Going down in (semi)lattices of finite Moore families and convex geometries

Gabriela Bordalo [*] Nathalie Caspard [†] Bernard Monjardet [‡]

Abstract

In this paper we first study what changes occur in the posets of irreducible elements when one goes from an arbitrary Moore family (respectively, a convex geometry) to one of its lower covers in the lattice of all Moore families (respectively, in the semilattice of all convex geometries) defined on a finite set. Then, we study the set of all convex geometries which have the same poset of join-irreducible elements. We show that this set – ordered by set inclusion – is a ranked join-semilattice and we characterize its cover relation. We prove that the lattice of all ideals of a given poset P is the only convex geometry having a poset of join-irreducible elements isomorphic to P if and only if the width of P is less than 3. Finally, we give an algorithm for computing all convex geometries having the same poset of join-irreducible elements.

Key words: Closure system, Moore family, convex geometry, (semi)lattice, algorithm.

1 Introduction

Finite Moore families (also called closure systems) are fundamental mathematical objects. For instance, they are the set representations of finite lattices. The set $\mathbb{M}$ of all these families defined on a set P and ordered by set inclusion is a lattice studied by many authors (see Caspard and Monjardet [7], 2003). The set $\mathbb{M}_P$ of all the Moore families whose poset of join-irreducible elements is isomorphic to a poset $(P, \leq)$, (where $\leq$ is any given partial order defined on P) is a lattice studied in Bordalo and Monjardet [4] (2002). Convex geometries are a significant class of Moore families appearing in many domains. They are the set representations of the so-called lower locally distributive (or meet-distributive) lattices studied as soon as 1940 by Dilworth [11], and they also are the families of closed sets of antiexchange closure operators (see, for instance, Edelman and Jamison [12] (1985) and Monjardet [15], 1990). Moreover, they are in a one-to-one correspondence with the so-called path-independent choice functions of the theory of choice functions in microeconomics (see Monjardet and Raderanirina [16], 2001). The set $\mathbb{G}$ of all convex geometries defined on a set P and ordered by set inclusion is a semilattice studied in Edelman and Jamison [12] (1985) and Caspard and Monjardet [8] (2004). One of the aims of the present paper is to study the poset $\mathbb{G}_P$ of all convex geometries defined on a finite set P and whose poset of join-irreducible elements is isomorphic to a given poset

[*]Departamento de Matematica da Faculdade de Ciencias e CAUL, Universidade de Lisboa, R. Prof. Gama Pinto, 2, 1699 Lisboa, Portugal. Email: mchauser@ptmat.lmc.fc.ul.pt.

[†]LACL, Université Paris 12 Val de Marne, 61 avenue du Général de Gaulle 94010 Créteil cedex, France. Email: caspard@univ-paris12.fr

[‡]CERMSEM, Université Paris I, Maison des Sciences Économiques, 106-112, boulevard de l'Hôpital, 75647 Paris Cedex 13, France. Email: monjarde@univ-paris1.

$(P, \leq)$. For instance, if $(P, \leq)$ is an antichain, $\mathbb{G}_P$ is the semilattice of all atomistic convex geometries with $|P|$ atoms (the number of such convex geometries is 87 for $|P| = 4$ and 16686 for $|P| = 5$, Nourine [19], 2003). Obviously, $\mathbb{G}_P$ is contained in the lattice $\mathbb{M}_P$, but it will appear that their behaviour is quite different. For instance, $\mathbb{G}_P$ is a semilattice which is generally not a lattice.

In the above (semi)lattices $\mathbb{M}$, $\mathbb{M}_P$, $\mathbb{G}$ and $\mathbb{G}_P$, one goes from an element $\mathcal{G}'$ of the semilattice to one of its lower covers $\mathcal{G}$ by deleting one (meet-irreducible) element of $\mathcal{G}'$. In this paper, we precise the characterization of those meet-irreducible elements which can be deleted and we study what changes occur between the irreducible elements of $\mathcal{G}'$ and those of $\mathcal{G}$.

In Section 2, we recall some generalities on posets, Moore families and convex geometries.

In Section 3, we recall and complete some results on (semi)lattices of Moore families. For instance, Propositions 7, 8 and Corollary 10 describe the changes occuring in the irreducible elements of a Moore family belonging to $\mathbb{M}$, $\mathbb{M}_P$ or $\mathbb{G}$, when one considers one of its lower covers in the corresponding semilattices.

Section 4 considers the particular case where the Moore family is the convex geometry – and distributive lattice – $\mathcal{O}(P)$ of all order ideals of a poset $(P, \leq)$. We study the lower covers of $\mathcal{O}(P)$ in $\mathbb{M}$, $\mathbb{M}_P$, $\mathbb{G}$ or $\mathbb{G}_P$. We point out the connection with the study of the maximal sublattices of a distributive lattice made by many authors (see Schmid [23] (2002) and the references there).

Section 5 is devoted to the study of $\mathbb{G}_P$. We first show that $\mathbb{G}_P$ is a join-subsemilattice of the semilattice $\mathbb{G}$ of all convex geometries (its maximum is $\mathcal{O}(P)$). Then, we study the cover relation of $\mathbb{G}_P$, by characterizing the so-called $\mathbb{G}_P$-deletable elements of a convex geometry of $\mathbb{G}_P$. In general, $\mathbb{G}_P$ is not a lattice and we study its coatoms and its minimal elements. We then show that $|\mathbb{G}_P| = 1$ (i.e. that the lattice $\mathcal{O}(P)$ of order ideals of $(P, \leq)$ is the unique convex geometry having a poset of join-irreducible elements isomorphic to $(P, \leq)$) if and only if the width of P is less than 3. At last, we raise several open problems and give an algorithm for computing all convex geometries having the same poset of join-irreducible elements.

The symbol $+$ denotes the disjoint union set, $P + x$ stands for $P + \{x\}$. $P \oplus Q$ denotes the ordinal sum of two posets P and Q (where $x < y$ for all x in P and y in Q).

2 Preliminaries

In this section, we recall or introduce some definitions, notations and results on posets, Moore families and convex geometries.

In this paper, P always denotes a finite set. When there is no ambiguity, P will also denote the poset $(P, \leq)$, where $\leq$ is a partial order defined on P.

When x is covered by y – denoted by $x \prec y$ – in the poset P, we say that x (respectively, y) is a *lower cover* (respectively, an *upper cover*) of y (respectively, of x).

An element of a poset P is *join-irreducible* (respectively, *meet-irreducible*) if it is not the join (respectively, the meet) of elements different from itself. An element is doubly-irreducible if it is join- and meet-irreducible. $J(P)$ (respectively, $M(P)$) denotes the set of join-irreducible (respectively, meet-irreducible) elements of the poset P. If P is a lattice, x^- (respectively, x^+) denotes the unique element covered by (respectively, covering) the join-irreducible (respectively, the meet-irreducible) element x.

An element x of a poset P is a *node* if x is comparable to every element of P.

An *(order) ideal* of a poset P is a subset I of P such that $x \in I$ and $y < x$ imply $y \in I$. One defines dually the notion of (*order*) *filter*. We denote by $\mathcal{O}(P)$ the distributive lattice of all ideals of P (ordered by set inclusion). For an element x of a poset P, $(x] = \{y \in P : y \leq x\}$ (respectively, $[x) = \{y \in P : x \leq y\}$) denotes the principal ideal (respectively, the principal filter) defined by x. Moreover, $(x[$ denotes the ideal $(x] \setminus \{x\}$ (where $\setminus$ is the symbol for set difference) and $]x)$ denotes the filter $[x) \setminus \{x\}$. We denote by $\mathcal{P}$ the poset $(\{(x], x \in P\}, \subseteq)$. So, the posets P and $\mathcal{P}$ are isomorphic. $\mathcal{P}$ is the poset of all join-irreducible elements of the lattice $\mathcal{O}(P)$. The meet-irreducible elements of $\mathcal{O}(P)$ are the ideals $P \setminus [x)$, for $x \in P$.

Let $\mathcal{G}$ denote a *Moore family* (also called a *closure system*) defined on a set P, i.e. a subset of the set 2^P of all subsets of P, that contains P and that is closed under set-intersection. Obviously, $(\mathcal{G}, \subseteq)$ is a lattice and its elements are called *closed sets*. Closed sets are the fixed points of the closure operator canonically associated with $\mathcal{G}$ and denoted by $\varphi_{\mathcal{G}}$. A preorder $R_{\mathcal{G}}$ is defined on P by setting $xR_{\mathcal{G}}y$ if $\varphi_{\mathcal{G}}(x) \subseteq \varphi_{\mathcal{G}}(y)$. Observe that a closed set G of $\mathcal{G}$ is an ideal of $R_{\mathcal{G}}$ (i.e. $x \in G$ and $yR_{\mathcal{G}}x$ imply $y \in G$), and that the principal ideals of $R_{\mathcal{G}}$ are the closures by $\varphi_{\mathcal{G}}$ of the elements of P: $\{y \in P : yR_{\mathcal{G}}x\} = \varphi_{\mathcal{G}}(x)$. In 1943, Öre [20] has characterized the join-irreducible elements of the lattice $(\mathcal{G}, \subseteq)$:

$$J(\mathcal{G}) = \{\varphi_{\mathcal{G}}(x) : \varphi_{\mathcal{G}}(x) \setminus \{y \in P : \varphi_{\mathcal{G}}(y) = \varphi_{\mathcal{G}}(x)\} \in \mathcal{G}\}.$$

Using this result, one easily proves the following:

Lemma 1 *Let $\mathcal{G}$ be a Moore family on P, $\varphi_{\mathcal{G}}$ the associated closure operator and $R_{\mathcal{G}}$ the associated preorder. The following two conditions are equivalent:*

1. *for every $x \in P$, $\varphi_{\mathcal{G}}(x) \setminus \{x\} \in \mathcal{G}$,*
2. *$J(\mathcal{G}) = \{\varphi_{\mathcal{G}}(x), x \in P\}$,and $\varphi_{\mathcal{G}}(x) = \varphi_{\mathcal{G}}(y)$ implies $x = y$.*

When these conditions are satisfied, $R_{\mathcal{G}}$ is an order denoted by $\leq_{\mathcal{G}}$ or, when no ambiguity can occur, by $\leq$. Then every element of $\mathcal{G}$ is an ideal of the poset $(P, \leq_{\mathcal{G}})$ and the closed sets of the form $\varphi_{\mathcal{G}}(x)$ are the principal ideals of $(P, \leq_{\mathcal{G}})$ as well as the join-irreducible elements of $\mathcal{G}$.

Remark 2 The two parts in Condition (2) are independent. When $\mathcal{G}$ is a Moore family such that $\varphi_{\mathcal{G}}(x) = \varphi_{\mathcal{G}}(y)$ implies $x = y$ (i.e. when $R_{\mathcal{G}}$ is an order), the above Condition (1) is not necessarily satisfied.

In this paper, we will generally consider Moore families satisfying Condition (1) (or (2)) of Lemma 1. In this case, a Moore family $\mathcal{G}$ on P will be considered as a set of ideals of the associated poset $(P, \leq_{\mathcal{G}})$ and, since the set $J(\mathcal{G})$ of its join-irreducible elements is the poset $\mathcal{P}$ defined by the principal ideals of $(P, \leq_{\mathcal{G}})$, $J(\mathcal{G})$ is isomorphic to $(P, \leq_{\mathcal{G}})$.

Observe that, for any given poset $(P, \leq)$, there exists at least one Moore family $\mathcal{G}$ such that $J(\mathcal{G})$ is isomorphic to $(P, \leq)$, namely the set $\mathcal{O}(P)$ of all ideals of $(P, \leq)$, but there exists in general several such Moore families (the so-called *strict completions of* P, see Bordalo and Monjardet [4], 2002).

A Moore family $\mathcal{G}$ defined on a set P is a *convex geometry* if it satisfies the two following properties:

1. the empty set $\emptyset$ is a closed set,

2. for every closed set C different from P, there exists $x \notin C$ such that $C + \{x\}$ is a closed set.

It is well-known that these properties imply that a convex geometry satisfies the above Condition (1) of Lemma 1, and so, any convex geometry $\mathcal{G}$ on P will be considered as a set of ideals of the associated poset $(P, \leq_{\mathcal{G}})$. The "abstract" lattices corresponding to convex geometries are the *lower locally distributive* (also called *meet-distributive*) lattices. In particular, they are lower semimodular (see Edelman and Jamison [12] (1985) or Monjardet [15] (1990) for a number of characterizations of convex geometries or lower locally distributive lattices). We will often use some properties of lower locally distributive lattices recalled in the following lemma. The first three ones are well-known characterizations of lower locally distributive lattices. The two others are true for lower semimodular lattices, since they are the duals of two properties of upper semimodular lattices proved in Bordalo and Monjardet ([3] (1996), Lemmas 7 and 8).

Lemma 3 *Let L be a lower locally distributive lattice. Then:*

1. *the set of lower covers of any non minimum element of L is the set of coatoms of a Boolean sublattice of L,*

2. *for any meet-irreducible x of L, there exists a unique join-irreducible element y such that $x \vee y = x^+$,*

3. *L is lower semimodular and every modular sublattice of L is distributive,*

4. *L does not contain a sublattice isomorphic to the lattice N_5 where the element y of the chain $0 \prec x \prec y \prec 1$ of N_5 is join-irreducible in L,*

5. *if t, y and z are three different elements of L such that $z \prec y$ and $z \prec t$, then there exists two elements ℓ, m such that $z \prec m \prec \ell$, and $y \prec \ell$. In particular, when z has only two upper covers, this implies $m = t$ and, when t is meet-irreducible, this implies $\ell = t^+$.*

3 Going down in Moore families: the changes in irreducible elements

In this section we describe what changes occur in the irreducible elements of a Moore family belonging to a poset of Moore families when one considers one of its lower covers in this poset. We consider the following four posets of Moore families, where the order is the set inclusion between families (observe that the Moore families of the last three posets satisfy Condition (1) of Lemma 1):

- $\mathbb{M}$ denotes the poset of all Moore families defined on a set P. This poset is a lattice which has been studied by several authors (see Caspard and Monjardet [7], 2003). For $\mathcal{G}, \mathcal{G}' \in \mathbb{M}$, $\mathcal{G} \prec \mathcal{G}'$ if and only if $\mathcal{G} = \mathcal{G}' \setminus \{I\}$, with I a meet-irreducible element of $\mathcal{G}'$. Moreover, $\mathcal{G} \wedge \mathcal{G}' = \mathcal{G} \cap \mathcal{G}'$ and $\mathcal{G} \vee \mathcal{G}' = \{G \cap G', G \in \mathcal{G}, G' \in \mathcal{G}'\}$.

- $\mathbb{M}_P$ denotes the poset of all Moore families $\mathcal{G}$ defined on a set P and such that $J(\mathcal{G}) = \mathcal{P}$ is isomorphic to a given poset $(P, \leq)$. The poset $\mathbb{M}_P$ is a lattice which has been studied in Bordalo and Monjardet [4] (2002). Its maximum is the lattice $\mathcal{O}(P)$ of all ideals of P. Its minimum is the lattice $\mathcal{D}(P)^*$ which is a meet-subsemilattice of the lattice $\mathcal{O}(P)$ consisting of the following ideals:

$$\mathcal{D}(P)^* = \{P\} \cup \{(x], x \in P\} \cup \{\bigcap(x_i], x_i \in S \subseteq P\} \cup \{(x[, x \in P\}.$$

In fact, $\mathbb{M}_P$ is exactly the interval $[\mathcal{D}(P)^*, \mathcal{O}(P)]$ of $\mathbb{M}$ consisting in all the meet-subsemilattices of $\mathcal{O}(P)$ containing $\mathcal{D}(P)^*$.

- $\mathbb{G}$ denotes the join-semilattice of all convex geometries defined on a set P. This semilattice has been studied in Edelman and Jamison [12] (1985) and Caspard and Monjardet [8] (2004). In particular, $\mathbb{G}$ is ranked.

- $\mathbb{G}_P$ denotes the poset of all convex geometries $\mathcal{G}$ defined on a set P and such that $J(\mathcal{G}) = \mathcal{P}$ is isomorphic to a given poset $(P, \leq)$. So $\mathbb{G}_P = \mathbb{M}_P \cap \mathbb{G}$. We shall study this poset in Section 5.

Observe that $\mathbb{M}_P$ is a sublattice of the lattice $\mathbb{M}$ of all Moore families, whereas $\mathbb{G}$ is only a join-subsemilattice of $\mathbb{M}$. In $\mathbb{G}$, $\mathcal{G} \wedge \mathcal{G}'$ (when it exists) can be strictly contained in $\mathcal{G} \cap \mathcal{G}'$. Observe also that the covering relation of $\mathbb{M}_P$ or $\mathbb{G}$ is the restriction of the covering relation of $\mathbb{M}$ to these posets. In fact, the following definitions and results allow us to make more precise the covering relation of $\mathbb{M}_P$ and $\mathbb{G}$ (the notions of forced and $\mathbb{M}_P$-deletable ideals are in Bordalo and Monjardet [4], 2002). In these definitions and results, $\mathcal{G}'$ (rather than $\mathcal{G}$) will denote an arbitrary Moore family belonging to one of our four posets of Moore families, and $\mathcal{G}$ will always denote a Moore family covered by $\mathcal{G}'$ in this poset.

Definition 4 *Let P be a poset and I an ideal of P.*

1. *I is* forced *if $I \in \mathcal{D}(P)^*$, i.e. if either $I = P$, or $I = (x]$ or $(x[$ for some x in P, or I is the intersection of a family of principal ideals of P. Observe that the empty set is a forced ideal. An ideal which is not forced is called* unforced.
2. *In a Moore family $\mathcal{G}' \in \mathbb{M}_P$ containing I, I is* $\mathbb{M}_P$-deletable *(*w.r.t. $\mathcal{G}'$*) if $\mathcal{G} = \mathcal{G}' \setminus \{I\} \in \mathbb{M}_P$.*
3. *In a Moore family $\mathcal{G}' \in \mathbb{G}_P$ containing I, I is* $\mathbb{G}$-deletable *(*w.r.t. $\mathcal{G}'$*) if $\mathcal{G} = \mathcal{G}' \setminus \{I\} \in \mathbb{G}$.*
4. *In a convex geometry $\mathcal{G}' \in \mathbb{G}_P$ containing I, I is* $\mathbb{G}_P$-deletable *(*w.r.t. $\mathcal{G}'$*) if $\mathcal{G} = \mathcal{G}' \setminus \{I\} \in \mathbb{G}_P$.*

Notations: When no ambiguity occurs, we will denote by J (respectively, M) the set $J(\mathcal{G})$ of join-irreducible elements (respectively, the set $M(\mathcal{G})$ of meet-irreducible elements) of a Moore family $\mathcal{G}$. Similarly for J' and M' (instead of $J(\mathcal{G}')$ and $M(\mathcal{G}')$) for a Moore family $\mathcal{G}'$. Using the notations given above for the irreducible elements of a lattice, I^- (respectively, I^+) denotes the unique element of $\mathcal{G}$ covered by (respectively, covering) a join-irreducible (respectively, a meet-irreducible) element I of $\mathcal{G}$.

The following theorem recalls and completes some useful results on the deletable elements of a Moore family $\mathcal{G}'$ belonging to $\mathbb{M}_P$ or to $\mathbb{G}$.

Theorem 5 *Let P be a poset.*

1. *Let $\mathcal{G}'$ be a Moore family in $\mathbb{M}_P$ and $I \in \mathcal{G}'$ an ideal of P. The three following properties are equivalent:*

a. I is $\mathbb{M}_P$-deletable w.r.t. $\mathcal{G}'$,

b. I is an unforced ideal of P and $I \in M'$,

c. $I \in M'$ and $I, I^+ \notin J'$.

2. *Let $\mathcal{G}' \in \mathbb{G}_P$ and $I \in \mathcal{G}'$ an ideal of P. Properties (a) and (b) below are equivalent and imply (c):*

a. I is $\mathbb{G}$-deletable w.r.t. $\mathcal{G}'$,

b. $I \in M'$, $I \neq \emptyset$ and I does not cover any meet-irreducible element of $\mathcal{G}'$,

c. $I \in M'$ and $I^+ \notin J'$ (or, equivalently, $I \in M'$ and is not a node of $\mathcal{G}'$).

Moreover, if $I \in J' \cap M'$, the following four properties are equivalent:

d. I is $\mathbb{G}$-deletable w.r.t. $\mathcal{G}'$,

e. $I^- \notin M'$,

f. $I^+ \notin J'$,

g. I is not a node of $\mathcal{G}'$,

and, in this case, I^+ has a unique other lower cover I' and $I^- \prec I'$.

Proof: (1) is proved in Bordalo and Monjardet [4] (2002).

(2) (a) implies (b): it it obvious that $\mathcal{G} = \mathcal{G}' \setminus \{I\} \in \mathbb{G}$ implies $I \in M'$ and $I \neq \emptyset$. If I covers $G \in M'$, then in $\mathcal{G}$, G is covered by I^+ with $|I^+| = |G| + 2$, a contradiction with $\mathcal{G}$ being a convex geometry.

(b) implies (a): we have to show that $\mathcal{G} = \mathcal{G}' \setminus \{I\}$ contains the empty set and that, for every element K of $\mathcal{G}$ (different from P), there exists $x \notin K$ such that $K + x$ belongs to $\mathcal{G}$. But this is obvious from the hypotheses made on I.

(b) implies (c): in order to show that $I^+ \notin J'$, apply Point (5) of Lemma 3 (with $\ell = I^+$, $z = G \prec I = y$ and $G \prec H = t$), which shows that there exists H' (different or not from H) with $H' \prec I^+$. We now prove that when $I \in M'$, $I^+ \notin J'$ if and only if I is not a node of $\mathcal{G}'$. If $I^+ \notin J'$, then $I^+ = I \vee K$ and so I is not a node of $\mathcal{G}'$. Conversely, assume that $I \in M'$ and is not a node of $\mathcal{G}'$. So, there exists $K \in \mathcal{G}'$ incomparable with I. If $I^+ \in J'$, then $H = I \vee K = I^+ \vee K \supset I^+$ and $G = I^+ \wedge K = I \wedge K \subset I$. Then, $\{G, I, K, I^+, H\}$ is a sublattice of $\mathcal{G}'$ isomorphic to the lattice N_5 and such that I^+ is join-irreducible in $\mathcal{G}'$. But this is a contradiction with Point (4) of Lemma 3.

Assume now that $I \in J \cap M'$ (which implies $I \neq \emptyset$). Then (a) equivalent to (b) implies (d) equivalent to (e). Since $I \in M'$, (f) and (g) are equivalent (by (c)) and, since (a) implies (c), (d) implies (f). In order to prove the equivalence between the four conditions (d), (e), (f) and (g), it remains to show that (f) implies (d). Let I' ($\neq I$) be covered by I^+ in $\mathcal{G}'$. Since the lattice $\mathcal{G}'$ is a convex geometry, it is lower semimodular, and, since $I \in J'$, one has $I^- \prec I'$. Now, for the same reason that in the above proof of [(b) implies (a)], $\mathcal{G} = \mathcal{G}' \setminus \{I\}$ is a convex geometry.

Since $I^+ \notin J'$, there exists I' ($\neq I$) covered by I^+. Assume that there exists I" ($\neq I', I$) covered by I^+. As above, one would have $I^- \prec I$". Then, $\{I^-, I, I', I", I^+\}$ would be a sublattice of J' isomorphic to the modular lattice M_5, a contradiction with J' being a lower locally distributive lattice (by Point (3) of Lemma 3). $\square$

Remark 6 1. Some of the above characterizations of an ideal $\mathbb{G}$-deletable w.r.t. a convex geometry are already in Caspard and Monjardet [8] (2004) with some other ones using the notion of extreme elements of a closed set. 2. Since Property (4) of Lemma 3 holds in a lower semimodular lattice, the above proof shows that in such a lattice, a meet-irreducible element m is a node if and only if m^+ is join-irreducible.

We now examine what changes occur in the posets of irreducible elements when one goes from a Moore family $\mathcal{G}'$ to one of its lower covers $\mathcal{G}$ in the posets $\mathbb{M}$, $\mathbb{M}_P$ or $\mathbb{G}$. The following propositions and corollaries answer completely this question.

In Proposition 7 we consider a lower cover $\mathcal{G} = \mathcal{G}' \setminus \{I\}$ of an arbitrary Moore family $\mathcal{G}'$ (with I a meet-irreducible of $\mathcal{G}'$) and we determine its poset of join-irreducible elements. The result depends on four disjoint possibilities for I and I^+: (a) $I, I^+ \notin J'$; (b) $I \notin J'$ and $I^+ \in J'$; (c) $I \in J'$, this last case leading to two different situations (c1) and (c2) according to different behaviours of I^+.

Proposition 7 *Let $\mathcal{G}'$ be a Moore family in $\mathbb{M}$ and $\mathcal{G} = \mathcal{G}' \setminus \{I\}$, where I is a meet-irreducible element of $\mathcal{G}'$. Then:*

1. $J = J'$ $\iff$ $I, I^+ \notin J'$,
2. $J = J' \setminus \{I^+\}$ $\iff$ $I \notin J'$ *and* $I^+ \in J'$,
3. $J = J' \setminus \{I\}$ $\iff$ $I \in J'$ *and either* $I^+ \in J'$, *or (*I^+ *has a unique other lower cover* I' *in* $\mathcal{G}'$ *and* I' *satisfies* $I^- \not\subset I'$*), or* I^+ *has at least three lower covers,*
4. $J = (J' \setminus \{I\}) + \{I^+\}$ $\iff$ $I \in J'$, I^+ *has a unique other lower cover* I' *in* $\mathcal{G}'$ *and this unique lower cover satisfies* $I^- \subset I'$.

In the first three cases, the order on J is the order induced by J'; in the last one, the order on $J = (J' \setminus \{I\}) + \{I^+\}$ is obtained by replacing I with I^+ in all the ordered pairs (K, I) and (I, L) of J', and by adding the ordered pairs (K, I^+), for $K \in J'$ and satisfying $K \subset I'$ and $K \not\subseteq I^-$.

Proof: Items (1) and (2) are straightforward.

For Item (3), if $J = J' \setminus \{I\}$ then $I \in J'$ and either $I^+ \in J'$ and remains join-irreducible in $\mathcal{G}$ (since, in $\mathcal{G}$, I^- is its unique lower cover), or $I^+ \notin J'$ and it does not become join-irreducible in $\mathcal{G}$, which is clearly equivalent to the condition [I^+ has a unique other lower cover I' and $I^- \not\subset I'$, or I^+ has at least three lower covers]. The converse implication is obvious.

For Item (4), if $(J' \setminus \{I\}) + \{I^+\}$, then $I \in J'$ and, since I^+ becomes join-irreducible in $\mathcal{G}$, it has a unique other lower cover I' in $\mathcal{G}'$ and $I^- \subset I'$ (otherwise, I^- and I' are lower covers of I^+ in $\mathcal{G}$). The converse implication is obvious.

Observing that, in this last case, the only elements K of J' such that $K \subset I^+$ and $(K, I) \notin J'$ are the elements K contained in I' and not contained in I^-, one gets the description of the order of $J = (J' \setminus \{I\}) + \{I^+\}$. □

In the next proposition, we consider a lower cover $\mathcal{G} = \mathcal{G}' \setminus \{I\}$ of an arbitrary Moore family $\mathcal{G}'$ (with I a meet-irreducible of $\mathcal{G}'$) and we determine its poset of meet-irreducible elements. We use the following notation, for $I \in M'$:

$$\mathcal{I}^-(I) = \{G \in \mathcal{G}' : G \prec I, G \text{ has a unique other upper cover } I', \text{ and } I' \prec I^+\}.$$

Proposition 8 *Let $\mathcal{G}'$ be a Moore family in $\mathbb{M}$ and $\mathcal{G} = \mathcal{G}' \setminus \{I\}$, where I is a meet-irreducible element of $\mathcal{G}'$. Then $M = (M' \setminus \{I\}) + \mathcal{I}^-(I)$.*

The order on M is obtained by adding to the induced order on $M' \setminus \{I\}$ the ordered pairs(G, H) and (L, G), where $G \in \mathcal{I}^-(I)$, $H, L \in M'$, $I' \subseteq H$ and $L \subset G$.

The proof is straightforward.

Remark 9 One can observe that, when one goes from a Moore family to one of its lower covers in $\mathbb{M}$, the number of join-irreducible elements decreases from at most one and never increases, whereas the number of meet-irreducible elements can decrease from at most one but can also increase.

Corollary 10 *Let P be a poset.*

1. *Let $\mathcal{G}' \in \mathbb{M}_P$ and let $\mathcal{G} = \mathcal{G}' \setminus \{I\}$, where I is an ideal of P which belongs to $\mathcal{G}'$ and is $\mathbb{M}_P$-deletable w.r.t. $\mathcal{G}'$.Then:*

   a. $J = J'$,

   b. $M = (M' \setminus \{I\}) + \mathcal{I}^-(I)$.

2. *Let $\mathcal{G}' \in \mathbb{G}_P$ and let $\mathcal{G} = \mathcal{G}' \setminus \{I\}$, where I is an ideal of P which belongs to $\mathcal{G}'$ and is $\mathbb{G}$-deletable w.r.t. $\mathcal{G}'$. Then:*

   a.
$$\begin{array}{lclcl} J = J' & \Longleftrightarrow & I \notin J', & & \\ J = (J' \setminus \{I\}) + \{I^+\} & \Longleftrightarrow & I \in J' & \Longleftrightarrow & I \in J' \cap M'. \end{array}$$

   When $I \in J'$, I^+ has a unique other lower cover I' in $\mathcal{G}'$ and I' satisfies $I^- \subset I'$.

   The order on $J = (J' \setminus \{I\}) + \{I^+\}$ is obtained by replacing I with I^+ in all the ordered pairs (K, I) or (I, L) of J' and by adding the unique ordered pair (K, I^+) where $K \in J'$, $K \subset I'$ and $K \not\subseteq I^-$.

   b. $M = (M' \setminus \{I\}) + \{G \in \mathcal{G}' : G \prec I$ *and G has a unique other upper cover* $I'\}$.

   In particular, when $I \in J' \cap M'$, either I^- has a unique other upper cover I' in $\mathcal{G}'$ and $M = (M' \setminus \{I\}) + \{I^-\}$, or I^- has at least two other upper covers in $\mathcal{G}'$, and $M = M' \setminus \{I\}$.

Proof: (1) is obvious by definition of $\mathbb{M}_P$ (observe also that if I is $\mathbb{M}_P$-deletable, it is unforced and so I, $I^+ \notin J'$).

(2)(a) results from Cases (1) and (4) of Proposition 7. Indeed, if I is a $\mathbb{G}$-deletable ideal, one has $I^+ \notin J'$ (according to the implication of (2)(c) by (2)(a) in Theorem 5). So, Case (1) of Proposition 7 corresponds to $I \notin J'$, and Case (2) of Proposition 7 does not occur. Moreover, when $I \in J' \cap M'$, I^+ has a unique other lower cover I' with $I^- \subset I'$ (by Point (2) of Theorem 5). So, Case (3) of Proposition 7 does not occur.

The result on the order on J is the same as in Proposition 7 except that, now, there exists a unique ordered pair (K, I^+) with $K \in J'$, $K \subset I'$ and $K \not\subseteq I^-$. Indeed, the last two conditions are equivalent to $K \vee I = I^+$ and, since a convex geometry is a lower locally distributive lattice, for any meet-irreducible I, there exists a unique $K \in J'$ such that $K \vee I = I^+$ (Point (2) of Lemma 3).

In order to prove (2)(b) and according to Proposition 8, we have to prove that, if $G \prec I$ and if G has a unique other upper cover I' in $\mathcal{G}'$, then $I' \prec I^+$. But, since $\mathcal{G}'$ is a lower locally distributive lattice, this results from Point (5) of Lemma 3. The final assertion results from Point (2) of Theorem 5. Indeed, when I is $\mathbb{G}$-deletable w.r.t. $\mathcal{G}'$ and $I \in J' \cap M'$, I^- cannot belong to M'. □

Remark 11 1. When $\mathcal{G}'$ is a convex geometry, we have just shown that $\mathcal{I}^-(I) = \{G \in \mathcal{G}' : G \prec I$ and G has a unique other upper cover $I'\}$. This is also true if $\mathcal{G}'$ is a Moore family without sublattice isomorphic to the lattice N_5.

2. When $\mathcal{G}'$ is a convex geometry and $\mathcal{G} = \mathcal{G}' \setminus \{I\} \prec \mathcal{G}'$ are two convex geometries such that $J = J'$, it is not difficult to see that, even if $|M| = |M'|$, M cannot be isomorphic to M'.

4 The lower covers of a distributive lattice

Let P be a poset, $\mathcal{O}(P)$ the distributive lattice of all ideals of P, $x \in P$ and $I = P \setminus [x)$ a meet-irreducible of $\mathcal{O}(P)$. The Moore family $\mathcal{O}(P) \setminus I$ may or may not belong to $\mathbb{M}_P$, $\mathbb{G}$ or $\mathbb{G}_P$. In this section we characterize the elements x of P corresponding to these different cases. We need to recall or define the following notions:

- x is *join-prime* if there exists $y \in P$ such that $P = [x) + (y]$. In this case, we also say that x is the *conjugate element of* y (Berman and Bordalo [2], 1998).

- x is *strongly meet-irreducible* if x is covered by a unique element, which we denote by x^+. Then, x is a meet-irreducible element of P.

One defines dually the notions of *meet-prime element* and of *strongly join-irreducible element*.

Let x, y be two elements of P. The ordered pair (x, y) is a *critical pair* (respectively, a *weak critical pair*) *for* P if x and y are incomparable (respectively, $x \not\leq y$) and, for all $z, t \in P$, $z < x$ implies $z \leq y$, and $y < t$ implies $x \leq t$. It is clear that (x, y) is a critical pair for P if and only if $P + (x, y)$ is a poset (cf. for instance Barbut and Monjardet [1], 1970, II, p.54, or Rabinovitch and Rival [21], 1979) and that, if (x, y) is a weak critical pair, then y is covered by x in P.

The following lemma gives some known or not known (but anyway obvious or easy to prove) relations between the above notions.

Lemma 12 *Let x be an element of a poset P. Then:*

- *if x is a node covering an element y, then y is a strong meet-irreducible of P.*
- *x is a join-prime element of P if and only $P \setminus [x)$ is a doubly-irreducible element of $\mathcal{O}(P)$.*

Let x be a join-prime element of a poset P, conjugate of the element y of P. Then:

- *x is join-irreducible, y is meet-prime and (x, y) is a weak critical pair for P.*
- *x is a node of P if and only if x covers a strong meet-irreducible – which is equal to y – if and only if $P \setminus [x) = (y]$ is a doubly-irreducible element and a node of $\mathcal{O}(P)$. In this case, $P = (y] \oplus [x)$, and so y is covered by x in P.*

- *if x is a node, (x, y) is a critical pair for P.*

In the following theorem, we characterize the elements x of P for which $\mathcal{O}(P)\setminus(P\setminus[x))$ belongs to $\mathbb{M}_P$, $\mathbb{G}$ or $\mathbb{G}_P$.

Theorem 13 *Let P be a poset, $\mathcal{O}(P)$ the distributive lattice of all ideals of P and $I = P \setminus [x)$ a meet-irreducible element of $\mathcal{O}(P)$. Then:*

1. *I is $\mathbb{M}_P$-deletable w.r.t. $\mathcal{O}(P)$ if and only if x is neither join-prime nor a node of P,*
2. *I is $\mathbb{G}$-deletable w.r.t. $\mathcal{O}(P)$ if and only if x does not cover any strong meet-irreducible of P and x is not the minimum of P (if this minimum exists),*
3. *I is $\mathbb{G}_P$-deletable w.r.t. $\mathcal{O}(P)$ if and only if x does not cover a strong meet-irreducible element of P and is neither join-prime nor the minimum of P (if this minimum exists).*

Proof: (1) is proved in Bordalo and Monjardet [4] (2002).

(2) By Point (2) of Theorem 5, $I = P \setminus [x)$ is $\mathbb{G}$-deletable w.r.t. $\mathcal{O}(P)$ if and only if $[I \neq \emptyset$ and I does not cover any meet-irreducible of $\mathcal{O}(P)]$. The ideal I is empty if and only if P has a minimum, which is then x. There exists $G \in M(\mathcal{O}(P))$ with $G \prec I$ if and only if $G = P \setminus [y) \prec P \setminus [x) = I$, if and only if $P \setminus [x) = (P \setminus [y)) + \{t\}$ for some t in P, if and only if $[x) =]y)$, if and only if $y = t$ is a strong meet-irreducible element covered by x.

(3) Since $\mathbb{G}_P = \mathbb{M}_P \cap \mathbb{G}$, we put together Conditions (1) and (2) and observe that, since x does not cover a strong meet-irreducible element of P and is not the minimum of P (when this minimum exists), it cannot be a node. □

An immediate consequence of this theorem (and of Lemma 12, for Item (3) below) is the following:

Corollary 14 *Let P be a poset, $\mathcal{O}(P)$ the distributive lattice of all ideals of P and $I = P \setminus [x)$ a meet-irreducible element of $\mathcal{O}(P)$. Then:*

1. *I is $\mathbb{M}$-deletable but not $\mathbb{M}_P$-deletable w.r.t. $\mathcal{O}(P)$ if and only if x is join-prime or a node of P,*
2. *I is $\mathbb{M}$-deletable but not $\mathbb{G}$-deletable w.r.t. $\mathcal{O}(P)$ if and only if x covers a strong meet-irreducible element of P or is the minimum of P (when this minimum exists),*
3. *I is $\mathbb{G}$-deletable but not $\mathbb{G}_P$-deletable w.r.t. $\mathcal{O}(P)$ if and only if x does not cover any strong meet-irreducible element of P and is join-prime.*

Considering as before $I = P \setminus [x)$, and using Lemma 12, it is easy to check that the above results induce a partition of the poset P into the following – some possibly empty – four classes P_i, $i = 1, 2, 3, 4$, where the last three ones cover the case where x is not a node:

- $P_1 = \{x \in P : x$ is a node$\} = \{x \in P : \mathcal{O}(P) \setminus I$ belongs neither to $\mathbb{M}_P$ nor to $\mathbb{G}\}$.
- $P_2 = \{x \in P : x$ covers a strong meet-irreducible and is not a node$\} = \{x \in P : \mathcal{O}(P) \setminus I$ belongs to $\mathbb{M}_P \setminus \mathbb{G}$ (and to $\mathbb{M}_P \setminus \mathbb{G}_P)\}$.
- $P_3 = \{x \in P : x$ does not cover a strong meet-irreducible of P and is join-prime$\} = \{x \in P : \mathcal{O}(P) \setminus I$ belongs to $\mathbb{G} \setminus \mathbb{G}_P\}$.

- $P_4 = \{x \in P : x$ does not cover a strong meet-irreducible element of P and is neither join-prime nor the minimum of P (if this minimum exists)$\} = \{x \in P : \mathcal{O}(P) \setminus I$ belongs to $\mathbb{G}_P\}$.

We illustrate this partition on the poset P represented on Figure 1. For this poset, $P_1 = \emptyset$, $P_2 = \{5\}$, $P_3 = \{1, 2\}$, and $P_4 = \{3, 4\}$. So the five lower covers of $\mathcal{O}(P)$ in $\mathbb{M}$ are :

$$\mathcal{O}(P) \setminus \{1234\}, \mathcal{O}(P) \setminus \{4\}, \mathcal{O}(P) \setminus \{1345\}, \mathcal{O}(P) \setminus \{124\}, \mathcal{O}(P) \setminus \{123\}.$$

The first one belongs to $\mathbb{M}_P \setminus \mathbb{G}_P$, the second and the third ones to $\mathbb{G} \setminus \mathbb{G}_P$ and the last two ones to $\mathbb{G}_P$.

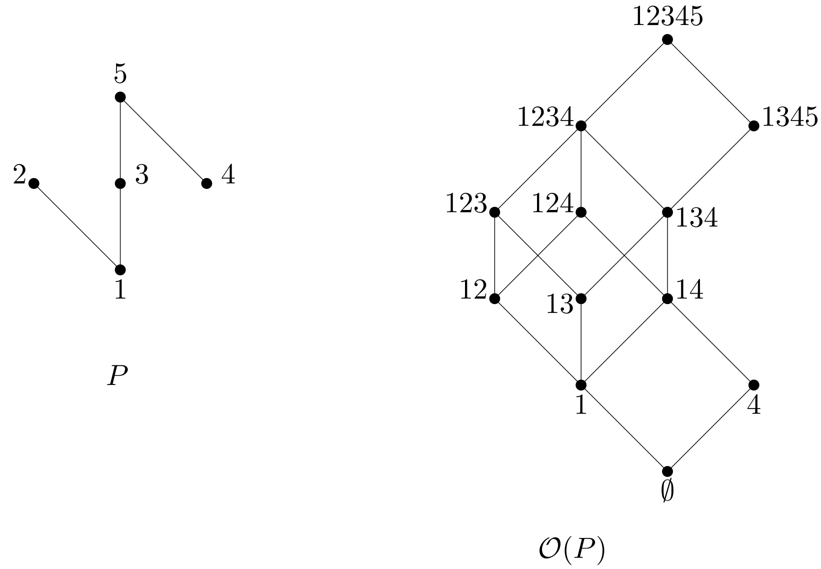

Figure 1:

We now aim to give an additional result for the case where $\mathcal{O}(P) \setminus I$ belongs to $\mathbb{G} \setminus \mathbb{G}_P$. To do so, we first prove the following lemma:

Lemma 15 *Let P be a poset and (x, y) a critical pair for P. Then $\mathcal{O}(P + (x, y))$ is a sublattice of $\mathcal{O}(P)$ obtained from $\mathcal{O}(P)$ by deleting the ideals $I \in \mathcal{O}(P)$ such that $y \in I$ and $x \notin I$.*

Proof: By the well-known Birkhoff's duality between posets and distributive lattices, we have $\mathcal{O}(P + (x, y))$ a sublattice of $\mathcal{O}(P)$. Now we have ($y \in I$ and $x \notin I$) if and only if $I \notin \mathcal{O}(P + (x, y))$. $\square$

Proposition 16 *Let P be a poset and x a join-prime of P which does not cover any strong meet-irreducible of P. Then $P = [x)+(y]$, (x, y) is a critical pair for P, $I = P \setminus [x) = (y]$ is a doubly-irreducible element of $\mathcal{O}(P)$ and $\mathcal{O}(P) \setminus I = \mathcal{O}(P+(x, y))$ is a (maximal) sublattice of $\mathcal{O}(P)$ which belongs to $\mathbb{G} \setminus \mathbb{G}_P$. Moreover, $J(\mathcal{O}(P) \setminus I) = (J(\mathcal{O}(P)) \setminus I) + \{I^+\}$ and $M(\mathcal{O}(P) \setminus I) = (M(\mathcal{O}(P)) \setminus I) + \{I^-\}$.*

Proof: If x is join-prime, $P \setminus [x)$ is a doubly-irreducible element of $\mathcal{O}(P)$ and (x, y) is a critical pair for P (by Lemma 12). Moreover, $(y]$ is the unique ideal of P which contains y and not x. So, by Lemma 15, $\mathcal{O}(P) \setminus I = \mathcal{O}(P + (x, y))$ is a maximal sublattice of $\mathcal{O}(P)$. Since x does not cover any strong meet-irreducible of P, it belongs to $\mathbb{G} \setminus \mathbb{G}_P$ (by Item (3) of Corollary 14). By Item (2)(a) of Corollary 10, $J(\mathcal{O}(P) \setminus I) = (J(\mathcal{O}(P)) \setminus I) + \{I^+\}$. Using the fact that in the distributive lattice $\mathcal{O}(P)$, I^- being the lower cover of a meet-irreducible element, can have only one other upper cover, Item (2)(b) of Corollary 10 implies that $M(\mathcal{O}(P) \setminus I) = (M(\mathcal{O}(P) \setminus I) + \{I^-\}$. □

By Lemma 12, we know that a join-prime element x of a poset P covers a strong meet-irreducible of P if and only if $P \setminus [x) = (y]$ is a doubly-irreducible element which is a node of $\mathcal{O}(P)$. Then we obtain:

Corollary 17 *There is a bijection between the doubly-irreducible elements which are not nodes of a distributive lattice $L = \mathcal{O}(P)$ and the family of convex geometries covered by $\mathcal{O}(P)$ in $\mathbb{G} \setminus \mathbb{G}_P$. Moreover, these convex geometries are maximal (distributive) sublattices of L.*

Remark 18 - Some of the above results can be compared with known results on the maximal sublattices of a distributive lattice (see Schmid [23] (2002) and the references there). The simplest way to get a maximal sublattice of a distributive lattice $L = \mathcal{O}(P)$ is to delete a doubly-irreducible element $I = P \setminus [x) = (y]$ of L (when such an element exists). In our study, this case splits into two sub-cases (see Lemma 12):

a. I is a node of $\mathcal{O}(P)$ (or, equivalently, by Lemma 12, x is a node of P, i.e. $x \in P_1$). Then, $\mathcal{O}(P) \setminus I$ belongs to $\mathbb{M} \setminus (\mathbb{M}_P \cup \mathbb{G})$ (but remains a distributive lattice).

b. I is not a node of $\mathcal{O}(P)$ or, equivalently, x is join-prime and does not cover a strong meet-irreducible element of P (i.e. $x \in P_3$). Then $\mathcal{O}(P) \setminus I$ is a maximal sublattice of $\mathcal{O}(P)$ which is also a convex geometry.

Using the Galois connection between families of subsets of a set X and binary relations on X (Lorrain [14] (1969), Chacron [9] (1971), Barbut and Monjardet [1], 1970), Schmid [23] (2002) has made a systematic study of the lattice of the (0-1) sublattices of a (finite or infinite) distributive lattice. The lattice of the finite $\cup$-stable Moore families has been studied in Caspard and Monjardet [8] (2004).

Previously we have defined a partition of a poset P into – some possibly empty – four classes P_i, $(i = 1, 2, 3, 4)$. We consider two cases where only one of these classes is non-empty.

First case:

Proposition 19 *For a poset P, the following properties are equivalent:*

1. *for every x in P, x is join-prime and does not cover a strong meet-irreducible element of P (i.e. $x \in P_3$),*
2. *for every x in P, x is join-prime and is not a node,*
3. *for all x in P and $I = P \setminus [x)$, $\mathcal{O}(P) \setminus I$ is a (maximal) sublattice of $\mathcal{O}(P)$,*
4. *P is an ordinal sum of 2-elements antichains.*

Remark 20 The class of posets which are ordinal sums of 2-elements antichains is a subclass of the class of posets P such that $|\mathbb{M}_P| = 1$ (see Proposition 16 in Bordalo and Monjardet [4], 2002). It is also a very special subclass of the class of *distributive posets* studied by several authors (Erné [13] (1991), Niederle [18] (1995) and Reading [22], 2002).

Second case: we consider a poset P such that, for all x in P and $I = P \setminus [x)$, $\mathcal{O}(P) \setminus I \in \mathbb{G}_P$ (i.e. $x \in P_4$). This is equivalent to say that, for every x in P, x does not cover a strong meet-irreducible element of P and is neither join-prime nor the minimum of P (if this minimum exists).

Observe that this property is also equivalent to the equality $|\{\text{coatoms of } \mathbb{G}_P\}| = |P|$ (and this implies that $|\{\text{coatoms of } \mathbb{M}_P\}| = |P|$).

It is easy to get examples of such posets. For instance, any complete bipartite poset $B_{p,q}$ (i.e. an ordinal sum of two antichains of sizes p and q) is such a poset if and only if $p, q \geq 3$. We give below another class of such posets. Say that a poset is *smi-free* if it has no strong meet-irreducible element. The following result shows that almost all the non connected smi-free posets satisfy this property.

Proposition 21 *Let P be a smi-free poset. If P is not connected, every meet-irreducible of $\mathcal{O}(P)$ is $\mathbb{G}_P$-deletable unless P is the (cardinal) sum of a singleton and a poset with a minimum.*

Proof: First observe that P is a smi-free poset if and only if every non maximal element of P has at least two upper covers. But then $|P| = 1$ or P has no maximum. Now, when P is not connected, no element of P is join-prime unless P is the disjoint union of a singleton and of a poset with a minimum (which is then the unique join-prime element of P). □

Remark 22 - The simplest example of a poset satisfying the conditions of Proposition 21 is the disjoint union of a singleton and of the complete bipartite poset $B_{2,2}$.

- The complete bipartite poset $B_{p,q}$ (with $p, q \geq 3$) is an example of a connected smi-free poset satisfying the conditions of Proposition 21.

5 The semilattice $\mathcal{G}_P$ of convex geometries with the same poset P of join-irreducible elements

We begin this section with an additional result on the changes in the poset of join-irreducible elements of a convex geometry when one goes to one of its lower covers in the semilattice $\mathbb{G}$ of convex geometries.

Proposition 23 *Let $\mathcal{G}' \in \mathbb{G}_P$ be a convex geometry, $\mathcal{G}$ one of its lower covers in the semilattice $\mathbb{G}$ of all convex geometries and J the poset of join-irreducible elements of $\mathcal{G}$. Then, either $J = J'$ and $\mathcal{G} \in \mathbb{G}_P$, or $J = (J' \setminus \{I\}) + \{I^+\}$ and $\mathcal{G} \in \mathbb{G}_{P+(x,y)}$, where $x = I^+ \setminus I$, $y = I \setminus I^-$ and (x, y) is a critical pair for P.*

Proof: By Item (2)(a) of Corollary 10, if $J \neq J'$, we have $J = J' \setminus \{I\} + \{I^+\}$ with $I \in J' \cap M'$ and I^+ having a unique other lower cover I' in $\mathcal{G}'$; moreover I^- is covered by I' in $\mathcal{G}'$. The order on J is the union of two orders. The first one is the order obtained by replacing I with I^+ in all the ordered pairs (K, I) and (I, L) of J'; this order is isomorphic to the order of J' and so to the poset P. The second order is the unique ordered pair (G, I^+) with $G \in J'$ and $G \vee I = I^+$. Then, in the convex geometries $\mathcal{G}'$ and $\mathcal{G}$, $G = \varphi_{\mathcal{G}'}(x) = \varphi_{\mathcal{G}}(x)$ where $x = I^+ \setminus I$ (see, for instance, Monjardet [15], 1990). Now, since I is join-irreducible in $\mathcal{G}'$, $I = \varphi_{\mathcal{G}'}(y)$, where $y = I \setminus I^-$ $(= I^+ \setminus I')$, and we have $I^+ = \varphi_{\mathcal{G}}(y)$ in $\mathcal{G}$. Then, adding the covering ordered pair $(G, I^+) = (\varphi_{\mathcal{G}}(x), \varphi_{\mathcal{G}}(y))$ to J' corresponds to adding the covering pair (x, y) to the isomorphic poset P. Moreover, since $P + (x, y)$ is a poset, (x, y) is a critical pair for P. □

Remark 24 The above proposition shows that, in the semilattice $\mathbb{G}$ of all convex geometries, one goes from a convex geometry $\mathcal{G}' \in \mathbb{G}_P$ to one of its lower covers $\mathcal{G} = \mathcal{G}' \setminus \{I\}$, either without changing its (up to isomorphism) poset P of join-irreducible elements or by adding to the cover relation of this poset a unique ordered pair $x \prec y$. The case where $\mathcal{G}' = \mathcal{O}(P)$ and $\mathcal{G} = \mathcal{O}(P) \setminus \{I\} \in \mathbb{G}_{P+(x,y)}$ has been already considered in Proposition 16.

Proposition 23 has two consequences:

Corollary 25 *Let P be a poset.*

1. *Let $\mathcal{G}' \in \mathbb{G}_P$ be a convex geometry with J' isomorphic to P. The number of critical pairs for P is an upper bound for the number of convex geometries $\mathcal{G}$ covered by $\mathcal{G}'$ in $\mathbb{G}$ and satisfying $J \neq J'$.*
2. *Let $\mathcal{G}, \mathcal{G}'$ be two convex geometries in $\mathbb{G}$, with $\mathcal{G} \subseteq \mathcal{G}'$ and $J = J'$. Then, for any convex geometry $\mathcal{G}''$ with $\mathcal{G} \subseteq \mathcal{G}'' \subseteq \mathcal{G}'$, we have $J = J" = J'$.*

Proof: (1) is obvious and (2) is immediate by considering in $\mathbb{G}$ a maximal chain between $\mathcal{G}$ and $\mathcal{G}'$ which contains $\mathcal{G}''$. □

Proposition 26 *Let P be a poset.*

1. *The poset $\mathbb{G}_P$ is a convex join-subsemilattice of the join-semilattice $\mathbb{G}$ and a join-subsemilattice of the lattice $\mathbb{M}_P$.*
2. *For $\mathcal{G}, \mathcal{G}' \in \mathbb{G}_P$, $\mathcal{G} \prec \mathcal{G}'$ in $\mathbb{G}_P$ if and only if $\mathcal{G} \prec \mathcal{G}'$ in $\mathbb{G}$, if and only if $\mathcal{G} \prec \mathcal{G}'$ in $\mathbb{M}_P$. Then, if $\mathcal{G} = \mathcal{G}' \setminus \{I\}$, $J = J'$ and $M = (M' \setminus \{I\}) + \{G \in \mathcal{G}' : G \prec I$ and G has a unique other upper cover$\}$.*
3. *$\mathbb{G}_P$ is a ranked semilattice.*

Proof: (1) It is clear that the poset $\mathbb{G}_P$ has a greatest element which is $\mathcal{O}(P)$. Let $\mathcal{G}$ and $\mathcal{G}'$ be two convex geometries with $J = J' = \mathcal{P}$. We must show that $J(\mathcal{G} \vee \mathcal{G}') = J$. The elements of $\mathcal{G}$ and $\mathcal{G}'$ are ideals of P as well as their intersections. So the join $\mathcal{G} \vee \mathcal{G}'$ of $\mathcal{G}$ and $\mathcal{G}'$ in $\mathbb{G}$ is contained in $\mathcal{O}(P)$. Now since $J(\mathcal{O}(P)) = \mathcal{P} = J$, and $\mathcal{G} \subseteq \mathcal{G} \vee \mathcal{G}' \subseteq \mathcal{O}(P)$, Item (2) of Corollary 25 shows that $J(\mathcal{G} \vee \mathcal{G}') = J$. This same corollary says that $\mathbb{G}_P$ is convex in $\mathbb{G}$.

(2) The first part is obvious since, for the covering relation $\prec$ in $\mathbb{M}_P$, $\mathbb{G}$ or $\mathbb{G}_P$, $\mathcal{G} \prec \mathcal{G}'$ if and only if $\mathcal{G} = \mathcal{G}' \setminus \{I\}$. The second part is an immediate consequence of Corollary 10.

(3) Since $\mathbb{G}_P$ is a convex subposet of the ranked semilattice $\mathbb{G}$, it is ranked. □

Remark 27 The semilattice $\mathbb{G}_P$ is generally not convex in $\mathbb{M}_P$. Consider, for instance, the atomistic convex geometry defined on the antichain $P = \{1, 2, 3, 4\}$ by taking all the intervals of the linear order $1 > 2 > 3 > 4$. Then, $\mathcal{G} = \{\emptyset, 1, 2, 3, 4, 12, 23, 34, 123, 234, 1234\}$ (where, for instance, 134 stands for $\{1, 3, 4\}$), and $\mathcal{G} + \{14\} \in \mathbb{M}_P \setminus \mathbb{G}_P$.

We are now going to characterize the ideals I of a poset P which are $\mathbb{G}_P$-deletable in a convex geometry $\mathcal{G}' \in \mathbb{G}_P$. In fact, since $\mathcal{G} = \mathcal{G}' \setminus \{I\}$ must belong to $\mathbb{G}_P = \mathbb{M}_P \cap \mathbb{G}$, Items (1)(c) and (2)(b) of Theorem 5 show that I is $\mathbb{G}_P$-deletable if and only if $I \in M' \setminus J'$, $I^+ \notin J'$ (which implies $I \neq \emptyset$) and I does not cover any meet-irreducible element of $\mathcal{G}'$. The following result gives another interesting charaterization.

Theorem 28 *Let P be a poset and $\mathcal{G}' \in \mathbb{G}_P$. An ideal I of P is $\mathbb{G}_P$-deletable w.r.t. $\mathcal{G}'$ if and only if it satisfies the following three conditions:*

1. *$I \in M'$,*
2. *for every $G \in \mathcal{G}'$ with $G \prec I$, $G \notin M'$,*
3. *I is a coatom of a Boolean sublattice $\underline{2}^3$ of $\mathcal{G}'$.*

Proof: We have just observed above that Conditions (1) and (2) together with $I, I^+ \notin J'$ are equivalent to $\mathcal{G}' \setminus \{I\} \in \mathbb{G}_P$. Since Condition (3) obviously implies $I, I^+ \notin J'$, we have to show that Conditions (1) and (2) together with $I, I^+ \notin J'$ imply Condition (3). Since I is not join-irreducible, there exist at least G_1 and G_2 distinct and covered by I. Since these elements are not meet-irreducibles (by Condition (2)), there exist K_1 and K_2 (different from I) with $G_1 \prec K_1$ and $G_2 \prec K_2$. It is impossible to have $K_1 = K_2$ (since one would have G_1 and G_2 covered by I and K_1). If K_1 and K_2 are covered by I^+, we are done since, in the convex geometry $\mathcal{G}'$, $\{K_1, K_2, I\}$ generate a Boolean sublattice $\underline{2}^3$ in which I is a coatom. Assume that K_1 is not covered by I^+. By Point (5) of Lemma 3, $G_1 \prec I$ and $G_1 \prec K_1$ imply the existence of $M_1, L_1 \in \mathcal{G}'$ with $G_1 \prec M_1 \prec L_1$ and $I \prec L_1$. Since $I \in M'$, one has $L_1 = I^+$ and M_1 must be different from K_1. If K_2 is covered by I^+, we are done (using $\{M_1, K_2, I\}$). If not, for the same reasons as given above, there exists M_2 ($\neq K_2$) with $G_2 \prec M_2 \prec L_2 = I^+$. But, $M_2 = M_1$ is impossible (since one would have G_1 and G_2 covered by I and M_1). Then, $\{M_1, M_2, I\}$ generate a Boolean sublattice $\underline{2}^3$ in which I is a coatom. □

The ideals $I = P \setminus [x)$ such that $\mathcal{O}(P) \setminus I$ is a coatom of the semilattice $\mathbb{G}_P$ have already been characterized by properties of x at Theorem 9 (Item (3)). Using the above theorem, we can also say that these ideals are the coatoms of a Boolean sublattice $\underline{2}^3$ of $\mathcal{O}(P)$ which are meet-irreducibles in $\mathcal{O}(P)$ and which do not cover any meet-irreducible element of $\mathcal{O}(P)$. Morever, such an ideal I covers $k = |\{\text{maximal elements of } I\}|$ ideals which all become new meet-irreducible elements in $\mathcal{O}(P) \setminus I$. Now, we focus on the minimal elements of $\mathbb{G}_P$. Using Theorem 5 and Corollary 10, one gets:

Corollary 29 *Let $\mathcal{G}$ be a convex geometry of $\mathbb{G}_P$. The following are equivalent:*

1. *$\mathcal{G}$ is a minimal element of $\mathbb{G}_P$,*
2. *for every $I \in M$, either there exists $G \prec I$ with $G \in M$, or $I \in M \cap J$,*
3. *the set M is a union of chains of M of the type $I_1 \prec I_2 \prec .... \prec I_k$, such that the minimum I_1 of each chain is doubly-irreducible.*

Remark 30 Let $\mathcal{G}$ and $\mathcal{G}'$ be two minimal elements of the semilattice $\mathbb{G}_P$. Since $\mathcal{G} \cap \mathcal{G}' \in \mathbb{M}_P \setminus \mathbb{G}_P$, the meet in $\mathbb{G}$ (when it exists) of $\mathcal{G}$ and $\mathcal{G}'$ satisfies $\mathcal{G} \wedge \mathcal{G}' \subseteq \mathcal{G} \cap \mathcal{G}'$. Observe also that $\mathcal{G}$ and $\mathcal{G}'$ are not necessarily isomorphic lattices. For instance, there are two types of minimal atomistic convex geometries on the set $P = \{1, 2, 3, 4\}$. The first is formed by all the intervals of a linear order defined on P, so is, for example, $\{\emptyset, 1, 2, 3, 4, 12, 23, 34, 123, 234, 1234\}$. The second is, for example $\{\emptyset, 1, 2, 3, 4, 13, 34, 23, 134, 234, 1234\}$.

We now characterize the posets P such that $|\mathbb{G}_P| = 1$, i.e. such that $\mathcal{O}(P)$ is the unique convex geometry $\mathcal{G}$ defined on P with $J(\mathcal{G}) = \mathcal{P}$. Recall that the *width* $w(P)$ of a poset P is the maximum size of an antichain in P.

Theorem 31 *For a poset P, the following are equivalent:*

1. $|\mathbb{G}_P| = 1$,
2. *for each x in P, either x is a node, or x is join-prime, or x covers a strong meet-irreducible element,*
3. *P is a poset of width at most* 2.

Proof: Since $\mathbb{G}_P| = 1$ if and only if $\mathbb{G}_P$ has no coatoms, the equivalence of (1) and (2) is an obvious consequence of Item (3) in Theorem 13.

Since $|\mathbb{G}_P| = 1$ if and only if $\mathcal{O}(P)$ has no $\mathbb{G}_P$-deletable element, the implication of (1) by (3) is an obvious consequence of Item (3) in Theorem 28.

Finally, we prove that (2) implies (3). Let P be a poset satisfying (2) and assume that $w(P) \geq 3$. Consider an antichain A of size $k \geq 3$, minimal in the classical order defined between antichains ($B \leq C$ if, for every $x' \in C$, there exists $x \in B$ such that $x \leq x'$). An element x in A cannot be a node of P (since A is an antichain) and cannot cover a strong meet-irreducible element, since then A would not be a minimal antichain. So x must be join-prime in P, i.e. $P = [x) + (y]$ for some y in P. This implies that the elements of $A \setminus [x)$ are all smaller than or equal to y. The same being true for every element of A, all the elements y form an antichain A' of size k such that A is smaller than A' in the order between antichains (and $A \cap A' = \emptyset$). Iterating this procedure, we would get an infinite chain of antichains of P, a contradiction with the fact that P is finite. □

Remark 32 The equivalence between (2) and (3) in this theorem gives a characterization of posets of width at most 2, that is new (to our knowledge).

We finally study the more general case where $\mathbb{G}_P$ has a unique coatom. We have the following result:

Proposition 33 *For a poset P, the following Properties (1) and (2) are equivalent and imply Property (3):*

1. *$\mathbb{G}_P$ has a unique coatom,*
2. *there exists a unique element in P which does not cover a strong meet-irreducible element of P and is neither join-prime nor the minimum of P (if this minimum exists),*
3. *P must contain at least one strong meet-irreducible element.*

Proof: The equivalence between Properties (1) and (2) comes from Point (3) of Theorem 13.

Assume that $\mathbb{G}_P$ has a unique coatom and that P is smi-free. Then, Property (2) becomes: "there exists a unique element in P which is neither a node nor join-prime". By Corollary 18 in Bordalo and Monjardet [4] (2002), this implies $|\mathbb{M}_P| = 2$, which in turns implies $w(P) = 2$ which, by Theorem 31, yields $|\mathbb{G}_P| = 1$ a contradiction. □

6 Conclusion

In Sections 2 to 5 of this paper, we have studied what changes occur in the posets of irreducible elements when one goes from an arbitrary Moore family (or from a convex geometry) to one of its lower covers in some (semi)lattices of Moore families or convex geometries. One knows that Moore families can be economically represented by implicational base (see, for instance, Wild [24], 1994). It would be interesting to study the changes in the implicational base – for instance, the Guigues-Duquenne canonical basis (see, for example, Caspard [6], 1999) – when one goes down in lattices of Moore families.

In Bordalo and Monjardet [4] (2002), we have studied the lattice $\mathbb{M}_P$ of all Moore families having the same poset P of join-irreducible elements. In particular, we have proved that $|\mathbb{M}_P| = 2$ if and only if $\mathbb{M}_P$ has a unique coatom, if and only if $\mathbb{M}_P$ has a unique atom, and if and only if P has a unique element wich is neither join-prime nor a node. So, if $\mathbb{M}_P$ is a chain, it is a 2-element chain. The situation is completely different for $\mathbb{G}_P$. Indeed, there exist posets P such that $\mathbb{G}_P$ is a chain of cardinality k, for any integer $k \geq 2$. An exemple of such a poset is given on Figure 2.

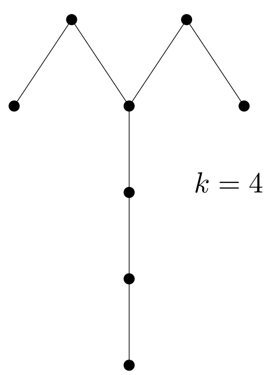

Figure 2:

As another illustration of the different behaviours of $\mathbb{M}_P$ and $\mathbb{G}_P$, one can observe that $\mathbb{G}_P$ can be the distributive lattice of cardinality 6 having a unique atom and a unique coatom (Figure 3(b)); this is the case when P is the poset given in Figure 3(a).

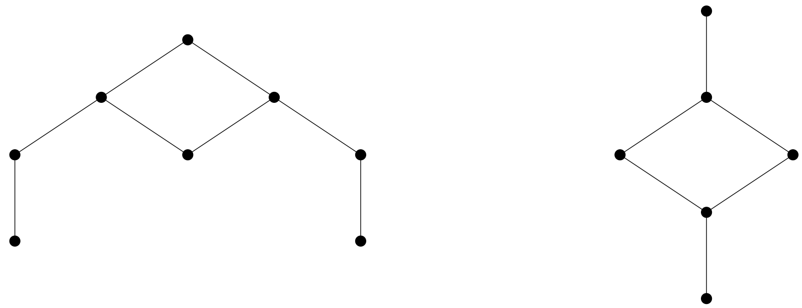

Figure 3:

Thus, a number of open problems remain and, in particular, the following one: what are the posets P such that $|\mathbb{G}_P| = 2$? Indeed, there exist many examples of such posets but it has been difficult so far to find some common characteristics.

This work also raises two more general questions:

1. What are the posets P such that $\mathbb{G}_P$ has a unique coatom ? In fact, we have given a characterization of these posets in Proposition 33, but is it possible to give a more precise description ?

2. What are the posets P such that $\mathbb{G}_P$ is a lattice ?

Last but not least, another interesting problem can be considered: if $\mathcal{G}$ and $\mathcal{G}'$ are minimal elements of the semilattice $\mathbb{G}_P$, does this imply $|\mathcal{G}| = |\mathcal{G}'|$?

7 Appendix: Algorithm

Theorem 28 easily leads to the following algorithm for computing all the convex geometries of $\mathbb{G}_P$.

Algorithm 1 Computing all convex geometries of $\mathbb{G}_P$.

Input: A poset P.
Output: The list CG_P of all convex geometries of $\mathbb{G}_P$.
begin
  1: Compute $(\mathcal{O}(P), \subseteq)$;
  $CG_P := \{(\mathcal{O}(P), \subseteq)\}$;
  2: **foreach** $L \in CG_P$ **do**
    Compute $M(L)$;
    Compute $J(L)$;
    Let $M^*_L := M(L) \setminus J(L)$;
    foreach $C \in M^*(L)$ **do**
      Compute $(C] := \{G \in L : G \prec C\}$;
      Compute $(C^+] := \{H \in L \setminus C : H \prec C^+\}$;
      if $(L \setminus C \notin CG_P)$ *and* $(|(C] \cap M(L)| = 0)$ *and* $(|(C^+]| \neq 0)$ **then**
        $CG_P := CG_P + \{(L \setminus C, \subseteq)\}$;
end

References

[1] M. Barbut and B. Monjardet, Ordre et Classification, Algèbre et Combinatoire, tomes I-II, (Hachette, Paris, 1970).

[2] J. Berman and G. Bordalo, Finite distributive lattices and doubly irreducible elements, Discrete Math., 178(1998), 237-243.

[3] G. Bordalo and B. Monjardet, Reducible classes of finite lattices, Order, 13(4)(1996), 379-390.

[4] G. Bordalo and B. Monjardet, The lattice of strict completions of a finite poset, Algebra Universalis, 47(2002), 183-200.

[5] G. Bordalo and B. Monjardet, Finite orders and their minimal strict completion lattices, Discuss. Math. Gen. Algebra Appl., 23(2003), 85-100.

[6] N. Caspard, A characterization theorem for the canonical basis of a closure operator, Order, 16(3)(1999), 227-230.

[7] N. Caspard and B. Monjardet, The lattice of closure systems, closure operators and implicational systems on a finite set: a survey, Discrete Appl. Math., 127(2)(2003), 241-269.

[8] N. Caspard and B. Monjardet, Some lattices of closure systems, Discrete Math. Theor. Comput. Sci., 6(2004), 163-190.

[9] J. Chacron, Nouvelles correspondances de Galois, Bull. Soc. Math. Belgique, 22(1971), 183-194.

[10] B.A. Davey and H.A. Priestley, Introduction to lattices and order, (Cambridge University Press, Cambridge, 1990).

[11] R.P. Dilworth, Lattices with unique irreducible representations, Ann. of Math., 41(1940), 771-777.

[12] P.H. Edelman and R.E. Jamison, The theory of convex geometries, Geom. Dedicata, 19(1985), 247-270.

[13] M. Erné, Bigeneration in complete lattices and principal separation in ordered sets, Order, 8(1991), 197-221.

[14] F. Lorrain, Notes on topological spaces with minimum neighborhoods, Amer. Math. Monthly, 76(6)(1969), 616-627.

[15] B. Monjardet, The consequences of Dilworth's work on lattices with unique irreducible decompositions. In: Bogart, K.P., Freese, R., Kung, J. (Eds), The Dilworth theorems Selected papers of Robert P. Dilworth. (Birkhaüser, Boston, 1990), 192-201.

[16] B. Monjardet and V. Raderanirina, The duality between the anti-exchange closure operators and the path independent choice operators on a finite set, Math. Social Sci., 41(2)(2001), 131-150.

[17] J.B. Nation and A. Pogel, The lattice of completions of an ordered set, Order, 14(1)(1997), 1-7.

[18] J. Niederle, Boolean and distributive ordered sets: characterization and representation by sets, Order, 12(2)(1995), 189-210.

[19] L. Nourine, Private communication (2003).

[20] O. Öre, Some studies on closure relations, Duke Math. J., 10(1943), 761-785.

[21] I. Rabinovitch and I. Rival, The rank of a distributive lattice, Discrete Math., 25(1979), 275-279.

[22] N. Reading, Order dimension, strong Bruhat order and lattice properties for posets, Order, 19(2002), 73-100.

[23] J. Schmid, Quasiorders and sublattices of distributive lattices, Order, 19(2002), 11-34.

[24] M. Wild, A theory of finite closure spaces based on implications, Adv. Math., 108(1)(1994), 118-139.