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The 0-1 inverse maximum stable set problem

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Résumé

Les problèmes inverses motivent de très nombreux travaux dans le cadre de l'optimisation continue, notamment en géophysique. Dans le cadre combinatoire, les problèmes inverses ont été étudiés depuis le début des années 90 et donnent lieu à de nombreux travaux depuis ces dernières années. Il s'agit, étant donné une instance d'un problème et une solution réalisable, de modifier le moins possible le système de paramètres (au sens d'une norme choisie) pour que la solution fixée devienne optimale.

Nous nous intéressons plus particulièrement à des problèmes inverses avec contraintes 0-1 pour lesquels il s'agit de modifier la structure du graphe instance (plutôt que ses paramètres) afin de rendre une solution fixée optimale. Ainsi, nous envisageons des problèmes inverses contre un algorithme spécifié.

Dans ce papier, on étudie le problème inverse de stable maximum en variables bivalentes, contre un algorithme spécifié A (optimal ou non), noté $IS_{\{0,1\}A}$. Le problème $IS_{\{0,1\}A}$ consiste, étant donné un graphe simple $G=(V,E)$, un stable S^*, et un algorithme A, à retirer un nombre minimum de sommets de G pour que S^* soit choisi par l'algorithme A dans le graphe modifié.

D'abord, nous étudions la difficulté du problème $IS_{\{0,1\}A}$ pour deux algorithmes très classiques, *Glouton* et *2-Opt*, ainsi que pour un algorithme optimal spécifié. Nous montrons que le rapport d'approximation strictement meilleur que 2 est garanti pour $IS_{\{0,1\}2\text{-}Opt}$. Dans la deuxième partie, nous étudions des classes de graphes pour lesquelles $IS_{\{0,1\}}$ est polynomial. Nous montrons que $IS_{\{0,1\}}$ est polynomial dans quelques classes de graphes parfaits telles que les graphes de comparabilité et les graphes chordaux (triangulés). Ainsi, nous comparons les difficultés de $IS_{\{0,1\}}$ et $IS_{\{0,1\}2\text{-}Opt}$ pour d'autres classes de graphes.

Mots de clés : Optimisation combinatoire inverse, Stable maximum, Rapport d'approximation, Graphes parfaits

Classification AMS : 90C27, 05C17.

The 0-1 inverse maximum stable set problem

Yerim CHUNG [*] Marc DEMANGE [†]

Abstract

In this paper we study the 0-1 inverse maximum stable set problem, denoted by $IS_{\{0,1\}}$. Given a graph and a stable set (not necessarily maximum), it is to delete a minimum number of vertices to make the given stable set maximum in the new graph. We also consider $IS_{\{0,1\}}$ against a specific algorithm such as Greedy and 2opt, which is denoted by $IS_{\{0,1\},greedy}$ and $IS_{\{0,1\},2opt}$, respectively. We prove the NP-hardness of these problems and an approximation ratio of $2 - \Theta(\frac{1}{\sqrt{log\Delta}})$ for $IS_{\{0,1\},2opt}$. In addition, we restrict $IS_{\{0,1\}}$ to some classes of perfect graphs such as comparability and chordal graphs, and we study its tractability. Finally, we compare the hardness of $IS_{\{0,1\}}$ and $IS_{\{0,1\},2opt}$ for some other classes of graphs.

Key words: Combinatorial inverse optimization, Maximum stable set problem, NP-hardness, Performance ratio, Perfect graphs.

1 Introduction

Given an instance of a weighted combinatorial optimization problem and its feasible solution, the usual inverse problem is to modify as little as possible (with respect to a fixed norm) the weight system to make the given solution optimal. This area has been extensively studied during the last decade [1, 11, 15]. Recall that a stable set in a graph $G = (V, E)$ is a vertex set $S \subset V$ of which every two vertices are non connected by an edge. The maximum (weight) stable set problem is to find a stable set of maximum size (weight); both problems are known to be NP-hard [7]. It is shown in [5] that the inverse maximum weight stable set problem is NP-hard. In this paper, we focus on its 0-1 version [5], called 0-1 inverse maximum stable set problem and denoted by $IS_{\{0,1\}}$, in which every vertex has a weight 0 or 1. This problem can be seen as to modify the structure of an instance of the original problem, since changing the weight of a vertex from 1 to 0 corresponds to removing this vertex from the graph instance.

We also consider $IS_{\{0,1\}}$ against a specific (optimal or not) algorithm. We denote this problem by $IS_{\{0,1\},A}$, where A is a fixed algorithm (this notion

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appeared first in [2]). It is to modify the instance (as in the usual inverse problem) to make A choose the fixed solution. More formally, it is defined as follows: *given an undirected graph G=(V,E), a stable set* S^* *and a specific algorithm A,* $IS_{\{0,1\},A}(G,S^*)$ *is to delete a minimum number of vertices of* $V \setminus S^*$ *such that* S^* *can be returned by A in the new instance.*

Algorithms Greedy and 2opt are both very natural and practical for approximating maximum stable set. The former repeatedly selects a vertex of minimum degree and removes it from the graph together with all of its neighbors. The latter is a local search algorithm that computes a 2-optimal stable set S, i.e. neither $\forall v \notin S$, $S \cup \{v\}$ nor $\forall u \in S$, $\forall v, w \notin S$, $(S \setminus \{u\}) \cup \{v, w\}$ is a stable set. In this work, we study $IS_{\{0,1\},opt}$, $IS_{\{0,1\},greedy}$ and $IS_{\{0,1\},2opt}$. $IS_{\{0,1\}}$ corresponds to the case where A is any optimal algorithm. Similarly, we define the strict problem of $IS_{\{0,1\},A}$, denoted by $\widehat{IS}_{\{0,1\},A}$, which is to modify the structure of a given instance to force S^* to be selected by A as an unique solution in the new instance.

In section 2, we prove the *NP*-hardness of $IS_{\{0,1\},opt}$, $IS_{\{0,1\},greedy}$ and $IS_{\{0,1\},2opt}$. In section 3, we show that the performance ratio $2 - \Theta(\frac{1}{\sqrt{log\Delta}})$ is guaranteed for $IS_{\{0,1\},2opt}$. In section 4, we restrict $IS_{\{0,1\}}$ to some classes of perfect graphs such as comparability graphs and chordal graphs. We study its tractability in these classes. Finally, in section 5, we compare the hardness of $IS_{\{0,1\}}$ and$IS_{\{0,1\},2opt}$ for some other classes of graphs.

Notation.

Graph theory notation

$\overline{G}$: the complement of a graph G
$G[V']$: the subgraph of G, induced by $V' \subset V$
(G, w): a weighted graph with weight w
G_w: the graph obtained from (G, w) by multiplication of vertices (to be defined in the text)
G^w: the graph obtained from (G, w) by co-multiplication of vertices (to be defined in the text)
$\Gamma(v)$: the set of the adjacent vertices of a vertex v
$\Delta(G)$: the maximum vertex degree of the graph G

Combinatorial problems notation

S: the maximum stable set problem
$\alpha(G)$: the stability number of G: the optimal value of the problem S
K: the maximum clique problem
$\omega(G)$: the clique number of G: the optimal value of the problem K
$\chi(G)$: the chromatic number of G: the fewest number of colors needed to cover the vertices of G
$\kappa(G)$: the clique cover number of G: the fewest number of cliques needed to cover the vertices of G

VC: the minimum vertex-covering problem
S_k: the maximum k-colorable subgraph problem
$\alpha_k(G)$: the size of the largest k-colorable subgraph of G: the optimal value of the problem S_k
PWS_k: the maximum weight k-colorable subgraph problem with polynomially bounded weights
$\alpha_{w,k}(G)$: the maximum weight of a k-colorable subgraph of G: the optimal value of the problem PWS_k
S^{S^*}: the problem of finding a maximum $|S^*|$-colorable subgraph containing S^*
$\alpha^{S^*}(G)$: the size of the largest $|S^*|$-colorable subset of G which contains S^*: the optimal value of the problem S^{S^*}

Inverse problems notations
IP: the inverse problem of a combinatorial optimization problem P
$\widehat{IP}$: the strict inverse problem of P
$IP_{\{0,1\}}$: the 0-1 inverse problem of P for any optimal algorithm
$\widehat{IP}_{\{0,1\}}$: the strict 0-1 inverse problem of P for any optimal algorithm
$IP_{\{0,1\},A}$: the 0-1 inverse problem of P against a specific (optimal or not) algorithm A
$\widehat{IP}_{\{0,1\},A}$: the strict 0-1 inverse problem of P against a specific algorithm A

Approximation theory notations
$\lambda_P(G)$: the value of the approximated solution of P on a graph G
$\beta_P(G)$: the value of the optimal solution of P on a graph G
$\rho_P(G) = \frac{\beta_P(G)}{\lambda_P(G)}$: the approximation ratio of P on a graph G
$P_1 \propto P_2$: a polynomial time reduction of P_1 to P_2.

Remark 1. In many cases, if P is polynomial, then IP is also polynomial [1]. Nevertheless, a counter example is given in [15]. Moreover, in most cases, if P is NP-hard, then IP is also NP-hard. In particular, the NP-completeness of the decision version of stable set problem (S) leads to the NP-hardness of the inverse maximum stable set problem (IS) by the following simple reduction. Let $(G = (V, E), k)$ be an instance of S. We construct an instance $(G' = (V', E'), S^*)$ of IS by adding to G a stable set S^* of size $k = |S^*|$ $(V' = V \cup S^*)$, and by connecting by an edge every vertex of S^* to all vertices of G $(E' = E \cup \{sv | \forall s \in S^*, \forall v \in V\})$. Then, $\alpha(G) \leq k \Leftrightarrow S^*$ is a maximum stable set of G' $\Leftrightarrow$ $IS(G', S^*)$ has an optimal value of 0. Consequently, IS is NP-hard in every class of graphs stable under this transformation and for which S is NP-hard. On the other hand, for the classes of graphs for which S is NP-hard, it is also pertinent to consider the 0-1 inverse maximum stable set problem against a specific approximated algorithm A, $IS_{\{0,1\},A}$. □

Remark 2. Another natural distinction may arise in inverse framework whether one aims for a fixed solution (as stated previously) or only for the optimal value in the new instance. In the frame of the inverse maximum stable set problem, both points of view are equivalent. Given a graph G and a fixed value k, one wants to remove the less possible number of vertices such that the new graph has an independence number not greater than k. We consider a similar reduction as in **Remark 1**: let us add to G an independent set S_k of size k completely connected to G. Then, if we denote by G' the new graph, the problem is exactly the same as the usual inverse problem $IS_{\{0,1\}}$ in G', S_k being fixed. So both problems are equivalent for any class of graphs which is stable under this reduction (this is in particular the case for permutation graphs). However, this fact is not always true for the other combinatorial inverse problems. □

Remark 3. A natural weighted generalization of inverse maximum independent set can be defined as follows: given a vertex-weighted graph, the inverse maximum weight stable set problem IWS consists in minimizing the total weight of vertices to delete so that the graph induced by the left vertices has a weighted independence number of k or less. **Remark 2** clearly holds; so the version where a solution is fixed is equivalent. This problem is NP-hard even if the graph instance is a stable set. Indeed, the *Partition* problem simply reduces to IWS in polynomial time. Given an instance of *Partition*, that is n numbers $a_1, \cdots, a_n$, we construct a weighted graph $(G = (V, \emptyset), w)$ of order $|V| = n$ and without any edge (a stable set). The weight function w is defined by $w(v_i) = a_i$ for $v_i \in V$, $i \in I = \{1, \cdots, n\}$; let $k = \frac{1}{2}\sum_{i \in I} a_i$. Then, IWS in this instance is clearly equivalent to the considered *Partition* instance. Note that this argument fails if weights are supposed to be polynomially bounded. This paper only focuses on the unweighted case. □

2 Some hardness results

Proposition 2.1 *$IS_{\{0,1\}}$, $IS_{\{0,1\},greedy}$ and $IS_{\{0,1\},2opt}$ are NP-hard, even if $|S^*| = 1$.*

Proof. We transform the maximum clique problem K to $IS_{\{0,1\},A}$ for $A \in \{\text{opt, greedy, 2-opt}\}$. Let $I = (G = (V, E), k)$ be an instance of K, where k is an integer and G is a graph of order $|V| = n$. We construct an instance $I' = (H = (V', E'), S^*, k')$ of $IS_{\{0,1\},A}$ as follows:

- $S^* = \{s^*\}$
- $k' = n - k$
- $V' = V \cup \{s^*\}$ and $E' = E \cup \{vs^* \mid v \in V\}$

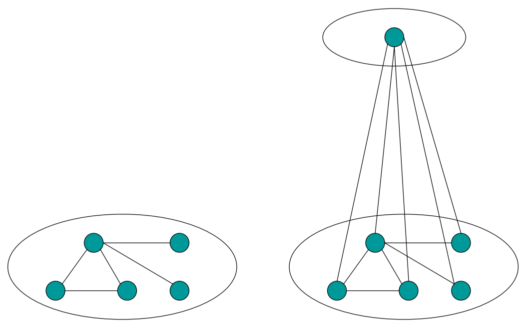

Figure 1: Construction of I' from I

Let us first point out that $V_0 \subset V$ is an optimal solution of $IS_{\{0,1\},A}$ for $A \in \{\text{opt, greedy, 2opt}\}$ if and only if $H' = H[V' \setminus V_0]$ is a clique, i.e. V_0 is a vertex-cover in $\overline{H}$.

Algorithm Greedy: Since Greedy selects a vertex of minimum degree and deletes all of its neighbors, it selects s^* if and only if every vertex in $H' = H[V' \setminus V_0]$ has at least the same degree as s^*, which means that H' is a clique.

Algorithm opt and 2opt: Similarly, S^* is optimal or 2-optimal in H' (in H' there exists no pair of vertices non connected to each other) if and only if H' is a clique.

To conclude the proof, we note that G contains a k-clique K_0 if and only if the vertex set to delete to make H' complete corresponds to $V' \setminus (K_0 \cup S^*)$ of size $k' = n - k$. Clearly, $H' = H[V' \setminus (V' \setminus (K_0 \cup S^*))] = H[K_0 \cup S^*]$ is a complete graph.

So, we have $K \propto IS_{\{0,1\},A}$ for $A \in \{\text{opt, greedy, 2-opt}\}$, and consequently $IS_{\{0,1\},opt}$, $IS_{\{0,1\},greedy}$ and $IS_{\{0,1\},2opt}$ are *NP*-hard, even if $|S^*| = 1$. ■

It is easy to verify that the strict problems $\widehat{IS}_{\{0,1\}}$, $\widehat{IS}_{\{0,1\},greedy}$ and $\widehat{IS}_{\{0,1\},2opt}$ are trivially solved in polynomial time for $|S^*| = 1$. However, these problems are *NP*-hard for $|S^*| \geq 2$. It can be shown in the very similar way as above if we replace H by $\widetilde{H} = (\widetilde{V}, \widetilde{E})$, where $\widetilde{V} = V \cup \{s_1^*\} \cup \{s_2^*\}$ and $\widetilde{E} = E \cup \{vs_i^* \mid i \in \{1, 2\},\ v \in V\})$. Note that the case $|S^*| > 2$ reduces to the case $|S^*| = 2$. Indeed, increasing the size of S^* by adding to $\widetilde{H}$ a set of $|S^*| - 2$ vertices non connected to any vertex of $\widetilde{H}$ does not affect the reduction.

Corollary 2.1 *$\widehat{IS}_{\{0,1\},opt}$, $\widehat{IS}_{\{0,1\},greedy}$ and $\widehat{IS}_{\{0,1\},2opt}$ are NP-hard, for $|S^*| \geq 2$.*

Finding the complement of the maximum clique in a graph G is equivalent to finding a minimum vertex-cover in $\overline{G}$. Consequently, the vertices to delete (the optimal solution of $IS_{\{0,1\},A}$) corresponds to the minimum vertex cover of $\overline{G}$. If we use the same notation of the proof for a graph H, then we have $IS_{\{0,1\},A}(H, S^*) \Leftrightarrow VC(\overline{G})$ for $A \in$\{opt, greedy, 2opt\}. Moreover, it is straightforward to verify that this reduction ($VC \propto IS_{\{0,1\},A}$) preserves approximation. So we have the following result:

Corollary 2.2 *Let n be the order of an instance of $IS_{\{0,1\},A}$.*
If $IS_{\{0,1\},opt}$, $IS_{\{0,1\},greedy}$ or $IS_{\{0,1\},2opt}$ is $\rho(n)$-approximated, then VC is $\rho(n+1)$-approximated.

3 Approximating $IS_{\{0,1\},2opt}$

In the previous section, we pointed out that VC reduces to $IS_{\{0,1\},2opt}$. In what follows, we show that $IS_{\{0,1\},2opt}$ reduces to VC.
For this, given k disjoint graphs $G_i = (V_i, E_i)$, $i \in \{1, \cdots, k\}$ ($\bigcap_{i=1}^k V_i = \phi$), we define their union $\bigcup_{i=1}^k G_i$ by a graph $G = (V, E)$ where $V = \bigcup_{i=1}^k V_i$ and $E = \bigcup_{i=1}^k E_i$. Note that a vertex-cover of $G = \bigcup_{i=1}^k G_i$ is just the union of vertex-covers of G_i, $i \in \{1, \cdots, k\}$.

Proposition 3.1 *Let n be the order of an instance of VC and Δ the maximum vertex degree of an instance of $IS_{\{0,1\},2opt}$.*
If there exists a $\rho(n)$-approximation algorithm for VC, then there exists a $\rho(\Delta)$-approximation algorithm for $IS_{\{0,1\},2opt}$.

Proof. Let $I = (G = (V, E), S^* = \{s_1, \cdots, s_k\} \subset V)$ be an instance of $IS_{\{0,1\},2opt}$ and Δ its maximum vertex degree. Without loss of generality, we can assume that S^* is maximal (i.e. $V \setminus S^* = \bigcup_{i=1}^k \Gamma(s_i)$ where $\Gamma(s_i)$ denotes a neighborhood of a vertex s_i). In the opposite case, every solution of $IS_{\{0,1\},2opt}$ contains $(V \setminus S^*) \setminus \bigcup_{i=1}^k \Gamma(s_i)$ with a better worse-case approximation ratio as for the restricted instance $G[(V \setminus S^*) \setminus \bigcup_{i=1}^k \Gamma(s_i)]$. We consider an instance $H = (V', E')$ of VC as follows:

- $k = |S^*|$
- $\forall i \in \{1, \cdots, k\}$, let $V'_i = \{u \in (V \setminus S^*) \mid \Gamma(u) \cap S^* = \{s_i\}\}$
  $(V'_i \cap V'_j = \phi,\ \forall i \neq j)$
- $V' = \bigcup_{i=1}^k V'_i$, $E' = \bigcup_{i=1}^k \overline{E_i}$, and $H = (V', E') = \overline{G}[V'_1] \cup \overline{G}[V'_2] \cup \cdots \overline{G}[V'_k]$

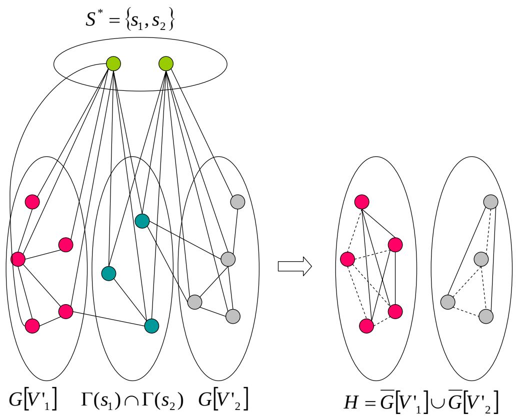

Figure 2: Construction of H (case $|S^*| = 2$)

It is easy to see that the minimum vertex-cover of H corresponds to the minimum vertex set to delete from G to make S^* 2-optimal, and conversely. Hence, we have $\beta_{IS_{\{0,1\},2opt}}(G,S^*)=\beta_{VC}(H)$. On the other hand, by applying an approximated algorithm for vertex-covering to each subgraph $\overline{G}[V_i']$, $i \in \{1,\cdots,k\}$ of H, we can obtain an approximated solution of $IS_{\{0,1\},2opt}$. Hence, $\lambda_{IS_{\{0,1\},2opt}}(G,S^*)=\lambda_{VC}(H)$. Consequently, we obtain $\rho_{IS_{\{0,1\},2opt}}(G,S^*) = \rho_{VC}(H)$. Since $|V_i'| \leq \Delta(G)$, the proposition holds. ■

Recently, Karakostas [12] improved the approximation factor for the vertex-covering problem to $2 - \Theta(\frac{1}{\sqrt{logn}})$, where n is the number of vertices. So we have:

Corollary 3.1 $IS_{\{0,1\},2opt}$ *can be approximated within ratio* $2-\Theta(\frac{1}{\sqrt{log\Delta}})$.

We also deduce from the proof of proposition 3.1:

Corollary 3.2 $IS_{\{0,1\},2opt}$ *is polynomially solved in triangle-free graphs.*

Proof. Let $G[V_i']$ be a subgraph of G defined in the proof of Proposition 3.1. If G is triangle-free (i.e. $G[V_i']$ is a stable set, otherwise G contains necessarily triangles), then $\overline{G}[V_i']$ is a clique. It is tractable in polynomial time to find a minimum vertex-cover in a clique. ■

Recall that a graph G is called perfect if G satisfies the following properties [8]:

$$\alpha(G[A]) = \kappa(G[A]) \text{ and } \omega(G[A]) = \chi(G[A]), \text{ for all } A \subseteq V$$

A graph G is perfect if and only if its complement $\overline{G}$ is perfect.

Corollary 3.3 $IS_{\{0,1\},2opt}$ *is polynomially solved in perfect graphs.*

Proof. If G is perfect, then each $\overline{G}[V_i']$ is also perfect. Furthermore, since perfectness is stable under disjoint union of graphs, H is also perfect. It is known [8] that the vertex-covering problem is solved in polynomial time in perfect graphs, thus the corollary holds. ■

Corollary 3.4 *$IS_{\{0,1\},2opt}$ is polynomially solved in degree-bounded graphs.*

Proof. If G is degree-bounded, each subgraph $\overline{G}[V_i']$ of H contains a bounded number of vertices. Thus, we can find exhaustively a minimum vertex-cover of each subgraph $\overline{G}[V_i']$. ■

4 $IS_{\{0,1\}}$ for some classes of perfect graphs

In section 2, we have shown that the 0-1 inverse maximum stable set problem against a specific algorithm is *NP*-hard for arbitrary graphs. Now, we turn our attention to identify classes of graphs for which $IS_{\{0,1\}}$ (against every optimal algorithm) is solvable in polynomial time.

Given a perfect graph $G = (V, E)$ and a stable set S^*, the 0-1 inverse maximum stable set problem for G can be written as follows:

$$\begin{cases} Min\ |V_0| \\ s.t.\ V_0 \subseteq (V \setminus S^*) \\ |S^*| = \alpha(G[V \setminus V_0]) \end{cases}$$

Since $G[V \setminus V_0]$ is perfect, we have $\alpha(G[V \setminus V_0]) = \kappa(G[V \setminus V_0])$. In addition, for any graph H, we have $\kappa(H) = \chi(\overline{H})$, $\alpha(H) = \omega(\overline{H})$ and $\omega(H) \leq \chi(H)$, $\alpha(H) \leq \kappa(H)$. So, we obtain the following equivalences:

$$\begin{cases} |S^*| = \alpha(G[V \setminus V_0]) \\ \Leftrightarrow |S^*| = \kappa(G[V \setminus V_0]) \\ \Leftrightarrow |S^*| \geq \chi(\overline{G[V \setminus V_0]}) \end{cases}$$

Thus, $IS_{\{0,1\}}$ in a perfect graph $G = (V, E)$ can be rewritten as follows:

$$\begin{cases} Max\ |V \setminus V_0| \\ s.t.\ S^* \subseteq V \setminus V_0 \\ |S^*| \geq \chi(\overline{G[V \setminus V_0]}) \end{cases}$$

That is, for a given instance (a perfect graph G and a stable set S^*), $IS_{\{0,1\}}(G, S^*)$ is equivalent to a problem of finding in $\overline{G}$ a maximum $|S^*|$-colorable subgraph containing S^*, which we denote by $S^{S^*}(\overline{G})$.

Proposition 4.1 *For any perfect graph G, $IS_{\{0,1\}}(G, S^*)$ is equivalent to $S^{S^*}(\overline{G})$.*

In what follows, we reduce S^{S^*} to PWS_k, the maximum weight k-colorable subgraph problem, where the weights are polynomially bounded and $k = |S^*|$.

Proposition 4.2 *S^{S^*} polynomially reduces to PWS_k.*

Proof. From the instance $(H = (V, E), S^* \subseteq V, k = |S^*| \geq 1)$ of S^{S^*}, we construct an instance $(H, w) = ((V, E), w)$ of PWS_k by assigning to nodes the polynomially bounded weight function w defined by:

$$w(x) = \begin{cases} n = |V| & \text{for } x \in S^* \\ 1 & \text{otherwise} \end{cases}$$

Let $V^{'}$ be a k-colorable subset of V which does not contain S^* (i.e. $\exists x \in S^*$ s.t. $x \notin V^{'}$), then the total weight of $V^{'}$ is at most equal to $n \times (|S^*| - 1) + (n - 1) = n|S^*| - 1$. Since S^* is a k-colorable subgraph of weight $n|S^*|$, it means that $V^{'}$ is not of maximum weight. Consequently, every maximum weighted k-colorable subgraph of (H, w) contains S^* and satisfies:

$$\alpha_{w,k}(H, w) = (n - 1)|S^*| + \alpha^{S^*}(H)$$

This equality implies that both problems have the same optimal solution and also S^{S^*} reduces to PWS_k in polynomial time. ■

Given a weighted graph (H, w) (weights are assumed to be integers), Golumbic defined in [8] a non-weighted graph H_w, obtained from (H, w) by the so-called *multiplication of vertices*: one replaces each vertex x_i of weight w_i by a stable set of w_i vertices $x_i^1, x_i^2, \cdots, x_i^{w_i}$ and joins x_i^s with x_j^t iff x_i and x_j are adjacent in (H, w). Similarly, we define co-multiplication of vertices, which is to replace each vertex x_i of weight w_i by a w_i-clique (a clique of size w_i). Let H^w be the graph constructed from (H, w) by co-multiplication of vertices. Note that $\overline{H^w} = (\overline{H})_w$ and $\overline{H_w} = (\overline{H})^w$, and that perfect graphs are stable under multiplication of vertices [8] and under co-multiplication of vertices.

The following transformation is well known for maximum stable set problem but also holds for maximum k-colorable subgraph problem (see [13]).

Proposition 4.3 *$PWS_k(H, w)$ is equivalent to $S_k(H_w)$ and the transformation is polynomial.*

Proof. We have $\alpha_{w,k}(H, w) = \alpha_k(H_w)$: since every maximal (for inclusion) k-colorable subgraph of size W in H_w corresponds to a maximal subgraph of weight W in (H, w), and conversely. ■

We deduce from the propositions 4.1, 4.2 and 4.3:

Theorem 4.1 *For a perfect graph G, $IS_{\{0,1\}}(G, S^*)$, $PWS_k(\overline{G}, w)$ and $S_k(\overline{G}_w)$ are equivalent to each other. Moreover, the transformations are polynomial.*

Since the maximum k-colorable subgraph problem is not known to be polynomially solvable for every class of perfect graphs, we cannot deduce the tractability of $IS_{\{0,1\}}$ for an arbitrary perfect graph. So, we study the complexity of $IS_{\{0,1\}}$ restricted to some particular classes of perfect graphs.
Recall that a comparability graph (a classical class of perfect graphs) is an undirected graph $G = (V, E)$ admitting a transitive orientation F, that is a binary relation on the vertices satisfying [8]: $F \cap F^{-1} = \phi$, $F \cup F^{-1} = E$ and $F^2 = \{ac \mid ab,\ bc \in F, \forall\ b \in V\} \subseteq F$.

Remark. Comparability graphs and their complements, co-comparability graphs are both stable under multiplication and co-multiplication of vertices. Let (H, w) be a weighted comparability graph and F be its associated transitive orientation. We define an orientation F_w of H_w (the graph obtained from (H, w) by multiplication of vertices) as follows:

$$\forall s \in \{1, \cdots, w_i\}, \forall t \in \{1, \cdots, w_j\}, x_i^s x_j^t \in F_w \text{ iff } x_i x_j \in F.$$

Due to the transitivity of F, $x_i^s x_j^t \in F_w$ and $x_j^t x_k^u \in F_w$ imply $x_i^s x_k^u \in F_w$, i.e. F_w is transitive and H_w is a comparability graph.
For the graph H^w, obtained from (H, w) by co-multiplication of vertices, we define the following orientation F^w: we first assign a transitive orientation to every w_i-clique (associated to the vertex x_i of weight w_i in (H, w)) and we complete it by F_w on the edges of H_w. By construction of H^w, if we have $x_i^s x_i^{s'} \in F^w$ and $x_i^{s'} x_j^t \in F^w$, then we also have $x_i^s x_j^t \in F^w$. So, the transitivity of F implies that F^w is transitive.
In addition, since $\overline{H^w} = (\overline{H})_w$ and $\overline{H_w} = (\overline{H})^w$, this argument holds for co-comparability graphs. $\square$

Proposition 4.4 $IS_{\{0,1\}}$ *is polynomially solvable for comparability graphs and co-comparability graphs.*

Proof. Frank, Greene and Kleitman proved the tractability of maximum k-colorable subgraph problem in comparability graphs and their complements [6], [9], [10]. Since comparability and co-comparability graphs are closed under multiplication of vertices, we conclude by applying theorem 4.1. ■

Recall that a permutation graph is a comparability graph whose complement is also a comparability graph, and an interval graph is a (chordal) co-comparability graph [8]. So, we have the following corollary:

Corollary 4.1 $IS_{\{0,1\}}$ *is polynomially solvable for permutation graphs and interval graphs.*

An undirected graph G is called chordal (or triangulated) if every cycle of length strictly greater than 3 has a chord. Since a chordal graph is perfect, we can use theorem 4.1: $IS_{\{0,1\}}(G, S^*) \Leftrightarrow PWS_k(\overline{G}, w)$. On the other hand, Yannakakis and Gavril proved in [14] that the maximum weight k-colorable subgraph problem is polynomially solvable in chordal graphs and their complements if k is fixed, and NP-complete if k is not fixed. This leads the following corollary:

Corollary 4.2 *If $k = |S^*|$ is fixed, $IS_{\{0,1\}}$ is polynomially solvable for chordal and co-chordal graphs .*

A graph is said to be $(1,2)$-colorable if its vertex set can be covered by one clique and two stable sets. We call K_1S_2 such a class of graphs; the problem of deciding whether a given graph belongs to this class is known to be polynomial [3].

Proposition 4.5 *If $k = |S^*|$ is not fixed, $IS_{\{0,1\}}$ is NP-hard for $(1,2)$-colorable co-chordal graphs.*

Proof. In fact, it is proved in [14] that S_k is NP-hard in split graphs (for an unbounded k). Let us consider an instance (G,k) of this problem where G is a split graph and add a stable set S^* of size k completely connected to the vertices of G. It is straightforward to verify that the resulting graph, $\widetilde{G}$, is $(1,2)$-colorable and co-chordal. Moreover, $S^{S^*}(\overline{\widetilde{G}})(\Leftrightarrow IS_{\{0,1\}}(\widetilde{G},S^*))$ corresponds exactly to finding a maximum size of k-colorable subgraph in G, which completes the proof. ■

Remark. Since interval graphs are not only chordal but also co-comparability graphs, $IS_{\{0,1\}}$ is polynomially solvable for interval graphs even if $k = |S^*|$ is not fixed. Anyway, Yannakakis and Gavril proved in [14] the tractability of the maximum weight k-colorable subgraph problem on interval graphs when k is not fixed. □

Let us now consider the 0-1 inverse maximum clique problem, denoted by $IK_{\{0,1\}}$. It is defined as follows: given an undirected graph $G = (V,E)$ and a clique K^* of G, delete as few vertices as possible from $V \setminus K^*$ so that the fixed clique K^* becomes maximum in the new instance.
Clearly, $IK_{\{0,1\}}(G,K^*)$ is equivalent to $IS_{\{0,1\}}(\overline{G},K^*)$, which leads the corollaries:

Corollary 4.3 *The problem $IK_{\{0,1\}}$ is NP-hard for arbitrary graphs.*

Corollary 4.4 *$IK_{\{0,1\}}(G,K^*)$ is polynomially solvable for perfect graphs such as comparability, co-comparability, permutation and interval graphs, and for chordal, co-chordal and split graphs if $k = |K^*|$ is fixed.*

5 Comparing $IS_{\{0,1\}}$ and $IS_{\{0,1\},2opt}$

The hardness of $IS_{\{0,1\}}$ and $IS_{\{0,1\},2opt}$ depends on the nature of the graph instance. It is interesting to identify classes of graphs for which $IS_{\{0,1\}}$ and $IS_{\{0,1\},2opt}$ are both polynomially solvable, or the ones for which $IS_{\{0,1\}}$ is *NP*-hard and $IS_{\{0,1\},2opt}$ is polynomial, and conversely.
For several classes of perfect graphs already mentioned in section 4, both $IS_{\{0,1\},2opt}$ and $IS_{\{0,1\}}$ can be solved in polynomial time. On the other hand, $IS_{\{0,1\}}$ is *NP*-hard in every graph for which S is *NP*-hard. In particular, $IS_{\{0,1\}}$ is *NP*-hard in degree-bounded graphs [7] and triangle-free graphs [8]. On the

contrary, for these graphs $IS_{\{0,1\},2opt}$ is proved (in section 3) to be polynomial. In what follows we point out that it is not true for every class of graphs that $IS_{\{0,1\}}$ is more difficult than $IS_{\{0,1\},2opt}$. We devise a class $\mathcal{G}$ of graphs for which $IS_{\{0,1\}}$ is polynomial and $IS_{\{0,1\},2opt}$ is NP-hard. From an arbitrary graph $G = (V, E)$, we construct a graph $G' = (V', E') \in \mathcal{G}$ as follows:

- $V' = V \cup \{s_1, s_2\} \cup C_1 \cup C_2,\ |C_i| > |V|\ \forall i \in \{1, 2\}$
- $E' = E \cup \{s_1 v | v \in V \cup C_1 \cup C_2\} \cup \{s_2 v | v \in C_1 \cup C_2\} \cup \{u_i v_i | (u_i, v_i) \in C_i \times C_i,\ i \in \{1, 2\}\}$

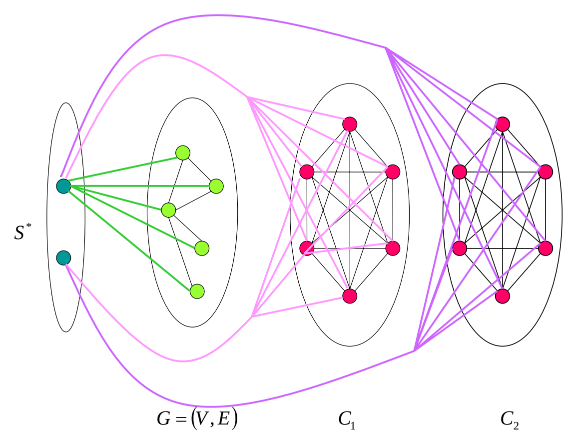

Figure 3: Instance $\mathcal{G}$

$\mathcal{G}$ is the class of all graphs defined by this way. Given a graph which is decomposed by $(G, \{s_1, s_2\}, C_1, C_2)$, it is polynomial to decide if it is in $\mathcal{G}$.

Proposition 5.1 *$IS_{\{0,1\},2opt}$ is NP-hard for the instance set of the form $(G', S^* = \{s_1, s_2\})$, where $G' = (G, \{s_1, s_2\}, C_1, C_2) \in \mathcal{G}$. On the other hand, $IS_{\{0,1\}}$ is polynomial on the same instance set.*

Proof. Let us consider an instance of $IS_{\{0,1\},2opt}$, $(I = (G', S^* = \{s_1, s_2\})$ where $G' = (G, \{s_1, s_2\}, C_1, C_2) \in \mathcal{G}$. Since every vertex of C_1 and C_2 is totally connected to S^*, the existence of C_1 and C_2 does not affect the 2-optimality of S^*. So, $IS_{\{0,1\},2opt}$ on I reduces to the problem of finding a vertex-cover in the graph $\overline{G}$, which concludes the NP-hardness of $IS_{\{0,1\},2opt}$ for this class of instances.
On the other hand, every three vertices $(u_1, u_2, v) \in C_1 \times C_2 \times V$ constitutes a stable set. If one removes less than $|V|$ vertices of V, then S^* cannot be a maximum stable set. Thus, an optimal solution of $IS_{\{0,1\}}$ in I is to remove all vertices of V (since $|C_i| > |V|\ \forall i \in \{1, 2\}$). ∎
Further research is needed to devise approximation algorithms for the problems

$IS_{\{0,1\},opt}$ or $IS_{\{0,1\},greedy}$ that guarantee a performance ratio, and to find the other classes of graphs for which $IS_{\{0,1\}}$ is tractable in polynomial time.

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