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Abstract

We propose in this note a simple non-parametric test of Richter-rationality which is the basic definition of rationality used in choice functions theory. Loosely speaking, the data set is rationalizable in the Richter' sense if there exists a complete-acyclic binary relation that rationalizes the data set. Hence a data set is rationalizable in the Richter' sense if there exists a variable intervals function which rationalizes this data set. Since an acyclic binary relation is not necessary transitive then the proposed Richter-rationality test is weaker than $GARP$. Finally the test is performed over Mattei's[1] data sets.

1 Introduction

The test of $GARP$ aims to test the existence of an utility function which rationalizes the data set. However from a mathematical standpoint, an utility function is an order isomorphism. Therefore (over a finite set) to require an agent's utility to be represented by a utility function is equivalent to requiring this agent's preference relation to be *complete* and *transitive*. Unfortunately there is no theoretical or empirical justification to require a preference to be transitive. This problem is an old one (see for instance Fechner [1860], Poincaré [1902], or Armstrong [1939]) and can be understood through the following example: *an agent with a preference relation R can be indifferent between 10g of sugar and 11g, between 11g of sugar and 12g of sugar, but not be indifferent between 10g and 12g.* Since this agent's indifference relation is not transitive and is a component of the agent's preference relation R, then R is not transitive. The theoreticians of preferences have therefore built several non-transitive preferences whose asymmetric component is transitive and whose symmetric component is not necessarily transitive. The most famous relation of such a type is the so-called semi-order whose functional representation has been widely studied by Luce [1956], Scott and Suppes [1958], Roberts [1970], Fishburn [1970], Fishburn [1973], Bridges [1983], Chateauneuf [1987], and many others. However the requirement of the transitivity of the asymmetric part of a binary relation is also not easy to justify and has been challenged by Kreweras [1961], Burros [1974], Bell [1982], Fishburn [1984], Anand [1987], Fishburn and Lavalle [1987], Loomes [1991], Dombi and Vincze [1994], and many others. Hence the modelling of preferences by acyclic preference relations (whose both symmetric and asymmetric components are not necessarily transitive). Now for most theoreticians of preferences the "appropriate" system of axioms over preference relations is the following: *preference is complete, preference is acyclic.* Abbas and Vincke [1993], Agaev and Alekserov [1993], Subiza [1994], Rodriguez–Palmero [1997], Diaye [1999] show that such a preference can be represented by a functional which is called *Variable Intervals Function.* Therefore (as in the case of a preorder preference), if there exists a complete-acyclic preference which rationalizes a data set, then there exists a correspondence demand which rationalizes this set. And of course an agent who uses (in order to choose) a correspondence demand derived from a variable intervals function is not less rational than an agent who uses a correspondence demand derived from an utility function. Both agents maximize their preferences and they can both choose their optimal elements over any subset of the main set of elements. Let us recall at this stage that by rationality, we mean the basic definition of rational choice by Richter [1971] which states that a choice is rational if there exists a binary relation which rationalizes this choice: the chosen elements over any set S corresponds to the most preferred elements in S with respect to a binary relation. The purpose of our paper is to provide a simple test of this Richter-Rationality in the case where the revealed preference relation is rationally equivalent to a complete-acyclic binary relation. This test can be achieved by testing an axiom called by us, $RARP$ (Richter Axiom of Revealed Preference). Since rationalization by a complete-acyclic preference is weaker than rationalization a preorder then $RARP$ is weaker than $GARP$ (and also weaker than $SARP$). Hence we expect

to find over data sets more individuals who respect $RARP$. Actually we think that the main reason why the $GARP$'s tests performed over experimental data sets[2](or over micro-economic data sets) find a significant number of violations is that some individuals who are rational in the Richter' sense are declared irrational by the test of $GARP$.

The paper is organized as follow. The second section sets some basic definitions in choice functions theory and demand theory. The third section is devoted to our main result which proves that a data set can be rationalized by a variable intervals function (i.e. by complete-acyclic preferences) i.f.f it satisfies $RARP$. In the fourth section, we test this axiom over Mattei's experimental data sets already used in Mattei [2000]. Finally the section 5 concludes.

2 Preliminary

2.1 Binary relations.

Let Q be a binary relation over a set X (i.e. Q is a subset of $X \times X$). Q can be divided into an asymmetric component, denoted P_Q, defined by $\forall x, y \in X, xP_Qy \Leftrightarrow xQy$ and $not(yQx)$; and a symmetric component, denoted I_Q, defined by $\forall x, y \in X, xI_Qy \Leftrightarrow xQy$ and yQx; we shall write $Q = P_Q + I_Q$.

Let us define the following properties of a binary relation Q on the set X:

- Q is reflexive if $\forall x \in X, xQx$.

- Q is complete if $\forall x, y \in X, xQy$ or yQx.

- Q is asymmetric if $\forall x, y \in X, xQy => not(yQx)$.

- Q is antisymmetric if $\forall x, y \in X, x \neq y, xQy => not(yQx)$.

- Q is transitive if $\forall x, y, z \in X, xQy$ and $yQz => xQz$.

- Q is acyclic if $x_1, \ldots, x_n \in X,\ not(x_1P_Qx_2P_Q \ldots . P_Qx_nP_Qx_1)$.

- Q is a weak order if Q is reflexive and transitive.

- Q is a preorder if Q is complete and transitive.

- Q is an order if Q is reflexive, antisymmetric and transitive.

- Q is a linear order if Q is asymmetric, transitive, and if $\forall\ x, y \in X, x \neq y$, xQy or yQx.

- Q is a complete order if Q is complete, antisymmetric and transitive.

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[2]For instance Sippel [1997], Mattei [2000], Février and Visser [2004]. See also Harbaugh et al.[2001], Andreoni and Miller [2002].

Let us set the following definitions:

- The dual relation of Q denoted Q^d is defined by:

$$Q^d = \{(x,y) \in X^2 : (y,x) \in Q\}$$

- The lower section associated with x denoted $Q(x \rightarrow)$ is the set:

$$\{y \in X : xQy\}$$

- The upper section associated with x denoted $Q(\rightarrow x)$ is the set:

$$\{y \in X : yQx\}$$

2.2 Choice Functions and Demand Theory

Definition 1 *Let X be a set of objects, $P(X)$ be the set of subsets of X, and F be a set of non-empty subsets of X. F is called a domain of choice and (X, F) is called a choice space.*

Definition 2 *A domain of choice F is selective if $F \neq P(X) \backslash \varnothing$ and it is abstract otherwise. If F is selective but it includes all finite non-empty subsets of X then it is said to be quasi-abstract.*

Definition 3 *A choice function is a function C defined from F to $P(X)$ with the condition that $C(S) \subseteq S$.*

We will restrict ourselves to the class of *decisive choice functions*, that is choice functions such that $C(S) \neq \varnothing$, $\forall\ S \in F$. We can derive the following binary relations (over X) from a choice function C:

$$\forall\ x,y \in X,\ xRy \Leftrightarrow \exists\ S \in F : x \in C(S)\ \ and\ \ y \in S \qquad (2.1)$$

$$\forall\ x,y \in X,\ xKy \Leftrightarrow \exists\ S \in F : x \in C(S)\ \ and\ \ y \in S\backslash C(S) \qquad (2.2)$$

Definition 4 *(Richter [1971]) A choice function C is rational if there is a binary relation Q (over X) such that*

$$\forall\ S \in F,\ C(S) = \{x \in S : xQy,\ \forall\ y \in S\}$$

Richter has also shown that any binary relation rationalizing a choice function C is rationally equivalent to R the so-called *revealed preference relation*. Therefore a choice function is rational i.f.f it is rationalizable by the revealed preference relation R.

Remark 1 *Of course the revealed preference relation R is not necessarily complete or transitive.*

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If we want the revealed preference relation which rationalizes a choice function to fulfil some properties like completeness or transitivity then the choice function C has to respect some well-known conditions like $WARP$ or $SARP$.

Criterion 1 (Weak Axiom of Revealed Preference) $\forall\ x, y \in X,\ xKy \Rightarrow not(yRx)$

Criterion 2 (Strong Axiom of Revealed Preference) $\forall\ x, y \in X,\ xK^*y \Rightarrow not(yRx)$, *where K^* is the transitive closure of K.*

For instance if the domain of choice F is abstract or quasi-abstract, then (see Arrow [1959]) $WARP$ and $SARP$ are equivalent and a choice function C respects $WARP$ i.f.f it is rational and R (the revealed preference) is a preorder.

2.3   Demand Functions

Let us now consider consumer theory which is, from a mathematical standpoint a sub-theory in the sense of Bourbaki [1954] of the theory of choice functions. In consumer theory, the set of objects is $X = \{\cdots, x_1, \cdots, x_i, \cdots\}$ with $x_i = \left(x_i^1, \cdots, x_i^k\right)$ where the x_i^j are quantities of goods $j = 1$ *to* k. X is included in $\mathbb{R}_+^k$.

$\boldsymbol{B}$ (a set of subsets of X) is the domain of choice, the elements of $\boldsymbol{B}$ are called *budgets*, and $(X, \boldsymbol{B})$ is called *budget space*.

Moreover the choice functions are called in consumer theory, *demand "functions"* denoted by h.

Let x be a quantity-vector, p be the price-vector at which x is available and let m be an income. Only the competitive budgets $\{x \in X : p.x \leq m\}$ are of interest to a theorist. Hence only the subset $\boldsymbol{C}$ (of $\boldsymbol{B}$) :

$$\boldsymbol{C} = \{\{x \in X : p.x \leq m\}, \forall p, \forall m\}$$

is of interest to us.

Let us denote by (p, m) such a competitive budget. The revealed preference relation R will be defined in consumer theory by :

$\forall\ x_i,\ x_j \in X,\ x_i\ R\ x_j \Leftrightarrow \exists\ (p, m) \in \boldsymbol{C} : x_i \in h(p, m)\ \ and\ \ x_j \in (p, m)$

That is :

$\forall\ x_i,\ x_j \in X,\ x_i\ R\ x_j \Leftrightarrow \exists\ (p, m) \in \boldsymbol{C} : x_i \in h(p, m)\ \ and\ \ p.x_j \leq m$

Under the locally nonsatiation hypothesis, we have the following property sometimes called *Walras law* (See Mas-Colell et al. 1995).

$$\forall\ (p, m) \in \boldsymbol{C}\ \ and\ \ x \in h(p, m),\ \ p.x = m$$

and the revealed preference R is therefore defined by :

$\forall\ x_i,\ x_j \in X,\ x_i\ R\ x_j \Leftrightarrow \exists\ (p, m) \in \boldsymbol{C} : x_i \in h(p, m)\ \ and\ \ p.x_j \leq p.x_i$ \quad (2.3)

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The binary relation K in equation (2.2) becomes in consumer theory (under the locally nonsatiation hypothesis):

$$\forall\, x_i, x_j \in X,\ x_i K\, x_j \Leftrightarrow \exists\, (p,m) \in \boldsymbol{C} : x_i \in h(p,m),\ p.x_j \leq p.x_i\ and\ x_j \notin h(p,m) \tag{2.4}$$

Therefore $WARP$ can be rewritten (in consumer theory) by :

$$\forall\, x_i,\ x_j \in X,\ x_i \neq x_j,\ \ x_i\ R\ x_j \Rightarrow not(x_j\ R\ x_i) \tag{2.5}$$

That is R is antisymmetric. And $SARP$ becomes :

$$\forall\, x_i,\ x_j \in X,\ x_i \neq x_j,\ \ x_i\ R^*\ x_j \Rightarrow not(x_j\ R^*\ x_i) \tag{2.6}$$

where R^* is the transitive closure of R.

2.4   The Non-parametric Tests of $WARP$, $SARP$ and $GARP$

Let $D = \{(x_i, p_i)\}_{i=1}^{N}$ be a data set including prices $p_i \in \mathbb{R}_+^n$ and bundles of goods $x_i \in \mathbb{R}_+^n$ purchased at price p_i. It is therefore possible to construct the revealed preference relation R from the data set D. This relation is easy to construct since for any x_i which belongs to the data set, there exists $(p_i, m) \in \boldsymbol{C}$ such that $x_i \in h(p_i, m)$. This is the reason why (see equation (2.3)) the revealed preference R is usually defined, in most papers devoted to non-parametric tests, by :

$$\forall\, x_i,\ x_j,\ \ \ x_i\ R\ x_j \Leftrightarrow p_i.x_j \leq p_i.x_i \tag{2.7}$$

and a binary relation denoted RS (called in the literature the strict revealed preference) is defined by:

$$\forall\, x_i,\ x_j,\ \ \ x_i\ RS\ x_j \Leftrightarrow p_i.x_j < p_i.x_i \tag{2.8}$$

Moreover in order to take into account errors of optimization and/or measurement that can affect total expenditure such that the "true" value of the total expenditure is $e \times (p_i.x_i)$ with $e \in [0,1]$ called the Afriat efficiency index, researchers construct the binary relations R and RS for several values of e. In this case (2.7) and (2.8) become respectively:

$$\forall\, x_i,\ x_j,\ \ \ x_i\ R\ x_j \Leftrightarrow p_i.x_j \leq e \times p_i.x_i$$

$$\forall\, x_i,\ x_j,\ \ \ x_i\ RS\ x_j \Leftrightarrow p_i.x_j < e \times p_i.x_i$$

Our tests in section 4 are performed for $e = 1$ ant for the *Afriat critical index*, that is to say the value of e such that there is no violation of the axiom tested. However in what follows and until section 4 we assume without loss of generality that $e = 1$, that is to say R and RS are defined by (2.7) and (2.8).

Now set the following definition :

Definition 5 [*Varian 1982*] *A utility function u rationalizes the data set* $D = \{(x_i, p_i)\}_{i=1}^{N}$ *if for any* x_i,

$$u(x_i) \geq u(x)\ whatever\ x \geq 0\ such\ that\ p_i.x \leq p_i.x_i$$

and set the Generalized Axiom of Revealed Preference (GARP).

Criterion 3 *(GARP [Varian 1982])* $\forall\ x_i,\ x_j \in X,\ \ x_i\ R^*\ x_j \Rightarrow not(x_j\ RS\ x_i)$

It is obvious that $SARP$ implies $GARP$ and $WARP$. However there is no relationship in general between $GARP$ and $WARP$. Let us now recall that for finite cases, the axioms and implications below are equivalent :

<u>Axiom</u>	<u>Implications</u>
$SARP$: eq.(2.6)	*Stable complete-order preference,* *there is a utility function that rationalizes the data set,* *there is a demand function that rationalizes the data set.*
$GARP$: crit.3	*Stable preorder preference,* *there is a utility function that rationalizes the data set,* *there is a demand correspondance that rationalizes the data set.*
$WARP$: eq.(2.5)	*Stable complete and antisymmetric preference,* *there is a function that rationalizes the data set.*

3 The main result: rationalization by a variable intervals function

In the papers which use non-parametric tests, the main issue is to test the rationalization of the data set by a utility function. Indeed the main motivation of these papers is to test if the agents' empirical behavior is the same as the one postulated in microeconomic theory : maximization of a utility function under a budget constraint. However if the criterion of optimization is not disputable (from our point of view) and is not disputed in general, the requirement of the preference relation to be transitive is disputed. Some arguments against the transitivity axiom can be found in Anand [1987], Fishburn [1988], Anand [1993] and especially in Burros [1974]. There is a consensus among preferences theory's researchers that a good candidate for preference consistency should be the acyclicity axiom. There are two main reasons for this consensus. First having an acyclic preference does not prevent an agent to choose : this agent can maximize his preference. The second reason is that this axiom is compatible with the choice function theory. Indeed as shown by Jamison and Lau [1973], when the domain of choice is abstract then a choice is (Richter-) rational i.f.f it is rationalizable by a complete acyclic preference relation. Let us therefore set that the agents' preference relations are complete and acyclic. Such a preference relation is representable by a function called *Variable Intervals Function* exactly as a preorder preference is representable by a utility function.

Definition 6 *Let Q be a binary relation over a set X. (X, Q) satisfies the Variable Intervals Model if there exist two functions (u, s) with $u : X \to \mathbb{R}$ and*

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$s : X \times X \to \mathbb{R}_+$ *such that :*

$$xP_Qy \quad \Leftrightarrow \quad J(x,y) > J(y,x) \tag{3.1}$$
$$xI_Qy \quad \Leftrightarrow \quad J(x,y) \cap J(y,x) \neq \emptyset$$

$$\tag{3.2}$$

where:

$$J(x,y) = [u(x), \ u(x) + s(x,y)] \ \textit{ is an interval of the real line.}$$

Over a finite set X, a binary relation Q is complete and acyclic i.f.f (X,Q) satisfies the variable intervals model (see for instance Diaye [1999]).

Remark 2 *Let us set the function $f(x,y) = u(x) + s(x,y)$. It is easy to see that*

$$J(x,y) \quad > \quad J(y,x) \Leftrightarrow u(x) > f(y,x)$$
$$J(x,y) \cap J(y,x) \quad \neq \quad \emptyset \Leftrightarrow u(x) \leq f(y,x) \ \ and \ \ u(y) \leq f(x,y)$$

If there is no risk of confusion, we will call such a function f, a *Variable Intervals Function*. The function u can be understood as the representation of the agent's underlying preorder preference. However the context of choice makes him deviate (through the threshold function s) from this underlying preference. It is nevertheless important to stress that s is definitively not an error term in the probabilistic sense. If we assume that s is symmetric then we get the below characterization of variable intervals functions by Abbas and Vincke [1993]:

$$xP_Qy \quad \Leftrightarrow \quad u(x) - u(y) > s(x,y)$$
$$xI_Qy \quad \Leftrightarrow \quad |u(x) - u(y)| \leq s(x,y)$$

which permits the following interpretation: the agent strictly prefers x to y if the difference of their utilities is greater than a threshold function s which depends on x and y.

Example 1 *Let $X \subseteq \mathbb{R}_+^2$ and $f(x,y) = u(x) + \|x - y\|^2$. Take for instance $x = (1,1), \quad y = (4,1), \quad z = (2.5, 2), \quad u(x) = 8.5, \quad u(y) = 1.5, \quad u(z) = 5$. x is strictly preferred to z which is strictly preferred to y, but the agent's is indifferent between x and y.*

Of course since acyclic preferences allow for thick indifference curves, local non-satiation is not fulfilled. Let us set the following weaker condition.

Definition 7 *A preference relation Q on X is weak locally non-satiated if for every $x \in X$ and every $\varepsilon, \varepsilon' > 0$, there exist $y, z \in X$ such that $\|x - y\| < \varepsilon$ and $\|y - z\| < \varepsilon'$ and $y \, Q \, z \, P_Q \, x$.*

If the agent maximizes a variable intervals function instead of a utility function then he can, over a data set, violate $GARP$, $SARP$ and $WARP$.

Example 2 *Let R be the agent's acyclic revealed preference over a data set :*

$$
\begin{array}{ccc}
x & \longrightarrow & z \\
 & \searrow & \downarrow \\
 & & y
\end{array}
$$

$$where \quad \longrightarrow \quad = RS = P_R$$
$$and \searrow \quad = \quad I_R$$

SARP is violated because R^* the transitive closure of R is not antisymmetric, WARP is violated because R is not antisymmetric, and GARP is violated because yR^*z and z RS y.

In the above example 2, the agent is "irrational" in the sense that he violates GARP, while he is able to choose the best element with respect to his preference (which is here x). This is why we propose instead to check the compatibility of the data set with a variable intervals function maximization.

Definition 8 *A variable intervals function $f = u + s$ rationalizes the data set $D = \{(x_i, p_i)\}_{i=1}^{N}$ if for any x_i,*

$$(i). \qquad u(x_i) > u(x) + s(x, x_i) \ \ or$$
$$(ii). \qquad u(x_i) \leq u(x) + s(x, x_i) \ \ and \ \ u(x) \leq u(x_i) + s(x_i, x)$$
$$whatever \ x \ \geq \ 0 \ \ such \ that \ \ p_i.x \leq p_i.x_i$$

As already stressed out by Varian [1982, page 946] in the case of rationalization by a utility function, only rationalization by a non degenerated variable intervals function is of interest to us:

Definition 9 *The data set $D = \{(x_i, p_i)\}_{i=1}^{N}$ is rationalizable by a non degenerated variable intervals function if:*

$$P_R \neq \emptyset \Rightarrow not\{u(x_i) \leq u(x_j) + s(x_j, x_i)$$

$$and$$

$$u(x_j) \leq u(x_i) + s(x_i, x_j)$$
$$\forall x_i, x_j, x_i \neq x_j\}$$

Where P_R is the asymmetric component of R the revealed preference relation in (2.7).

Criterion 4 ($RARP$) *A data set satisfies the Richter Axiom of Revealed Preference (RARP) if*

$$\forall \ i \neq j, \quad x_i \ P_R^* \ x_j \Rightarrow not(x_j \ P_R \ x_i)$$

Where P_R^* is the transitive closure of P_R.

Theorem 1 (The main result) [3] *The following two conditions are equivalent.*

1. *The data set satisfies RARP.*

2. *There exists a weakly locally non-satiated, variable intervals function which rationalizes the data set.*

$RARP$ is easy to test and its algorithm complexity is exactly the same as the one of $GARP$.

4 Tests over Mattei's experimental data sets

We want now to test the $RARP$, $GARP$, $SARP$ and $WARP$ over Mattei's experimental sets.

The purpose of our tests is first to distinguish over the data sets the individuals who are Richter-rational in the sense that they respect $RARP$ from those who are not rational, and second to distinguish among the rational individuals those who are complete-order preference maximizers ($SARP$), preorder preference maximizers ($GARP$) or complete-acyclic preference maximizers ($RARP$).

4.1 The data sets and the tests.

The three experimental data sets constructed by Mattei [2000] include respectively 20, 100, and 320 individuals. They have to choose among 8 goods[4] in 20 different budget situations.

Note that in order to compute the transitive closure of the revealed preference relation (GARP and SARP), and that of its asymmetric component (RARP), we have used an algorithm which determines all the paths between two vertex in the matrix associated to the revealed preference relation (GARP and SARP) and the matrix associated to its asymmetric component (RARP). The results are the same as those of the algorithms of minimum cost path like Warshall's algorithm or Dijkstra algorithm.

The approximate Bronars power have been computed using the second Bronars algorithm which is described in appendix 3.

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[3]The proof is given in appendix 1.

[4]In the first two data sets, the goods are: milk chocolate, salted peanuts, biscuits, text maker, ball-point pen, plastic folder, writing pad, post it. In the third data set, the goods are: milk chocolate, biscuit, orange juice, iced tea, post it, audio cassette c90, ball point pen, battery(R6, 1.5V).

4.2 Results

Tables 1 to 3 report the number of subjects who violate at least once the considered axiom for several values of the *Afriat Index*. We can read these results in the following way.

Over the first data set (table 1):

- 15 (75 percent) subjects are complete-order preference maximizers and preorder preference maximizers (their behavior satisfies SARP and GARP).

- None of the subjects is irrational in the Richter sense.

Over the second data set (table 2):

- 41 (41 percent) subjects are complete-order preference maximizers (their behavior satisfies SARP).

- 56 (56 percent) subjects are preorder preference maximizers (their behavior satisfies GARP).

- 97 (97 percent) subjects are complete-acyclic preference maximizers (their behavior satisfies RARP).

- 3 are irrationals in the Richter sense.

Over the third data set (table 3):

- 155 (48.75 percent) subjects are complete-order preference maximizers (their behavior satisfies SARP).

- 219 (68.43 percent) subjects are preorder preference maximizers (their behavior satisfies GARP).

- 304 (95 percent) subjects are complete-acyclic preference maximizers (their behavior satisfies RARP).

- 16 are irrationals in the Richter sense.

Table 1
Number of WARP, SARP, GARP, and RARP irrationals subjects over the first data set (20 subjects).

Afriat Index	GARP	SARP	WARP	RARP
1.00	5	5	5	0
0.99	2	2	2	0
0.98	1	1	1	0
0.97	1	1	1	0
0.96	0	0	0	0

Table 2
Number of WARP, SARP, GARP, and RARP irrationals subjects over the second
experimental data set (100 subjects).

Afriat Index	GARP	SARP	WARP	RARP
1.00	59	59	44	3
0.99	30	30	30	2
0.98	15	16	16	1
0.97	11	11	11	1
0.96	8	8	8	2
0.95	5	5	5	2
0.94	3	3	3	2
0.93	3	3	3	0
0.83	0	0	0	0

Table 3
Number of WARP, SARP, GARP, and RARP irrationals subjects over the third
experimental data set (320 subjects).

Afriat Index	GARP	SARP	WARP	RARP
1.00	164	165	101	16
0.99	65	66	66	10
0.98	50	50	50	5
0.97	35	35	35	6
0.96	22	22	22	3
0.95	17	17	17	2
0.94	14	14	14	3
0.93	11	11	11	0
0.66	0	0	0	0

Let us now call *Absolute Power in the Sense of Bronars*, the standard Power Index in the sense of Bronars (see. Appendix 2). However we define also a *Relative Power Index* in the sense of Bronars. In order to understand why, let us recall that power in the sense of Bronars aims to compare the behavior of the individuals in the data sets with that of individuals who choose their consumption bundles randomly from their budget sets. Therefore, what is important in this notion of power is not only its absolute value but the relative increase of the number of "irrational" (in the sense of RARP, WARP, GARP or SARP) individuals when testing the axioms over the initial data sets and the random consumption data sets. For instance suppose that when testing GARP over a given data set, the (absolute) Bronars Power Index is 100%. What can we conclude if at the same time the percentage of GARP violating individuals on this data set is 98% ? Let us compare this example with the following one where when testing GARP over a data set, the (absolute) Bronars Power Index is 60%; but the percentage of GARP-violating individuals on this data set is 1%. Since the relative increase of GARP-violating individuals is greater in the second example than in the first one, we will say that the relative Bronars Power Index (that is the relative increase of GARP-violating individuals between the initial data sets and the random consumption data sets) is bigger in the second example than in the first one. Thus over the second and the third experimental data sets (respectively 100 and 320 subjects), the (absolute) Bronars Power Index of RARP is on average about 55 per cent for an Afriat Index of 1 and is weaker than those of WARP, GARP and SARP. But the Relative Bronars Power Index of RARP is on average about 13.5, about ten times greater than

those of WARP, GARP, SARP. Finally, the reason why the (absolute) power in the sense of Bronars of RARP is "small" compared to those of GARP for instance, is that RARP is a very very weak axiom and the use of other definitions of (absolute) power (see Harbaugh and Adreoni [2005]) will change nothing to this fact.

Table 4
Bronars Power Index (method 2) and Relative Bronars Power Index (result in brackets) over the first experimental data set

Afriat Index	Warp	Sarp	Garp	Rarp
1	0.968 (2.872)	0.983 (2.932)	0.9745 (2.898)	0.513 (.)
0.99	0.917 (8.17)	0.9445 (8.44)	0.9465 (8.465)	0.412 (.)
0.98	0.834 (15.68)	0.861 (16.22)	0.8765 (16.53)	0.3295 (.)
0.97	0.741 (13.82)	0.759 (14.18)	0.748 (13.96)	0.235 (.)

Table 5
Bronars Power Index (method 2) and Relative Bronars Power Index (result in brackets) over the second experimental data set

Afriat Index	Warp	Sarp	Garp	Rarp
1.00	0.9882 (0.6749)	0.9917 (0.68)	0.9896 (1.24)	0.5617 (17.72)
0.99	0.9475 (2.15)	0.9595 (2.19)	0.9606 (2.20)	0.4477 (21.38)
0.98	0.8542 (4.69)	0.8861 (4.53)	0.8836 (4.52)	0.3381 (32.81)
0.97	0.7271 (5.61)	0.7449 (5.77)	0.7619 (5.92)	0.2263 (21.63)

Table 6
Bronars Power Index (method 2) and Relative Bronars Power Index (result in brackets) over the third experimental data set

Afriat Index	Warp	Sarp	Garp	Rarp
1.00	0.9680 (0.88)	0.9830 (0.90)	0.98888 (2.13)	0.55572 (10.11)
0.99	0.9170 (3.51)	0.9445 (3.57)	0.95716 (3.64)	0.44106 (13.11)
0.98	0.8340 (4.33)	0.8610 (4.51)	0.88119 (4.63)	0.32809 (19.99)
0.97	0.7410 (5.77)	0.7590 (5.93)	0.75109 (5.86)	0.22184 (10.83)

5 conclusion

The purpose of $GARP$ testing is to see if agents in the "real world" maximize a utility function as postulated by economic theory. According to various tests over experimental data sets from many researchers, it seems that the answer is negative. We think that this answer is quite normal. Indeed to require an agent to maximize a utility function is equivalent to requiring him to maximize a complete and transitive preference relation. However the basic definition of rationality in choice functions theory (recall that demand theory is a sub-theory in the sense of Bourbaki [1954] of choice functions theory) states that a rational agent is the one who maximizes a preference relation. There is no need for this preference to be transitive. Nevertheless we can require the preference relation

to be (at least) acyclic. The question (set by our paper) is the following : do agents in the "real world" maximize a complete-acyclic preference relation, that is, do they maximize a generalized utility function called variable intervals function? The answer provided by our tests on Professor Mattei's experimental data sets seems to be yes. Indeed these tests have been checked for consistency with an axiom we called $RARP$ equivalent to the rationalization by a variable intervals function (who can represent a complete-acyclic preference), and we found that the number of individuals who are complete-acyclic preference maximizers (RARP consistent) represents over the three data sets respectively, 100 percent (first data set), 93 percent (second data set), and 84 percent (third data set), of individuals who are not preorder maximizers (who are not GARP consistent). Thus although over the three data sets more than 30 percent individuals are not preorder maximizers, a great part of them (more than ninety percent on average) are rational in Richter sense.

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<center>**Appendix 1: Proof of Theorem 1.**</center>

1. **Definition: Generalized Afriat Inequalities (GAI).**

   GAI is fulfilled if there exist numbers $u_i > 0$, $s_{ij} = s_{ji} \geq 0$ $\forall i = 1, ...n$, and some numbers λ_i, α_{ij}, $\alpha_{ji} > 0$, $i, j = 1, ...n$ such that:

   $$u_i + s_{ij} < u_j + \lambda_j p_j (x_i - x_j) \tag{5.1}$$

   or

   $$|u_i - u_j| = s_{ij} + \alpha_{ji} p_j (x_i - x_j) \tag{5.2}$$
   $$\forall i, j = 1, ...n$$

   We can remark that it is not possible for any i, j to fulfill both (5.1) and (5.2) (in the above definition). Moreover, if $p_j x_j \geq p_j x_i$ and $not(p_i x_i \geq p_i x_j)$ then it is impossible to have (5.2). And when $p_j x_j \geq p_j x_i$ and $p_i \geq p_i x_j$ then it is impossible to have (5.1).

2. **Proof of theorem** 1.

   (a) <u>$(2) \Rightarrow (1)$</u>

   Suppose that $x_i P_R^* x_j$ and $x_j P_R x_i$. It must therefore be the case that:
   $$x_i P_R x_j \Rightarrow u(x_i) > u(x_j) + s(x_j, x_i)$$
   is impossible (otherwise we have $u(x_i) > ... > u(x_j) > u(x_i)$).

   Since $f = u + v$ rationalizes (see Definition 8) the data set D and $u(x_j) > u(x_i) + s(x_j, x_i)$ is impossible then:

   $$x_i P_R x_j \Rightarrow u(x_i) \leq u(x_j) + s(x_j, x_i)$$

   and

   $$u(x_j) \leq u(x_i) + s(x_i, x_j)$$

   Moreover, $x_i I_R x_j$ implies necessarily (because I_R is symmetric by definition):

   $$u(x_i) \leq u(x_j) + s(x_j, x_i)$$

   and

   $$u(x_j) \leq u(x_i) + s(x_i, x_j)$$

   Hence we have:

   $$P_R \neq \emptyset$$

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and $x_i R x_j \Rightarrow$

$$u(x_i) \leq u(x_j) + s(x_j, x_i) \quad and \quad u(x_j) \leq u(x_i) + s(x_i, x_j)$$

$$\text{whatever } x_i, \ x_j.$$

That is f is degenerated.
Well this case is excluded by hypothesis. Hence:

$$x_i P_R^* x_j \Rightarrow not(x_j P_R x_i): \ Rarp.$$

(b) $\underline{(1) \Rightarrow (2)}$

We will first show that (1) implies GAI.

Let X be the support of D. If (1) then R is acyclic over X.

Let

$$L = P_L \cup I_L$$

with

$$P_L = P_R \cup T$$

and

$$I_L = I_R$$

Where $T = \{(x, y) \in J_R : (y, x) \notin T$ and the acyclicity of L is preserved$\}$

Let H be a preorder [5] such that:

$$P_L \subseteq P_H \subset H \subseteq L$$

Since X is finite then H is representable by a numerical function v such that:
$$\forall x_i, x_j \in X, x_i H x_j \Leftrightarrow v(x_i) \geq v(x_j)$$

We will now construct the numbers u_i, λ_i, s_{ij}, and $\alpha_{i,j}$:

i. Set (without loss of generality) $s_{ii} = 0$.
ii. If $x_i P_L x_j$, $i \neq j$, then take

$$s_{ji} \in [0, a[$$

with $a = |v(x_i) - v(x_j)|$
iii. Set (without loss of generality): $s_{ij} = s_{ji}$.

---

[5]Such a preorder necessarily exists. For instance we can construct H in the following way:

i. Take the transitive closure of P_L denoted P_L^*.

ii. Extend P_L^* into a linear order denoted $\widetilde{L}$.

iii. H is the dual relation of $\widetilde{L}$.

iv. Let[6]
$$\overline{\lambda}_i = \max_{x_j \in P_L(\rightarrow x_i)} \left\{ \frac{v(x_j) - v(x_i) + sij}{p_i(x_j - x_i)} \right\}$$

with $P_L(\rightarrow x_i) = \{x_j \in X : x_j P_L x_i\}$

v.
$$\lambda_i = \inf]\overline{\lambda}_i, +\infty[$$

vi. Let[7]
$$\underline{\lambda}_i = \min_{x_j \in P_L(x_i \rightarrow)} \left\{ \frac{v(x_i) - v(x_j) + sji}{p_i(x_i - x_j)} \right\}$$

with $P_L(x_i \rightarrow) = \{x_j \in X : x_i P_L x_j\}$

vii. *If* $P_L(\rightarrow x_i) \neq \emptyset$ *and* $P_L(x_i \rightarrow) \neq \emptyset$ *then do:*

A.
$$\beta_i = \begin{cases} \sup[0, \underline{\lambda}_i[& if \ \underline{\lambda}_i \neq +\infty \\ \lambda_i & otherwise \end{cases}$$

B. • *If* $\lambda_i > \beta_i$ *then set:* [8]

$$u_i = v(x_i) + \lambda_i p_i(x_i - x_j^*)$$

where

$$x_j^* = \arg\max p_i(x_i - x)$$
$$x \in P_L(x_i \rightarrow)$$

• Otherwise *(if* $\lambda_i \leq \beta_i$*)* set:

$$u_i = v(x_i)$$

viii. *If* $P_L(\rightarrow x_i) \neq \emptyset$ *and* $P_L(x_i \rightarrow) \neq \emptyset$ *then:*

$$\lambda_i = \beta_i = \begin{cases} \sup[0, \underline{\lambda}_i[& if \ \underline{\lambda}_i \neq +\infty \\ \eta > 0 & otherwise \end{cases}$$

ix. *If* $x_i I_L x_j$ *then set:*

$$s_{ji} \in [\max(a, b), +\infty[$$

with $b = |u_i - u_j|$.

x. Set(without lost if generality) $s_{ij} = s_{ji}$.

xi.
$$\alpha_{ij} = \begin{cases} \frac{|u_i - u_j| - s_{ij}}{p_i(x_j - x_i)} & if \ p_i(x_j - x_i) \neq 0 \\ a > 0 & otherwise \end{cases}$$

---

[6]$\forall x_j \in P_L(\rightarrow x_i)$, $p_i(x_j - x_i) > 0$.

[7]$\forall x_j \in P_L(x_i \rightarrow)$, $v(x_i) - v(x_j) - s_{ji} > 0$.

[8]Indeed *if* $\lambda_i > \beta_i$ then it must be the case that $\forall x_j \in P_L(x_i \rightarrow)$, we have $v(x_j) + s_{ji} \geq v(x_i) + \lambda_i p_i(x_j - x_i)$

Such u_i, s_{ij}, λ_i, α_{ij}, fulfill GAI:

Let $i, j = 1, ...n$. Then we have;

either $x_i P_R x_j$, or $x_i I_R x_j$, or $x_i J_R x_j$.

If $x_i P_R x_j$ then $x_i P_L x_j$ and by construction we have:

$$u_i + \lambda_i p_i(x_i - x_j) > u_j + s_{ij}$$

$$u_j + \lambda_j p_j(x_j - x_i) > u_i + s_{ij}$$

If $x_i I_R x_j$ then $x_i I_L x_j$ and we get:

$$|u_i - u_j| = s_{ij} + \alpha_{ji} p_j(x_i - x_j)$$

If $x_i J_R x_j$ then either $x_i P_L x_j$ or $x_j P_L x_i$.

Without lost of generality, suppose that $x_i P_L x_j$, then we have:

$$u_i + \lambda_i p_i(x_i - x_j) > u_j + s_{ij}$$

$$u_j + \lambda_j p_j(x_j - x_i) > u_i + s_{ij}$$

We will now construct a variable intervals function which rationalizes the data set D.

Let f defined from $\mathbb{R} \times X$ into $\mathbb{R}$ with:

$f(x, x_j) =$

$$u(x) + s(x, x_j) = \begin{cases} u_i + s_{ij} & if \ x = x_i \in X \\ \min_i \{u_i + s_{ij} + \lambda_i p_i(x - x_i)\} & if \ p_j x_j > p_j x \ and \ x \notin X \\ \max_i \{u_i + s_{ij} + \alpha_{ji} p_i(x - x_i)\} & if \ p_j x_j \leq p_j x \ and \ x \notin X \end{cases}$$

Let $x \geq 0$ with $p_j x_j \geq p_j x$, then 2 cases can occur:

 i. $\underline{x \in X}$.
   In this case, by construction of the u_i, s_{ij}, λ_i, α_{ij}, f rationalizes x.
 ii. $\underline{x \notin X}$.

   Then:
   - *If $p_j x_j > p_j x$ then:*

$$f(x, x_j) = u(x) + s(x, x_j) \leq u_j + s_{jj} + \lambda_j p_j(x - x_j)$$

   Since $s_{jj} = 0$ and $\lambda_j p_j(x - x_j) < 0$ we have:

$$u(x) + s(x, x_j) < u_j$$

- *If $p_j x_j = p_j x$ then*

$$f(x, x_j) = u(x) + s(x, x_j) \geq u_j + s_{jj} + \alpha_{jj} p_j(x - x_j)$$

Since $s_{jj} = 0$ *and* $\lambda_j p_j(x - x_j) = 0$ we get

$$u(x) + s(x, x_j) \geq u_j$$

Finally the weakly locally non satiation property is trivially fulfilled.

3. **Remark.**

Let us remark that in our proof (in order to improve the algorithmic complexity of our algorithm) we do not need to have a number λ_i for any $i = 1, ...n$, or to have numbers α_{ij} for any $i, j = 1, ...n$. Actually, we have if x_i is such that $P_L(x_i \rightarrow)$ *and* $P_L(\rightarrow x_i)$ are empty sets then we do not create λ_i. Likewise if x_i is such that $I_L(x_i \rightarrow)$ is empty then we do not need to create the α_{ij}. Nevertheless, for any i, if λ_i is not computed then at least , one α_{ij} is computed (because if λ_i is not computed then x_i is indifferent to any x_j with respect to binary relation L).

Appendix 2: The approximate Power of the test.

From a statistical viewpoint the power of a test between two hypotheses is given by the probability of rejecting the null hypothesis when the alternative hypothesis is true. In our case the former is that the consumer behavior satisfies the axiom we test, and the latter that it does not. But these tests being non probabilistic, their power is unknown. However there seems to exist a relative consensus consisting in trying to compute an approximate power of the non-parametric test called *power in the sense of Bronars* .
Indeed, S. Bronars compute the approximate power of the test by taking as an alternative hypothesis Becker's notion of irrational behavior. In this case the consumer is assumed to choose consumption bundles randomly from his budget set and the power of the tests is given by the fraction of data sets in which violations of the axiom occurs. So as to do this, S. Bronars constructs algorithms which generate random consumption data exhausting the budget set in each period. In our case we have used Bronars second algorithm. It works as follows:
   -In a first time we draw N i.i.d uniform random variables in each year, named $z_{1,t},...z_{N,t}$, $t = 1,...T$.
   -Using these random variables the (random) budget shares are then given by:

$$w_{i,t} = \frac{z_{i,t}}{\sum_{i=1}^{N} z_{i,t}}$$

It is straightforward to show that the consumption bundles computed from these budget shares exhaust the budget set line in each year, indeed:

$$\sum_{i=1}^{N} p_{i,t} \times x_{i,t} = \sum_{i=1}^{N} w_{i,t} \times p_t x_t$$

$$= p_t x_t$$

Where $p_t x_t$ is the actual total expenditure for period t.

These budget shares are then used to compute consumption bundles (in each period they are multiplied by total expenditure and divided by the actual price of the corresponding commodity) from which we test the axioms.