all $x \in U$, $V \supset \varphi(x)$. The correspondence φ is said to be upper semicontinuous if it is upper semicontinuous at each point of X.

It is said continuous at x_0 if φ is lower and upper semicontinuous en x and φ is continuous on X if it is continuous at each point of X.

It is important to notice that the concept of lower (or upper) semicontinuity of a function do not coincide with the concept of lower (upper) semicontinuity of a correspondence. In the special case of a single-valued correspondence, it is possible to associate to this correspondence a function; in this special case, the two notions of semicontinuity for correspondences coincide and are equivalent to the classical continuity of the associated function.

Proposition A.1 *Let $\varphi : X \to Y$ be a correspondence. The following properties are equivalent:*
i) φ is scs on X,
ii) For all open set $V \subset Y$, $\varphi^+(V)$ is open,
iii) For all closed $V \subset Y$, $\varphi^-(V)$ is closed.

Proposition A.2 *Let $\varphi : X \to Y$ be a correspondence. The following properties are equivalent:*
i) φ is sci on X,
ii) For all open set $V \subset Y$, $\varphi^-(V)$ is open,
iii) For all closed $V \subset Y$, $\varphi^+(V)$ is closed.

Since the topology is entirely characterized by the notion of convergent sequences in metric spaces, it is possible to characterize the notions of semicontinuity by sequences. In the first result, we only use the fact that X is metric.

Proposition A.3 *Let $\varphi : X \to Y$ be a correspondence, $x \in X$.*
a) φ is upper semicontinuous at x if and only if for all convergent sequence $\{x_k\}_k$ convergent sequence to x, and for all V open neighborhood of $\varphi(x)$, there exists n such that for all $k \geq n$, $\varphi(x_k) \subset V$.
b) φ is lower semicontinuous at x if and only if for all $\{x_k\}_k$ convergent sequence to x, and V open set such that $\varphi(x) \cap V \neq \emptyset$, there exists n such that for all $k \geq n$, $\varphi(x_k) \cap V \neq \emptyset$.

In the case where the two spaces are metric spaces, we state the following theorem.

Theorem A.1 *: Let $\varphi : X \to Y$ be a correspondence.*
a) φ is lower semicontinuous at x if and only if for all $\{x_k\}_k$ convergent sequence to x, and all $y \in \varphi(x)$, there exists a sequence $\{y_k\}_{k \geq k_0}$ in Y such that $\{y_k\} \to y$ and for all $k \geq k_0$, $y_k \in \varphi(x_k)$.
b) If $\varphi(x)$ is compact, then φ is upper semicontinuous at x if and only if for all sequences $\{x_k\}_k$, $\{y_k\}_k$ such that $\{x_k\} \to x$, $y_k \in \varphi(x_k)$, there exists a subsequence $\{y_{k_n}\}$ of $\{y_k\}$ and $y \in \varphi(x)$ such that $\{y_{k_n}\} \to y$.

Either from Proposition A.1 or from Theorem A.1 it is possible to adapt for the correspondence case the Weierstrass theorem.

Proposition A.4 *Let X be a compact space and $\varphi : X \to Y$ be a correspondence. If for all x in X, φ is upper semicontinuous at x and $\varphi(x)$ is compact, then the set $\varphi(X)$ is compact.*

A.2   Related concepts

Definition A.4 *Let $\varphi : X \to Y$ be a correspondence. φ is said to be open (respectively closed in $X \times Y$ if $gr(\varphi)$ is open (respectively closed).*

Proposition A.5 *Let $\varphi : X \to Y$ be a correspondence. One has : φ open $\Longrightarrow \varphi$ is sci.*

It is also possible to consider locally the concept of closed correspondence.

Definition A.5 *Let $\varphi : X \to Y$ be a correspondence. φ is said closed at x if $y \notin \varphi(x)$ implies the existence of a neighborhood U_x, (respectively V_y) of x (respectively of y) such that $(U_x \times V_y) \cap gr(\varphi) = \emptyset$.*

Proposition A.6 *Let $\varphi : X \to Y$ be a correspondence.*
a) If φ is closed at x, $\varphi(x)$ is a closed subset of Y.
b) φ is closed in $X \times Y$ if and only if φ is closed at each point of X.

Proposition A.7 *Let $\varphi : X \to Y$ be a correspondence.*
φ is closed at x if and only if for all $\{x_k\}_k$ and $\{y_k\}_k$ such that $\{x_k\}_k$ converges to x, and $\{y_k\}_k$ converges to y, with $y_k \in \varphi(x_k)$, for all k, one has $y \in \varphi(x)$.

As shown by the following two propositions, the concepts of upper semicontinuity and closed correspondence are very closed.

Proposition A.8 *Let $\varphi : X \to Y$ be a correspondence.*
a) If φ is upper semicontinuous at $x \in X$ with $\varphi(x)$ closed, then φ is closed at x.
b) If φ is upper semicontinuous on X with closed values, then φ is closed in $X \times Y$.

Proposition A.9 *Let $\varphi : X \to Y$ be a correspondence.*
a) If φ is closed at x and if there exists a neighborhood U_x of x, and a compact set V such that $\varphi(U_x) \subset V$, then φ is upper semicontinuous at x.
b) If φ is closed and if there exists a compact set V such that $\varphi(X) \subset V$, then φ is upper semicontinuous on X.

A.3 Operations with correspondences

Definition A.6 *(composition of correspondences) Let $\varphi : X \to Y$, and $\psi : Y \to Z$ be two correspondences. We define $\psi \circ \varphi$ by $\psi \circ \varphi(x) = \psi(\varphi(x))$.*

Roughly speaking, the property of semicontinuity is stable by composition. Formally one has :

Proposition A.10 *Let $\varphi : X \to Y$ and $\psi : Y \to Z$ be two correspondences, $x \in X$,*
a) If φ is upper semicontinuous at x and ψ is upper semicontinuous on $\varphi(x)$, then $\psi \circ \varphi$ is upper semicontinuous at x.
b) If φ is lower semicontinuous at x and ψ is lower semicontinuous on $\varphi(x)$, then $\psi \circ \varphi$ is lower semicontinuous at x.

In our setting of study, the property of semicontinuity is stable by the operation of closure.

Definition A.7 *(closure of a correspondence) Let φ be a correspondence, its closure is the correspondence $\operatorname{cl}\varphi$ defined by : $\operatorname{cl}\varphi(x) = \operatorname{cl}(\varphi(x))$.*

Proposition A.11 *Let $\varphi : X \to Y$ be a correspondence.*
a) If φ is upper semicontinuous at x, then $\operatorname{cl}\varphi$ is upper semicontinuous at x.
b) If φ is lower semicontinuous at x, then $\operatorname{cl}\varphi$ is lower semicontinuous at x.

Definition A.8 *(intersection of correspondences): Let φ and $\psi : X \to Y$ be two correspondences. We define $\varphi \cap \psi$ by $\varphi \cap \psi(x) = \varphi(x) \cap \psi(x)$.*

Proposition A.12 *Let φ and $\psi : X \to Y$ be two correspondences, $x \in X$*
a) If φ and ψ are upper semicontinuous at x and $\varphi(x)$ and $\psi(x)$ are closed, then $\varphi \cap \psi$ is upper semicontinuous at x .
b) If φ is closed at x, ψ is upper semicontinuous at x, and $\psi(x)$ is compact, then $\varphi \cap \psi$ is upper semicontinuous at x and the set $\varphi \cap \psi(x)$ is compact.
c) If φ is lower semicontinuous at x and ψ is open, then $\varphi \cap \psi$, is lower semicontinuous at x.

Definition A.9 *(product of correspondences): Let $(\varphi_i)_{i=1}^n : X \to Y_i$, be n correspondences. We define $\prod_{i=1}^n \varphi_i : X \to \prod_{i=1}^n Y_i$ by $(\prod_{i=1}^n \varphi_i)(x) = \prod_{i=1}^n \varphi_i(x)$ for all $x \in X$.*

Proposition A.13 *Let $(\varphi_i)_{i=1}^n, : X \to Y$ be n correspondences, $x \in X$.*
a) If for all i, φ_i is lower semicontinuous at x, then $\prod_{i=1}^n \varphi_i$ is lower semicontinuous at x,
b) If for all i, φ_i is upper semicontinuous at x and $\varphi_i(x)$ is compact, $\prod_{i=1}^n \varphi_i$ is upper semicontinuous at x.

Definition A.10 *(Sum and convex hull of correspondences) Let $(\varphi_i)_{i=1,\ldots,n}$ be n correspondences, $x \in X$. We define:*
the sum of $(\varphi_i)_{i=1,\ldots,n}$: $(\sum_{i=1}^{n} \varphi_i)(x) = \sum_{i=1}^{n} \varphi_i(x)$
the convex hull of φ : $\operatorname{co} \varphi(x) = \operatorname{co}(\varphi(x))$.

Proposition A.14 *: Let $(\varphi_i)_{i=1}^{n}, : X \to Y$, be n correspondences, $x \in X$.*
a) If $(\varphi_i)_{i=1,\ldots,n}$ are lower semicontinuous at x, their sum is lower semicontinuous at x.
b) If $(\varphi_i)_{i=1,\ldots,n}$ are upper semicontinuous at x, and for all i, $\varphi_i(x)$ is compact, then their sum is upper semicontinuous at x.

Proposition A.15 *Let Y be a convex subset of $\mathbb{R}^{\ell'}$ and $\varphi : X \to Y$, be a correspondence, and $x \in X$,*
a) If φ is lower semicontinuous at x, $\operatorname{co} \varphi$ is lower semicontinuous at x.
b) If φ is upper semicontinuous at x and $\varphi(x)$ is compact, then $\operatorname{co} \varphi$ is upper semicontinuous at x.

A.4   Main Theorems

In Mathematical Economics, one has frequently to deal with optimization problems and more precisely one has to take care about the topological dependence of both the set of optimal solutions and the optimal value of the problem when the domain of optimization vary.

Let us consider the problem of optimization:

$P_x : g(x) = \max \{ f(x,y) \mid y \in \varphi(x) \}$

where $f : X \times Y \to \mathbb{R}$ is continuous function and $\varphi : X \to Y$ is a correspondence. The real number $g(x)$ is the optimal value of P_x and the correspondence of optimal solutions $\psi : X \to Y$, is defined by: $\psi(x) = \{ y \in \varphi(x) \mid f(x,y) = g(x) \}$.

Theorem A.2 (BERGE) *With the previous notations, if f is a continuous function and φ is lower semicontinuous and upper semicontinuous, with nonempty compact values, then*
a) the correspondence of optimal solutions ψ is upper semicontinuous, with compact values.
b) the value function g is continuous.

Kakutani's theorem is a very central result in fixed-point theory that extended to correspondences the result of Brouwer. It can be proved from Brouwer's theorem by many ways, one of them uses partitions of unity. Conversely, Brouwer's theorem is an obvious corollary of Kakutani's theorem. Both of them are proved by combinatorial techniques.

Theorem A.3 (KAKUTANI) *Let $X = \prod_{i=1}^{n} X_i$, where, for every i, X_i is a non-empty, compact, convex subset of a Euclidean space R^{ℓ_i}. Let F_i ($i = 1, \ldots, n$) be n upper semicontinuous correspondences from X to X_i with nonempty, convex and compact values, then there exists a fixed point x^* of F; $x^* = (x_i^*)$ in X, such that : for every i, $x_i^* \in F_i(x^*)$.*

Theorem A.4 (BROUWER) *Let $X = \prod_{i=1}^{n} X_i$, where, for every i, X_i is a non-empty, compact, convex subset of a Euclidean space R^{ℓ_i}. Let f_i ($i = 1, \ldots, n$) be n continuous functions from X to X_i, then there exists a fixed point x^* of f; $x^* = (x_i^*)$ in X, such that : for every i, $x_i^* = f_i(x^*)$.*

The following fixed point like theorem can be found in Gale and Mas-Colell (1975-1979) and its proof is based on a selection theorem due to Michael (1956) and on Kakutani's theorem.

Lemma A.1 (GALE AND MAS-COLELL) *Let $X = \prod_{i=1}^{n} X_i$, where, for every i, X_i is a non-empty, compact, convex subset of a Euclidean space R^{ℓ_i}. Let F_i ($i = 1, \ldots, n$) be n lower semicontinuous correspondences from X to X_i (possibly empty valued), then there exists $x^* = (x_i^*)$ in X, such that :*
for every i, either $x_i^* \in \mathrm{co}\, F_i(x^*)$ or $F_i(x^*)$ is empty.

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