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Manh Hung NGUYEN, CERMSEM
San NGUYEN VAN, CERMSEM

**2005.41**

CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

Maison des Sciences Économiques, 106-112 boulevard de L'Hôpital, 75647 Paris Cedex 13
http://mse.univ-paris1.fr/Publicat.htm
ISSN : 1624-0340

# The Lagrange Multipliers and Existence of Competitive Equilibrium in an Intertemporal Model with Endogenous Leisure*

Nguyen Manh Hung†, San Nguyen Van‡

CERMSEM,Université de Paris 1,

MSE,106-112 Bd de l'Hopital, 75647 Paris, France.

January 25, 2005

**Abstract**

This paper proves the existence of competitive equilibrium in a single-sector dynamic economy with elastic labor supply. The method of proof relies on some recent results (see Le Van and Saglam [2004]) concerning the existence of Langrange multipliers in infinite dimensional spaces and their representation as a summable sequence.

*JEL Classification: C61, C62, D51, E13, O41*

*Keywords*: Optimal Growth Model, Lagrange Multipliers, Competitive Equilibrium, Elastic labor supply.

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*We are grateful to Cuong Le Van for suggesting the problem to us and proposing the method of using Lagrange multipliers. We are also grateful to Yiannis Vailakis for comments and suggestions.

† *E-mail*: manh-hung.nguyen@malix.univ-paris1.fr

‡ *E-mail:* san.nguyen-van@malix.univ-paris1.fr.

# 1 Introduction

Since the seminal work of Ramsey [1928], optimal growth models have played a central role in modern macroeconomics. Classical growth theory relies on the assumption that labor is supplied in fixed amounts, although the original paper of Ramsey did include the disutility of labor as an argument in consumers' utility functions. Subsequent research in applied macroeconomics (theories of business cycles fluctuations) have reassessed the role of labor-leisure choice in the process of growth. Nowadays, intertemporal models with elastic labor continue to be the standard setting used to model many issues in applied macroeconomics.

Lagrange multipliers techniques have facilitated considerably the analysis of constraint optimization problems. The applications of those techniques in the analysis of intertemporal models inherits most of the tractability found in a finite setting. However, the passage to an infinite dimensional setting raises additional questions. These questions concerned both with the extension of the Lagrangean in an infinite dimensional setting as well as the representation of the Lagrange multipliers as a summable sequence.

Our purpose is to prove existence of competitive equilibrium for the basic neoclassical model with elastic labor using some recent results (see Le Van and Saglam [2004]) concerning the existence of Lagrange multipliers in infinite dimensional spaces and their representation as a summable sequence.

The issue of endogenous labor supply in intertemporal models have been analyzed before. See, for example,Greenwood and Huffman [1995], Coleman(1997), and Datta et al. [2002]. These models analyze economies with distortions and exploit the existence and uniqueness of stationary equilibrium paths. Recently, Le Van and Vailakis [2004] have proved the existence of a competitive equilibrium in a version of a Ramsey model in which leisure enters the utility function by exploiting the link between Pareto optima and competitive equilibria. Their method of proof is in the line of Le Van and Vailakis [2003] for the optimal growth model with inelastic labor supply. To develop methods for studying the existence of competitive equilibrium in a one sector growth model with leisure, this paper takes somewhat different approach than attempting to impose Inada conditions on the utility function.

Following the early work of Peleg and Yaari [1970], Bewley [1972] studied the existence of equilibrium in an economy in which $l^\infty$ is the commodity space and the method of using the limit of equilibria of finite dimensional economies. The most important development since Bewley's work was provided by Mas-Collel [1986], by using Negishi's approach when the commodity space is a topological vector lattice. Many others works can be found in Florenzano [1983], Aliprantis et al. [1990], Mas-Collel and Zame [1991], Dana and Le Van [1991], Becker and Boyd [1997]... These methods constitute a general and elegant approach to the question of optimality and existence but they require a high level of abstraction. Our approach here extends the results of Le Van and Saglam [2004] to a model with endogenous leisure. The proof of existence of equilibrium we give is more simple than in Le Van and Vailakis [2004] and require less stringent assumptions ( no Inada conditions for the utility function and the production function, no constant return to scale for the production function).

The organization of the paper is as follows. Section 2 provides the sufficient conditions on the objective function and the constraint functions so that Lagrangean multipliers can be presented by an $l^1_+$ sequence of multipliers in optimal growth model with the leisure in the utility function. In section 3, we prove the existence of competitive equilibrium by using these multipliers as sequences of price and wage systems.

# 2 The Lagrange Multipliers in Optimal Growth Model

Consider an economy in which a representative consumer has preferences defined over processes of consumption and leisure described by the utility function

$$\sum_{t=0}^{\infty} \beta^t u(c_t, l_t).$$

In each period, the consumer faces two resource constraints given by

$$\begin{aligned} c_t + k_{t+1} &\leq F(k_t, L_t) + (1-\delta)k_t, \\ l_t + L_t &= 1, \ \forall t \end{aligned}$$

where $F$ is the production function, $\delta \in (0,1)$ is the depreciation rate of capital stock and $L_t$ is labor. These constraints restrict allocations of commodities and time for the leisure.

Formally, the problem of the representative consumer is stated as follows:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$
$$\begin{aligned}
\text{s.t. } c_t + k_{t+1} &\leq F(k_t, 1-l_t) + (1-\delta)k_t, \ \forall t \geq 0 \\
c_t &\geq 0, k_t \geq 0, l_t \geq 0, 1-l_t \geq 0, \ \forall t \geq 0 \\
k_0 &\geq 0 \text{ is given.}
\end{aligned}$$

We make the following assumptions:

**A1**: $u(c,l) \in \mathbb{R}_+$ if $(c,l) \in \mathbb{R}^2_+$, $u(c,l) = -\infty$ if $(c,l) \notin \mathbb{R}^2_+$ and $u(0,0) = 0$.

**A2**:The function $u$ is strictly increasing, concave and differentiable in $\mathbb{R}^2_{++}$.

**A3**: $F(k,L) \in \mathbb{R}_+$ if $(k,L) \in \mathbb{R}^2_+$, $F(k,L) = -\infty$ if $(k,L) \notin \mathbb{R}^2_+$ and $F(0,L) = F(k,0) = 0$.

**A4**: The function $F$ strictly increasing, concave and differentiable in $\mathbb{R}^2_{++}$. Further, $F_k(0,1) > \delta$ and $F_k(+\infty, 1) = 0$.

We say that a sequence $(c_t, k_t, l_t)_{t=0,1,...,\infty}$ is feasible from $k_0$ if it satisfies the constraints

$$\begin{aligned}
\forall t &\geq 0, \ c_t + k_{t+1} \leq F(k_t, 1-l_t) + (1-\delta)k_t \\
c_t &\geq 0, k_t \geq 0, l_t \geq 0, 1-l_t \geq 0, \\
k_0 &> 0 \text{ is given.}
\end{aligned}$$

It is easy to check that, for any initial condition $k_0 > 0$, a sequence $\mathbf{k} = (k_0, k_1, k_2, ..., k_t, ...)$ is feasible iff $0 \leq k_{t+1} \leq F(k_t, 1) + (1-\delta)k_t$ for all $t$. The class of feasible capital paths is denoted by $\Pi(k_0)$.A pair of consumption-leisure sequences $(\mathbf{c}, \mathbf{l})$ =$((c_0, l_0), (c_1, l_1), ...)$ is feasible from $k_0 > 0$ if there exists a sequence $\mathbf{k} \in \Pi(k_0)$ that satisfies $0 \leq c_t + k_{t+1} \leq F(k_t, 1-l_t) + (1-\delta)k_t$ and $0 \leq l_t \leq 1$ for all $t$.

Define $f(k_t, L_t) = F(k_t, L_t) + (1-\delta)k_t$. Assumption **A4** implies that

$$\begin{aligned}
f_k'(+\infty, 1) &= F_k(+\infty, 1) + (1-\delta) = 1-\delta < 1 \\
f_k'(0,1) &= F_k(0,1) + (1-\delta) > 1.
\end{aligned}$$

From above, it follows that there exists $\overline{k} > 0$ such that: (i) $f(\overline{k}, 1) = \overline{k}$ , (ii) $k > \overline{k}$ implies $f(k,1) < k$, (iii) $k < \overline{k}$ implies $f(k,1) > k$. Therefore for any $\mathbf{k} \in \Pi(k_0)$, we have $0 \leq k_t \leq \max(k_0, \overline{k})$. Thus, $\mathbf{k} \in l^\infty_+$ which in turn implies $\mathbf{c} \in l^\infty_+$, if $(\mathbf{c}, \mathbf{k})$ is feasible from $k_0$.

Denote $\mathbf{x} = (\mathbf{c}, \mathbf{k}, \mathbf{l})$ and define $\mathcal{F}(\mathbf{x}) = -\sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$,

$$\Phi_t^1(\mathbf{x}) = c_t + k_{t+1} - F(k_t, 1 - l_t) - (1 - \delta)k_t, \ \forall t$$

$$\Phi_t^2(\mathbf{x}) = -c_t, \ \Phi_t^3(\mathbf{x}) = -k_t, \ \Phi_t^4(\mathbf{x}) = -l_t, \Phi_t^5(\mathbf{x}) = l_t - 1, \ \forall t,$$

$$\Phi_t = (\Phi_t^1, \Phi_t^2, \Phi_{t+1}^3, \Phi_t^4, \Phi_t^5), \ \forall t.$$

The social planner's problem $(\mathbf{P})$ can be written as:

$$\begin{aligned} & \min \mathcal{F}(\mathbf{x}) \\ \text{s.t.}\Phi(\mathbf{x}) \ & \leq \ \mathbf{0}, \mathbf{x} \in l_+^\infty \times l_+^\infty \times l_+^\infty \end{aligned}$$

where:

$$\begin{aligned} \mathcal{F} \ &: \ l_+^\infty \times l_+^\infty \times l_+^\infty \to \mathbb{R} \cup \{+\infty\} \\ \Phi \ &= \ (\Phi_t)_{t=0\ldots\infty} : l_+^\infty \times l_+^\infty \times l_+^\infty \to \mathbb{R} \cup \{+\infty\} \end{aligned}$$

Let:

$$\begin{aligned} C \ &= \ dom(\mathcal{F}) = \{\mathbf{x} \in l_+^\infty \times l_+^\infty \times l_+^\infty | \mathcal{F}(\mathbf{x}) < +\infty\} \\ \Gamma \ &= \ dom(\Phi) = \{\mathbf{x} \in l_+^\infty \times l_+^\infty \times l_+^\infty | \Phi_t(\mathbf{x}) < +\infty, \ \forall t\} \end{aligned}$$

.The following Theorem is due to Le Van and Saglam [2004].

**Theorem 1** *Let* $\mathbf{x} \in l^\infty$, $\mathbf{y} \in l^\infty$, $T \in N$.

*Define* $x_t^T(\mathbf{x}, \mathbf{y}) = \begin{cases} x_t & if \quad t \leq T \\ y_t & if \quad t > T \end{cases}$

*Suppose that two following assumptions are satisfied:*

***T1****: If* $\mathbf{x} \in C$, $\mathbf{y} \in l^\infty$ *satisfy* $\forall T \geq T_0$, $\mathbf{x}^T(\mathbf{x}, \mathbf{y}) \in C$ *then* $F(\mathbf{x}^T(\mathbf{x}, \mathbf{y})) \to F(\mathbf{x})$ *when* $T \to \infty$.

***T2****: If* $\mathbf{x} \in \Gamma$, $\mathbf{y} \in \Gamma$ *and* $\mathbf{x}^T(\mathbf{x}, \mathbf{y}) \in \Gamma$, $\forall T \geq T_0$.

*Then,*

*a)* $\Phi_t(\mathbf{x}^T(\mathbf{x}, \mathbf{y})) \to \Phi_t(\mathbf{x})$ *as* $T \to \infty$

*b)* $\exists M$ *s.t.* $\forall T \geq T_0$ , $\|\Phi_t(\mathbf{x}^T(\mathbf{x}, \mathbf{y}))\| \leq M$

*c)* $\forall N \geq T_0$, $\lim_{t \to \infty} [\Phi_t(\mathbf{x}^T(\mathbf{x}, \mathbf{y})) - \Phi_t(\mathbf{y})] = 0$

*Let* $\mathbf{x}^*$ *be a solution to* ***(P)*** *and* $\mathbf{x}^0 \in C$ *satisfy the Slater condition:*

$$\sup_t \Phi_t(\mathbf{x}^0) < 0.$$

*Suppose* $\mathbf{x}^T(\mathbf{x}^*, \mathbf{x}^0) \in C \cap \Gamma$. *Then, there exists* $\mathbf{\Lambda} \in l^1_+$ *such that*

$$\forall \mathbf{x} \in l^\infty,\ \mathcal{F}(\mathbf{x}) + \mathbf{\Lambda}\Phi(\mathbf{x}) \geq \mathcal{F}(\mathbf{x}^*) + \Lambda\Phi(\mathbf{x}^*),\ \forall \mathbf{x} \in (C \cap \Gamma)$$

*and* $\Lambda\Phi(\mathbf{x}^*) = 0$.

**Proof.** See Theorems 1 and 2 in Le Van and Saglam [2004]. ■

We make use of Theorem 1 and obtain :

**Proposition 2** *If* $\mathbf{x}^* = (\mathbf{c}^*, \mathbf{k}^*, \mathbf{l}^*)$ *is a solution to the following problem:*

$$\begin{aligned}
&\min -\sum_{t=0}^{\infty} \beta^t u(c_t, l_t) && \textbf{\textit{(Q)}}\\
&s.t.\ c_t + k_{t+1} - F(k_t, 1 - l_t) - (1-\delta)k_t \leq 0\\
&-c_t \leq 0\\
&-k_t \leq 0\\
&-l_t \leq 0\\
&l_t - 1 \leq 0
\end{aligned}$$

*Then there exists* $\boldsymbol{\lambda} = (\boldsymbol{\lambda}^1, \boldsymbol{\lambda}^2, \boldsymbol{\lambda}^3, \boldsymbol{\lambda}^4, \boldsymbol{\lambda}^5) \in l^1_+ \times\ l^1_+ \times l^1_+ \times l^1_+ \times l^1_+$ *such that:* $\forall \mathbf{x} = (\mathbf{c}, \mathbf{k}, \mathbf{l}) \in l^\infty_+ \times l^\infty_+ \times l^\infty_+$

$$\begin{aligned}
&\sum_{t=0}^{\infty}\beta^t u(c^*_t, l^*_t) - \sum_{t=0}^{\infty}\lambda^1_t(c^*_t + k^*_{t+1} - F(k^*_t, 1 - l^*_t) - (1-\delta)k^*_t)\\
&+\sum_{t=0}^{\infty}\lambda^2_t c^*_t + \sum_{t=0}^{\infty}\lambda^3_t k^*_t + \sum_{t=0}^{\infty}\lambda^4_t l^*_t + \sum_{t=0}^{\infty}\lambda^5_t(1 - l^*_t)\\
\geq\ &\sum_{t=0}^{\infty}\beta^t u(c_t, l_t) - \sum_{t=0}^{\infty}\lambda^1_t(c_t + k - F(k_t, 1 - l_t) - (1-\delta)k_t)\\
&+\sum_{t=0}^{\infty}\lambda^2_t c_t + \sum_{t=0}^{\infty} \lambda^3_t k_t + \sum_{t=0}^{\infty} \lambda^4_t l_t + \sum_{t=0}^{\infty}\lambda^5_t(1 - l_t) \qquad (1)
\end{aligned}$$

*and*

$$\lambda^1_t(c^*_t + k^*_{t+1} - F(k^*_t, 1 - l^*_t) - (1-\delta)k^*_t) = 0 \qquad (2)$$

$$\lambda_t^2 c_t^* = 0 \tag{3}$$

$$\lambda_t^3 k_t^* = 0 \tag{4}$$

$$\lambda_t^4 l_t^* = 0 \tag{5}$$

$$\lambda_t^5 (1 - l_t^*) = 0 \tag{6}$$

$$0 \in \beta^t \partial_1 u(c_t^*, l_t^*) - \{\lambda_t^1\} + \{\lambda_t^2\} \tag{7}$$

$$0 \in \beta^t \partial_2 u(c_t^*, l_t^*) - \lambda_t^1 \partial_2 F(k_t^*, L_t^*) + \{\lambda_t^4\} - \{\lambda_t^5\} \tag{8}$$

$$0 \in \lambda_t^1 \partial_1 F(k_t^*, L_t^*) + \{(1-\delta)\lambda_t^1\} + \{\lambda_t^3\} - \{\lambda_{t-1}^1\} \tag{9}$$

*where* $\partial_i u(c_t^*, l_t^*), \partial_i F(k_t^*, L_t^*)$ *respectively denote the projection on the* $i^{th}$ *component of the subdifferential of function* $u$ *at* $(c_t^*, l_t^*)$ *and the function* $F$ *at* $(k_t^*, L_t^*)$.

**Proof.** We first check that Slater condition holds. Indeed, since $f_k'(0,1) > 1$, then for all $k_0 > 0$, there exists some $0 < \widehat{k} < k_0$ such that: $0 < \widehat{k} < f(\widehat{k}, 1)$ and $0 < \widehat{k} < f(k_0, 1)$.Thus, there exists two small positive numbers $\varepsilon$, $\varepsilon_1$ such that:

$$0 < \widehat{k} + \varepsilon < f(\widehat{k}, 1 - \varepsilon_1) \text{ and } 0 < \widehat{k} + \varepsilon < f(k_0, 1 - \varepsilon_1).$$

Denote $\mathbf{x}^0 = (\mathbf{c}^0, \mathbf{k}^0, \mathbf{l}^0)$ such that $\mathbf{c}^0 = (\varepsilon, \varepsilon, \varepsilon, ...)$, $\mathbf{k}^0 = (k_0, \widehat{k}, \widehat{k}, ...)$, $\mathbf{l}^0 = (\varepsilon_1, \varepsilon_1, \varepsilon_1, ...)$.

We have

$$\begin{aligned}
\Phi_0^1(\mathbf{x}^0) &= c_0 + k_1 - F(k_0, 1 - l_0) - (1-\delta)k_0 \\
&= \varepsilon + \widehat{k} - f(k_0, 1 - \varepsilon_1) < 0 \\
\Phi_1^1(\mathbf{x}^0) &= c_1 + k_2 - F(k_1, 1 - l_1) - (1-\delta)k_1 \\
&= \varepsilon + \widehat{k} - f(\widehat{k}, 1 - \varepsilon_1) < 0 \\
\Phi_t^1(\mathbf{x}^0) &= \varepsilon + \widehat{k} - f(\widehat{k}, 1 - \varepsilon_1) < 0, \ \forall t \geq 2
\end{aligned}$$

$$\Phi_t^2(\mathbf{x}^0) = -\varepsilon < 0, \ \forall t \geq 0$$

$$\begin{aligned}\Phi_0^3(\mathbf{x}^0) &= -k_0 < 0;\\ \Phi_t^3(\mathbf{x}^0) &= -\widehat{k} < 0 \ \ \forall t \geq 1.\end{aligned}$$

$$\Phi_t^4(\mathbf{x}^0) = -\varepsilon_1 < 0, \ \forall t \geq 0$$

$$\Phi_t^5(\mathbf{x}^0) = \varepsilon_1 - 1 < 0, \ \forall t \geq 0$$

Therefore the Slater condition is satisfied. Now, it is obvious that, $\forall T$, $\mathbf{x}^T(\mathbf{x}^*, \mathbf{x}^0)$ belongs to $l_+^\infty \times l_+^\infty \times l_+^\infty$.

As in Le Van-Saglam 2004, Assumption **T2** is satisfied. We now check Assumption **T1**.

For any $\widetilde{\mathbf{x}} \in C, \widetilde{\widetilde{\mathbf{x}}} \in l_+^\infty \times l_+^\infty \times l_+^\infty$ such that for any $T$ $\mathbf{x}^T(\widetilde{\mathbf{x}}, \widetilde{\widetilde{\mathbf{x}}}) \in C$ we have

$$\mathcal{F}(\mathbf{x}^T(\widetilde{\mathbf{x}}, \widetilde{\widetilde{\mathbf{x}}})) = -\sum_{t=0}^{T} \beta^t u(\widetilde{c}_t, \widetilde{l}_t) - \sum_{t=T+1}^{\infty} \beta^t u(\widetilde{\widetilde{c}}_t, \widetilde{\widetilde{l}}_t).$$

As $\widetilde{\widetilde{\mathbf{x}}} \in l_+^\infty \times l_+^\infty \times l_+^\infty$, $\sup\limits_t |\widetilde{\widetilde{c}}_t| < +\infty$ , there exists $m > 0, \forall t, |\widetilde{\widetilde{c}}_t| \leq m$. Since $\beta \in (0,1)$ we have

$$\sum_{t=T+1}^{\infty} \beta^t u(m,1) = u(m,1) \sum_{t=T+1}^{\infty} \beta^t \to 0 \text{ as } T \to \infty.$$

Hence, $\mathcal{F}(\mathbf{x}^T(\widetilde{\mathbf{x}}, \widetilde{\widetilde{\mathbf{x}}})) \to \mathcal{F}(\widetilde{\mathbf{x}})$ when $T \to \infty$.Taking account of the Theorem 1, we get (1) - (6)

Finally, we obtain (7) - (9) from the Kuhn-Tucker first-order conditions.

■

# 3 Competitive Equilibrium

**Definition 3** *A competitive equilibrium for this model consists of an allocation* $\{\mathbf{c}^*, \mathbf{l}^*, \mathbf{k}^*, \mathbf{L}^*\} \in l_+^\infty \times l_+^\infty \times l_+^\infty \times l_+^\infty$, *price sequence* $\mathbf{p}^* \in l_+^1$ *for the consumption good, a wage sequence* $\mathbf{w}^* \in l_+^1$ *for labor and a price* $r > 0$ *for the initial capital stock* $k_0$ *such that:*

*i)* $(\mathbf{c}^*, \mathbf{l}^*)$ *is a solution to the problem*

$$\begin{aligned} \max \quad & \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \\ s.t. \quad & \mathbf{p}^*\mathbf{c} \leq \mathbf{w}^*\mathbf{L} + \boldsymbol{\pi}^* + r k_0 \end{aligned}$$

*where* $\boldsymbol{\pi}^*$ *is the maximum profit of the firm.*

*ii)* $(\mathbf{k}^*, \mathbf{L}^*)$ *is a solution to the firm's problem*

$$
\begin{aligned}
\boldsymbol{\pi}^* &= \max \sum_{t=0}^{\infty} p_t^* [F(k_t, L_t) + (1-\delta)k_t - k_{t+1}] - \sum_{t=0}^{\infty} w_t^* L_t - r k_0 \\
s.t. \quad 0 &\leq k_{t+1} \leq F(k_t, L_t) + (1-\delta)k_t, \ \ L_t \geq 0, \forall t
\end{aligned}
$$

*iii)Markets clear*

$$
\begin{aligned}
\forall t, \ c_t^* + k_{t+1}^* &= F(k_t^*, L_t^*) + (1-\delta)k_t^* \\
l_t^* + L_t^* &= 1 \ \textit{and} \ k_0^* = k_0
\end{aligned}
$$

**Theorem 4** *Let* $(\mathbf{c}^*, \mathbf{k}^*, \mathbf{l}^*)$ *solve Problem (**Q**). Take*

$$p_t^* = \lambda_t^1 \ \textit{for any } t \ \textit{and } r > 0.$$

*There exists* $f_t^2(k_t^*, L_t^*) \in \partial_2 F(k_t^*, L_t^*)$ *such that* $\{\mathbf{c}^*, \mathbf{k}^*, \mathbf{L}^*, \mathbf{p}^*, \mathbf{w}^*, r\}$ *is a competitive equilibrium with* $w_t^* = \lambda_t^1 f_t^2(k_t^*, L_t^*)$*.*

**Proof.** Consider $\boldsymbol{\lambda} = (\boldsymbol{\lambda}^1, \boldsymbol{\lambda}^2, \boldsymbol{\lambda}^3, \boldsymbol{\lambda}^4, \boldsymbol{\lambda}^5)$ of Proposition2. Conditions (7),(8),(9) in Proposition2 show that $\partial u(c_t^*, l_t^*)$ and $\partial F(k_t^*, L_t^*)$ are nonempty. Moreover,$\forall t$, there exists $u_t^1(c_t^*, l_t^*) \in \partial_1 u(c_t^*, l_t^*)$, $u_t^2(c_t^*, l_t^*) \in \partial_2 u(c_t^*, l_t^*)$, $f_t^1(k_t^*, L_t^*) \in \partial_1 F(k_t^*, L_t^*)$and $f_t^2(k_t^*, L_t^*) \in \partial_2 F(k_t^*, L_t^*)$ such that

$$\beta^t u_t^1(c_t^*, l_t^*) - \lambda_t^1 + \lambda_t^2 = 0 \tag{10}$$

$$\beta^t u_t^2(c_t^*, l_t^*) - \lambda_t^1 f_t^2(k_t^*, L_t^*) + \lambda_t^4 - \lambda_t^5 = 0 \tag{11}$$

$$\lambda_t^1 f_t^1(k_t^*, L_t^*) + (1-\delta)\lambda_t^1 + \lambda_t^3 - \lambda_{t-1}^1 = 0 \tag{12}$$

Define $w_t^* = \lambda_t^1 f_t^2(k_t^*, L_t^*) < +\infty$.We now prove that $\mathbf{w}^* \in l_+^1$.

We have

$$+\infty > \sum_{t=0}^{\infty} \beta^t u(c_t^*, l_t^*) - \sum_{t=0}^{\infty} \beta^t u(0,0) \geq \sum_{t=0}^{\infty} \beta^t u_t^1(c_t^*, l_t^*) c_t^* + \sum_{t=0}^{\infty} \beta^t u_t^2(c_t^*, l_t^*) l_t^*,$$

which implies

$$\sum_{t=0}^{\infty} \beta^t u_t^2(c_t^*, l_t^*) l_t^* < +\infty, \tag{13}$$

and

$$+\infty > \sum_{t=0}^{\infty} \lambda_t^1 F(k_t^*, L_t^*) - \sum_{t=0}^{\infty} \lambda_t^1 F(0,0) \geq \sum_{t=0}^{\infty} \lambda_t^1 f_t^1(k_t^*, L_t^*) k_t^* + \sum_{t=0}^{\infty} \lambda_t^1 f_t^2(k_t^*, L_t^*) L_t^*$$

which implies

$$\sum_{t=0}^{\infty} \lambda_t^1 f_t^2(k_t^*, L_t^*) L_t^* < +\infty. \tag{14}$$

Given $T$, we multiply (11) by $L_t^*$ and sum from 0 to $T$. We then obtain

$$\forall T, \ \sum_{t=0}^{T} \beta^t u_t^2(c_t^*, l_t^*) L_t^* = \sum_{t=0}^{T} \lambda_t^1 f_t^2(k_t^*, L_t^*) L_t^* + \sum_{t=0}^{T} \lambda_t^5 L_t^* - \sum_{t=0}^{T} \lambda_t^4 L_t^*. \tag{15}$$

Observe that

$$0 \leq \sum_{t=0}^{\infty} \lambda_t^5 L_t^* \leq \sum_{t=0}^{\infty} \lambda_t^5 < +\infty. \tag{16}$$

$$0 \leq \sum_{t=0}^{\infty} \lambda_t^4 L_t^* \leq \sum_{t=0}^{\infty} \lambda_t^4 < +\infty \tag{17}$$

Thus, since $L_t^* = 1 - l_t^*$, from (15), we get

$$\begin{aligned} \sum_{t=0}^{T} \beta^t u_t^2(c_t^*, l_t^*) &= \sum_{t=0}^{T} \beta^t u_t^2(c_t^*, l_t^*) l_t^* + \sum_{t=0}^{T} \lambda_t^1 f_t^2(k_t^*, L_t^*) L_t^* \\ &+ \sum_{t=0}^{T} \lambda_t^5 L_t^* - \sum_{t=0}^{T} \lambda_t^4 L_t^* \end{aligned}$$

Using (13),(14),(16),(17) and letting $T \to \infty$, we obtain

$$\begin{aligned} 0 &\leq \sum_{t=0}^{\infty} \beta^t u_t^2(c_t^*, l_t^*) = \sum_{t=0}^{\infty} \beta^t u_t^2(c_t^*, l_t^*) l_t^* + \sum_{t=0}^{\infty} \lambda_t^1 f_t^2(k_t^*, L_t^*) L_t^* \\ &+ \sum_{t=0}^{\infty} \lambda_t^5 L_t^* - \sum_{t=0}^{\infty} \lambda_t^4 L_t^* < +\infty \end{aligned}$$

Consequently, from (11), $\sum_{t=0}^{\infty}\lambda_t^1 f_t^2(k_t^*, L_t^*) < +\infty$ i.e. $\mathbf{w}^* \in l_+^1$.
So, we have $\{\mathbf{c}^*, \mathbf{l}^*, \mathbf{k}^*, \mathbf{L}^*\} \in l_+^\infty \times l_+^\infty \times l_+^\infty \times l_+^\infty$, with $\mathbf{p}^* \in l_+^1$ and $\mathbf{w}^* \in l_+^1$.
We show that $(\mathbf{k}^*, \mathbf{L}^*)$ is solution to the firm's problem.
Since $p_t^* = \lambda_t^1$, $w_t^* = \lambda_t^1 f_t^2(k_t^*, L_t^*)$, we have

$$\boldsymbol{\pi}^* = \sum_{t=0}^{\infty}\lambda_t^1[F(k_t^*, L_t^*) + (1-\delta)k_t^* - k_{t+1}^*] - \sum_{t=0}^{\infty}\lambda_t^1 f_t^2(k_t^*, L_t^*)\ L_t^* - rk_0$$

Let :

$$\begin{aligned}\Delta_T &= \sum_{t=0}^{T}\lambda_t^1[F(k_t^*, L_t^*) + (1-\delta)k_t^* - k_{t+1}^*] - \sum_{t=0}^{T}\lambda_t^1 f_t^2(k_t^*, L_t^*)\ L_t^* - rk_0 \\ &\quad - \left(\sum_{t=0}^{T}\lambda_t^1[F(k_t, L_t) + (1-\delta)k_t - k_{t+1}] - \sum_{t=0}^{T}\lambda_t^1 f_t^2(k_t^*, L_t^*)\ L_t - rk_0\right)\end{aligned}$$

By the concavity of $F$ , we get

$$\begin{aligned}\Delta_T &\geq \sum_{t=1}^{T}\lambda_t^1 f_t^1(k_t^*, L_t^*)(k_t^* - k_t) + (1-\delta)\sum_{t=1}^{T}\lambda_t^1(k_t^* - k_t) - \\ &\quad \sum_{t=0}^{T}\lambda_t^1(k_{t+1}^* - k_{t+1}) = [\lambda_1^1 f_t^1(k_1^*, L_1^*) + (1-\delta)\lambda_1^1 - \lambda_0^1](k_1^* - k_1) + ... \\ &\quad + [\lambda_T^1 f_t^1(k_T^*, L_T^*) + (1-\delta)\lambda_T^1 - \lambda_{T-1}^1](k_T^* - k_T) - \lambda_T^1(k_{T+1}^* - k_{T+1}).\end{aligned}$$

By (4) and (12), we have: $\forall t = 1, 2, ..., T$

$$[\lambda_t^1 f_t^1(k_t^*, L_t^*) + (1-\delta)\lambda_t^1 - \lambda_{t-1}^1](k_t^* - k_t) = -\lambda_t^3(k_t^* - k_t) = \lambda_t^3 k_t \geq 0.$$

Thus,

$$\Delta_T \geq -\lambda_T^1(k_{T+1}^* - k_{T+1}) = -\lambda_T^1 k_{T+1}^* + \lambda_T^1 k_{T+1} \geq -\lambda_T^1 k_{T+1}^*.$$

Since $\boldsymbol{\lambda}^1 \in l_+^1$, $\sup_T k_{T+1}^* < +\infty$, we have

$$\lim_{T\to+\infty}\Delta_T \geq \lim_{T\to+\infty} -\lambda_T^1 k_{T+1}^* = 0.$$

We have proved that the sequences $(\mathbf{k}^*, \mathbf{L}^*)$ maximize the profit of the firm.

We now show that $c^*$ solves the consumer's problem.

Let $\{\mathbf{c}, \mathbf{L}\}$ satisfy

$$\sum_{t=0}^{\infty}\lambda_t^1 c_t \leq \sum_{t=0}^{\infty} w_t^* L_t + \pi^* + rk_0 \tag{18}$$

By the concavity of $u$, we have:

$$\Delta = \sum_{t=0}^{\infty}\beta^t u(c_t^*, l_t^*) - \sum_{t=0}^{\infty}\beta^t u(c_t, l_t)$$

$$\geq \sum_{t=0}^{\infty}\beta^t u_t^1(c_t^*, l_t^*)(c_t^* - c_t) + \sum_{t=0}^{\infty}\beta^t u_t^2(c_t^*, l_t^*)\ (l_t^* - l_t).$$

Combining (3 ),(6),(10),(11) yields that

$$\Delta \geq \sum_{t=0}^{\infty}(\lambda_t^1 - \lambda_t^2)(c_t^* - c_t) + \sum_{t=0}^{\infty}(\lambda_t^1 f_t^2(k_t^*, 1 - l_t^*) + \lambda_t^5 - \lambda_t^4)(l_t^* - l_t)$$

$$\begin{aligned}
&= \sum_{t=0}^{\infty}\lambda_t^1(c_t^* - c_t) + \sum_{t=0}^{\infty}\lambda_t^2 c_t - \sum_{t=0}^{\infty}\lambda_t^2 c_t^* + \sum_{t=0}^{\infty}(w_t^* + \lambda_t^5)((l_t^* - l_t) \\
&\quad -\sum_{t=0}^{\infty}\lambda_t^4 l_t^* + \sum_{t=0}^{\infty}\lambda_t^4 l_t \\
&\geq \sum_{t=0}^{\infty}\lambda_t^1(c_t^* - c_t) + \sum_{t=0}^{\infty}(w_t^* + \lambda_t^5)((l_t^* - l_t) = \\
&\quad \sum_{t=0}^{\infty}\lambda_t^1(c_t^* - c_t) + \sum_{t=0}^{\infty} w_t^*(l_t^* - l_t) + \sum_{t=0}^{\infty}\lambda_t^5(1 - l_t) \\
&\geq \sum_{t=0}^{\infty}\lambda_t^1(c_t^* - c_t) + \sum_{t=0}^{\infty} w_t^*(L_t - L_t^*)
\end{aligned}$$

Since

$$\pi^* = \sum_{t=0}^{\infty} \lambda_t^1 [F(k_t^*, L_t^*) + (1-\delta)k_t^* - k_{t+1}^*] - \sum_{t=0}^{\infty} w_t^* L_t^* - rk_0$$

and

$$k_{t+1}^* - (1-\delta)k_t^* = F(k_t^*, L_t^*) - c_t^*,$$

it follows from ( 18) that

$$\begin{aligned} \sum_{t=0}^{\infty} \lambda_t^1 c_t &\leq \sum_{t=0}^{\infty} w_t^* L_t + \sum_{t=0}^{\infty} \lambda_t^1 [F(k_t^*, L_t^*) - F(k_t^*, L_t^*) + c_t^*] - \sum_{t=0}^{\infty} w_t^* L_t^* \\ &\leq \sum_{t=0}^{\infty} w_t^* (L_t - L_t^*) + \sum_{t=0}^{\infty} \lambda_t^1 c_t^*. \end{aligned}$$

Consequently, $\Delta \geq 0$ that means $c^*$ solves the consumer's problem.

Finally, the market clears at every period, since $\forall t, c_t^* + k_{t+1}^* - (1-\delta)k_t^* = F(k_t^*, L_t^*)$ and $1 - l_t^* = L_t^*$ ■

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