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THE TAXATION OF CAPITAL RETURNS IN OVERLAPPING GENERATIONS ECONOMIES WITHOUT FINANCIAL ASSETS

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Abstract. I show in this paper that in an overlapping generations economy with production à la Diamond (1970) in which the agents can only save in terms of capital (i.e. with no asset bubbles à la Tirole (1985) or public debt as in Diamond (1965)), there is a period-by-period balanced fiscal policy supporting a steady state allocation that Pareto-improves upon the laissez-faire competitive equilibrium steady state (without having to resort to intergenerational transfers) if there is no first generation or the economy starts there. A transition from the competitive equilibrium steady state to this other allocation is also Pareto-improving if the former is dynamically inefficient, but even in the dynamically efficient case if the elasticity of output to capital is high enough. This intervention allows every subsequent generation to attain, as a competitive equilibrium outcome, the highest utility attainable at a steady state through the existing markets for the consumption good and the production factors. The active fiscal policy consists of taxing (or subsidizing, in the dynamically efficient case) linearly the returns to capital, while balancing the budget period by period through a lump-sum transfer (or tax, respectively) on second period income. This policy does not finance any public spending, since there is none in the model. The only purpose of the intervention is to decentralize as a competitive equilibrium the steady state allocation that maximizes the utility of the representative agent among all steady state allocations attainable through the existing markets.

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1. Introduction

Whether the taxation of capital returns is a good or bad idea is a recurrent issue in the economic literature.[1] Arguments against and in favor are put forward by, respectively, those who would like to see taxes on capital income eliminated (or at least reduced) because of the inefficiencies they may introduce in the allocation of resources (as in Chamley (1986), Judd (1987)), and those who think that taxes on capital income serve a purpose if only because they may help undo some inefficiencies due to the incompleteness of markets (e.g. oversaving as a self-insurance against uninsurable risks, as in Aiyagari (1995), Chamley (2001)). Actually, the conclusions depend crucially on the framework in which the question is addressed, namely the neoclassical growth model or the overlapping generations model. In effect, in the ideal case in which there is no uncertainty, while in the neoclassical growth model the taxation of capital returns induces a distortion of the competitive equilibrium factor prices that can only create inefficiencies (not surprisingly, given the Pareto optimality of the laissez-faire competitive equilibrium allocations of this model), it turns out that the breakdown of the First Welfare Theorem in the overlapping generations economy with production in Diamond (1965) prevents to replicate straightforwardly this kind of argument in that setup.

Diamond (1970) argued nevertheless that, even in the overlapping generations setup of Diamond (1965), an increase in the tax rate of capital returns can decrease the steady state utility of the representative agent, so that a reduction or outright elimination of taxes on capital returns would be Pareto-improving, even if the taxes raised were given back as a lump-sum to the same generation. Still, Diamond (1970) established this negative impact of the taxation (even compensated) of capital returns on the steady state representative agent's utility only for the case in which the capital returns are actually taxed (as opposed to subsidized) and the after-tax rate of return *does exceed* the rate of growth of the population. One can add that, although Diamond (1970) did not consider the symmetric case, it follows from his analysis too that, if the returns to capital are actually linearly subsidized (by means of a lump-sum tax raised from the same generation), a decrease in the rate at which they are subsidized decreases the steady state utility of the representative agent if the after-subsidy rate of return *does not exceed* the rate of growth of the population.

The result provided in Diamond (1970) is actually a partial answer to a more gen-

[1]See, for instance, Atkeson, Chari, and Kehoe (1999) and Conesa, Kitao, and Krueger (2007)

eral question that can be addressed in the framework of the overlapping generations economy with production of Diamond (1965). This question is whether the laissez-faire competitive equilibrium steady state maximizes or not the utility of the representative agent *among the steady states that are attainable through the existing markets* for the consumption good and the factors of production. The answer to this question is negative: even among just the steady states in which factors are remunerated by their marginal productivities and without transfers across agents, the laissez-faire competitive equilibrium steady state does not maximize the representative agent's utility.[2] Indeed, given the typical inefficiency of the laissez-faire competitive equilibrium steady state, all the agents could be better-off if they saved differently from what they choose to save at that steady state, even if their labor and savings are still priced by their marginal productivities. In effect, since the return to the savings of any given generation is the marginal productivity of the aggregate capital next period, and hence is determined by the aggregate savings today, all the members of a generation could in principle coordinate to manipulate the returns to their own savings in order to implement such improvement.[3] Of course under competitive conditions no agent has incentives to deviate from the competitive behavior for lack of a coordination device, and this room from improvement is thus left unexploited at competitive equilibria.

But then the question arises of whether some adequate government intervention would allow to decentralize as a competitive equilibrium the steady state that provides all the agents the highest possible utility among the steady states attainable through the good and factor markets. The answer to this second question is yes, and the right intervention just requires —in the case in which the laissez-faire competitive equilibrium steady state over-accumulates capital compared to the steady state that maximizes the representative agent among all feasible steady states, and is hence dynamically inefficient— to tax linearly each generation's capital returns (at a rate depending on the savings of the previous generation) and, simultaneously, to make to the same generation a second period lump-sum transfer equal to the amount that would have been raised from the previous generation if it had been taxed at the current generation rate.[4] In doing so, at the steady state no

[2]Except for the knife-edge case in which the laissez-faire competitive equilibrium steady state happens to maximize it already among *all* feasible steady states.

[3]Note that, for an economy running from $-\infty$ to $+\infty$ (and hence without a first generation), if all generations behaved this way, then each generation's utility could be improved upon the laissez-faire steady state competitive equilibrium, even in the dynamically efficient case.

[4]Contrarily, in the case in which the laissez-faire competitive equilibrium steady state under-accumulates capital, the returns to savings need rather to be subsidized and a second period lump-sum tax needs to be raised.

resources are redistributed across agents or generations, and the government does never incur in any deficit or superavit. Actually, a similar policy is considered in Diamond (1970)[5] in order to asses the impact of taxes on capital returns in this setup. Nevertheless, it turns out that, in the dynamically inefficient case, the tax rate that implements as a competitive equilibrium the steady state maximizing the representative agent utility (among those attainable through the existing markets) is such that the after-tax return to savings *does not* exceed the rate of growth of the population, so that the result in Diamond (1970) does not apply. Similarly, in the dynamically efficient case, the subsidy rate that implements the utility maximizing steady state attainable through the existing markets is such that the after-subsidy return to savings *does* exceed the rate of growth of the population, so that Diamond (1970) does not apply either.

It is worth noting that in Diamond (1970) no other market exists for any asset (other than capital) in which the agents could save as well, and that might allow for intergenerational transfers. More specifically, no asset bubble à la Tirole (1985) or public debt as in Diamond (1965), that would support the first-best steady state exists. I choose to maintain this assumption in this paper for the following reasons. Firstly, the absence of financial markets allowing for intergenerational transfers may reflect of an economy with an under-developped financial system and may therefore be of interest to address development issues. But second and more importantly, a conceptual problem seems to undermine the implementation of the first-best steady state through the introduction of fiat money or debt. In effect, as it is well known in the overlapping generations economy with production of Diamond (1965) the first-best steady state can in principle be attained as a competitive outcome if the agents can put part of their savings into some intrinsically worthless asset (a bubble à la Tirole (1985) or public debt as in Diamond (1965)) that allows to implement the necessary intergenerational transfers. Nevertheless, since in the absence of uncertainty the returns to physical capital as well as to the financial asset have to be equal at equilibrium (in effect, the first-best steady state requires typically strictly positive holdings of both assets), the agents are actually indifferent about the composition of their savings portfolio, and hence the latter is not determined by their decisions. Only by chance would the agents choose precisely the portfolio that supports the Pareto efficient steady state.[6] In other words, the implementation of the first-best steady state by means of, say, fiat money is not an equilibrium outcome

[5]In Diamond (1970) the lump-sum transfer equals in each period *exactly* the amount raised from that same generation, and hence contingent to the decisions of the very agents that will receive it. On the contrary, here the tax rates and transfers are determined each period by past decisions and cannot therefore be subject to manipulation. At the steady state both nonetheless coincide.

[6]That the right amount of savings in terms of physical capital is the one that makes the marginal

in the sense of being the result of a collection of compatible optimal *decisions*, since the composition of the agents saving portfolio is decided by no one *in the model* at the first-best steady state.[7] This poses a problem for the conceptual soundness of the decentralization of the first-best steady state by means of an intrinsically worthless financial asset that is beyond the scope of this paper, and that I leave to be addressed elsewhere (see Dávila (2008)). As a consequence, in this paper (as in Diamond (1970)) agents are assumed to be able to save in terms of physical capital only.

The rest of the paper proceeds as follows. Section 2 characterizes the competitive equilibria of the economy (mainly to fix notation), characterizes the competitive equilibrium steady state and shows that, even without resorting to the introduction of asset bubbles, there exists a reallocation that Pareto-improves over the competitive steady state whenever the latter is dynamically inefficient, but also in the dynamically efficient case if output is sufficiently elastic to capital. Section 3 establishes that any competitive equilibrium steady state providing the representative agent a utility smaller than that of the first-best steady state does not even maximize the agent's utility among the steady states attainable trough the existing good and factors markets (Proposition 1). Section 4 characterizes the steady state that achieves this maximization (Proposition 2), and establishes that it is a laissez-faire competitive equilibrium outcome if, and only if, it coincides with the first-best steady state. As a matter of fact, the first-best steady state, the utility-maximizing steady state attainable through the good and factor markets, and the competitive equilibrium steady state are either all identical or all distinct (Proposition 3). Section 5 characterizes the fiscal policy that allows to decentralize the utility-maximizing steady state attainable through the good and factor markets as a competitive equilibrium (Proposition 4). The policy requires taxing or subsidizing the returns on savings depending on whether this steady state over-accumulates or under-accumulates capital with respect to the first-best (Proposition 5).

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productivity of capital equal to the rate of growth of the population does not follow from any agent's (or firm's) decision problem.

[7]It has been suggested to me that this problem may be of the same nature than that of the innocuous indeterminacy at equilibrium of the production plan of a firm with a constant returns to scale technology, which is widely assumed to result in production adjusting to a demand that is well determined by prices. Nevertheless, it is my view that the indeterminacy of the savings portfolio I am referring to is of a different, more disturbing nature, since in this case in the money market, each period, *both* sides of the market are identical, they are both the same representative agent, and face then the same indeterminacy. As a result, contrarily to what happens in the constant return to technology case, there is no well-determined other side of the market able to anchor here the indeterminate side. Rather, in the savings portfolio problem the decisions of both sides of the money market are indeterminate each period at the first-best steady state equilibrium.

2. Competitive equilibria of the productive overlapping generations economy without financial assets

Consider the economy in Diamond (1965,1970), i.e. consider first an agent living for two periods, t and $t+1$, in which he is, say, young and old respectively. When young he can work l hours four a real hourly wage of w_t. His real income when young $w_t l$ can then be either consumed immediately or saved for consumption when old. No other way of saving is available to him. Let c_t^t denote the share of his income $w_t l$ that he consumes when young, and k_{t+1} be his share of income saved for consumption when old. He can lend his savings as capital for a rate of return of r_{t+1}, and then consume the returns $(1 + r_{t+1})k_{t+1}$ when old. Let c_{t+1}^t denote his consumption when old. The agent evaluates consumption by a utility function u and discounts future utilities by a discount factor $\beta \in (0, 1]$.[8] This agent faces then the problem of deciding how much of his income to save when young. Formally, the agent's problem is

$$\begin{aligned} \max_{0 \leq c_t^t, c_{t+1}^t, k_{t+1}} \quad & u(c_t^t) + \beta u(c_{t+1}^t) \\ c_t^t + k_{t+1} &= w_t l \\ c_{t+1}^t &= (1 + r_{t+1})k_{t+1} \end{aligned} \tag{1}$$

for given values of w_t and r_{t+1} (without loss of generality, capital is supposed not to depreciate at all, as in Diamond (1965,1970)). Under standard assumptions, the agent's optimal saving k_{t+1} is then completely characterized to be a function of w_t and r_{t+1} through the condition

$$\frac{1}{\beta} \frac{u'(w_t l - k_{t+1})}{u'((1 + r_{t+1})k_{t+1})} = 1 + r_{t+1} \tag{2}$$

that equalizes the marginal rate of substitution of old-age consumption to young-age consumption and the rate of return to savings. The agent's optimal consumption when young and old are then determined by k_{t+1} through his budget constraints above.

Suppose now this agent is one of the many members of the generation born in period t of an economy whose population grows at a rate $1 + n > 0$, and whose many firms produce consumption good out of capital and labor through a constant returns to

[8] Just for notational convenience, separability and discounting do not play any role whatsoever in what follows.

scale Cobb-Douglas production function $F(K, L) = K^{\alpha}L^{1-\alpha}$, with $0 < \alpha < 1$. Under perfect competition, the wage and the rental rate of capital are determined, at each period t, by the marginal productivities of labor and capital respectively. Since the capital K_t available at any given period t consists of the previous period aggregate savings $(1+n)^{t-1}k_t$, and aggregate labor L_t is $(1+n)^t l$, then the wage and rental rate that the agents living in periods t and $t+1$ face are actually

$$\begin{aligned} r_{t+1} =& F_K(\frac{k_{t+1}}{1+n}, l) \\ w_t =& F_L(\frac{k_t}{1+n}, l) \end{aligned} \tag{3}$$

(given the homogeneity of degree 1 of the production function) where k_t is the amount that each of the agents born in the previous period $t-1$ decided to save.

Note that the agents' budget constraints guarantee the feasibility of the allocation of resources, since adding up, at any date t, the budget constraints of the $(1+n)^{t-1}$ old agents

$$c_t^{t-1} = (1+r_t)k_t \tag{4}$$

and the $(1+n)^t$ contemporaneous young agents

$$c_t^t + k_{t+1} = w_t l \tag{5}$$

it follows (from the homogeneity of degree 1 of the production function) that

$$\begin{aligned} c_t^t + \frac{1}{1+n}c_t^{t-1} + k_{t+1} = & \\ (1+r_t)\frac{k_t}{1+n} + w_t l = & (1 + F_K(\frac{k_t}{1+n}, l))\frac{k_t}{1+n} + F_L(\frac{k_t}{1+n}, l)l \\ = & \frac{k_t}{1+n} + F(\frac{k_t}{1+n}, l). \end{aligned} \tag{6}$$

The competitive equilibrium allocations are, therefore, completely characterized by the following per capita savings dynamics that results from each agent's utility maximization and the determination of the factors prices by their marginal productivities

$$\frac{1}{\beta}\frac{u'(F_L(\frac{k_t}{1+n}, l)l - k_{t+1})}{u'((1 + F_K(\frac{k_{t+1}}{1+n}, l))k_{t+1})} = 1 + F_K(\frac{k_{t+1}}{1+n}, l). \tag{7}$$

More specifically, a competitive equilibrium *steady state* of this overlapping generations economy is characterized by a constant sequence of per capita savings k^c satisfying the dynamics (7) above, i.e. solving the equation

$$\frac{1}{\beta}\frac{u'(F_L(\frac{k^c}{1+n},l)l-k^c)}{u'((1+F_K(\frac{k^c}{1+n},l))k^c)}=1+F_K(\frac{k^c}{1+n},l). \tag{8}$$

Thus, if at some date t the available per capita level of savings —made by generation $t-1$— happens to be k^c,[9] then the only allocation the agents will be able to attain under laissez-faire through the existing markets for the good and factors of production is the allocation in which every generation $t'\geq t$ obtains a consumption profile $(c_0^c,c_1^c)=(F_L(\frac{k^c}{1+n},l)l-k^c,(1+F_K(\frac{k^c}{1+n},l))k^c)$. Nevertheless, this competitive equilibrium steady state allocation is Pareto-dominated by the following one if $k^c>k^g$, where k^g is the level of per capital savings that maximizes the net output per capita $\frac{k}{1+n}+F(\frac{k}{1+n},l)-k$ each period:

(1) generations $t'\leq t-1$ get the same consumption as before,

(2) generation t obtains the consumption profile[10]

$$\begin{aligned}\tilde{c}_0&=F_L(\frac{k^c}{1+n},l)l-k^*\\ \tilde{c}_1&=(1+F_K(\frac{k^*}{1+n}))k^*\end{aligned} \tag{9}$$

where k^* is the solution to

$$\begin{aligned}&\max_{0\leq c_0,c_1,k} u(c_0)+\beta u(c_1)\\ &c_0+k=F_L(\frac{k}{1+n},l)l\\ &c_1=(1+F_K(\frac{k}{1+n},l))k\end{aligned} \tag{10}$$

(3) and subsequent generations $t\geq t+1$ obtain

$$\begin{aligned}c_0^*&=F_L(\frac{k^*}{1+n},l)l-k^*\\ c_1^*&=(1+F_K(\frac{k^*}{1+n}))k^*.\end{aligned} \tag{11}$$

[9] Or, equivalently, generation $t-1$ is actually a first generation born old at date t that happens to be endowed with k^c units of capital —and consumes hence $(1+F_K(\frac{k^c}{1+n},l))k^c$.

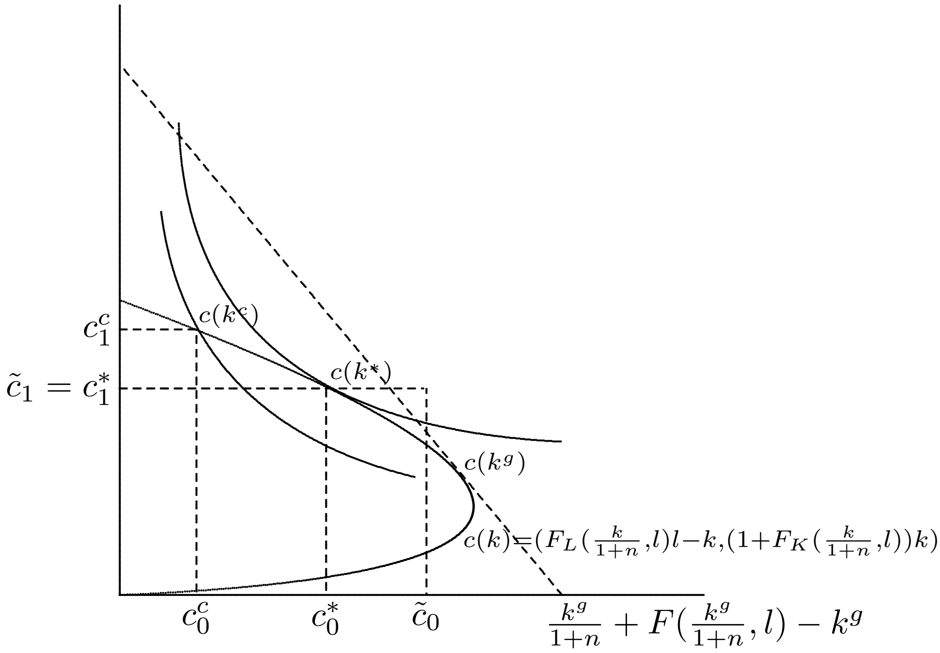
[10] Note that $\tilde{c}_0>0$ because of $k^c>k^g$, since this implies $k^c>k^*$ (as established in the appendix) and hence that $\tilde{c}_0>c_0^*\geq 0$.

Note that, by construction, the new allocation above is feasible. Also, note that (i) generations $t' \leq t-1$ see their utility remain unchanged, and are therefore indifferent between the two allocations; (ii) the new allocation provides nevertheless to all generations $t' \geq t+1$ the highest utility they could ever get at a steady state trading in the existing markets (and hence not smaller than the one they get in the competitive equilibrium steady state); and (iii) for generation t the utility is higher than at the competitive equilibrium steady state as long as $k^c > k^*$, which is the case when $k^c > k^g$ (as established in the appendix), i.e. when k^c is said to be dynamically inefficient (actually, generation t gets an utility even higher than that of all the other subsequent generations):

$$\begin{aligned}
&u(c_0^c) + \beta u(c_1^c) \leq \\
&u(c_0^*) + \beta u(c_1^*) = u(F_L(\frac{k^*}{1+n}, l)l - k^*) + \beta u((1 + F_K(\frac{k^*}{1+n}, l))k^*) \\
&\qquad < u(F_L(\frac{k^c}{1+n}, l)l - k^*) + \beta u((1 + F_K(\frac{k^*}{1+n}, l))k^*) \\
&\qquad = u(\tilde{c}_0) + \beta u(\tilde{c}_1).
\end{aligned} \tag{12}$$

This fact is illustrated in Figure 1 where, for a Cobb-Douglass technology, have also been depicted the curve of possible steady state profiles of consumption $c(k) = (F_L(\frac{k}{1+n}, l)l - k, (1 + F_K(\frac{k}{1+n}, l))k)$ attainable through the good and factors markets (parametrized by the initial level of per capita savings k) and the feasibility line corresponding to the highest possible net output (attained for the per capita level of savings k^g).

Figure 1



In the case $k^g > k^c$, necessarily $k^* > k^c$ (see the lemma in the appendix) and therefore (12) above does not go through. As a matter of fact, the change in the utility of generation t receiving the consumptions $(F_L(\frac{k^c}{1+n},l)l-k^*,(1+F_K(\frac{k^*}{1+n}))k^*)$ can, in principle, in this case go either way, since[11]

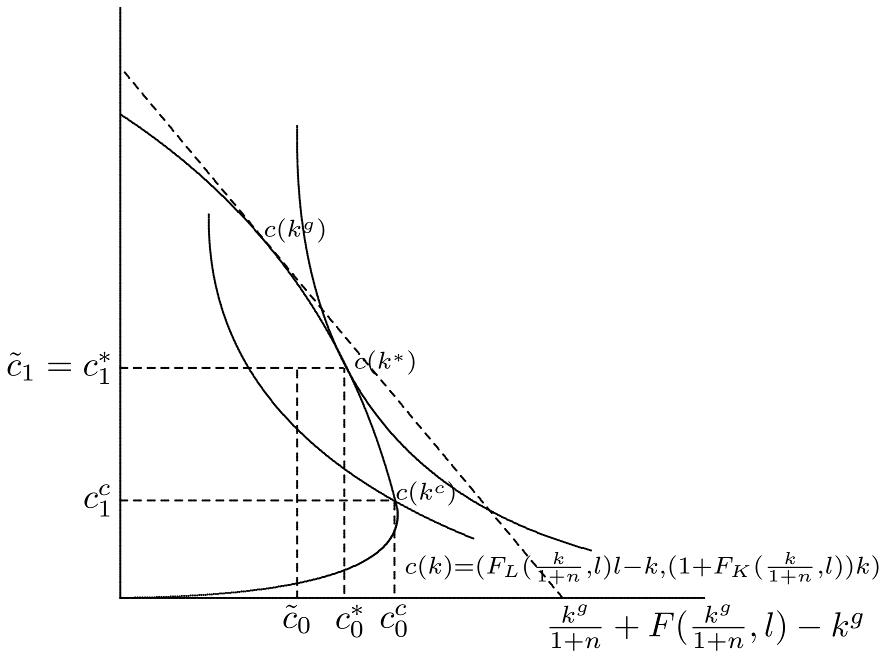
$$F_L(\frac{k^c}{1+n},l)l-k^* < F_L(\frac{k^c}{1+n},l)l-k^c \tag{13}$$

$$(1+F_K(\frac{k^*}{1+n}))k^* > (1+F_K(\frac{k^c}{1+n}))k^c \tag{14}$$

so that the change in utility for generation t depends on which of the these two opposite changes in consumption, when young and old, overcomes the other (see Figure 2). For instance, for an α in the Cobb-Douglass production function close enough to 1 the productivity of labor will be insensitive enough to the level of capital for $\tilde{c}_0$ to be arbitrarily close to c_0^* and make, hence, the utility of generation t increase in this case as well.

[11]Note that, as depicted in Figure 2, the inequality $F_L(\frac{k^c}{1+n},l)l-k^* < F_L(\frac{k^*}{1+n},l)l-k^*$ holds also.

Figure 2



Of course, the previous allocation is not a competitive equilibrium allocation under laissez-faire, since the only competitive equilibrium starting from the level of per capita savings k^c is precisely the steady state k^c. So there is no hope that the economy will attain this Pareto-improving allocation without public intervention. The question remains nonetheless whether the economy can attain it with some minimal public intervention, more specifically within the framework of the existing markets (i.e. without resorting to redistributions).[12] The answer to this question is yes, and in order to establish this (and to provide the policy that implements this allocation), I consider first in the next section the inter-temporal allocation problem as if there were no first generation. Indeed, recall that the per capita level of savings k^* with which we have constructed the allocation Pareto-improving over the competitive equilibrium steady state allocation, is in fact the per capita level of savings that maximizes the utility of the representative agent among all the steady state allocations attainable in such an economy with no first generation.

[12] As it is well known, the introduction of asset bubbles allows to do even better, but this possibility is not considered here for the reasons explained in the introduction.

3. Comparing steady states

Consider the question of which is the best technologically feasible steady state if we are free to choose the initial level of savings for the first generation.[13] It is well known that the competitive equilibrium steady state level of per capita savings k^c solution to (8) needs not be the one that allows to maximize the utility of the representative agents among all feasible steady states. That is to say, k^c is typically distinct from the first-best per capita savings k^g that follows from solving

$$\begin{aligned} &\max\ u(c_1) + \beta u(c_2) \\ c_1 + \frac{c_2}{1+n} + k &= \frac{k}{1+n} + F(\frac{k}{1+n}, l) \end{aligned} \tag{15}$$

and that is characterized by the condition

$$F_K(\frac{k^g}{1+n}, l) = n. \tag{16}$$

Whenever $k^c > k^g$ the laissez-faire competitive equilibrium steady state allocation is said to over-accumulate capital with respect to the first-best steady state. On the contrary, if $k^g > k^c$ holds, then the free market inefficiently under-accumulates capital.

Only in a knife-edge case would the competitive equilibrium steady state coincide with the first-best steady state. As a matter of fact, implementing the first-best steady state requires a lot of power since it allocates freely the output produced each period among the young and old alive then, without any consideration about the productivity of the factors of production they provide to obtain it (labor and capital respectively). It is not surprising therefore that with so much power one could in general do better then the laissez-faire competitive outcome. Nevertheless, the next proposition establishes (maybe more surprisingly) that whenever the competitive equilibrium steady state of this economy is sub-optimal among all technologically feasible steady states (i.e. whenever k^c is distinct from the first-best steady state

[13] This amounts to consider the problem for the whole family of economies with first generation for all possible levels of initial capital savings, and is equivalent, to assume that time runs from $-\infty$ to $+\infty$. With the first interpretation above, we compare in fact steady states of a continuum of different economies, while with this second interpretation one would be comparing the continuum of steady states of a given economy. Whichever interpretation is kept, such steady states correspond to the so-called golden age paths in Diamond (1965).

level k^g), then it is sub-optimal even among just those steady states that can be attained through the existing markets, i.e. that remunerate factors by their marginal productivities and require no transfers. This is interesting because it indicates that, typically, there is room for improving upon the laissez-faire allocation even without interfering with the working of markets or without resorting to redistributing income across generations, as implementing the first-best steady state typically requires.

Proposition 1. *In the Diamond (1965) overlapping generations economy, the competitive equilibrium steady state allocation of resources is sub-optimal among the steady states attainable through the output and factors markets, whenever it is sub-optimal among all the technologically feasible steady states.*

Proof. The utility u^c that every agent obtains at the competitive equilibrium steady state with per capita savings k^c solution to (8) is $u(F_L(\frac{k^c}{1+n}, l)l - k^c) + \beta u((1 + F_K(\frac{k^c}{1+n}, l))k^c)$, so that (k^c, u^c) is in the graph of the function ϕ defined by

$$\phi(k) = u(F_L(\frac{k}{1+n}, l)l - k) + \beta u((1 + F_K(\frac{k}{1+n}, l))k). \tag{17}$$

Note however that, for a constant returns to scale neoclassical production function the derivative of ϕ at the competitive equilibrium steady state level of per capita savings k^c is strictly negative (respectively positive) whenever at this steady state there is over-accumulation (resp. under-accumulation) of capital with respect to the representative agent utility-maximizing steady state level of per capita savings k^g characterized by (16). In effect, note that

$$\begin{aligned}\phi'(k^c) = u'(c_0^c)[F_{LK}(\frac{k^c}{1+n}, l)\frac{l}{1+n} - 1] \\ + \beta u'(c_1^c) \cdot [1 + F_K(\frac{k^c}{1+n}, l) + F_{KK}(\frac{k^c}{1+n}, l)\frac{k^c}{1+n}]\end{aligned} \tag{18}$$

where $c_0^c = F_L(\frac{k^c}{1+n}, l)l - k^c$ and $c_1^c = (1 + F_K(\frac{k^c}{1+n}, l))k^c$ are the competitive equilibrium steady state consumptions when young and old respectively. But at the competitive equilibrium steady state it holds that

$$-u'(c_0^c) + \beta u'(c_1^c)[1 + F_K(\frac{k^c}{1+n}, l)] = 0 \tag{19}$$

so that the derivative $\phi'(k^c)$ simplifies to

$$\phi'(k^c) = u'(c_0^c)F_{LK}(\frac{k^c}{1+n}, l)\frac{l}{1+n} + \beta u'(c_1^c)F_{KK}(\frac{k^c}{1+n}, l)\frac{k^c}{1+n}. \tag{20}$$

Therefore, $\phi'(k^c) < (>) 0$ holds if, and only if,

$$\frac{1}{\beta}\frac{u'(c_0^c)}{u'(c_1^c)}\frac{1}{1+n} < (>) - \frac{F_{KK}(\frac{k^c}{1+n}, l)\frac{k^c}{1+n}}{F_{LK}(\frac{k^c}{1+n}, l)l} \tag{21}$$

or, equivalently —since the right-hand side is 1 because of the homogeneity of degree 1 of the neoclassical production function F, and the marginal rate of substitution $\frac{1}{\beta}\frac{u'(c_0^c)}{u'(c_1^c)}$ in the left-hand side is $1 + F_K(\frac{k^c}{1+n}, l)$ at the competitive steady state levels of consumption— if, and only if,

$$F_K(\frac{k^c}{1+n}, l) < (>)\, n = F_K(\frac{k^g}{1+n}, l). \tag{22}$$

That is to say, $\phi'(k^c) < (>) 0$ holds if, and only if,

$$k^c > (<)\, k^g \tag{23}$$

because of the decreasing marginal productivity of capital. Q.E.D.

In other words, what Proposition 1 is saying is that, whenever the competitive equilibrium steady state is not outright the first-best one, there is another way of saving, for all generations,[14] that gives every agent a higher utility than the competitive equilibrium steady state. Which is then the best steady state allocation that is attainable through the existing markets for output, capital and labor? This question is addressed in the next section.

4. The optimal steady state attainable through the good and factors markets

If the laissez-faire competitive equilibrium steady state can be improved upon through the existing markets without redistributing income, but rather (as established in Proposition 1 above) saving less in case the competitive equilibrium steady state over-accumulates capital, or saving more in case it under-accumulates, which level of per capita savings is such that, if chosen by all agents, would make everyone strictly better off? The next proposition characterizes first the steady state that provides the highest utility to the representative agent among those attainable through the good and factor markets.

[14]Provided it is the initial one, if the interpretation with a first generation is chosen.

Proposition 2. *In the Diamond (1965) overlapping generations economy, the optimal steady state level of per capita savings k^*, among those attainable through the good and factors markets, is characterized by the condition*

$$\frac{1}{\beta}\frac{u'(F_L(\frac{k^*}{1+n},l)l-k^*)}{u'((1+F_K(\frac{k^*}{1+n},l))k^*)}=(1+n)\frac{1+F_K(\frac{k^*}{1+n},l)+F_{KK}(\frac{k^*}{1+n},l)\frac{k^*}{1+n}}{(1+n)+F_{KK}(\frac{k^*}{1+n},l)\frac{k^*}{1+n}} \tag{24}$$

as long as $0<k^*<F_L(\frac{k^*}{1+n},l)l$.[15]

Before we proceed with the proof of Proposition 2, a remark on condition (24) is maybe in order. Recall that the competitive equilibrium steady state could be Pareto-improved upon because in it the agents failed to take into account the impact that their own saving decisions had on their returns. In condition (24) this impact is accounted for by the derivative with respect to savings of the marginal productivity of capital, i.e. $F_{KK}(\frac{k^*}{1+n},l)$ in the right-hand side. As a matter of fact, should one drop these derivatives, condition (24) would revert to condition (8).

Proof of Proposition2. In effect, firstly the steady state utility ϕ defined in (17) is everywhere strictly concave because

$$\begin{aligned}\phi''(k)=&u''(c_0)[F_{LK}(\frac{k}{1+n},l)\frac{l}{1+n}-1]^2+u'(c_0)F_{LKK}(\frac{k}{1+n},l)\frac{l}{(1+n)^2}\\&+\beta u''(c_1)[1+F_K(\frac{k}{1+n},l)+F_{KK}(\frac{k}{1+n},l)\frac{k}{1+n}]^2\\&+\beta u'(c_1)[\frac{2}{1+n}F_{KK}(\frac{k}{1+n},l)+F_{KKK}(\frac{k}{1+n},l)\frac{k}{(1+n)^2}]<0\end{aligned} \tag{25}$$

—where $c_0=F_L(\frac{k}{1+n},l)l-k$ and $c_1=(1+F_K(\frac{k}{1+n},l))k$— given that all the terms are negative since

$$F_{LKK}(K,L)=(1-\alpha)\alpha(\alpha-1)K^{\alpha-2}L^{-\alpha}<0 \tag{26}$$

and

$$2F_{KK}(K,L)+F_{KKK}(K,L)K=\alpha^2(\alpha-1)K^{\alpha-2}L^{1-\alpha}<0. \tag{27}$$

[15]This is guaranteed if the utility function has the boundary behavior implied by the assumption that the representative agent needs to consume a positive amount in each period in order to stay alive. Additive separability has been assumed only for notational convenience.

So that k^* such that $\phi'(k^*) = 0$ maximizes the representative agent market steady state utility $\phi(k)$. Then the optimal market steady state level of capital k^* is characterized by the condition $\phi'(k^*) = 0$, or equivalently by (24) above as long as $0 < k^* < F_L(\frac{k^*}{1+n}, l)l$, given that

$$\begin{aligned} (1+n) + F_{KK}(\frac{k^*}{1+n}, l)\frac{k^*}{1+n} = \\ (1+n) + \alpha(\alpha-1)\left(\frac{k^*}{(1+n)l}\right)^{\alpha-1} > 0 \end{aligned} \tag{28}$$

In effect, since k^* maximizes

$$\phi(k) = u(F_L(\frac{k}{1+n}, l)l - k) + \beta u((1 + F_K(\frac{k}{1+n}, l))k) \tag{29}$$

and

$$\frac{d}{dk}[F_L(\frac{k}{1+n}, l)l - k] = 0 \tag{30}$$

for $\frac{k}{(1+n)l} = \left(\frac{\alpha(1-\alpha)}{1+n}\right)^{\frac{1}{1-\alpha}}$ while

$$\frac{d}{dk}[1 + F_K(\frac{k}{1+n}, l)k] = \alpha^2\left(\frac{k}{(1+n)l}\right)^{\alpha-1} > 0, \tag{31}$$

then $\phi'(k) > 0$ for $\frac{k}{(1+n)l} = \left(\frac{\alpha(1-\alpha)}{1+n}\right)^{\frac{1}{1-\alpha}}$, that is to say, necessarily,

$$\frac{k^*}{(1+n)l} > \left(\frac{\alpha(1-\alpha)}{1+n}\right)^{\frac{1}{1-\alpha}} \tag{32}$$

as stated above.

Finally, the steady state allocation giving a consumption $F_L(\frac{k^*}{1+n}, l)l - k^*$ when young and a consumption $(1 + F_K(\frac{k^*}{1+n}, l))k^*$ when old is feasible since

$$F_L(\frac{k^*}{1+n}, l)l - k^* + \frac{1}{1+n}(1 + F_K(\frac{k^*}{1+n}, l))k^* + k^* = \frac{k^*}{1+n} + F(\frac{k^*}{1+n}, l). \tag{33}$$

Q.E.D.

Note however that the optimal steady state level of per capita savings k^* among those attainable through the existing markets is not a laissez-faire competitive equilibrium outcome, unless it is actually optimal among all feasible steady states.

In effect, k^* does not satisfy the condition characterizing the competitive equilibrium steady state

$$\frac{1}{\beta}\frac{u'(F_L(\frac{k}{1+n},l)l-k)}{u'((1+F_K(\frac{k}{1+n},l))k)} = 1+F_K(\frac{k}{1+n},l) \tag{34}$$

unless

$$1+F_K(\frac{k^*}{1+n},l) = (1+n)\frac{1+F_K(\frac{k^*}{1+n},l)+F_{KK}(\frac{k^*}{1+n},l)\frac{k^*}{1+n}}{(1+n)+F_{KK}(\frac{k^*}{1+n},l)\frac{k^*}{1+n}} \tag{35}$$

i.e. unless

$$F_K(\frac{k^*}{1+n},l) = n \tag{36}$$

that is to say, only if k^* is equal to the first-best level of per capita savings k^g. But whenever the level of per capita savings k^* is distinct from the first-best level k^g, then both are distinct from the competitive equilibrium level k^c as well. Proposition 3 below states the precise way in which the three steady states relate to each other.

Proposition 3. *In the Diamond (1965) overlapping generations economy, either all k^g, k^*, and k^c coincide, or they all are distinct, i.e.*

$$k^c = k^g \Leftrightarrow k^g = k^* \Leftrightarrow k^* = k^c \tag{37}$$

with k^c, k^*, and k^g being the per capita level of savings at, respectively, the competitive equilibrium steady state, the steady state maximizing the representative agent utility through the good and factors markets, and the first-best steady state.

Proof. Assume $k^c = k^g$. Then

$$\begin{aligned}\frac{1}{\beta}\frac{u'(F_L(\frac{k^g}{1+n},l)l-k^g)}{u'((1+F_K(\frac{k^g}{1+n},l))k^g)} &= \\ 1+F_K(\frac{k^g}{1+n},l) &= 1+n \\ &= (1+n)\frac{1+F_K(\frac{k^g}{1+n},l)+F_{KK}(\frac{k^g}{1+n},l)\frac{k^g}{1+n}}{(1+n)+F_{KK}(\frac{k^g}{1+n},l)\frac{k^g}{1+n}}\end{aligned} \tag{38}$$

so that $k^g = k^*$.

Assume $k^g = k^*$. Then

$$\frac{1}{\beta}\frac{u'(F_L(\frac{k^*}{1+n},l)l - k^*)}{u'((1+F_K(\frac{k^*}{1+n},l))k^*)} = \\ 1+n = 1+F_K(\frac{k^g}{1+n},l) \\ = 1+F_K(\frac{k^*}{1+n},l) \tag{39}$$

so that $k^* = k^c$.

Assume $k^* = k^c$. Then

$$1+F_K(\frac{k^c}{1+n},l) = \\ \frac{1}{\beta}\frac{u'(F_L(\frac{k^c}{1+n},l)l - k^c)}{u'((1+F_K(\frac{k^c}{1+n},l))k^c)} = (1+n)\frac{1+F_K(\frac{k^c}{1+n},l)+F_{KK}(\frac{k^c}{1+n},l)\frac{k^c}{1+n}}{(1+n)+F_{KK}(\frac{k^c}{1+n},l)\frac{k^c}{1+n}} \tag{40}$$

from which

$$F_K(\frac{k^c}{1+n},l) = n \tag{41}$$

so that $k^c = k^g$. Q.E.D.

The question now is therefore whether some government intervention can make of the optimal steady state level of per capita savings k^* attainable through the existing markets a competitive equilibrium outcome, whenever it happens to be distinct from the first-best one. This question is addressed in the next section.

5. Decentralization of the optimal steady state attainable through the good and factors markets

Consider now a government with the ability to tax (or subsidize) linearly the agents capital income as well as to distribute them a lump-sum transfer (or raise a lump-sum tax) when old. In particular, suppose the government taxes the returns from savings that the agent living at t and $t+1$ gets when old, letting $\tau_{t+1} > 0$ be one minus the tax rate (it is a subsidy if bigger than one), while distributing to him at the same time a lump-sum transfer T_{t+1} (a lump-sum tax if negative). Then that agent's problem becomes

$$\begin{aligned} &\max_{0 \le c_t^t, c_{t+1}^t, k_{t+1}} u(c_t^t) + \beta u(c_{t+1}^t) \\ &c_t^t + k_{t+1} = w_t l \\ &\quad c_{t+1}^t = (1 + \tau_{t+1} r_{t+1}) k_{t+1} + T_{t+1} \end{aligned} \tag{42}$$

and his optimal saving k_{t+1} is characterized by the condition

$$\frac{1}{\beta} \frac{u'(w_t l - k_{t+1})}{u'((1 + \tau_{t+1} r_{t+1}) k_{t+1} + T_{t+1})} = 1 + \tau_{t+1} r_{t+1} \tag{43}$$

for a given real wage w_t, a return to savings r_{t+1}, a capital income tax (or subsidy) rate $1 - \tau_{t+1}$, and a lump-sum transfer (or tax) T_{t+1}. The per capita level of savings competitive equilibrium dynamics, for given tax and transfer policy $\{\tau_t, T_t\}_t$ is then given by

$$\frac{1}{\beta} \frac{u'(F_L(\frac{k_t}{1+n}, l) l - k_{t+1})}{u'((1 + \tau_{t+1} F_K(\frac{k_{t+1}}{1+n}, l)) k_{t+1} + T_{t+1})} = 1 + \tau_{t+1} F_K(\frac{k_{t+1}}{1+n}, l) \tag{44}$$

The next proposition characterizes the specific policy that allows to implement as a competitive equilibrium the best steady state attainable through the existing markets

Proposition 4. *In the Diamond (1965) overlapping generations economy, if the government taxes linearly the return to savings of each generation t at a rate τ_{t+1} (a subsidy if $1 < \tau_{t+1}$) and subsidizes the second period income by a lump-sum transfer T_{t+1} (a lump-sum tax if negative), determined as functions of the previous generation per capita savings k_t according to*

$$\begin{aligned} \tau_{t+1} &= \frac{1}{F_K(\frac{k_t}{1+n}, l)} \left[(1+n) \cdot \frac{1 + F_K(\frac{k_t}{1+n}, l) + F_{KK}(\frac{k_t}{1+n}, l) \frac{k_t}{1+n}}{(1+n) + F_{KK}(\frac{k_t}{1+n}, l) \frac{k_t}{1+n}} - 1 \right] \\ T_{t+1} &= (1 - \tau_{t+1}) F_K(\frac{k_t}{1+n}, l) k_t \end{aligned} \tag{45}$$

if τ_{t+1} in (45) is positive, and $\tau_{t+1} = 0$ otherwise, then the competitive equilibrium steady state is the optimal steady state k^* attainable through the good and factors *markets. At such steady state the government keeps moreover a balanced budget every period.*

Note, incidentally, that for the computation at any given period t of both the distortionary and lump-sum tax and subsidy, the fiscal authority uses only information already known at the time t of choosing τ_{t+1} and T_{t+1}. Note also that no generation can manipulate the determination of the government's policy, since it is determined by what the previous generation did.

Proof of proposition 4. If the government determines in each period t the tax rate and the lump-sum transfer as a function of the current level of capital k_t (saved by the previous generation in $t-1$) according to (45) above, then in the competitive equilibrium dynamics (44) for the level of capital k_t the net of tax returns in right-hand side becomes, whenever $\tau_{t+1} > 0$,

$$1 + \frac{F_K(\frac{k_{t+1}}{1+n}, l)}{F_K(\frac{k_t}{1+n}, l)} \left[(1+n) \cdot \frac{1 + F_K(\frac{k_t}{1+n}, l) + F_{KK}(\frac{k_t}{1+n}, l)\frac{k_t}{1+n}}{(1+n) + F_{KK}(\frac{k_t}{1+n}, l)\frac{k_t}{1+n}} - 1 \right] \tag{46}$$

and the old-age consumption within the marginal utility in the denominator of the left-hand side is

$$\big(1 + \tau_{t+1} F_K(\frac{k_{t+1}}{1+n}, l)\big)k_{t+1} + (1 - \tau_{t+1})F_K(\frac{k_t}{1+n}, l)k_t \tag{47}$$

so that the steady state of the dynamics (44) is characterized precisely by the same condition (24) than the optimal steady state attainable through the good and factors markets, i.e.

$$\frac{1}{\beta}\frac{u'(F_L(\frac{k^*}{1+n}, l)l - k^*)}{u'((1 + F_K(\frac{k^*}{1+n}, l))k^*)} = (1+n) \cdot \frac{1 + F_K(\frac{k^*}{1+n}, l) + F_{KK}(\frac{k^*}{1+n}, l)\frac{k^*}{1+n}}{(1+n) + F_{KK}(\frac{k^*}{1+n}, l)\frac{k^*}{1+n}} \tag{48}$$

Note also that at the steady state the net of tax return to savings $1 + \tau^* F_K(\frac{k^*}{1+n}, l)$ is indeed positive, which guarantees that the agents' problem is well defined (in particular that their budget set is compact). In effect, the solution to the condition

(24) above satisfies

$$
\begin{aligned}
1+\tau^* F_K(\frac{k^*}{1+n},l) &= (1+n)\cdot\frac{1+F_K(\frac{k^*}{1+n},l)+F_{KK}(\frac{k^*}{1+n},l)\frac{k^*}{1+n}}{(1+n)+F_{KK}(\frac{k^*}{1+n},l)\frac{k^*}{1+n}} \\
&= \frac{1}{\beta}\frac{u'(F_L(\frac{k^*}{1+n},l)l-k^*)}{u'(F_K(\frac{k^*}{1+n},l)k^*)} \\
&> 0
\end{aligned}
\tag{49}
$$

as requested.

From the definitions of the second period lump sum transfer T_{t+1} it follows trivially that, at the steady state, what the government withdraws (respectively, injects) from each generation in a distortionary way is exactly offset by the resources it injects (resp. withdraws) to that same generation in a non-distortionary way, so that the government's budget is balanced every period. Q.E.D.

If the economy finds itself at the competitive equilibrium steady state (or, equivalently, starts with a first generation born old that is endowed with the competitive equilibrium steady state per capita level of capital k^c), there exists a fiscal policy that allows to make a transition to an allocation in which every generation from the date t of implementation of the policy onwards obtains the highest utility attainable through the good and factor markets. The new allocation will be Pareto-superior for sure in the case $k^c > k^g$, because the members of generation t in which the intervention takes place also see their utility increase. This is not necessarily the case when $k^g > k^c$, but it will also be so if, for instance the elasticity α of output with respect to capital is close enough to 1, as shown in Section 2. This policy is the following:

(1) announce at t that interest income will be taxed at date $t+1$ at a rate $1-\tau_{t+1}$ with

$$
\tau_{t+1} = \frac{1}{F_K(\frac{k^*}{1+n},l)}\left[\frac{1}{\beta}\frac{u'(F_L(\frac{k^c}{1+n},l)l-k^*)}{u'(F_K(\frac{k^*}{1+n},l)k^*)}-1\right] \tag{50}
$$

and that a lump-sum transfer T_{t+1} will be distributed to every old agent at $t+1$, with

$$
T_{t+1} = (1-\tau_{t+1})F_K(\frac{k^*}{1+n},l)k^* \tag{51}
$$

(2) tax interest income at every date $t' \geq t+2$ at a rate $1-\tau_{t'}$ with

$$\tau_{t'} = \frac{1}{F_K(\frac{k^*}{1+n}, l)} \left[\frac{1}{\beta} \frac{u'(F_L(\frac{k^*}{1+n}, l)l - k^*)}{u'(F_K(\frac{k^*}{1+n}, l)k^*)} - 1 \right] \tag{52}$$

and make a lump-sum transfer $T_{t'}$ to every old agent at t', with

$$T_{t'} = (1-\tau_{t'})F_K(\frac{k^*}{1+n}, l)k^* \tag{53}$$

It is straightforward to check that this policy makes all generations $t' \geq t$ choose to save exactly k^*. Moreover, the government budget stays balanced every period. Of course, the implementation of this policy requires to be able to know the marginal rates of substitution of old-age consumption to young-age consumption for the representative agent at different profiles of consumption. This is therefore more demanding than the implementation of the policy sustaining k^* (without any transition) as a steady state competitive equilibrium provided in Proposition 4 above, since that policy only required to be able to know at each period the per capita savings decided in the previous period.

Note also that, at the optimal steady state level of per capita savings k^* attainable through the good and factor markets, if $\tau^* < 1$ (respectively, $\tau^* > 1$), the net of tax (resp. of subsidy) interest rate $\tau^* F_K(\frac{k^*}{1+n}, l)$ *does not* (resp. *does*) exceed the population growth rate n when $k^* < k^g$ (resp. $k^* > k^g$). In effect, since

$$\tau^* = \frac{1}{F_K(\frac{k^*}{1+n}, l)} \left[(1+n) \cdot \frac{1 + F_K(\frac{k^*}{1+n}, l) + F_{KK}(\frac{k^*}{1+n}, l)\frac{k^*}{1+n}}{(1+n) + F_{KK}(\frac{k^*}{1+n}, l)\frac{k^*}{1+n}} - 1 \right] \tag{54}$$

then

$$\begin{aligned} \tau^* F_K(\frac{k^*}{1+n}, l) &= (1+n) \cdot \frac{1 + F_K(\frac{k^*}{1+n}, l) + F_{KK}(\frac{k^*}{1+n}, l)\frac{k^*}{1+n}}{(1+n) + F_{KK}(\frac{k^*}{1+n}, l)\frac{k^*}{1+n}} - 1 \\ &< (>) n \end{aligned} \tag{55}$$

because $F_K(\frac{k^*}{1+n}, l) < (>) n$ when $k^* > (<) k^g$, and $(1+n) + F_{KK}(\frac{k^*}{1+n}, l)\frac{k^*}{1+n} > 0$, as established in Proposition 2. As a consequence, the result in Diamond (1970) does not apply to the tax (if $\tau^* < 1$) or subsidy (if $\tau^* > 1$) rate that implements steady state level of per capita savings k^* that maximizes the representative agent utility among those attainable through the good and factor markets, both when $k^* < k^g$ and when $k^* > k^g$.

6. Tax or subsidize savings?

Proposition 4 above established that the optimal steady state attainable through the good and factors markets can be attained in a decentralized way with the appropriate tax rate (positive or negative) on savings returns coupled with some lump-sum transfer or tax. But when exactly attaining through the existing markets the best possible steady requires taxing savings and when subsidizing?

As the next proposition establishes, if the optimal steady state level of per capita savings k^* attainable though the good and factors markets requires over-accumulating capital with respect to the first-best steady state level of per capita savings k^g, then k^* can only be attained taxing savings returns and distributing second period lump-sum transfers. Conversely, if k^* requires under-accumulating capital with respect to the first-best steady state, it can only be attained subsidizing savings returns and raising a second period lump-sum tax.

Proposition 5. *In the Diamond (1965) overlapping generations economy, the decentralization of the optimal steady state level of per capita savings k^* attainable through the good and factor markets as a competitive equilibrium requires taxing (resp. subsidizing) linearly the capital income, coupled with a second period lump-sum transfer (resp. tax) if, and only if k^* is bigger (resp. smaller) than the first-best steady state level of per capita savings k^g, i.e.*

$$\tau^* < (>) 1 \Leftrightarrow k^g < (>) k^*. \tag{56}$$

Proof. Note that, at the steady state level of per capita savings k^*, capital revenue is taxed (resp. subsidized) if the constant rate

$$\tau^* = \frac{1}{F_K(\frac{k^*}{1+n}, l)} \left[(1+n) \cdot \frac{1 + F_K(\frac{k^*}{1+n}, l) + F_{KK}(\frac{k^*}{1+n}, l)\frac{k^*}{1+n}}{(1+n) + F_{KK}(\frac{k^*}{1+n}, l)\frac{k^*}{1+n}} - 1 \right] \tag{57}$$

is smaller (respectively, bigger) than 1, i.e. if, and only if,

$$(1+n) \cdot \frac{1 + F_K(\frac{k^*}{1+n}, l) + F_{KK}(\frac{k^*}{1+n}, l)\frac{k^*}{1+n}}{(1+n) + F_{KK}(\frac{k^*}{1+n}, l)\frac{k^*}{1+n}} < 1 + F_K(\frac{k^*}{1+n}, l) \tag{58}$$

which (given that the denominator in the right-hand side is positive, as established in Proposition 2) holds if, and only if,

$$F_K(\frac{k^*}{1+n}, l) < n = F_K(\frac{k^g}{1+n}, l) \tag{59}$$

i.e. if, and only if,

$$k^* > k^g. \tag{60}$$

Q.E.D.

In case it may seem counterintuitive that taxing in a distortionary may allow to improve upon the laissez-faire competitive steady state, note that what the taxation of capital income aims at is the reduction of the overaccumulation of capital (with respect to the unattainable first-best) from the laissez-faire competitive equilibrium level of per capita savings k^c to a smaller level k^*.[16] Reducing per capita savings further below k^* is not efficient if factors are to be remunerated by their marginal productivities and no redistribution can take place. Similarly, subsidizing savings returns, but not up to the firs-best level, allows to improve upon the laissez-faire in case it leads to excessive under-accumulation.

Appendix

Lemma. *In the Diamond (1965) overlapping generations economy, it holds that*

$$k^c > k^* \Leftrightarrow k^c > k^g$$

and

$$k^c < k^* \Leftrightarrow k^c < k^g.$$

with k^c, k^*, and k^g being the per capita level of savings at, respectively, the competitive equilibrium steady state, the steady state maximizing the representative agent utility through the good and factors markets, and the first-best steady state.

[16]But not its complete elimination, as the first-best would require, given the lack of a mechanism to implement the intergenerational transfers necessary to attain it.

Proof. Regarding claim that $k^c > k^*$ holds if, and only if, $k^c > k^g$, note that a necessary and sufficient condition for $k^c > k^*$ is

$$1 + F_K(\frac{k^c}{1+n}, l) > (1+n)\frac{1 + F_K(\frac{k^c}{1+n}, l) + F_{KK}(\frac{k^c}{1+n}, l)\frac{k^c}{1+n}}{(1+n) + F_{KK}(\frac{k^c}{1+n}, l)\frac{k^c}{1+n}} \tag{13}$$

—where the left-hand side is the slope of the representative agent indifference curve at the competitive equilibrium steady state profile of consumptions (c_0^c, c_1^c) (see Figure 1), and the right-hand side is the slope at that same point of the curve $c(k)$ of constant consumption profiles attainable through the existing markets, both in absolute value— and condition (13) holds if, and only if,

$$F_K(\frac{k^c}{1+n}, l) < n = F_K(\frac{k^g}{1+n}, l) \tag{14}$$

i.e. if, and only if, $k^c > k^g$. This follows from the fact that $(1+n)+F_{KK}(\frac{k^c}{1+n}, l)\frac{k^c}{1+n} > 0$. In effect, the alternative inequality $(1+n) + F_{KK}(\frac{k^c}{1+n}, l)\frac{k^c}{1+n} < 0$ (in the knife-edge case of equality, in which the slope of $c(k)$ at k^c is vertical, it cannot be that $k^c > k^g$) would imply that the marginal rate of substitution of old-age consumption to young-age consumption is, in absolute value, bigger than the negative of the slope of the curve of steady state market consumptions at $c(k^c)$ (so that $k^c > k^*$) if, and only if, $k^g > k^c$, which implies $k^g > k^c > k^*$. Nevertheless, the optimal policy to implement k^* (see Proposition 5) requires, in the case $k^g > k^*$, subsidizing the return to savings in order to *increase* —rather than reduce— savings from its laissez-faire competitive level k^c towards the first-best level k^g, i.e. $k^g > k^* > k^c$ instead of $k^g > k^c > k^*$. Finally, from $k^c = k^* \Leftrightarrow k^c = k^g$ in Proposition 3 and the previous result the rest of the statement follows. Q.E.D.

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