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Values on regular games under Kirchhoff's laws

Fabien LANGE

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En théorie des jeux coopératifs, la valeur de Shapley est une notion centrale permettant de définir d'une manière rationnelle le moyen de partager la valeur de la grande coalition entre tous les joueurs. Dans le cadre général de ce papier, l'ensemble des coalitions faisables où est défini le jeu forme un ensemble ordonné (par l'inclusion) dont toutes les chaînes maximales ont la même longueur. Nous montrons d'abord que certaines définitions et axiomatisations précédemment étudiées par Faigle et Kern de la valeur de Shapley restent valables. Notre principale contribution est de proposer une nouvelle axiomatisation qui évite l'*axiome de force hiérarchique* de Faigle et Kern (difficilement interprétable), considérant un nouveau moyen de généraliser l'axiome d'anonymat entre les joueurs.

Des idées de la théorie des réseaux électriques sont ensuite empruntées, où nous montrons que notre axiome d'anonymat (regularity axiom) ainsi que l'axiome bien connu d'efficacité (efficiency axiom) correspondent en fait aux deux lois de Kirchhoff d'un circuit électrique résistif (les noeuds étant données par les coalitions faisables et les branches par les couples de coalitions se précédant). Plus précisément, des analogies sont données entre l'axiome d'efficacité et la *loi des noeuds* et entre l'axiome d'anonymat et la *loi des mailles*.

Nous établissons enfin une forme plus faible de l'axiome de monotonie qui est satisfait par la valeur proposée.

Mots clés :

Système d'ensemble ; jeu régulier ; valeur de Shapley ; valeur probabiliste ; axiome d'anonymat ; axiome d'efficacité ; lois de Kirchhoff

JEL Classification : C71

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Abstract. In cooperative game theory, the Shapley value is a central notion defining a rational way to share the total worth of a game among players. In this paper, we address a general framework, namely regular set systems, where the set of feasible coalitions forms a poset where all maximal chains have the same length. We first show that previous definitions and axiomatizations of the Shapley value proposed by Faigle and Kern, and Bilbao and Edelman still work. Our main contribution is then to propose a new axiomatization avoiding the hierarchical strength axiom of Faigle and Kern, and considering a new way to define the symmetry among players. Borrowing ideas from electric networks theory, we show that our symmetry axiom and the classical efficiency axiom correspond actually to the two Kirchhoff's laws in the resistor circuit associated to the Hasse diagram of feasible coalitions. We finally work out a weak form of the monotonicity axiom which is satisfied by the proposed value.

Key words. Regular set systems; regular games; Shapley value; probabilistic efficient values; regular values; Kirchhoff's laws.

1 Introduction

The value or solution concept of a game is a key concept in cooperative game theory, since it defines a rational imputation given to the players if they join the grand coalition. In this respect, the Shapley value remains the best known solution concept [16, 17] applied also to more general notions of game, like multichoice games [12].

In the above cited classical works, it is assumed that any coalition of players can form. However, this assumption is often unrealistic, for various reasons (incompatibilities between players, precedence constraints, etc.). A great deal of work has been done in order to consider weaker assumptions on the set of *feasible* coalitions. Along this line, we may cite Faigle [10, 11] who introduced the idea of precedence constraints among players, and Bilbao and Edelman, considering that the set of feasible coalitions is a convex geometry. Due to well known results in lattice representation, the construction of Faigle amounts to have a distributive lattice as the set of feasible coalitions, and hence is a particular case of Bilbao and Edelman's construction. We may also cite the recent work of Bilbao, who introduced cooperative games under augmenting systems [2], which are particular structures where the grand coalition is not necessarily feasible.

Despite the mathematical interest of convex geometries, we may argue if they fit or not to the framework of game theory. Specifically, feasible coalitions of a convex geometry should

satisfy two conditions: (1) if S is a feasible coalition, then it is possible to find a player i such that $S \cup i$ is still feasible, (2) if S, T are feasible, then their intersection too should be feasible. The first condition is a natural and very weak one in a context where the grand coalition can form, since it says that from a given coalition, it is possible to augment it gradually to reach the grand coalition. On the contrary, it is more difficult to accept the second one.

We propose to consider more general structures, avoiding the closure under intersection, which we call *regular set systems*, which more or less amounts to take condition (1) above and a symmetric one, saying that from a given coalition S , it is possible to withdraw one player while remaining feasible. Regular set systems have been proposed by Honda and Grabisch [13], and have all their maximal chains of same length. One of their main mathematical advantages is that they allow to keep many classical notions defined for games, capacities [5] and other set functions [13], such as the Möbius transform, the core, the Shapley value, the entropy, etc., since all these notions can be defined through maximal chains. A general view of regular set systems, giving connections with more classical ordered structures, is given in Section 2.

Our main aim is the axiomatization of a solution concept for games defined on regular set systems—which we call *regular games*—, close to the Shapley value. In Section 3, we begin by considering probabilistic and marginalist values, and we generalize results obtained by Bilbao and Edelman. In Section 4, we propose a substitute for the classical symmetry axiom, which cannot be straightforwardly generalized in such general coalition structures. Our proposal, called the *regularity axiom*, has a more natural interpretation than the hierarchical strength axiom of Faigle and Kern [11], which is merely a combinatorial axiom. Our main achievement is Theorem 6, which shows that there is a unique marginalist value satisfying the regularity axiom and efficiency. This is done through an analogy with networks and electrical circuits, explained in Section 5. The efficiency and regularity axioms are shown to be respectively equivalent to the first and second Kirchhoff’s laws. A last section is devoted to the study of monotonicity. It is shown that our value does not satisfy the monotonicity axiom in general, but a weaker form of monotonicity, which is the aggregate monotonicity.

In the paper, $N := \{1, 2, \dots, n\}$ refers to the finite set of players. In order to avoid heavy notations, we will often omit braces for subsets, by writing i instead of $\{i\}$ or 123 for $\{1, 2, 3\}$. Furthermore, cardinalities of subsets S, T, \dots will be denoted by the corresponding lower case letters s, t, \dots

2 Regular games

Let us consider \mathcal{N} a subcollection of the power set 2^N of N . Then we call (N, \mathcal{N}) a *set system* on N if \mathcal{N} contains \emptyset and N . In the sequel, (N, \mathcal{N}) always denotes a set system.

Elements of \mathcal{N} are called (*feasible*) *coalitions*. For any two coalitions A, B of \mathcal{N} , we say that A is *covered* by B , and write $A \prec B$, if $A \subsetneq B$ and $A \subseteq C \subsetneq B$, with $C \in \mathcal{N}$, implies $C = A$.

Definition 1 (N, \mathcal{N}) is a regular set system if it satisfies the following property:

$$\forall S, T \in \mathcal{N} \text{ such that } S \prec T \text{ in } \mathcal{N}, \text{ then } |T \setminus S| = 1.$$

If in addition, the regular set system has a lattice structure, then we call it a *regular set lattice*.

For any two coalitions S, T in a set system (N, \mathcal{N}) , we call *maximal chain from S to T* any sequence (S_0, S_1, \dots, S_m) of elements of \mathcal{N} such that $S_0 = S$, $S_m = T$, and $S_i \prec S_{i+1}$ for every $0 \leq i \leq m - 1$. If S and T are not specified, maximal chains are understood to be from \emptyset to N . Note that we find in [14] Definition 1 under the equivalent form (ii) below:

Proposition 1 Let (N, \mathcal{N}) be a set system. Then the following assertions are equivalent:

(i) (N, \mathcal{N}) is a regular set system.

(ii) All maximal chains of (N, \mathcal{N}) have length n , i.e., all maximal chains have exactly $n + 1$ elements.

Proof: Assuming that (N, \mathcal{N}) is regular, let C be a maximal chain of (N, \mathcal{N}) . Every element of C covers the previous one, and then contains only one extra player. Thus C contains $n + 1$ elements. Conversely, if (N, \mathcal{N}) is not regular, i.e. there are two elements S, T such that $S \prec T$ and $|T \setminus S| \geq 2$, then any maximal chain going through S and T has necessarily less than $n + 1$ elements. ■

Note that regular set systems also satisfy the following properties, which straightforwardly derive from the definition:

(iii) **One-point extension:** $\forall S \in \mathcal{N}, S \neq N, \exists i \in N \setminus S$ such that $S \cup i \in \mathcal{N}$.

(iv) **Accessibility:** $\forall T \in \mathcal{N}, T \neq \emptyset, \exists j \in T$ such that $T \setminus j \in \mathcal{N}$.

These properties are not sufficient to characterize regular set systems, and are actually used by Labreuche as an underlying structure of games in [15].

What is interesting for the sequel, the set of regular set systems is a general class embodying some classical structures such as distributive lattices and convex geometries [3]. We now present them.

A *Jordan-Dedekind poset* is any poset such that all its maximal chains between any two elements have the same length. Note that if the Jordan-Dedekind poset has least and greatest elements, it is sufficient to verify that all its maximal chains between them have the same length. Thus we call *Jordan Dedekind set system* any Jordan-Dedekind poset which is a set system. A *convex geometry* is any set system (N, \mathcal{N}) satisfying

(C1) **One-point extension property.**

(C2) **Intersection closure:** $\forall A, B \in \mathcal{N}, A \cap B \in \mathcal{N}$.

The dual set system of the convex geometry is called *antimatroid*, that is to say any set system satisfying

(A1) **Accessibility property.**

(A2) **Union closure.**

A lattice is *distributive* when the infimum and the supremum obey the distributivity law. For any poset (P, \leq) , a subset $Q \subseteq P$ is called a *downset* of (P, \leq) if $x \leq y$ and $y \in Q$ imply $x \in Q$. We denote by $\mathcal{O}(P)$ the set of all downsets of P . Besides, a *join-irreducible element* x of a lattice (L, \leq) is an element that is not the least one, and for which $(x = y \vee z)$ implies $(x = y$ or $x = z)$. It is known that the set of all downsets of (P, \leq) endowed with the inclusion relation is a distributive lattice. Conversely, a fundamental Theorem due to Birkhoff [4] says that any distributive lattice (L, \leq) is isomorphic to the set $\mathcal{O}(\mathcal{J})$ of all downsets of the set \mathcal{J} of join-irreducible elements of L . Consequently, for any distributive lattice (L, \leq) , there is a poset (P, \leq) such that (L, \leq) has the isomorphic form $\mathcal{O}(P)$. Moreover, it is also known that distributive lattices which are set systems coincide with the class of set systems closed under intersection and union. Finally, we will call *distributive regular set system* any distributive

lattice given under the form $\mathcal{O}(P)$, where P is endowed with the appropriate partial order relation \leq .

We present now the following inclusion diagram where these set systems structures fit into each other (see Fig. 1).

Proposition 2

- (1) *The class of Jordan-Dedekind set systems strictly includes regular set systems.*
- (2) *The class of regular set systems strictly includes regular set lattices.*
- (3) *The class of regular set lattices strictly includes convex geometries and antimatroids.*
- (4) *The intersection of the classes of convex geometries and antimatroids coincides with the class of distributive regular set systems.*
- (5) *The class of distributive regular set systems strictly includes distributive regular set systems isomorphic to direct products of linear lattices.*
- (6) *The class of direct products linear lattices strictly includes Boolean lattices.*

Proof: We show the successive inclusions (1) to (4), (5) and (6) being well known or evident.

(1) is clear since for any regular set system, for any two coalitions S and T such that $S \subseteq T$, all maximal chains from S to T have clearly $t - s + 1$ elements. However, the reverse property is not true. Indeed, it is self-evident that for any $n \geq 3$, the poset $\{\emptyset, 1, 2, \dots, n, N\}$ is a Jordan-Dedekind set system but is not regular, since its maximal chains have length 2 and should have length n .

About (2), we have only to show that the inclusion is strict: for $n = 4$, we easily see that the set system $\mathcal{N} := \{\emptyset, 1, 2, 13, 23, 14, 24, 123, 124, 1234\}$ is regular but $\{1\}$ and $\{2\}$ have no supremum thus (N, \mathcal{N}) is not a lattice.

By a simple induction using (C1), we show that any maximal chain of a convex geometry (N, \mathcal{N}) has necessarily length n . By Proposition 1, (N, \mathcal{N}) is a regular set system. This holds for an antimatroid, by the duality principle. Besides, the convex geometry and the antimatroid are lattices $(\mathcal{N}, \subseteq, \vee, \cap, N, \emptyset)$ and $(\mathcal{N}, \subseteq, \cup, \wedge, N, \emptyset)$ where $A \vee B := \cap\{C \in \mathcal{N} \mid A \cup B \subseteq C\}$ and $A \wedge B := \cup\{C \in \mathcal{N} \mid A \cap B \subseteq C\}$, respectively. Conversely, the set system $\{\emptyset, 1, 2, 13, 23, 123\}$ is a regular lattice but is neither a convex geometry nor an antimatroid. Thus, (3) is shown.

Now, let (N, \leq) be any poset and $L := \mathcal{O}(N)$ be the distributive lattice of all downsets of (N, \leq) . As said above, it is known that the union and the intersection of any two downsets is also a downset. Furthermore, it is clear that condition (C1) holds since from any downset $S \neq N$ of (N, \leq) , adding a minimal element of the restricted poset $(N \setminus S, \leq)$ in S leaves S a downset. By withdrawing a maximal element of $S \neq \emptyset$, the dual condition (A1) holds. Conversely, if a set system is a convex geometry and an antimatroid, then supremum and infimum laws are union and intersection, which immediately implies the distributivity law. Thus (4) is shown.

Remark that (5) and (6) are seen as set system inclusions in the sense that for any direct product of linear lattices L , there is a regular set system (N, \mathcal{N}) that is isomorphic to L . In addition, for any Boolean lattice \mathcal{B} , there is an integer n such that \mathcal{B} is isomorphic to 2^N . ■

We call *regular game* any game defined on a regular set system, that is to say, any mapping v defined over a regular set system (N, \mathcal{N}) such that $v(\emptyset) = 0$. We denote by $\mathcal{G}(\mathcal{N})$ the \mathbb{R} -vector space of games over the set system (N, \mathcal{N}) . Remark that for the Boolean lattice $\mathcal{N} := 2^N$, $\mathcal{G}(\mathcal{N})$ is the set of classical cooperative games.

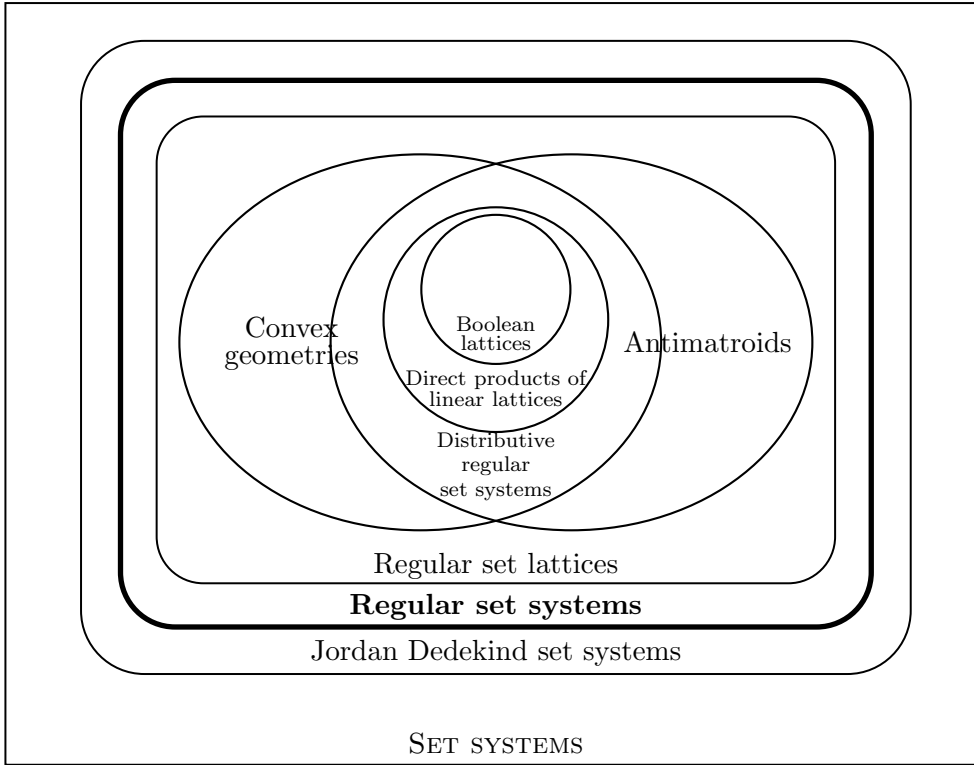


Figure 1: Inclusion diagram of set systems

Considering a regular set system (N, \mathcal{N}) , the following mappings

$$\delta_S : \mathcal{N} \rightarrow \mathbb{R}$$

$$A \mapsto \begin{cases} 1 & \text{if } A = S, \\ 0 & \text{otherwise,} \end{cases}$$

form a special collection of games in $\mathcal{G}(\mathcal{N})$, that are called *identity games*, for $S \in \mathcal{N}$. Note that the mapping δ_\emptyset is not a game since $\delta_\emptyset(\emptyset) = 1$.

We also introduce *symmetric games*, whose worths depend only on the cardinality of the coalitions, and *equidistributed games* of $\mathcal{G}(\mathcal{N})$, being regular games v that are both symmetric and additive, that is to say, worths $v(S)$ are proportional to s :

$$\exists \nu \in \mathbb{R} \text{ such that } \forall S \in \mathcal{N}, v(S) = \nu \cdot s.$$

As a consequence of Proposition 2, the material we propose in what follows, is convenient as well for games on convex geometries [1, chap.7], and thus for games with precedence constraints [11], where feasible coalitions of players are the only ones that respect a given precedence structure on the set of players: let (N, \leq) be a partially ordered set of players, where \leq is a relation of *precedence* in the sense that $i \leq j$ if the presence of j enforces the presence of i in any coalition $S \subseteq N$. Hence, a *coalition* of N is a subset S of N such that $i \in S$ and $j \leq i$ entails $j \in S$. Consequently, the collection $\mathcal{C}(N)$ of all coalitions of N is the collection of all downsets of (N, \leq) , which is a distributive regular set system.

3 Probabilistic and efficient values

From now on, (N, \mathcal{N}) refers to a regular set system. A *value* on $\mathcal{G}(\mathcal{N})$ is a mapping $\Phi : \mathcal{G}(\mathcal{N}) \rightarrow \mathbb{R}^n$ that associates to each game v a vector $(\Phi^1(v), \dots, \Phi^n(v))$, where the real number $\Phi^i(v)$ represents the payoff to player i in the game v . The Shapley value for cooperative games Φ_{Sh} is well known [16].

$$\forall v \in \mathcal{G}(2^N), \forall i \in N, \quad \Phi_{Sh}^i(v) := \sum_{S \subseteq N \setminus i} \frac{s!(n-s-1)!}{n!} (v(S \cup i) - v(S)). \quad (1)$$

Following the work of Weber [17], Bilbao has defined and axiomatized a class of values for games defined over convex geometries, the *probabilistic values*. It is possible to define such values for regular games.

First, we denote by $S+i$ the coalition $S \cup i$ whenever $S \not\ni i$. Thus, writing $S+i \in \mathcal{N}$ infers two relations: $i \notin S$ and $S \cup i \in \mathcal{N}$. Similarly, $S-i$ denotes the coalition $S \setminus i$ and infers $S \ni i$.

Definition 2 A value Φ on $\mathcal{G}(\mathcal{N})$ is a probabilistic value if there exists for each player i , a collection of real numbers $\{p_S^i \mid S \in \mathcal{N}, S+i \in \mathcal{N}\}$ satisfying $p_S^i \geq 0$ and $\sum_{S \in \mathcal{N} \mid S+i \in \mathcal{N}} p_S^i = 1$ such that

$$\Phi^i(v) = \sum_{S \in \mathcal{N} \mid S+i \in \mathcal{N}} p_S^i (v(S \cup i) - v(S)), \quad (2)$$

for every game $v \in \mathcal{G}(\mathcal{N})$.

If no condition is required for real numbers p_S^i , then we call Φ a *marginalist value*.

Observe that for a probabilistic value, the participation of player i is assessed to be a weighted average of his marginal contribution $v(S \cup i) - v(S)$ whenever i joins coalition S (provided that $S \cup i$ is a feasible coalition), p_S^i being the subjective probability that i joins S .

In a cooperative game, it is assumed that all players decide to cooperate among them and form the grand coalition N . This leads to the problem of distributing the amount $v(N)$ among them. In this case, a value Φ is *efficient* if it satisfies:

$$\mathbf{Efficiency axiom (E):} \quad \forall v \in \mathcal{G}(\mathcal{N}), \quad \sum_{i=1}^n \Phi^i(v) = v(N).$$

We consider also the following axioms.

$$\mathbf{Linearity axiom (L):} \quad \forall i \in N, \forall v, w \in \mathcal{G}(\mathcal{N}), \forall \alpha \in \mathbb{R}, \quad \Phi^i(\alpha v + w) = \alpha \Phi^i(v) + \Phi^i(w).$$

Player i is a *null player* when his contribution to all coalitions $S \cup i \in \mathcal{N}$ formed with his incorporation to $S \in \mathcal{N}$ has no effect.

Definition 3 A player $i \in N$ is null for $v \in \mathcal{G}(\mathcal{N})$ if

$$\forall S \in \mathcal{N} \text{ such that } S+i \in \mathcal{N}, \quad v(S \cup i) = v(S).$$

Player i is *dummy* for $v \in \mathcal{G}(\mathcal{N})$ if

$$\forall S \in \mathcal{N} \text{ such that } S+i \in \mathcal{N}, \quad v(S \cup i) - v(S) = \begin{cases} v(i), & \text{if } i \in \mathcal{N} \\ 0, & \text{else.} \end{cases}$$

Null axiom (N): If player i is null for v , then $\Phi^i(v) = 0$.

The dummy axiom of Bilbao introduced to axiomatize games on convex geometries, is:

Dummy axiom (D): If player i is dummy for v , then $\Phi^i(v) = v(i)$, whenever $i \in \mathcal{N}$ and 0 otherwise.

Monotonicity axiom (M): If the game $v \in \mathcal{G}(\mathcal{N})$ is monotonic, that is to say, $S \subseteq T$ implies $v(S) \leq v(T)$ for all $S, T \in \mathcal{N}$, then the values Φ^i are nonnegative.

Let us present the axiomatization of probabilistic values for games on regular set systems, as already seen in [1].

Proposition 3 *Let Φ a value on $\mathcal{G}(\mathcal{N})$. Under axioms (L) and (N), Φ is a marginalist value.*

Proof: First, under (L), for all $i \in N$, there is a unique collection of real numbers $\{c_S^i \mid S \in \mathcal{N}, S \neq \emptyset\}$ such that

$$\Phi^i(v) = \sum_{S \in \mathcal{N}, S \neq \emptyset} c_S^i v(S),$$

for every game $v \in \mathcal{G}(\mathcal{N})$. Indeed, the collection of identity games is clearly a basis of $\mathcal{G}(\mathcal{N})$ since every game v can be written as

$$v = \sum_{S \in \mathcal{N}, S \neq \emptyset} v(S) \delta_S,$$

in a unique way. By axiom (L),

$$\Phi^i(v) = \sum_{S \in \mathcal{N}, S \neq \emptyset} v(S) \Phi^i(\delta_S).$$

Now, this formula can also write

$$\Phi^i(v) = \sum_{\substack{S \in \mathcal{N} \\ S-i \in \mathcal{N} \text{ or } S+i \in \mathcal{N}}} v(S) \Phi^i(\delta_S) + \sum_{\substack{S \in \mathcal{N} \\ S-i \notin \mathcal{N} \text{ or } S+i \notin \mathcal{N}}} v(S) \Phi^i(\delta_S).$$

Next, assume that $S \in \mathcal{N}$, $i \notin S$ and $S+i \notin \mathcal{N}$ (resp. $S \in \mathcal{N}$, $i \in S$ and $S-i \notin \mathcal{N}$). Thus i is null for δ_S and, by (N), $\Phi^i(\delta_S) = 0$. Then, the second part of the above sum vanishes. Therefore

$$\begin{aligned} \Phi^i(v) &= \sum_{S \in \mathcal{N} \mid S+i \in \mathcal{N}} [v(S) \Phi^i(\delta_S) + v(S \cup i) \Phi^i(\delta_{S \cup i})] \\ &= \sum_{S \in \mathcal{N} \mid S+i \in \mathcal{N}} [v(S \cup i) - v(S)] \Phi^i(\delta_{S \cup i}) + \sum_{S \in \mathcal{N} \mid S+i \in \mathcal{N}} v(S) [\Phi^i(\delta_S) + \Phi^i(\delta_{S \cup i})] \\ &= \sum_{S \in \mathcal{N} \mid S+i \in \mathcal{N}} \Phi^i(\delta_{S \cup i}) [v(S \cup i) - v(S)] + \sum_{S \in \mathcal{N} \mid S+i \in \mathcal{N}} v(S) \Phi^i(\delta_S + \delta_{S \cup i}). \end{aligned}$$

Since i is null for $\delta_S + \delta_{S \cup i}$ whenever $S \in \mathcal{N}$ and $S+i \in \mathcal{N}$, we conclude that $\Phi^i(\delta_S + \delta_{S \cup i}) = 0$ by (N). Thus

$$\Phi^i(v) = \sum_{S \in \mathcal{N} \mid S+i \in \mathcal{N}} \Phi^i(\delta_{S \cup i}) [v(S \cup i) - v(S)]. \quad (3)$$

■

Observe now that the dummy axiom implies the null axiom since a null player i is a particular dummy player satisfying $v(i) = 0$. Bilbao has shown that values for games over convex geometries (that are particular regular games) which satisfy axioms **(L)**, **(D)**, **(M)** and **(E)**, are precisely the efficient probabilistic values [1, chap.7].

We improve now this result by weakening the set of axioms and considering more general structures.

Theorem 4 *Let Φ be a value on $\mathcal{G}(\mathcal{N})$. Under axioms **(L)**, **(N)**, **(M)** and **(E)**, Φ is a probabilistic and an efficient value.*

Proof: Whenever $i \notin S$, we denote by p_S^i the coefficient $\Phi^i(\delta_{S \cup i})$ of formula (3) above.

Let choose some $T \in \mathcal{N}$ and define the game of $\mathcal{G}(\mathcal{N})$

$$\hat{u}_T(S) = \begin{cases} 1, & \text{if } S \supseteq T \\ 0, & \text{else.} \end{cases}$$

By definition, \hat{u}_T is monotonic. Letting $i \in N$ and $T \in \mathcal{N}$ such that $T + i \in \mathcal{N}$, under **(L)**, **(N)** and **(M)**, we get:

$$\begin{aligned} \Phi^i(\hat{u}_T) &= \sum_{S \in \mathcal{N} | S+i \in \mathcal{N}} p_S^i (\hat{u}_T(S \cup i) - \hat{u}_T(S)) \\ &= p_T^i \geq 0. \end{aligned}$$

Lastly, it remains to show under the efficiency axiom that for all $i \in N$, the collections of number $\{p_S^i \mid S \in \mathcal{N} : S + i \in \mathcal{N}\}$ form probability distributions, so that we could conclude to the result. For any $i \in N$, let us consider the game

$$u_i(S) = \begin{cases} 1, & \text{if } S \ni i \\ 0, & \text{else.} \end{cases}$$

Then, on the one hand, we have under **(E)**,

$$\sum_{j=1}^n \Phi^j(u_i) = u_i(N) = 1.$$

And on the other hand ,

$$\begin{aligned} \sum_{j=1}^n \Phi^j(u_i) &= \sum_{j=1}^n \sum_{S \in \mathcal{N} | S+j \in \mathcal{N}} p_S^j (u_i(S \cup j) - u_i(S)) \\ &= \sum_{S \in \mathcal{N} | S+i \in \mathcal{N}} p_S^i (u_i(S \cup i) - u_i(S)) + \sum_{j \in N \setminus i} \sum_{S \in \mathcal{N} | S+j \in \mathcal{N}} p_S^j (u_i(S \cup j) - u_i(S)) \\ &= \sum_{S \in \mathcal{N} | S+i \in \mathcal{N}} p_S^i, \end{aligned}$$

since differences of the first sum always worth 1 whereas differences of the second one vanish. This achieves the proof. \blacksquare

We present now an important result about marginalist values, already known for convex geometries [1, chap.7].

Proposition 5 Let Φ be a marginalist value on $\mathcal{G}(\mathcal{N})$, defined by

$$\Phi^i(v) = \sum_{S \in \mathcal{N} | S+i \in \mathcal{N}} p_S^i (v(S \cup i) - v(S)),$$

for every game v and for all $i \in N$, where p_S^i are real numbers. Then Φ satisfies the efficiency axiom if and only if

$$\sum_{i \in N | i \in \mathcal{N}} p_{\emptyset}^i = \sum_{i \in N | N \setminus i \in \mathcal{N}} p_{N \setminus i}^i = 1, \quad (4)$$

$$\sum_{i \in N | S-i \in \mathcal{N}} p_{S \setminus i}^i = \sum_{i \in N | S+i \in \mathcal{N}} p_S^i, \quad (5)$$

for all $S \in \mathcal{N} \setminus \{\emptyset, N\}$.

Proof: For every $v \in \mathcal{G}(\mathcal{N})$, we compute the sum of the values $\Phi^i(v)$.

$$\begin{aligned} \sum_{i=1}^n \Phi^i(v) &= \sum_{i=1}^n \sum_{S \in \mathcal{N} | S+i \in \mathcal{N}} p_S^i (v(S \cup i) - v(S)) \\ &= \sum_{S \in \mathcal{N}} \left(\sum_{i \in N | S-i \in \mathcal{N}} p_{S \setminus i}^i - \sum_{i \in N | S+i \in \mathcal{N}} p_S^i \right) v(S) \\ &= \sum_{\substack{S \in \mathcal{N} | \\ S \neq \emptyset, N}} \left(\sum_{i \in N | S-i \in \mathcal{N}} p_{S \setminus i}^i - \sum_{i \in N | S+i \in \mathcal{N}} p_S^i \right) v(S) + \left(\sum_{i \in N | N-i \in \mathcal{N}} p_{N \setminus i}^i \right) v(N). \end{aligned}$$

If the coefficients satisfy (4) and (5), then it is clear that Φ satisfies the efficiency axiom.

Conversely, fix $T \in \mathcal{N}$ such that $T \neq \emptyset, N$ and consider the identity game δ_T . The efficiency axiom straightforwardly implies that $\sum_{i=1}^n \Phi^i(\delta_T) = 0$. Applying the above equality to δ_T , we have

$$\sum_{i=1}^n \Phi^i(\delta_T) = 0 = \sum_{i \in N | T-i \in \mathcal{N}} p_{T \setminus i}^i - \sum_{i \in N | T+i \in \mathcal{N}} p_T^i,$$

that is to say (5) is proven. If $T = N$, then the equality becomes

$$\sum_{i=1}^n \Phi^i(\delta_N) = 1 = \sum_{i \in N | N-i \in \mathcal{N}} p_{N \setminus i}^i,$$

which partially proves (4). Finally, consider the game $\hat{u}_{\emptyset} := \sum_{T \in \mathcal{N}, T \neq \emptyset} \delta_T$. Then

$$\begin{aligned} \sum_{i=1}^n \Phi^i(\hat{u}_{\emptyset}) &= 1 = \sum_{i=1}^n \sum_{S \in \mathcal{N} | S+i \in \mathcal{N}} p_S^i (\hat{u}_{\emptyset}(S \cup i) - \hat{u}_{\emptyset}(S)) \\ &= \sum_{i \in N | i \in \mathcal{N}} p_{\emptyset}^i \hat{u}_{\emptyset}(i) = \sum_{i \in N | i \in \mathcal{N}} p_{\emptyset}^i, \end{aligned}$$

which achieves the proof. ■

4 The Shapley value for regular games

If we focus now on the particular case of classical cooperative games, we know that Weber has characterized the Shapley value on $\mathcal{G}(2^N)$ as the unique probabilistic value satisfying the well known *symmetry axiom*, assuming that the coefficients of the value should not depend on the labelling of the elements of N , that is a very natural property.

The fundamental idea of the symmetry axiom rests on permutations of players. Symmetry could be naturally defined in regular games. Unfortunately, this generalization has a very limited interest: apart from particular cases of regular set systems, players generally cannot be permuted, which leaves this axiom ineffective.

Faigle and Kern attempted to generalize the Shapley value for their *games under precedence constraints* [11], which are games over distributive regular set systems (cf. Proposition 2), and thus particular regular games: in their framework, maximal chains correspond to what they call feasible ranking of players. For that, the *hierarchical strength axiom* is introduced, which is actually difficult to interpret.

We propose a different approach for the axiomatization of the Shapley value on $\mathcal{G}(\mathcal{N})$. Consider first the classical Shapley value on $\mathcal{G}(2^N)$ where $p_S^i := \frac{s!(n-s-1)!}{n!}$ (cf. (1)).

Observing that for any subsets $A, A+i+j \subseteq N$, the equality $p_A^i + p_{A \cup i}^j = p_A^j + p_{A \cup j}^i$ holds, one may wonder if this property is sufficient to form the Shapley value from a probabilistic and efficient value. Actually, the answer is positive. Furthermore, one can generalize it to any case of game defined on a regular set system.

In this perspective, we introduce the following material. Let A, B be any two coalitions of \mathcal{N} such that $A \subseteq B$, and $\mathcal{C} := (S_a, S_{a+1}, \dots, S_b)$ be a maximal chain from A to B . Thus we denote by $\sigma^{\mathcal{C}}$ the mapping defined over $\{a+1, \dots, b\}$ by $\sigma^{\mathcal{C}}(i) := S_i \setminus S_{i-1}$. Remark that σ is a permutation of N if $A = \emptyset$ and $B = N$. Besides, for any marginalist value Φ whose coefficients are given by (2) and any game $v \in \mathcal{G}(\mathcal{N})$, we denote the cumulative sum of marginal contributions of players of $B \setminus A$ along \mathcal{C} by:

$$m_{\Phi}^{\mathcal{C}}(v) := \sum_{i=a+1}^b p_{S_{i-1}}^{\sigma^{\mathcal{C}}(i)} (v(S_i) - v(S_{i-1})).$$

Let us consider any equidistributed game (Section 2). Thus, *for such a game, the cumulative sum of expected marginal contributions of players of $B \setminus A$ involved should not depend on the considered maximal chain from A to B* , since the path taken from A to B has no effect on the successive increasing worth $v(C)$, $A \subseteq C \subseteq B$.

Indeed, if we assume that \mathcal{N} contains the coalitions 1, 12, 2, 23 and 123, for instance, then the following equalities should hold for every equidistributed game v :

$$\begin{aligned} p_{\emptyset}^1 v(1) + p_1^2 (v(12) - v(1)) + p_{12}^3 (v(123) - v(12)) &= p_{\emptyset}^2 v(2) + p_2^3 (v(23) - v(2)) + p_{23}^1 (v(123) - v(23)) \\ , p_2^1 (v(12) - v(2)) + p_{12}^3 (v(123) - v(12)) &= p_2^3 (v(23) - v(2)) + p_{23}^1 (v(123) - v(23)), \end{aligned}$$

that are respectively equivalent to

$$\begin{aligned} p_{\emptyset}^1 + p_1^2 + p_{12}^3 &= p_{\emptyset}^2 + p_2^3 + p_{23}^1, \\ p_2^1 + p_{12}^3 &= p_2^3 + p_{23}^1, \end{aligned}$$

since v is equidistributed.

In this spirit, we propose the following axiom.

Regularity axiom (R): For any equidistributed game $v \in \mathcal{G}(\mathcal{N})$, for any couple of maximal chains $\mathcal{C}_1, \mathcal{C}_2$ of (N, \mathcal{N}) , then $m_{\Phi}^{\mathcal{C}_1}(v) = m_{\Phi}^{\mathcal{C}_2}(v)$.

We call *regular value* any marginalist value satisfying the regularity axiom.

Actually, the regularity axiom may be seen as a generalization of the Shapley's symmetry axiom. The next result confirms this view, and gives a generalization of the Shapley's result asserting the unicity of the value Φ under the linearity axiom, the null axiom, the efficiency axiom and the regularity axiom.

Theorem 6 *Let (N, \mathcal{N}) be a regular set system. Then there is a unique efficient regular value Φ_K on $\mathcal{G}(\mathcal{N})$.*

To show this result, we need to introduce new material and definitions, that is done in the next section and in Appendix A.

5 The Shapley value in the framework of network theory

Considering results of previous section, let us fix a regular set system (N, \mathcal{N}) and a marginalist value Φ on $\mathcal{G}(\mathcal{N})$, and let $\{p_S^i \mid i \in N, S \in \mathcal{N} \text{ s.t. } S+i \in \mathcal{N}\}$ be the set of associated coefficients. We associate to any regular set system (N, \mathcal{N}) an *electrical network*, that is, an interconnection of electrical components such as *resistors*. Precisely, we consider the mapping

$$(N, \mathcal{N}) \mapsto \mathcal{E}(N, \mathcal{N}),$$

where $\mathcal{E}(N, \mathcal{N})$ is built in this way: *nodes* of $\mathcal{E}(N, \mathcal{N})$ are simply the elements of \mathcal{N} , whereas its *branches* are directed wires given by the couples $b_S^i := (S, S+i)$ of (N, \mathcal{N}) . Note that at this stage, $\mathcal{E}(N, \mathcal{N})$ may be seen as the Hasse diagram of (N, \mathcal{N}) , since the b_S^i 's are given by the covering relation \prec of (N, \mathcal{N}) . We complete the building by adding another branch $b_{N, \emptyset}$ by connecting the node N with the node \emptyset .

In $\mathcal{E}(N, \mathcal{N})$, we call *circuit*¹, any sequence (b_1, \dots, b_m) of branches such that the b_j 's are different and two consecutive edges are incident, as well as b_1 and b_m . For convenience, we also may write a circuit as a sequence of m nodes $(S_0, S_1, \dots, S_m = S_0)$.

Now, one can attribute to the branches b_S^i (resp. $b_{N, \emptyset}$) some weights $I_{S \rightarrow S \cup i}$ and $V_{S \rightarrow S \cup i}$ (resp. $I_{N \rightarrow \emptyset}$ and $V_{N \rightarrow \emptyset}$), where I_b 's are the worths of a directed commodity flowing in the branches called *electrical current*, and where V_b 's are worths proportional to currents I_b 's: V_b 's are called *potential drops* and satisfy the well-known *Ohm's law*: $V_b = R_b \cdot I_b$, where R_b is said to be the *resistance* of the non-oriented branch b . In our framework, we assign for any electrical network a unitary resistance to every branch b_S^i , so that $V_b = I_b$. Precisely, the necessary and sufficient following conditions for the electrical current and potential drops must be satisfied:

First Kirchhoff's law: The sum of all currents entering a node is equal to the sum of all currents leaving the node.

Second Kirchhoff's law: The directed sum of the electrical potential drops around a circuit must be zero.

Therefore, let us now assign to any branch b_S^i the coefficient p_S^i of the marginalist value Φ , with $I_{N \rightarrow \emptyset} := 1$. Thus Proposition 5 asserts that Φ satisfies the efficiency axiom if and only if the first Kirchhoff's law is satisfied in $\mathcal{E}(N, \mathcal{N})$.

We also establish a straight link between the second Kirchhoff's law and the regularity axiom in Corollary 8. In this respect, for any circuit $\mathcal{M} := (b_1, \dots, b_m)$ on (N, \mathcal{N}) , for $j = 1, \dots, m$, if

¹This terminology shall also be used for (N, \mathcal{N})

b_j represents the couple $(S, S + i)$, we denote by $\bar{p}_j = \bar{p}_S^i$ the signed coefficient of Φ associated to b_j relatively to its orientation in \mathcal{M} :

$$\bar{p}_S^i := \begin{cases} p_S^i & \text{if in } \mathcal{M}, b_j \text{ is directed in accordance with } \subseteq, \\ -p_S^i & \text{otherwise.} \end{cases}$$

As a consequence, the second Kirchhoff's law fitted for the marginalist value Φ may be expressed by

Circuit property: For any circuit (b_1, \dots, b_m) of (N, \mathcal{N}) , $\sum_{j=1}^m \bar{p}_j = 0$.

We call *potential* over (N, \mathcal{N}) any real-valued mapping \mathbf{V} defined on \mathcal{N} satisfying for any coefficient p_S^i

$$p_S^i = \mathbf{V}(S + i) - \mathbf{V}(S). \quad (6)$$

Proposition 7 *Let Φ be a marginalist value on $\mathcal{G}(\mathcal{N})$ and $\{p_S^i \mid i \in N, S \in \mathcal{N} \mid S + i \in \mathcal{N}\}$ be the set of associated coefficients. Then the circuit property holds if and only if there exists a potential over (N, \mathcal{N}) .*

Moreover, for a given instance of coefficients $p := (p_S^i)_{(S, S+i) \in \mathcal{N}^2}$ satisfying the circuit property, there is a unique potential \mathbf{V} vanishing at \emptyset . We call it the potential associated to p and grounded on \emptyset , and denote it by \mathbf{V}_0^p .

Proof: The sufficiency condition is clear. Indeed, let (b_1, \dots, b_m) be a circuit of (N, \mathcal{N}) , where the b_j 's belong to E and $(S_0, S_1, \dots, S_m = S_0)$ be the same circuit expressed in terms of coalitions. Then

$$\begin{aligned} \sum_{j=1}^m V_{b_j} &= \sum_{j=1}^m (\mathbf{V}(S_j) - \mathbf{V}(S_{j-1})) \\ &= \mathbf{V}(S_m) - \mathbf{V}(S_0) \\ &= 0. \end{aligned}$$

Conversely, assume that the circuit property holds. Let us define by induction on the cardinality of the coalitions, the following correspondance:

$$\begin{aligned} \mathbf{V}(\emptyset) &:= 0, \\ \forall S \in \mathcal{N} \setminus \{\emptyset\}, \text{ and } i \in S, \mathbf{V}(S) &:= \mathbf{V}(S \setminus i) + p_{S \setminus i}^i. \end{aligned}$$

Then by definition, \mathbf{V} has the required property of a potential. Thus it remains to show that this mapping is properly defined. Indeed, let $S \in \mathcal{N}$ and i, k two distinct players in S . By an inductive argument, we make the assumption that \mathbf{V} is properly defined on $\{T \in \mathcal{N} \mid t < s\}$. Thus we must show that $\mathbf{V}(S \setminus i) + p_{S \setminus i}^i = \mathbf{V}(S \setminus k) + p_{S \setminus k}^k$. Let S_0 be any maximal element of the set $\{T \in \mathcal{N} \mid T \subseteq S \setminus i, T \subseteq S \setminus k\}$ and $m := s - s_0$ ($m \geq 2$). Thus there is a circuit $\mathcal{M} = (b_1, \dots, b_{2m})$ such that b_1 is directed from S_0 , $b_m = b_{S \setminus i}^i$ and $b_{m+1} = b_{S \setminus k}^k$. By induction, we easily compute $\mathbf{V}(S \setminus i) = \mathbf{V}(S_0) + \sum_{j=1}^{m-1} p(b_j)$ and $\mathbf{V}(S \setminus k) = \mathbf{V}(S_0) + \sum_{j=m+2}^{2m} p(b_j)$. Besides, if we denote by $(S_0, S_1, \dots, S_{2m} = S_0)$ the same circuit in terms of coalitions, we get

by the circuit property

$$\begin{aligned}
& \sum_{j=1}^{2m} V_{b_j} = 0 \\
& \text{iff } \sum_{j=1}^m p(b_j) = \sum_{j=m+1}^{2m} p(b_j) \\
& \text{iff } \mathbf{V}(S_0) + \sum_{j=1}^{m-1} p(b_j) + p_{S \setminus i}^i = \mathbf{V}(S_0) + \sum_{j=m+2}^{2m} p(b_j) + p_{S \setminus k}^k \\
& \text{iff } \mathbf{V}(S \setminus i) + p_{S \setminus i}^i = \mathbf{V}(S \setminus k) + p_{S \setminus k}^k.
\end{aligned}$$

Now, we show that if there are two mappings \mathbf{V}_1 and \mathbf{V}_2 satisfying (6), then they are the same up to an additive constant. Let us denote by c the real number $\mathbf{V}_2(\emptyset) - \mathbf{V}_1(\emptyset)$. For any $S \in \mathcal{N}$, there is a maximal chain $(T_0 = \emptyset, T_1, \dots, T_s = S)$ from \emptyset to S . Thus, by denoting i_j the singleton $T_j \setminus T_{j-1}$, we get by (6)

$$\mathbf{V}_k(S) = \mathbf{V}_k(T_s) = \mathbf{V}_k(T_{s-1}) + p_{T_{s-1}}^{i_s} = \mathbf{V}_k(T_{s-2}) + p_{T_{s-2}}^{i_{s-1}} + p_{T_{s-1}}^{i_s} = \dots = \mathbf{V}_k(\emptyset) + \sum_{j=1}^s p_{T_{j-1}}^{i_j},$$

for $k = 1, 2$. Thus, $\mathbf{V}_2(S) - \mathbf{V}_1(S) = \mathbf{V}_2(\emptyset) - \mathbf{V}_1(\emptyset) = c$. As a consequence, by fixing \mathbf{V} on any vertex, we have a unique possibility for the potential, which gives \mathbf{V}_0^p if \emptyset is assigned to 0. ■

Lastly, we deduce the following result, asserting that Φ satisfies the regularity axiom if and only if the second Kirchhoff's law is satisfied in $\mathcal{E}(N, \mathcal{N})$.

Corollary 8 *Let Φ be a marginalist value on $\mathcal{G}(N)$:*

$$\Phi^i(v) = \sum_{S \in \mathcal{N} \setminus S+i \in \mathcal{N}} p_S^i (v(S \cup i) - v(S)),$$

where p_S^i are real numbers, for all $i \in N$. Then Φ satisfies the regularity axiom if and only if the circuit property holds.

Proof: The regularity axiom means that along any maximal chain of (N, \mathcal{N}) , for an equidistributed game v , the cumulative sum of marginal contributions of players do not depend on the considered maximal chain. Since v is equidistributed, worths $v(S)$ only depend on a multiplicative constant, thus it is equivalent to only consider coefficients along the chain.

Assuming firstly that the circuit property holds, we consider any potential \mathbf{V} associated to the coefficients p_S^i 's. Therefore, if $\mathcal{C} := (S_0 = \emptyset, S_1, \dots, S_n = N)$ is any maximal chain of (N, \mathcal{N}) and v is any equidistributed game defined by $v(S) := \nu \cdot s$, $\forall S \in \mathcal{N}$, we obtain

$$\begin{aligned}
m_{\Phi}^{\mathcal{C}}(v) &= \nu \sum_{j=1}^n (\mathbf{V}(S_j) - \mathbf{V}(S_{j-1})) \\
&= \nu \cdot (\mathbf{V}(N) - \mathbf{V}(\emptyset)),
\end{aligned}$$

which does not depend on \mathcal{C} , and thus the regularity axiom holds.

Conversely, if the regularity axiom holds, we build the same correspondance than in the proof of Proposition 7, that is to say

$$\mathbf{V}(\emptyset) := 0,$$

$$\forall S \in \mathcal{N} \setminus \{\emptyset\}, \text{ and } i \in S, \mathbf{V}(S) := \mathbf{V}(S \setminus i) + p_{S \setminus i}^i.$$

Then by a similar argument than in the proof of Proposition 7, we show that \mathbf{V} is a potential, which will implies that the circuit property holds. Once again, we just have to show that \mathbf{V} is properly defined, that is to say, by making again the assumption that \mathbf{V} is properly defined on $\{T \in \mathcal{N} \mid t < |S|\}$, where $S \in \mathcal{N}$ and $i, k \in S$ ($i \neq k$), we must show that $\mathbf{V}(S \setminus i) + p_{S \setminus i}^i = \mathbf{V}(S \setminus k) + p_{S \setminus k}^k$. Let $\mathcal{C}^1 := (b_1^1, \dots, b_n^1)$ and $\mathcal{C}^2 := (b_1^2, \dots, b_n^2)$ be any two maximal chains such that $b_s^1 = (S \setminus i, S)$, $b_s^2 = (S \setminus k, S)$ and $b_j^1 = b_j^2$ for every $j > s$. We easily verify that $\mathbf{V}(S \setminus i) = \sum_{j=1}^{s-1} p(b_j^1)$ and $\mathbf{V}(S \setminus k) = \sum_{j=1}^{s-1} p(b_j^2)$. Besides, by the regularity axiom applied on \mathcal{C}^1 and \mathcal{C}^2

$$\begin{aligned} \sum_{j=1}^n p(b_j^1) &= \sum_{j=1}^n p(b_j^2) \\ \text{iff } \mathbf{V}(S \setminus i) + p_{S \setminus i}^i + \sum_{j=s+1}^n p(b_j^1) &= \mathbf{V}(S \setminus k) + p_{S \setminus k}^k + \sum_{j=s+1}^n p(b_j^2) \\ \text{iff } \mathbf{V}(S \setminus i) + p_{S \setminus i}^i &= \mathbf{V}(S \setminus k) + p_{S \setminus k}^k. \end{aligned}$$

■

To sum up, we could say with a slight abuse of language that the unique efficient regular value on $\mathcal{G}(\mathcal{N})$, is also the unique marginalist value satisfying the Kirchhoff's laws in the sense that it satisfies (4), (5) and the circuit property. That is why we may call Φ_K the *Kirchhoff's value* or also the *Shapley-Kirchhoff value*.

This being introduced, we have now a sufficient material to show Theorem 6, that is made in Appendix A.

6 Shapley-Kirchhoff value and monotonicity axioms

At this point of the work, a natural question arises about a last property of the Shapley-Kirchhoff value. Indeed, Φ_K satisfies linearity axiom, null axiom, efficiency axiom and regularity axiom. However, we have no information about the monotonicity axiom, that would make of Φ_K a probabilistic value. With this in mind, we present the following handy short result.

Lemma 9 *Let Φ be a marginalist value on $\mathcal{G}(\mathcal{N})$, defined by*

$$\Phi^i(v) = \sum_{S \in \mathcal{N} \mid S+i \in \mathcal{N}} p_S^i (v(S \cup i) - v(S)),$$

for every game v and for all $i \in N$, where p_S^i are real numbers. Then Φ satisfies the monotonicity axiom if and only if all p_S^i 's are nonnegative.

Proof: Let v be a game on $\mathcal{G}(\mathcal{N})$. If all p_S^i 's are nonnegative and v monotonic, $\Phi^i(v)$ is clearly nonnegative for any $i \in N$.

Conversely, let $i \in N$ and $S \in \mathcal{N}$ with $S + i \in \mathcal{N}$, such that $p_S^i < 0$. Let $\hat{u}_S \in \mathcal{G}(\mathcal{N})$ be the *unanimity game* defined by

$$\forall T \in \mathcal{N}, \quad \hat{u}_S(T) = \begin{cases} 1 & \text{if } T \supseteq S \\ 0 & \text{otherwise.} \end{cases}$$

Thus, u_S is clearly monotonic, nevertheless $\Phi^i(v) = p_S^i < 0$. ■

In the light of this result, for Φ_K to satisfy monotonicity axiom, we should show that coefficients p_S^i 's are nonnegative. Actually, it turns out that there are regular set systems for which the Shapley-Kirchhoff value does not satisfy the monotonicity axiom, as the counterexample given in Appendix B proves it.

Nevertheless, it appears that there are other kinds of monotonicity axioms in the framework of cooperative games, as Young worked out in [18]. Indeed, monotonicity is a general principle of fair division which states that as the underlying data of a problem change, the solution should change in parallel fashion. We give the following monotonicity axioms adapted to our framework (cf. p.7).

A frequently encountered form of monotonicity is *aggregate monotonicity*. This principle states that if the worth of the coalition of the whole increases, while the worth of all other coalitions remains fixed, then no player should get less than before.

Aggregate monotonicity axiom (AM): Let v, w two games in $\mathcal{G}(\mathcal{N})$ such that $w(N) \geq v(N)$ and $w(S) = v(S)$ for other coalitions $S \in \mathcal{N}$. Then, $\forall i \in N, \quad \Phi^i(w) \geq \Phi^i(v)$.

Coalitional monotonicity is satisfied if an increase in the worth of a particular coalition implies no decrease in the allocation to any member of that coalition. Thus the following axiom is stronger than the previous one.

Coalitional monotonicity axiom (CM): Let $S \in \mathcal{N}$ and v, w two games in $\mathcal{G}(\mathcal{N})$ such that $w(S) \geq v(S)$ and $w(T) = v(T)$ for other coalitions $T \in \mathcal{N}$. Then, $\forall i \in S, \quad \Phi^i(w) \geq \Phi^i(v)$.

Coalitional monotonicity refers to monotonic changes in the *absolute* worth of the coalitions a given player. There are also situations where the worth of coalitions containing a given player i increase *relatively* to the worth of coalitions not containing i :

Strong monotonicity axiom (SM): Let v, w two games in $\mathcal{G}(\mathcal{N})$ and a player i such that for every $S \in \mathcal{N}$ satisfying $S + i \in \mathcal{N}$, $w(S \cup i) - w(S) \geq v(S \cup i) - v(S)$. Then $\Phi^i(w) \geq \Phi^i(v)$.

Having described these monotonicity axioms, we give results in context of games over regular set systems.

Proposition 10 *Under linearity and null axioms, (M), (SM) and (CM) are equivalent. In addition, these axioms are strictly stronger than (AM), whatever the regular set system is.*

Proof: We successively show that $(M) \Leftrightarrow (SM)$ and $(M) \Leftrightarrow (CM)$.

- Due to Lemma 9, if **(M)** holds, then **(SM)** also holds since $\Phi^i(v)$ depends only on the marginal contributions of player i . Conversely, let $i \in N$ and $S \in \mathcal{N}$ with $S + i \in \mathcal{N}$, such that $p_S^i < 0$, so that **(M)** is not true. Let v, w be any two games such that $w(S \cup i) - w(S) = v(S \cup i) - v(S) + 1$, and for any other coalition $T \in \mathcal{N}$ satisfying $T + i \in \mathcal{N}$, $w(T \cup i) - w(T) = v(T \cup i) - v(T)$. Then assumption of **(SM)** is satisfied but $\Phi^i(w) - \Phi^i(v) = p_S^i < 0$, that is to say **(SM)** does not hold.
- Due to Lemma 9, if **(M)** holds then **(CM)** clearly holds. Indeed, if only the worth of a coalition S increases in the game v , then for any player i of the coalition, in $\Phi^i(v)$, the associated marginal contribution $p_{S \setminus i}^i(v(S) - v(S \setminus i))$ increases, and other marginal contribution remain the same. Conversely, let us assume that **(M)** is not satisfied, that is to say there is a player $i \in N$ and a coalition $S \in \mathcal{N}$ with $S + i \in \mathcal{N}$, such that $p_S^i < 0$. Let v, w be any two games such that $w(S \cup i) = v(S \cup i) + 1$, and $w(T) = v(T)$ for any other coalition $T \in \mathcal{N}$. Then $\Phi^i(w) - \Phi^i(v) = p_S^i < 0$, that is to say **(SM)** does not hold.

Second part of the result is clear and is an immediate consequence of Lemma 11. ■

Thus, as the aggregate monotonicity axiom is weaker than the classical one, one may wonder if the Shapley-Kirchhoff values satisfies it. The answer is actually positive and rests on the following characterization.

Lemma 11 *Let Φ be a marginalist value on $\mathcal{G}(\mathcal{N})$, defined by*

$$\Phi^i(v) = \sum_{S \in \mathcal{N} \mid S+i \in \mathcal{N}} p_S^i (v(S \cup i) - v(S)),$$

for every game v and for all $i \in N$, where p_S^i are real numbers. Then Φ satisfies the aggregate monotonicity axiom if and only if for every i in N such that $N \setminus i \in \mathcal{N}$, $p_{N \setminus i}^i$ is nonnegative.

Proof: The sufficient condition derives from Lemma 9. Indeed, if all $p_{N \setminus i}^i$'s are nonnegative and v, w any two games having the same worths, except for N with $w(N) \geq v(N)$, then $\Phi^j(w) - \Phi^j(v) = p_{N \setminus j}^j (w(N) - v(N))$ or vanishes, depending on whether $N \setminus j$ is a coalition of \mathcal{N} or not. Conversely, let us assume that there is a player $i \in N$ satisfying $N \setminus i \in \mathcal{N}$, such that $p_{N \setminus i}^i < 0$. Let v, w be any two games such that $w(N) = v(N) + 1$ and $w(T) = v(T)$ for any other coalition. Then $\Phi^i(w) - \Phi^i(v) = p_{N \setminus i}^i < 0$. Thus the necessary condition is satisfied. ■

Finally, we show this final result, whose proof uses the *Maximum Principle* applied to a valued graph (see Appendix A).

Theorem 12 *For any regular set system (N, \mathcal{N}) , the Shapley-Kirchhoff value Φ_K satisfies the aggregate monotonicity axiom.*

APPENDICES

A Dirichlet problem on a graph and potentials

The problem of finding the currents on the branches of a resistor network, an entering current being given, is easy to solve if seen as the solution of a Dirichlet problem associated to that

network [8, 6]. Thus, we introduce now a few rudimentary notions of discrete *potential theory*, in order to prove Theorems 6 and 12.

Let $G = (V, E)$ be a non-oriented connected graph where the set of vertices V is randomly divided into two distinct categories, a non-empty set of *boundary points* V_0 and the set of *interior points* $V_1 = V \setminus V_0$. For any interior point x , we denote by $d(x)$ the degree of vertex x in G , that is to say $d(x) := |\{y \in V \mid \{x, y\} \in E\}|$. A function f defined on V is said to be *harmonic* on G if, for points x in V_1 , it has the averaging property

$$f(x) = \frac{\sum_{\{x,y\} \in E} f(y)}{d(x)}, \quad (7)$$

with no restriction on the values of f at the boundary points.

Now the problem of finding a harmonic function given its boundary values is called the *Dirichlet problem*, and the *Uniqueness Principle* for the Dirichlet problem asserts that there cannot be two different harmonic functions having the same boundary values. We approach the Uniqueness Principle by way of the *Maximum Principle* for harmonic functions.

Maximum Principle. A harmonic function f defined on V takes on its maximum value M and its minimum value m on the boundary.

Proof: If M is the maximum value of f and if $f(x) = M$ for x an interior point, then since $f(x)$ is the average of the values of f at its neighbors, these values must all equal M also. By working our way, repeating this argument at every step, we eventually reach a boundary point x_0 for which we can conclude that $f(x_0) = M$. That same argument works for the minimum value m . ■

Uniqueness Principle. If f and g are harmonic on V such that $f = g$ on V_0 , then $f(x) = g(x)$ for all $x \in V$.

Proof: Let $h := f - g$. Then if x is any interior point,

$$\frac{\sum_{\{x,y\} \in E} h(y)}{d(x)} = \frac{\sum_{\{x,y\} \in E} f(y)}{d(x)} - \frac{\sum_{\{x,y\} \in E} g(y)}{d(x)} = f(x) - g(x) = h(x).$$

Therefore h is a harmonic function which vanishes on V_0 , and hence, by the Maximum Principle, the maximum and minimum values of h are 0. Thus h is identically null, and $f = g$. ■

Once again, let us consider graph $G = ((V_0, V_1), E)$ as an electrical network, where $V = V_0 \cup V_1$ is the set of nodes, with $|V_0| = 2$, and E is the set of branches, each of them having a unit resistance. If a voltage \mathbf{V} is given on V_0 , and satisfies the first Kirchhoff's law for interior points:

$$\forall x \in V_1, \quad \sum_{\{x,y\} \in E} (\mathbf{V}(y) - \mathbf{V}(x)) = 0, \quad (8)$$

then \mathbf{V} is precisely a harmonic function. Indeed, since all resistances of the branches are unitary, the current in the oriented branch (x, y) is expressed by the potential drop $\mathbf{V}(y) - \mathbf{V}(x)$.

Let x be an interior node, then solving (8) for $\mathbf{V}(x)$ straightforwardly gives the averaging property (7), and by the Uniqueness Principle, we conclude that $f = \mathbf{V}$. It remains to show that such a function exists, which rests on basic linear algebra. Indeed, by expressing (8) (or

(7)) for each interior point, we easily obtain a linear system $\Gamma(\mathbf{V}_1) = B$, Γ being a $v_1 \times v_1$ morphism, \mathbf{V}_1 being the vector of unknown variables $(\mathbf{V}(x))_{x \in V_1}$, and where the right-hand member B depends only on the values taken by \mathbf{V} on the boundary points. Γ being an injective endomorphism, then it is also bijective. Thus by

$$\mathbf{V}_1 = \Gamma^{-1}(B), \quad (9)$$

the existence of \mathbf{V} follows.

Proof: [Theorem 6]

Let $V_0 := \{\emptyset, N\}$ and $V_1 := \mathcal{N} \setminus V_0$. By the Uniqueness Principle, given a real number R , there is a unique potential \mathbf{V} defined on \mathcal{N} , such that $\mathbf{V}(\emptyset) = 0$, $\mathbf{V}(N) = R$, and satisfying the first Kirchhoff's law on V_1 . Moreover, we know by (9) that \mathbf{V} is linearly dependent on R , that is to say, proportional to R :

$$\forall S \in \mathcal{N}, \mathbf{V}(S) = \mathbf{V}_0^1(S) \cdot R,$$

where \mathbf{V}_0^1 is the required potential for $R = 1$. Thus one can adjust the value of R so that

$$\sum_{S \succ \emptyset} \mathbf{V}(S) = 1. \quad (10)$$

Indeed, let $C_{\mathcal{N}} := \sum_{S \succ \emptyset} \mathbf{V}_0^1(S)$ and $R_{\mathcal{N}} := C_{\mathcal{N}}^{-1}$ (note that $C_{\mathcal{N}}$ is non null since by the argument used for Maximum Principle, all $\mathbf{V}_0^1(S)$'s, $S \neq \emptyset$, are necessarily strictly positive). Thus for $R = R_{\mathcal{N}}$, (10) holds. We denote by \mathbf{V}_0 the associated potential, that is to say, satisfying $\mathbf{V}_0(\emptyset) = 0$ and $\mathbf{V}_0(N) = R_{\mathcal{N}}$.

Now, for any $(S, S+i) \in \mathcal{N}$, define $p_S^i := \mathbf{V}_0(S+i) - \mathbf{V}_0(S)$. Therefore, the marginalist value associated to the coefficients p_S^i 's is an efficient regular value. Indeed, by Proposition 7, the circuit property holds, and thus by Corollary 8, the regularity axiom also holds. Besides, by Proposition 5, the efficiency axiom holds, (5) being equivalent to (8), and (4) being expressed by (10) on the one hand, and by conservation of the flow pattern $(p_S^i)_{(S, S+i) \in \mathcal{N}^2}$ on the other hand.

Lastly, coefficients p_S^i 's being determined by the unique potential grounded on \emptyset and satisfying (10), the unicity of Φ_K is also shown. ■

Proof: [Theorem 12]

Let \mathbf{V} be a potential associated to the coefficients p_S^i 's of Φ_K . The source node of the electrical network being \emptyset , \mathbf{V} is necessarily greater on bound N than bound \emptyset . Then by the Maximum Principle, $\forall i \in N$ such that $N \setminus i \in \mathcal{N}$, $p_{N \setminus i}^i = \mathbf{V}(N) - \mathbf{V}(N \setminus i)$ is nonnegative. By Lemma 11, the result follows. ■

B Example of regular set system for which Φ_K does not satisfy the monotonicity axiom

Let $N := \{\alpha, \beta, \gamma, 1, 2, \dots, n'\}$ be the set of players, with $N' := \{1, \dots, n'\}$ and $n' \geq 1$. We make the assumption that the set of coalitions \mathcal{N} is defined by

$$\mathcal{N} := \left\{ \emptyset, N, N \setminus \beta, N \setminus \gamma \right\} \cup \bigcup_{i=0}^{n'} \left\{ \{\alpha, 1, 2, \dots, i\} \right\} \cup \bigcup_{S \in 2^{N'}} \left\{ S \cup \beta \right\}.$$

Then (N, \mathcal{N}) is a regular set system. The associated Shapley-Kirchhoff value defined over $\mathcal{G}(\mathcal{N})$ satisfies the monotonicity axiom if and only if $n' < 5$. Indeed, we show that if it is not the case, coefficient $p_{N' \cup \alpha}^\beta$ is negative, which corresponds to a negative current in the associated directed branch $(N' \cup \alpha, N' \cup \alpha\beta)$ (cf. Section 5). Fig. 2 is given with $n' = 5$, where the bold line represents the above mentioned branch.

Proof: The first point is to verify that (N, \mathcal{N}) is a regular set system. By checking that each coalition S of \mathcal{N} has all its successors (for inclusion order) T satisfying $|T \setminus S| = 1$, we have the result.

We compute now the different coefficients p_S^i of the Shapley-Kirchhoff Φ_K value associated to the regular set system (N, \mathcal{N}) . Let us denote by x the coefficient p_\emptyset^β and y the coefficient $p_{N' \cup \alpha}^\beta$. On the one hand, we consider the sub-order \mathcal{N}'_β induced by all vertices associated to coalitions including β and included in $N' \cup \beta$. \mathcal{N}'_β being isomorphic to the Boolean lattice $2^{N'}$, any permutation of N' leaves unchanged \mathcal{N}'_β , and since the circuit property is a symmetric rule (in the sense that labels of players have no importance for it), the coefficients computed for the edges of \mathcal{N}'_β are proportional to the coefficients of the classical Shapley value over $\mathcal{G}(\mathcal{N})$ whenever $\mathcal{N} = 2^{N'}$. Indeed, for any subset S of N' and any $i \in N' \setminus S$, let us denote by x_s^1 the coefficient $p_{S \cup \beta}^i$. By Proposition 5, we get

$$\begin{aligned} x_0^1 &= \frac{x}{n'}, \\ s \cdot x_{s-1}^1 &= (n' - s) \cdot x_s^1, \quad \forall s \in \{1, \dots, n' - 1\}, \end{aligned}$$

which immediately conducts by induction on s to

$$x_s^1 = \frac{s!(n' - s - 1)!}{n'!} \cdot x, \quad \forall s \in \{0, \dots, n' - 1\}.$$

Remark that the left coefficient also known under the form $p_s^1(n')$, is well-known since being a coefficient of Shapley for games with n' players (cf. (1)).

Naturally, we also get $p_{N' \cup \beta}^\alpha = \sum_{i=1}^{n'} x_{n'-1}^1 = x$.

On the other hand, also by Proposition 5, we successively deduce:

- On node \emptyset , $p_\emptyset^\alpha = 1 - x$.
- On nodes $\alpha 1 \dots i$, $i \in N'$, $p_\alpha^1 = p_{\alpha 1}^2 = \dots = p_{\alpha 1 2 \dots (n'-1)}^{n'} = 1 - x$.
- On nodes $N' \cup \alpha, N \setminus \beta, N \setminus \gamma$, $p_{N' \cup \alpha}^\gamma = p_{N \setminus \beta}^\beta = 1 - x - y$, and $p_{N \setminus \gamma}^\gamma = x + y$.

Now, let \mathcal{M}_1 be the circuit $(\emptyset, \alpha, \alpha 1, \alpha 1 2, \dots, N' \cup \alpha, N \setminus \gamma, N' \cup \beta, \dots, \beta 1 2, \beta 1, \beta, \emptyset)$ and \mathcal{M}_2 be the circuit $(N, N \setminus \gamma, N' \cup \alpha, N \setminus \beta, N)$. Then by the circuit property applied to these two circuits, we have the system:

$$\begin{cases} (n' + 1)(1 - x) + y - x - \sum_{s=0}^{n'-1} x_s^1 - x & = 0 \\ -(x + y) - y + 2(1 - x - y) & = 0 \end{cases}.$$

By denoting $\varsigma(n')$ the sum of the coefficients of Shapley $\sum_{s=0}^{n'-1} p_s^1(n')$, the above system writes

$$\begin{cases} (n' + 3 + \varsigma(n'))x - y & = n' + 1 \\ 3x + 4y & = 2 \end{cases},$$

and has for determinant the strictly positive number $\Delta := 4(n' + \zeta(n')) + 15$. Thus, we obtain

$$x = \frac{4n' + 6}{\Delta} \quad \text{and} \\ y = \frac{2\zeta(n') - n' + 3}{\Delta}.$$

For all the above p_s^i 's given in terms of x and y , except for $p_{N' \cup \alpha}^\beta = y$, we straightforwardly verify that $p_s^i > 0$, for all $n' \geq 1$. Thus it remains to find condition on n' so that y is nonnegative. For $n' \leq 4$, it can be checked that $2\zeta(n') - n' + 3$ is positive. Besides, $\forall n' \geq 1, \forall s \in \{0, \dots, n' - 1\}$, $p_s^1(n') = \frac{s!(n'-s-1)!}{n'!} \leq \frac{(s+(n'-s-1))!}{n'!} = \frac{1}{n'}$, the inequality being strict whenever $s \neq 0, n' - 1$. Thus, $\forall n' > 2$, $\zeta(n') = \sum_{s=0}^{n'-1} p_s^1(n') < 1$. Moreover,

$$\Delta \cdot y/2 = \zeta(n') - (n' - 3)/2,$$

where $(n' - 3)/2 \geq 1, \forall n' \geq 5$. As a consequence, $\zeta(n') - (n' - 3)/2 < 0, \forall n' \geq 5$, and so is y . The result finally follows, by Lemma 9.

In particular, whenever $n' = 5$, which corresponds to the regular system of Fig. 2, Φ_K is not monotonic, with $p_{\alpha 12345}^\beta = -\frac{14}{557}$. ■

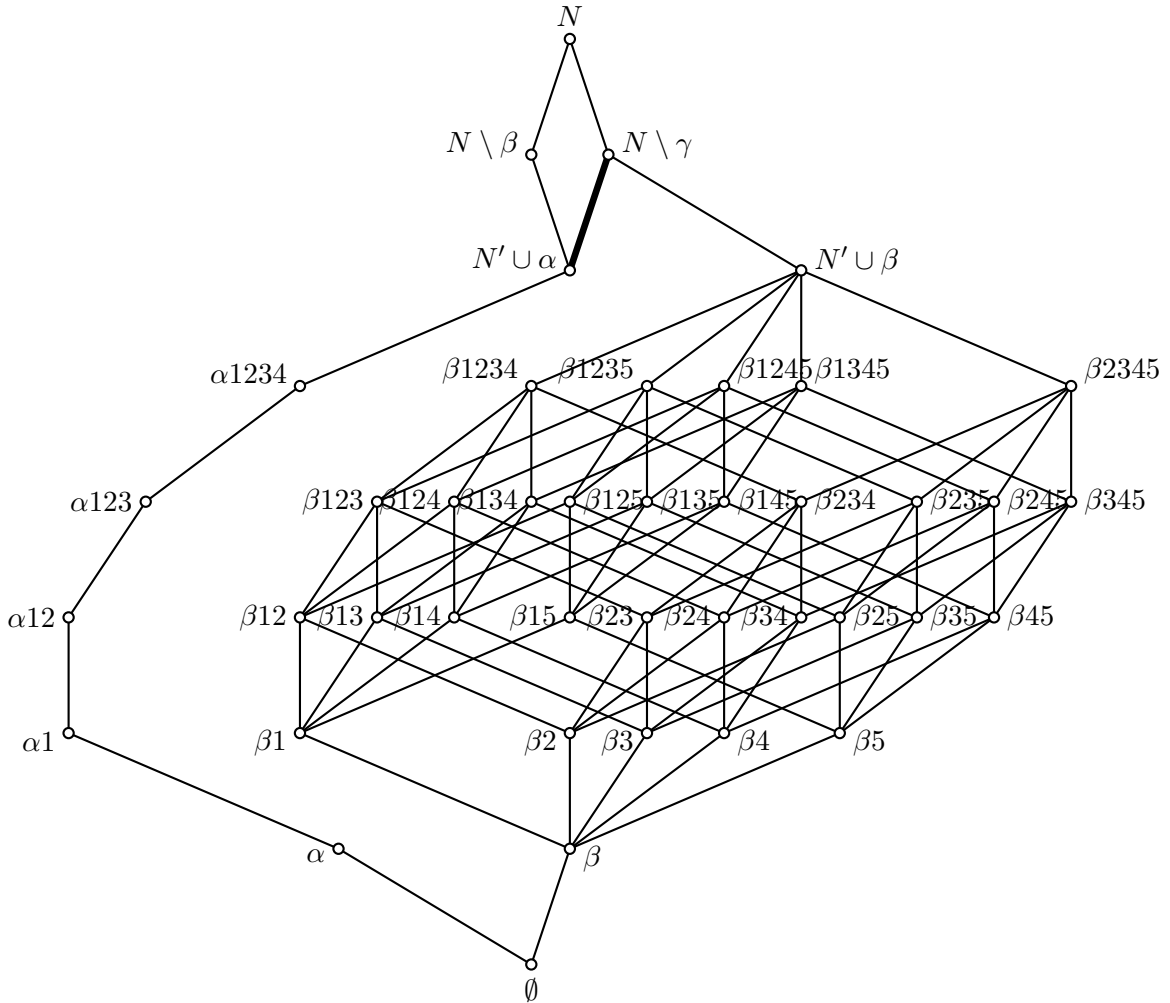


Figure 2: A regular set system where (\mathbf{M}) does not hold for Φ_K

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