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INCOMPLETE MARKETS AND MONETARY POLICY

P. GOURDEL AND L. TRIKI

We consider an extension of a general equilibrium model with incomplete markets that considers cash-in-advance constraints. The total amount of money is supplied by an authority, which produces at no cost and lends money to agents at short term nominal rates of interest, meeting the demand. Agents have initial nominal claims, which in the aggregate, are the counterpart of an initial public debt. The authority covers its expenditures, including initial debt, through public revenues which consists of taxes and seignorage, and distributes its eventual budget surpluses through transfers to individuals, while no further instruments are available to correct eventual budget deficits. We define a concept of equilibrium in this extended model, and prove that there exists a monetary equilibrium with no transfers. Moreover, we show that if the price level is high enough, a monetary equilibrium with transfers exists.

Keywords: Cash-in-advance constraints, incomplete markets, nominal assets, monetary equilibrium, money, nominal interest rate

JEL Classification: C62, D52, E40, E50, G10.

1. Introduction

In the canonical general equilibrium model, all trade takes place in a barter economy, precluding the role of money as a medium of exchange. In 1965, Frank Hahn [10] has argued that it was difficult to justify a positive price for fiat money (i.e. paper money) (this is known as the "Hahn Paradox") which stipulates by a backward induction reasoning in a finite-period economy that money cannot have positive value. As discussed by Dubey–Geanakoplos [5], there are several ways to overcome this paradox. Among them, one can consider an infinite-horizon model (Samuelson [15], Grandmont–Younes [9]), and in these cases, money has value because it is a store of value. Another way to overcome Hahn's Paradox is to introduce an external agent, who stands ready to trade commodities for money (Lucas [13], Magill–Quinzii [14]). Alternatively, following Lerner [11], one could postulate the existence of a government that is owed in taxes. In these two latter cases, money has value because an external agent gives something in exchange for it.

The present paper considers the presence of an external agent, an authority, and the theoretical work underlying it, Drèze–Polemarchakis [4], [3], consists in formulating an intertemporal general equilibrium model with money, introducing reasonable assumptions that guarantee the existence of equilibria in this extended model. In order to define a general competitive equilibria in a monetary economy, we modify the canonical Walrasian model by introducing an incomplete financial market and money balances that facilitate transactions. Fiat money produced at

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no cost by banks serves as medium of exchange. An authority lends money to agents against promise of reimbursement with interest rate, or equivalently, in exchange for interest bearing bonds. All initial holdings of money are the counterparts of debts to banks. In the monetary vocabulary of monetary economies, this is a model of "inside money"[1]. It is a model appropriate for economies where an authority issues money in exchange for offsetting claims. There is no default, and the authority raises revenue from taxes and seignorage. It distributes its eventual surpluses as lump-sum transfers to agents. The demand for money at given commodity prices and interest rates results from the preference maximizing choice of individuals. As store of value, non-interest-bearing fiat money is dominated by interest-bearing nominal assets. Balances, prices and rates of interests do not enter as arguments of preferences of the agents.

Over a finite horizon with no public debt, and no taxes, Drèze and Polemarchakis [3] proved the existence of equilibria for arbitrarily set nominal rates of interest and price levels at all terminal nodes. In a recent joint paper with Bloize [1], they proved the existence over an infinite horizon economy under uncertainty and complete asset markets. The primitive of the model include nominal claims held by individuals (that in the aggregate are the counterpart of initial public debt). Their work extends for Woodford [17] in the case of heterogeneous agents, which is in term similar to cash-in-advance economies with a representative agent as in Lucas–Stockey [12]. Woodford [17] asserts that the price level is determinate so as to balance the initial public debt and public revenu from taxes and seignorage. Similarly, Dubey–Geanakoplos [6] obtain deterministic equilibria considering the case of a given initial shock of outside money. On the other hand, Bloize–Drèze–Polemarchakis [1] obtain indeterminacy of equilibria since they assume that the public authority can redistribute its eventual surpluses.

In this paper, we propose an extension of Bloize–Drèze–Polemarchakis [1], in an incomplete market setting, and over a finite, two-period horizon. The main results are:

- The existence of a monetary equilibrium with no transfers, under reasonnable assumptions.
- The existence of a monetary equilibrium above a lower bound of the overall price level, when the authority faces a budget surplus. Two alternative assumptions on the public portfolio (the portfolio that the authority supplies) are proposed and the results are compared.

The paper is organized as follows: we begin by introducing the primitive of the model, as well as the time and uncertainty setup (Section 2). We also define an appropriate notion of monetary equilibria, and state the assumptions under which existence will be proved. Section 3 proves the existence of equilibria with no transfers. Finally, Section 4 proves existence when the authority faces a budget surpluses.

2. A 2-Period Monetary Economy

We consider a finite set I of agents, two periods $t = 0$ and $t = 1$ with a finite set S of states of the world at the second period. We denote $\Sigma = \{0\} \cup S$, where 0 is the state of the world known with certainty at $t = 0$. The state of the world $\sigma \in \Sigma$ is called a date-event state. There

[1] Several authors have studied the implications of integrating outside money in a general equilibrium model with incomplete markets. Main contributions are Dubey–Geanakoplos [6].

is a finite set of goods L available for trades at both periods, a finite set J of 1-period maturity nominal assets that agents can buy at $t=0$ and which yield monetary returns at $t=1$.
We denote by $y^\bullet$ the family:

$$y^\bullet = (y^i,\ i \in I)$$

- The commodity market $\mathcal{E}^c$ is described by $\mathcal{E}^c = (X^\bullet, u^\bullet, e^\bullet, g^\bullet, \xi^\bullet)$ where, for each agent $i \in I$, $X^i \subset \mathbb{R}_+^{L\Sigma}$ is the consumption set of agent i. A vector $x^i \in X^i$ is a consumption plan. The utility function $u^i : X^i \longrightarrow \mathbb{R}$ describes the preferences of agent $i \in I$. The initial endowments are given by $e^i \in \mathbb{R}_+^{L\Sigma}$ and every agent i pays taxes $g^i \in \mathbb{R}_+^{L\Sigma}$ to the authority. Notice that in particular, it can be assumed that $g^i = \eta^i e^i$, for some $0 \leqslant \eta^i < 1$ with $\eta^\bullet \geqslant 0$ being some tax rates across individuals. Our commodity taxes then reduce to a wealth tax. The public authority also issues transfers t which are elements of $\mathbb{R}^\Sigma$. These transfers are distributed to individuals according to given shares $\xi^i \in]0,1[$ such that $\sum_{i\in I} \xi^i = 1$ and each agent receives the amount $\xi^i t$.
- The financial market $\mathcal{E}^f$ is described by $\mathcal{E}^f = (R, \Theta^\bullet, \theta)$ where $R \in R^{S\times J}$ is the return matrix, $J \leqslant S$, and for every $(s,j) \in S \times J$, $R(s,j) \in \mathbb{R}$ is denominated in units of account. For each agent $i \in I$, $\Theta^i \subset \mathbb{R}^J$ is the portfolio set of agent i. Given an agent $i \in I$ and a portfolio $\theta^i \in \Theta^i$, $(R(\sigma)\theta^i, \sigma \in \Sigma) \in \mathbb{R}^\Sigma$ denotes the image of θ^i by R. Finally, the portfolio $\theta \in \mathbb{R}^J$ is the total amount of each asset available for trade, fixed by the authority.
- The money market is described by $\mathcal{E}^m = (w^\bullet, r)$ where $w^\bullet \in \mathbb{R}^{\Sigma I}$ and for each $i \in I$, $w^i(0) \in \mathbb{R}_+$ are initial individual nominal claims against the authority (corresponding to the public debt). For convenience, we introduce the following notation: for every agent $i \in I$, $w^i = (w^i(\sigma), \sigma \in \Sigma)$ where $w^i(s) = 0$, and for all states $s \in S$. We set$w = \sum_{i\in I} w^i$. Short term nominal rates of interest r are positive element of $\mathbb{R}^\Sigma$ exogenously given.

Finally, a monetary economy is the triplet

$$\mathcal{E} = (\mathcal{E}^c, \mathcal{E}^f, \mathcal{E}^m).$$

2.1. **The transactions demand for money.** Let us begin by introducing these notations: Let $p \in \mathbb{R}_+^{L\Sigma}$ the commodity price vector. We define the payoff matrix $V \in \mathbb{R}^{\Sigma\times J}$ by $V = (-q \quad R)$. This operator summarizes the financial structure of the economy, given that $q \in R^J$ is the asset price vector. Let an agent $i \in I$. We denote net trades by $z^i = (x^i - e^i)$, where $x^i \in X^i$ and e^i initial endowments, $z_+^i = (x^i - e^i)^+$ net purchases, $z_-^i = (x^i - e^i)^-$ net sales[2]. We will denote by $\widehat{m}^i \in \mathbb{R}_+^\Sigma$ initial money balance, and by $m^i \in \mathbb{R}_+^\Sigma$ terminal money balance.

An important modeling choice concerns the treatment of time. There are two periods. Formally, a date is a point of time. For purpose of interpretation, the length of time period is thought as non-trivial. Precise timing of transactions does not affect preferences while it does affect money balances and accounting. Taking this into account, we follow the convention that budget constraints will be written at beginning-of-period, given a path of interest rate $r \in \mathbb{R}^\Sigma$

[2]For a scalar, $z^+ = \max\{z, 0\}$ and $z^- = \max\{-z, 0\}$; for a vector, $z^+ = (\ldots, z_{k+}, \ldots)$ and $z^- = (\ldots, z_{k-}, \ldots)$. Notice that $z = z^+ - z^-$. Moreover, recall that the functions $z \longrightarrow z^+$ and $z \longrightarrow z^-$ are convex.

and a vector $h \in \mathbb{R}^{\Sigma}$, we introduce, for every state $\sigma \in \Sigma$:

$$\widetilde{h}(\sigma) = \frac{1}{1+r(\sigma)} h(\sigma) \tag{2.1}$$

Consider a date-event $\sigma \in \Sigma$. The transaction demand for money follows the scheme of cash-in-advance constraints introduced by Clower [2]. An agent $i \in I$ acquires cash balances $\widehat{m}^i(\sigma)$ by borrowing initially from the authority in exchange for bonds at the rate of interest $r(\sigma) \in \mathbb{R}_+$, according to the constraint

$$b^i(\sigma) + \widehat{m}^i(\sigma) = 0.$$

Subsequently, he purchases commodities according to the constraint:

$$p(\sigma) \cdot z^i_+(\sigma) \leqslant \widehat{m}^i(\sigma), \tag{2.2}$$

He accumulates end-of-period balances from the sale of commodities according to the constraint

$$p(\sigma) \cdot z^i_-(\sigma) = m^i(\sigma), \tag{2.3}$$

At the end of the period, or at the beginning of a subsequent, fictitious period that serves for accounting purposes, the agent settles his debt according to the constraint:

$$(1+r(\sigma))\widehat{m}^i(\sigma) + p(\sigma) \cdot g^i(\sigma) - (V\theta^i)(\sigma) - \xi^i t(\sigma) \quad \leqslant \quad m^i(\sigma) + (1+r(\sigma))w^i(\sigma)$$

where $g^i(\sigma) \in \mathbb{R}^L_+$ are commodity taxes payed to the authority, $w^i(\sigma) \in \mathbb{R}_+$ are nominal initial claims against the authority that agent receive at beginning-of-period, θ^i is the portfolio he chooses to acquire, and $\xi^i t(\sigma) \in \mathbb{R}$ is the amount of transfers that his share ξ^i allows him to obtain. According to equations 2.2 and 2.3, the budget equation of agent i at date-event $\sigma \in \Sigma$ is summarized by:

$$\begin{aligned}
(1+r(\sigma))[p(\sigma) \cdot z^i_+(\sigma)] + p(\sigma) \cdot g^i(\sigma) &\leqslant p(\sigma) \cdot z^i_-(\sigma) + (V\theta^i)(\sigma) + \xi^i t(\sigma) + (1+r(\sigma))w^i(\sigma) \\
p(\sigma) \cdot [z^i_+(\sigma) + \widetilde{g}^i(\sigma)] &\leqslant \frac{1}{1+r(\sigma)}\left(p(\sigma) \cdot z^i_-(\sigma)\right) + (\widetilde{V}\theta^i)(\sigma) + w^i(\sigma) + \xi^i t(\sigma) \\
p(\sigma) \cdot (z^i(\sigma) + \widetilde{g}^i(\sigma)) + p(\sigma) \cdot z^i_-(\sigma) &\leqslant \frac{1}{1+r(\sigma)}\left(p(\sigma) \cdot z^i_-(\sigma)\right)(\widetilde{V}\theta^i)(\sigma) + w^i(\sigma) + \xi^i \widetilde{t}(\sigma) \\
p(\sigma) \cdot (z^i(\sigma) + \widetilde{g}^i(\sigma)) + \frac{r(\sigma)}{1+r(\sigma)} p(\sigma) \cdot z^i_-(\sigma) &\leqslant (\widetilde{V}\theta^i)(\sigma) + w^i(\sigma) + \xi^i \widetilde{t}(\sigma)
\end{aligned}$$

For each commodity price $p \in \mathbb{R}^{\Sigma L}_+$ and each consumption plan $x \in \mathbb{R}^{\Sigma L}_+$, we define the vector $p \Box x \in \mathbb{R}^{\Sigma}$ by

$$p \Box x = (p(\sigma) \cdot x(\sigma), \sigma \in \Sigma) \in \mathbb{R}^{\Sigma}$$

where the operator $\cdot$ is the scalar product in $\mathbb{R}^{\Sigma L}$.
For each interest rate $r \in \mathbb{R}^{\Sigma}_+$ and each money balance $m \in \mathbb{R}^{\Sigma}_+$, we define the vector $r \circ m \in \mathbb{R}^{\Sigma}$ by

$$r \circ m = (r(\sigma) m(\sigma), \sigma \in \Sigma) \in \mathbb{R}^{\Sigma}$$

We get to the overall budget constraints:

$$p \Box (x^i - e^i + \widetilde{g}^i) + \widetilde{r} \circ (p \Box (x^i - e^i)^-) \leqslant \widetilde{V}\theta^i + w^i + \xi^i \widetilde{t}. \tag{2.4}$$

2.2. **Authority.** The authority enters a date-event 0 with a given public liability $w(0)$ and covers this beginning-of-period expenditure and end-of-period supply of security $\theta \in \mathbb{R}^J$ by collecting commodity taxes $\widetilde{g}(0) \in \mathbb{R}^L_+$, given that money balances $m(0) \in \mathbb{R}_+$ are supplied so as to accommodate the market demand, where

$$m(0) = r(0)p(0) \cdot \sum_{i \in I} \left(x^i(0) - e^i(0)\right)^-$$

At the end-of-period, the authority distributes its eventual budget surpluses as transfers to individuals $t(0) \in \mathbb{R}$ determined by the beginning-of-period constraint:

$$\widetilde{t}(0) = \widetilde{r}(0)p(0) \cdot \sum_{i \in I} \left(x^i(0) - e^i(0)\right)^- + p(0) \cdot \widetilde{g}(0) + \widetilde{q} \cdot \theta - w(0) \tag{2.5}$$

These transfers are distributed among agents according to their exogenous shares $\xi^i \in [0,1]$, and vary accordingly to different consumption allocation $x^\bullet \in \mathbb{R}^{\Sigma LI}_+$.

At date-event $s \in S$, given end-of-period returns of assets, and collected taxes $\widetilde{g}(\sigma) \in \mathbb{R}^L_+$ the eventual budget surpluses distributed among agents amount to:

$$\widetilde{t}(s) = \widetilde{r}(s)p(s) \cdot \sum_{i \in I} \left(x^i(s) - e^i(s)\right)^- + p(s) \cdot \widetilde{g}(s) - (\widetilde{R}\theta)(s) \tag{2.6}$$

The overall constraint faced by the authority sums up to:

$$\widetilde{t} = \widetilde{r} \circ p \,\square \sum_{i \in I} (x^i - e^i)_- + p \,\square\, \widetilde{g} - \widetilde{V}\theta - w \tag{2.7}$$

We can now go through the definition of an equilibrium and state the main result of the paper.

2.3. **Definitions and notations.** Given a commodity price vector $p \in \mathbb{R}^{L\Sigma}_+$ and an asset price $q \in \mathbb{R}^J$, we introduce the budget set of an agent $i \in I$ by:

$$B^i(p,q,t) := \left\{ (x^i, \theta^i) \in X^i \times \Theta^i \;:\; p \,\square\, (x^i - e^i + \widetilde{g}^i) + \widetilde{r} \circ [(p \,\square\, (x^i - e^i)^-] \leqslant \widetilde{V}\theta^i + w^i + \xi^i \widetilde{t} \right\}$$

A consumption plan $x^i \in X^i$ and a composition of portfolio $\theta^i \in \Theta^i$ are budget feasible for agent $i \in I$ if these actions belong to budget set $B^i(p,q,t)$.

Given a commodity price vector $p \in \mathbb{R}^{L\Sigma}_+$, agent i's behavior in this economy is summarized by the demand correspondence $d^i(p,q,t)$ defined by:

$$d^i(p,q,t) := \left\{ (x^i, \theta^i) \in B^i(p,q,t),\; B^i(p,q,t) \cap [P^i(x^i) \times \Theta^i] = \emptyset \right\}$$

where $P^i(x^i) := \{y \in X^i \;:\; u^i(y) > u^i(x^i)\}$.

Definition 2.1. A collection $(x^\bullet, \theta^\bullet, p, q, t) \in \mathbb{R}^{L\Sigma I}_+ \times \mathbb{R}^{JI} \times \mathbb{R}^{L\Sigma}_+ \times \mathbb{R}^J_+ \times \mathbb{R}^\Sigma$ is a monetary equilibrium of a monetary economy $\mathcal{E} = (\mathcal{E}^c, \mathcal{E}^f, \mathcal{E}^m)$ if

(i) For each agent $i \in I$, $(x^i, \theta^i) \in d^i(p,q,t)$,
(ii) The public plan t satisfies the authority's budget constraints: $\widetilde{t} = \widetilde{r} \circ \left(p \,\square \sum_{i \in I} (x^i - e^i)^-\right) + p \,\square\, \widetilde{g} - \widetilde{V}\theta - w$.
(iii) Commodity and asset markets clear: $\sum_{i \in I} x^i = \sum_{i \in I} e^i$ and $\sum_{i \in I} \theta^i = \theta$.

A monetary equilibrium is said to be with no-transfers if $\widetilde{\widetilde{t}} = 0$.

2.4. **Assumptions.** Before stating the assumptions, let us introduce the following notation: A vector $v = (v(\sigma),\ \sigma \in \Sigma))$ in $\mathbb{R}^{\Sigma}$ is said to be positive, denoted by $v > 0$, if $\forall \sigma \in \Sigma,\ v(\sigma) \geqslant 0,\ v \neq 0$, and it is said to be strictly positive, denoted by $v \gg 0$ if, $\forall \sigma \in \Sigma,\ v(\sigma) > 0$.

The commodity market $\mathcal{E}^c = (X^\bullet, u^\bullet, e^\bullet, g^\bullet, \xi^\bullet)$ is subject to the following assumptions: for each agent $i \in I$,
C1 The consumption set X^i is a closed, convex subset of the positive orthant of $\mathbb{R}^{L\Sigma}$, and $e^i \geqslant 0$.
C2 The utility function is continuous, strictly monotone and strictly quasi-concave.[3]
C3 There exists a consumption plan $x^i \in X^i$ such that $x^i - e^i + g^i \ll 0$.
This is a strong survival assumption in this extended model. After paying his taxes, agent i can still consume.

The financial market $\mathcal{E}^f = (R, \Theta^\bullet, \theta)$ is subject to the following assumptions:
F1 For each agent $i \in I$, the portfolio set Θ^i is equal to $\mathbb{R}^J$.
F2 The return matrix R has full rank. For convenience, we assume $R > 0$[4].
F3 The public portfolio is non-negative, i.e. $\theta \geqslant 0$.

Non-risky asset
NRA The public portfolio is a non-risky portfolio:

$$R\theta \gg 0.$$

Assumptions (C1) to (F2) are the standard assumptions considering an incomplete market framework. We provide hereafter specific assumptions due to the extension of the incomplete market framework that we consider.

Transfers
T1 Transfers t are distributed among agents through given shares $\xi^\bullet$, i.e. each agent receives the amount $\xi^i t$.

Public Revenue
PR Aggregate taxes $g = \sum_{i \in I} g^i$ are strictly positive.

Definition 2.2. A monetary economy $\mathcal{E} = (\mathcal{E}^c, \mathcal{E}^f, \mathcal{E}^m)$ is said to be standard if it satisfies the above assumptions

Initial public debt
M1 The total amount of initial liabilities is positive: $w(0) > 0$.

Finally, we propose in the following two additional assumptions on the financial and money market $\mathcal{E}^f = (R, \Theta^\bullet, \theta)$:

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[3]The utility function u^i is strictly quasi-concave if:
$\forall x^i, \forall\ y^i \in X^i,$ and $\forall \lambda \in [0,1],\ u^i(x^i) > u^i(y^i)$ implies $u^i(\lambda x^i + (1-\lambda) y^i) > u^i(y^i)$

[4]Assuming that the public portfolio is a non-risky portfolio (refer to Assumption (NRA) defined later), there is no loss of generality in considering $R > 0$. One may refer to Lemma 4.3 for the proof of this result.

Public Portfolio

PP The public portfolio consists only in safe bonds[5], i.e. $\theta = \mathbb{1}_J$[6].

Neither (PP) implies (NRA), nor is the converse true. Assumption (PP) is a restrictive assumption, but allows us to precise properties on first period price levels that is lost when one considers only (NRA) (refer to Theorem 2.2, or Theorem 2.3). The results to be proved are the following:

Theorem 2.1. *Let $\mathcal{E}$ be a standard monetary economy. Under assumptions (M1), for every path of rate of interest $r \geqslant 0$ fixed by the authority, there exists a monetary equilibrium with no transfers $(x^\bullet, \theta^\bullet, p, q)$ of $\mathcal{E}$.*

Remark 2.1. In the previous theorem, we may consider a weaker version of Assumption (PR), namely, requiring $g(\sigma) > 0$ in all states $\sigma \in \Sigma$.

Before stating the existence of a monetary equilibrium with transfers, let us introduce the following notations: We endow the dimensional space $\mathbb{R}^n$ with norm 1: for any vector $h \in \mathbb{R}^n$, $\|h\| = \sum_{d=1}^{n} |h_d|$. And we denote by $B(n,k)$ the closed ball on $\mathbb{R}^n$ of radius $k > 0$, with center 0. Let $d \in \mathbb{R}^\Sigma_{++}$. We call $d(\sigma)$ the overall price level at date-event $\sigma \in \Sigma$ when d is defined by:

$$d(0) = \|p(0)\| + \|\widetilde{q}\| \quad \text{and} \quad d(s) = \|p(s)\|\,,\ s \in S.$$

Theorem 2.2. *Let $\mathcal{E}$ be a standard monetary economy. Under assumption (PP), for every path of rate of interest $r \geqslant 0$ fixed by the authority, there is $d^\star \in \mathbb{R}^\Sigma_+$, such that, for every $d \geqslant d^\star$, $d \gg 0$, there exists a monetary equilibrium with transfers $(x^\bullet, \theta^\bullet, p, q, t)$ of $\mathcal{E}$ with $\|p(0)\| + \|\widetilde{q}\| = d(0)$ and $\|p(s)\| = d(s)$ for every date-event $s \in S$ of the second period.*

We also prove that by choosing a higher price level ($c^\star \geqslant d^\star$), we show that, at equilibrium, transfers are positive.

Theorem 2.3. *Let $\mathcal{E}$ be a standard monetary economy. Under assumption (PP), for every path of rate of interest $r \geqslant 0$ fixed by the authority, there is $c^\star \in \mathbb{R}^\Sigma$, $c^\star \gg 0$ such that, for every $c \geqslant c^\star$ there exists a monetary equilibrium with positive transfers $(x^\bullet, \theta^\bullet, p, q, t)$ of $\mathcal{E}$ with $\|p(0)\| + \|\widetilde{q}\| = c(0)$ and $\|p(s)\| = c(s)$ for every date-event $s \in S$ of the second period.*

If we drop assumption (PP), we get the following corollaries of Theorem 2.2 and Theorem 2.3:

Corollary 2.1. *Let $\mathcal{E}$ be a standard monetary economy. For every path of rate of interest $r \geqslant 0$ fixed by the authority, there is $e^\star \in \mathbb{R}^S$, $e^\star \gg 0$ such that, for every $e \geqslant e^\star$ there exists a monetary equilibrium with transfers $(x^\bullet, \theta^\bullet, p, q, t)$ of $\mathcal{E}$ with $\|p(s)\| = e(s)$ for every date-event $s \in S$ of the second period.*

Corollary 2.2. *Let $\mathcal{E}$ be a standard monetary economy. For every path of rate of interest $r \geqslant 0$ fixed by the authority, there is $e^\star \in \mathbb{R}^S$, $e^\star \gg 0$ such that, for every $e \geqslant e^\star$ there exists a monetary equilibrium with positive transfers $(x^\bullet, \theta^\bullet, p, q, t)$ of $\mathcal{E}$ with $\|p(s)\| = e(s)$ for every date-event $s \in S$ of the second period.*

[5] Given a dimensional space $\mathbb{R}^n$, we denote $\mathbb{1}_n$ the vector in $\mathbb{R}^n$ with all components equal to one.

[6] Notice that it is equivalent to consider any public portfolio $\theta \gg 0$ in assumption (PP), given an adequate corresponding choice of the return matrix R.

It is important to notice here that Corollary 2.1 is a consequence of Theorem 2.2, while Corollary 2.2 is a consequence of Theorem 2.3. A more reasonable assumption on the public portfolio (in particular (NRA)) prevents us to get precise information on first period price levels.

The next section is devoted for the proof of the case of no transfers. Section 4 will study the case of transfers.

3. Existence of Monetary Equilibrium with No Transfers

The proof follows the usual scheme considering an incomplete market setting, The general method of proof is the usual incomplete market arguments as in Duffie [7], Florenzano [8], Werner [16], among others.

We begin by identifying compact, convex sets for consumption sets and portfolio sets. Adapting the work of Bloize–Drèze–Polemarchakis [1] in an incomplete market framework, we modify budget sets by introducing an index $\mu \in \mathbb{R}^{\Sigma}_{+}$ of the reciprocal of the overall price level leading to *well-behaved* correspondences. Applying Kakutani's fixed point theorem in $(p, q, x^\bullet, \theta^\bullet, \mu)$ leads to the existence of an *abstract equilibrium*, an equilibrium concept which is defined below (Definition 3.2). The last step of the proof consists in showing that under (PR)–(NRA)–(M1), the introduced index is strictly positive, and the abstract equilibrium is achieved as an monetary equilibrium with no transfers.

3.1. **Truncating the economy.** Given assumptions (C2) and (F2), we may restrict ourselves to positive commodity and asset prices. We consider the following compact, convex set for commodity and asset price vectors:

$$\Pi = \left\{(p, q) \in \mathbb{R}^{L\Sigma}_{+} \times \mathbb{R}^{J}_{+} : \quad \|p(0)\| + \|\widetilde{q}\| = 1 \quad \text{and} \quad \|p(s)\| = 1, \ \forall s \in S\right\}$$

We provide hereafter the definition of a truncated monetary economy. The following lemma establishes that in order to prove Theorem 2.1, we can suppose without any loss of generality that commodity and financial sets are compact.

Definition 3.1. If $\mathcal{E}^c = (X^\bullet, u^\bullet, e^\bullet, g^\bullet, \xi^\bullet)$ is a commodity market, and $\mathcal{E}^f = (R, \Theta^\bullet, \theta)$ is a financial market, then for any $k > 0$, we let $\mathcal{E}^c_k$ and $\mathcal{E}^f_k$ defined by

$$\mathcal{E}^c_k = (X^\bullet_k, u^\bullet_k, e^\bullet, g^\bullet, \xi^\bullet) \qquad \mathcal{E}^f_k = (R, \Theta^\bullet_k, \theta)$$

where, for each $i \in I$, $X^i_k = X^i \cap B(\Sigma \times L, k)$. We set u^i_k as the restriction of u^i to X^i_k, $\Theta^i_k = \Theta^i \cap B(J, k)$.

Let $\widehat{X}$ the set of attainable commodity allocations, i.e.

$$\widehat{X} := \left\{ x \in \prod_{i \in I} X^i \ : \ \sum_{i \in I} (x^i - e^i) = 0 \right\} \tag{3.1}$$

For every $i \in I$, $\widehat{X}^i$ is the projection of $\widehat{X}$ on X^i.
The sets $(\widehat{\Theta}^i, \ i \in I)$ of attainable portfolios are defined as follows:

$$\widehat{\Theta}^i := \{\theta^i \in \Theta^i \ : \ \exists (p, q) \in \Pi, \ \exists x^i \in \widehat{X}^i, \ p \,\Box\, (x^i - e^i + \widetilde{g}^i) + \widetilde{r} \circ (p \,\Box\, (x^i - e^i)^-) = \widetilde{V}\theta^i + w^i\}.$$

We set $\widehat{\Theta} = \prod_{i \in I} \widehat{\Theta}^i$.

Lemma 3.1. *Let $\mathcal{E}^c$ commodity market and $\mathcal{E}^f$ a financial market. Then*

(a) *There exists $k > 0$ such that*

$$\forall i \in I,\ \widehat{X}^i \subset \operatorname{int} B(\Sigma \times L, k), \quad \widehat{\Theta}^i \subset \operatorname{int} B(J, k) \tag{3.2}$$

(b) *If $k > 0$ is sufficiently large such that 3.2 is satisfied, then for each money market $\mathcal{E}^m$, any monetary equilibrium of the truncated economy $(\mathcal{E}^c_k, \mathcal{E}^f_k, \mathcal{E}^m)$ is a monetary equilibrium of the initial economy $(\mathcal{E}^c, \mathcal{E}^f, \mathcal{E}^m)$.*

The proof of Lemma 3.1 is referred to the appendix (A1). We can now fix $k > 0$. Following Lemma 3.1, we can suppose without any loss of generality that for each $i \in I$, the sets X^i and Θ^i are compact. Let us introduce the following notation: Consider a set $V \subset \mathbb{R}^n$. We recall that the convex hull of V, denoted by $\operatorname{co}(V) \subset \mathbb{R}^n$ is the smallest convex set containing V.

For convenience of notation, we set $v(x) \in \mathbb{R}^\Sigma_+$ such that:

$$v(x, \sigma) = \sum_{i \in I} \left(x^i(\sigma) - e^i(\sigma)\right)^-, \quad \forall \sigma \in \Sigma.$$

Since $\mu \in \mathbb{R}^\Sigma$ is also a variable involved in our fixed-point argument, we propose the following convex set:

$$M := \operatorname{co}\left\{ \mu \in \mathbb{R}^\Sigma_+ \;\middle|\; \begin{array}{lcl} \exists x \in \prod_{i \in I} X^i,\ \exists (p, q) \in \Pi, & & \\ \mu(0) w(0) & = & \widetilde{r}(0) p(0) \cdot v(x, 0) + p(0) \cdot \widetilde{g}(0) + \widetilde{q}\theta \\ \mu(s)(\widetilde{R}(s)\theta) & = & \widetilde{r}(s) p(s) \cdot v(x, s) + p(s) \cdot \widetilde{g}(s) \end{array} \right\}$$

Claim 3.1. *The set $M \subset \mathbb{R}^\Sigma$ is compact.*

Proof. The compactness of M follows from assumptions (M1)–(F2)–(NRA) and from the compactness of X and Π. Let (x^i_ν, p_ν, q_ν) be a sequence in $X^i \times \Pi$ and (μ_ν) a sequence in M. Then, for each $\nu \in \mathbb{N}$, we have

$$\begin{array}{lcl} \mu_\nu(0) w(0) & = & \widetilde{r}(0) \left(p(0) \cdot \sum_{i \in I} \left(x^i_\nu(0) - e^i(0)\right)^- \right) + p_\nu(0) \cdot \widetilde{g}(0) + q_\nu \theta^i \\ \mu_\nu(s)(\widetilde{R}(s)\theta) & = & \widetilde{r}(s) \left(p_\nu(s) \cdot \sum_{i \in I} \left(x^i_\nu(s) - e^i(s)\right)^- \right) + p_\nu(s) \cdot \widetilde{g}(s) \end{array}$$

According to assumption (M1), $w(0) > 0$, and according to (NRA), $(R(s)\theta) > 0,\ \forall s \in S$, thus we can suppose that μ_ν converges μ. Note that $\mu \in M$. □

3.2. **Modifying budget sets.** For each $(p, q, \mu) \in \Pi \times M$, we define $\beta^i(p, q, \mu)$ the following modified budget set of agent $i \in I$ defined by the set of actions $(x^i, \theta^i) \in X^i \times \Theta^i$ such that:

$$\begin{array}{lcl} p(0) \cdot (x^i(0) - e^i(0) + \widetilde{g}^i(0)) + \widetilde{r}(0) \left(p(0) \cdot (x^i(0) - e^i(0))^-\right) & \leqslant & -\widetilde{q}\theta^i + \mu(0) w^i(0) \\ p(s) \cdot (x^i(s) - e^i(s) + \widetilde{g}^i(s)) + \widetilde{r}(s) \left(p(s) \cdot (x^i(s) - e^i(s))^-\right) & \leqslant & \mu(s)(\widetilde{R}\theta^i)(s),\ \forall s \in S \end{array}$$

The associated demand correspondence is defined by:

$$\delta^i(p, q, \mu) := \left\{(x^i, \theta^i) \in \beta^i(p, q, \mu),\ \beta^i(p, q, \mu) \cap [P^i(x^i) \times \Theta^i] = \emptyset\right\}$$

Let us begin by introducing the notion of abstract equilibrium

Definition 3.2. An abstract equilibrium consists in a collection $(\overline{p},\overline{q},\overline{x}^\bullet,\overline{\theta}^\bullet)$ and an index of the reciprocal overall price levels $\overline{\mu}\in\mathbb{R}_+^\Sigma$, such that:

(i) For every agent $i\in I$, $(\overline{x}^i,\overline{\theta}^i)\in\delta^i(\overline{p},\overline{q},\overline{\mu})$
(ii) The authority's constraints are satisfied

$$\begin{aligned}\widetilde{r}(0)\overline{p}(0)\cdot\sum_{i\in I}(\overline{x}^i(0)-e^i(0))^-+\overline{p}(0)\cdot\widetilde{g}(0) &= \mu(0)w(0)-\widetilde{q}\cdot\theta\\ \widetilde{r}(s)\overline{p}(s)\cdot\sum_{i\in I}(\overline{x}^i(s)-e^i(s))^-+\overline{p}(s)\cdot\widetilde{g}(s) &= \mu(s)(\widetilde{R}\theta)(s),\quad \forall s\in S\end{aligned}$$

(iii) Markets clear: $\sum_{i\in I}\overline{x}^i=\sum_{i\in I}e^i$ and $\sum_{i\in I}\theta^i=\theta$,

Remark 3.1. If $\mu\gg 0$, and $(p,q,x^\bullet,\theta^\bullet,\mu)$ is an abstract equilibrium of $\mathcal{E}$, then $(p',q',x^\bullet,\theta^\bullet)$ is a monetary equilibrium given that

$$p'(\sigma)=\frac{p(\sigma)}{\mu(\sigma)},\forall\sigma\in\Sigma\qquad\text{and}\qquad q'=\frac{q}{\mu(0)}.$$

For each $i\in I$, for each $(p,q,\mu)\in\Pi\times M$, we denote by β'^i the interior of the set β^i on $\Pi\times M$.

Lemma 3.2. *For every agent $i\in I$, the correspondence β'^i has non-empty values on $\Pi\times M$.*

The proof of this lemma is referred in appendix (A2). We have the following properties for the modified correspondences:

Claim 3.2. *For each agent $i\in I$,*

(i) *the correspondence β^i is u.s.c. on $\Pi\times M$ with compact convex values.*
(ii) *the correspondence β^i is l.s.c. on $\Pi\times M$.*
(iii) *the demand correspondence δ^i is u.s.c. on $\Pi\times M$ with non-empty compact, convex values.*

The proof of this claim is given in appendix (A3).

3.3. **Applying Kakutani's fixed point theorem.** Let us define the correspondence

$$F\ :\ \Pi\times\prod_{i\in I}X^i\times\prod_{i\in I}\Theta^i\times M\longrightarrow\Pi\times\prod_{i\in I}X^i\times\prod_{i\in I}\Theta^i\times M$$

such that:

$$F(p,q,x^\bullet,\theta^\bullet,\mu)=\Phi(x^\bullet,\theta^\bullet)\times\prod_{i\in I}\ \delta^i(p,q,\mu)\times\Gamma(x^\bullet,p,q)$$

where

$$\Phi(x^\bullet,\theta^\bullet)=\left\{(p,q)\in\Pi\colon\forall(p',\widetilde{q}')\in\Pi,(p-p')\cdot\sum_{i\in I}\left(\overline{x}^i-e^i\right)+(\widetilde{q}-\widetilde{q}')\cdot\left(\sum_{i\in I}\theta^i-\theta\right)\geqslant 0)\right\}$$

and

$$\Gamma(x^\bullet,p,q)=\left\{\mu\in M\ :\ \begin{array}{lll}\mu(0)w(0) & = & \widetilde{r}(0)p(0)\cdot\sum_{i\in I}\left(x^i(0)-e^i(0)\right)^-+p(0)\cdot\widetilde{g}(0)+\widetilde{q}\theta\\ \mu(s)(\widetilde{R}\theta)(s) & = & \widetilde{r}(s)p(s)\cdot\sum_{i\in I}\left(x^i(s)-e^i(s)\right)^-+p(s)\cdot\widetilde{g}(s)\end{array}\right\}$$

Correspondence F is u.s.c. with non-empty convex compact values. Applying Kakutani's fixed point theorem, there exists $(\overline{p}, \overline{q}, \overline{x}^\bullet, \overline{\theta}^\bullet, \overline{\mu}) \in \Pi \times \prod_{i\in I} X^i \times \prod_{i\in I} \Theta^i \times M$ such that:

$$\forall i \in I, (\overline{x}^i, \overline{\theta}^i) \in \delta^i(\overline{p}, \overline{q}, \overline{\mu}), \tag{3.3}$$

$$\forall (p,q) \in \Pi,\ (p - \overline{p}) \cdot \sum_{i\in I} (\overline{x}^i - e^i) + (\widetilde{q} - \widetilde{\overline{q}}) \cdot (\sum_{i\in I} \overline{\theta}^i - \theta) \leqslant 0, \tag{3.4}$$

$$\overline{\mu}(0) w(0) \ = \ \widetilde{r}(0) \left(\overline{p}(0) \cdot \sum_{i\in I} \left(\overline{x}^i(0) - e^i(0) \right)^- \right) + \overline{p}(0) \cdot \widetilde{g}(0) + \widetilde{\overline{q}} \theta \tag{3.5}$$

$$\overline{\mu}(s)(\widetilde{R}\theta)(s) \ = \ \widetilde{r}(s) \left(\overline{p}(s) \cdot \sum_{i\in I} \left(\overline{x}^i(s) - e^i(s) \right)^- \right) + \overline{p}(s) \cdot \widetilde{g}(s) \tag{3.6}$$

We will now show that the obtained fixed point is an abstract equilibrium. In order to do this, we need only to prove that commodity and asset markets clear. This follows from Claims 3.3 to 3.8.

For convenience, we introduce the following sets $\Delta(\sigma)$, $\forall \sigma \in \Sigma$:

$\Delta(0) := \{(p(0), q) \in \mathbb{R}^L_+ \times \mathbb{R}^J_+ :\ \|p(0)\| + \|q\| = 1\}$ $\quad \Delta(s) := \{(p(s), s \in S) \in \mathbb{R}^{SL}_+ :\ \|p(s)\| = 1\}$

Note that $\Delta(\sigma)$ for all date-event $\sigma \in \Sigma$ are simply projections of Π. For a given set $U \in \mathbb{R}^n$, we denote by U° the negative polar cone of U, i.e. the cone of vectors $\eta \in \mathbb{R}^n$ such that $\eta \cdot u \leqslant 0$, for every $u \in U$.

Claim 3.3. *At the first period, we have the following property:*

$$\sum_{i\in I} \overline{x}^i(0) \leqslant \sum_{i\in I} e^i(0), \qquad \text{and} \qquad \sum_{i\in I} \bar{\theta}^i \leqslant \theta$$

Proof. Taking $p(s) = \overline{p}(s)$ for every date-event $s \in S$ in fixed point property (3.4), one has, for all prices $(p(0), q) \in \Delta(0)$:

$$p(0) \cdot \sum_{i\in I} (\overline{x}^i(0) - e^i(0)) + q \cdot (\sum_{i\in I} \bar{\theta}^i - \theta) \leqslant \overline{p}(0) \cdot \sum_{i\in I} (\overline{x}^i(0) - e^i(0)) + \widetilde{\overline{q}} \cdot (\sum_{i\in I} \bar{\theta}^i - \theta) \tag{3.7}$$

Moreover, fixed point property (3.3) states that $(\overline{x}^i, \overline{\theta}^i) \in \beta^i(\overline{p}, \overline{q}, \overline{\mu})$. Summing first period constraints among all agents and recalling fixed point property (3.5), we get:

$$\overline{p}(0) \cdot \sum_{i\in I} \left(\overline{x}^i(0) - e^i(0) \right) + \widetilde{\overline{q}} \cdot (\sum_{i\in I} \bar{\theta}^i - \theta) \leqslant 0 \tag{3.8}$$

According to inequalities (3.7) and (3.8), one has:

$$p(0) \cdot \sum_{i\in I} \left(\overline{x}^i(0) - e^i(0) \right) + q \cdot (\sum_{i\in I} \bar{\theta}^i - \theta) \leqslant 0, \quad \forall (p(0), q) \in \Delta(0).$$

Thus $\left[\sum_{i\in I} \left(\overline{x}^i(0) - e^i(0) \right); (\sum_{i\in I} \bar{\theta}^i - \theta)\right] \in [\Delta(0)]^\circ = \mathbb{R}^{LJ}_-$, i.e.

$$\sum_{i\in I} \overline{x}^i(0) \leqslant \sum_{i\in I} e^i(0) \quad \text{and} \quad \sum_{i\in I} \bar{\theta}^i \leqslant \theta.$$

$\square$

Claim 3.4. *At the second period, commodity markets satisfy:*

$$\sum_{i\in I}\overline{x}^i(s)\leqslant\sum_{i\in I}e^i(s),\quad \forall s\in S$$

Proof. Consider a date-event $s\in S$. According to fixed point property (3.4), by choosing $p(\sigma)=\overline{p}(\sigma)$ for all date-event $\sigma\in\Sigma\setminus\{s\}$ and $q=\overline{q}$, one has:

$$p(s)\cdot\sum_{i\in I}\left(\overline{x}^i(s)-e^i(s)\right)\leqslant\overline{p}(s)\cdot\sum_{i\in I}\left(\overline{x}^i(s)-e^i(s)\right),\ \forall\ p(s)\in\Delta(s).$$

Moreover, fixed point property (3.3) states that $(\overline{x}^i,\overline{\theta}^i)\in\beta^i(\overline{p},\overline{q},\overline{\mu})$. Summing second period constraints among all agents and recalling fixed point property (3.6), we get:

$$\overline{p}(s)\cdot\sum_{i\in I}\left(\overline{x}^i(s)-e^i(s)\right)\leqslant\mu(s)R(s)(\sum_{i\in I}\bar{\theta}^i-\theta)\tag{3.9}$$

Since $\mu\geqslant 0$, $R>0$ (assumption (F2)), and $(\sum_{i\in I}\bar{\theta}^i-\theta)\leqslant 0$ (Claim 3.3), focusing on date-event s, one has:

$$\overline{p}(s)\cdot\sum_{i\in I}\left(\overline{x}^i(s)-e^i(s)\right)\leqslant 0,\quad\forall p(s)\in\Delta(s)\tag{3.10}$$

This means that $\left(\overline{x}^i(s)-e^i(s)\right)\in[\Delta(s)]^\circ=\mathbb{R}^L_-$, i.e. $\sum_{i\in I}\overline{x}^i(s)\leqslant\sum_{i\in I}e^i(s)$. □

Claim 3.5. *Budget constraints of all agents $i\in I$ are saturated:*

$$\overline{p}(0)\cdot(\overline{x}^i(0)-e^i(0)+\widetilde{g}^i(0))+\widetilde{r}(0)\left(\overline{p}(0)\cdot(\overline{x}^i(0)-e^i(0))^-\right)+\widetilde{\overline{q}}\theta^i = \mu(0)w^i(0)\tag{3.11}$$

$$\overline{p}(s)\cdot(\overline{x}^i(s)-e^i(s)+\widetilde{g}^i(s))+\widetilde{r}(s)\left(\overline{p}(s)\cdot(\overline{x}^i(s)-e^i(s))^-\right) = \mu(s)(\widetilde{R}\theta^i)(s)\tag{3.12}$$

Proof. We will only consider the case where $s\in S$ (for $s=0$, the proof is similar). Assume on the contrary that (3.12) does not hold, i.e. there exists $i\in I$, and a date-event $s\in S$ where the equality is not true, i.e.

$$\overline{p}(s)\cdot(\overline{x}^i(s)-e^i(s)+\widetilde{g}^i(s))+\widetilde{r}(s)\left(\overline{p}(s)\cdot(\overline{x}^i(s)-e^i(s))^-\right)<\mu(s)(\widetilde{R}\theta^i)(s).$$

In view of Claims (3.3)–(3.4) and of our choice of k, there exists some consumption plan $x^i\in B(\Sigma L,k)$ satisfying $x^i>\overline{x}^i$ and

$$\overline{p}(s)\cdot(x^i(s)-e^i(s)+\widetilde{g}^i(s))+\widetilde{r}(s)(\overline{p}(s)\cdot(x^i(s)-e^i(s))^-)\leqslant(\widetilde{V}\overline{\theta}^i)(s)+w^i(s).$$

Thus, $(x^i,\overline{\theta}^i)\in\beta^i(\overline{p},\overline{q},\overline{\mu})$. Following assumption (C1), $u^i(x^i)>u^i(\overline{x}^i)$, which yields a contradiction to the fixed point property 3.3. □

Claim 3.6. *Commodity prices are strictly positive, i.e.* $\overline{p}\gg 0$.

Proof. Indeed, if not, there exists a date-event $\sigma\in\Sigma$ and a good $\ell\in L$ such that $\overline{p}(\sigma,\ell)=0$. Let us consider an agent $i\in I$. In view of our choice of k and Claims (3.3)–(3.4), we can find some consumption plan $x^i\in B(\Sigma L,k)$ such that $x^i(\sigma)>\overline{x}^i(\sigma)$ and $(x^i,\overline{\theta}^i)\in\beta^i(\overline{p},\overline{q},\overline{\mu})$. Following assumption (C1), $u^i(x^i)>u^i(\overline{x}^i)$, which yields a contradiction to the fixed point property (3.3). □

Claim 3.7. *At first period, we have the following property:*

$$\sum_{i\in I}\overline{x}^i(0)=\sum_{i\in I}e^i(0)\qquad\textit{and}\qquad\widetilde{\overline{q}}\cdot\left(\sum_{i\in I}\overline{\theta}^i-\theta\right)=0$$

Proof. Indeed, according to Claim (3.5), summing among all agents equalities (3.11), one has

$$\overline{p}(0)\cdot\sum_{i\in I}\left(\overline{x}^i(0)-e^i(0)\right)+\widetilde{\overline{q}}\cdot\left(\sum_{i\in I}\theta^i-\theta\right)=0$$

Since $(\overline{p}(0),\widetilde{\overline{q}})\in\mathbb{R}_+^{LJ}$, $\left(\overline{x}^i(0)-e^i(0)\right)\leqslant 0$, and $\left(\sum_{i\in I}\theta^i-\theta\right)\leqslant 0$, one has:

$$\overline{p}(0)\cdot\sum_{i\in I}\left(\overline{x}^i(0)-e^i(0)\right)\;=\;0\quad\text{and}\quad\widetilde{\overline{q}}\cdot\left(\sum_{i\in I}\theta^i-\theta\right)=0 \tag{3.13}$$

Since $\overline{p}(0)\gg 0$, we get $\sum_{i\in I}\overline{x}^i(0)=\sum_{i\in I}e^i(0)$.

□

Claim 3.8. *Asset markets clear and second period commodity markets clear:*

$$\sum_{i\in I}\overline{\theta}^i=\theta\quad\textit{and}\quad\sum_{i\in I}\overline{x}^i(s)=\sum_{i\in I}e^i(s)$$

Proof. Indeed, according to the previous claim, one has $\widetilde{\overline{q}}\cdot\left(\sum_{i\in I}\theta^i-\theta\right)=0$. Moreover, referring to claim (3.4), inequality (3.8) tells us that $R(s)\left(\sum_{i\in I}\overline{\theta}^i-\theta\right)\leqslant 0$. Thus, by setting $\overline{\overline{\theta}}=-\left(\sum_{i\in I}\overline{\theta}^i-\theta\right)$ one has $\widetilde{V}\overline{\overline{\theta}}\geqslant 0$. Assume that there exists a date-event $\sigma\in\Sigma$ such that

$$\widetilde{V}(\sigma)\overline{\overline{\theta}}>0. \tag{3.14}$$

Let an agent $i\in I$. According to Claim 3.5, budget constraints of agents are saturated at fixed point. Consider an agent $i\in I$, One has,

$$\overline{p}\,\square\,(\overline{x}^i-e^i+\widetilde{g}^i)+\widetilde{r}\circ(\overline{p}\,\square\,(\overline{x}^i-e^i)^-)=\widetilde{V}\overline{\theta}^i$$

Hence, for $\lambda>0$, one has $(\overline{x}^i,\overline{\theta}^i+\lambda\overline{\overline{\theta}})\in\beta^i(\overline{p},\widetilde{\overline{q}},\overline{\mu})$. Moreover, recalling inequality (3.14), one has at date-event $\sigma\in\Sigma$,

$$\overline{p}(\sigma)\cdot(\overline{x}^i(\sigma)-e^i(\sigma)+\widetilde{g}^i(\sigma))+\widetilde{r}(\sigma)(\overline{p}(\sigma)\cdot(\overline{x}^i(\sigma)-e^i(\sigma))^-)<\widetilde{V}(\sigma)(\overline{\theta}^i+\lambda\overline{\overline{\theta}})$$

One can find an allocation $x^i\in B(\Sigma L,k)$ such that $(x^i,(\overline{\theta}^i+\lambda\overline{\overline{\theta}}))\in\delta^i(\overline{p},\widetilde{\overline{q}},\overline{\mu})$ which contradicts fixed point property (3.3). As for the clearance of second period commodity markets, it is straightforward by summing among all agents second period's saturated budget constraints (3.12) and the fact that $p(s)\gg 0$ for all date-event $s\in S$ (Claim 3.6). □

We have proved that there exists an abstract equilibrium. In order for the abstract equilibrium to be achieved as an equilibrium, we need to show that at every abstract equilibrium, $\overline{\mu}\gg 0$. The following lemma shows under what condition this is satisfied.

Lemma 3.3. *Under (M1)–(NRA)–(PR), at every abstract equilibrium $\overline{\mu}\gg 0$, and the abstract equilibrium is achieved as an equilibrium.*

Proof. Assume that (PR) is satisfied. Referring to 3.5, one has:

$$\overline{\mu}(0)w(0)\geqslant\overline{p}(0)\widetilde{g}(0)+\widetilde{\overline{q}}\cdot\theta.$$

Since $\overline{p}(0) \gg 0$, the fact that $\widetilde{g}(0) \gg 0$ and recalling assumption (M1), $w(0) > 0$, one has $\overline{\mu}(0) > 0$.
Moreover, referring to 3.6, one has at a state $s \in S$ of the second period:

$$\overline{\mu}(s)(R\theta)(s) \geqslant \overline{p}(s)\widetilde{g}(s) > 0.$$

Since θ is a non-risky portfolio, one has $\overline{\mu}(s) > 0$, $\forall s \in S$.

□

Hence, we have proved the existence of an abstract equilibrium with strictly positive index of the reciprocal of the overall price level. Following remark 3.1, we get that there exists a monetary equilibrium $(\overline{p}', \overline{q'}, \overline{x}^\bullet, \overline{\theta}^\bullet)$ with no transfers, where

$$\overline{p}'(\sigma) = \frac{\overline{p}(\sigma)}{\mu(\sigma)},\ \forall \sigma \in \Sigma \qquad \text{and} \qquad \overline{q'} = \frac{\overline{q}}{\overline{\mu}(0)}.$$

4. Existence of Monetary Equilibrium with Transfers

Let $\mathcal{E}$ be a standard economy satisfying assumption (PP) and (NRA). Let $d \in \mathbb{R}^{\Sigma}_{++}$. Given assumptions (C1) and (F2), we may restrict ourselves to positive commodity and asset prices. We consider the following convex, compact set for commodity and asset price vectors:

$$\Pi_d = \left\{(p,q) \in \mathbb{R}^{L\Sigma}_{+} \times \mathbb{R}^{J}_{+} :\ \|p(0)\| + \|\widetilde{q}\| = d(0) \quad \text{and} \quad \|p(s)\| = d(s),\ \forall s \in S\right\}$$

4.1. **Truncating the economy.** We provide hereafter the definition of a truncated monetary economy. The following lemma establishes that in order to prove Theorem 2.2 and Theorem 2.3, we can suppose without any loss of generality that commodity and financial sets are compact. Transfers are also considered to belong to an adequate compact, convex set.

Definition 4.1. If $\mathcal{E}^c = (X^\bullet, u^\bullet, e^\bullet, g^\bullet, \xi^\bullet)$ is a commodity market, and $\mathcal{E}^f = (R, \Theta^\bullet, \theta)$ is a financial market, then for any $h > 0$, we let $\mathcal{E}^c_h$ and $\mathcal{E}^f_h$ defined by

$$\mathcal{E}^c_h = (X^\bullet_h, u^\bullet_h, e^\bullet, g^\bullet, \xi^\bullet) \qquad \mathcal{E}^f_h = (R, \Theta^\bullet_h, \theta)$$

where, for each $i \in I$, $X^i_h = X^i \cap B(\Sigma \times L, h)$. We set u^i_h as the restriction of u^i to X^i_k, $\Theta^i_h = \Theta^i \cap B(J,h)$, and $T_h = B(\Sigma, h)$.

Let $\widehat{X}$ is the set of attainable commodity allocations defined in (3.1). We set $\widehat{T}$ the set of attainable transfers, i.e.

$$\widehat{T} = \left\{t \in \mathbb{R}^{\Sigma}\ :\ \exists x^\bullet \in \widehat{X},\ \exists (p,\widetilde{q}) \in \Pi_d,\ \widetilde{t} = \widetilde{r} \circ \left(p \square \sum_{i \in I} \left(x^i - e^i\right)^-\right) + p \square \widetilde{g} - \widetilde{V}\theta - w\right\}$$

Let the set of attainable portfolios be defined by:

$$\widehat{\Theta}^i = \{\theta^i \in \Theta^i\ :\ \exists (p,\widetilde{q}) \in \Pi_d,\ \exists x^i \in \widehat{X}^i,\ \exists t \in \widehat{T},\ p \square (x^i - e^i + \widetilde{g}^i) + \widetilde{r} \circ (p \square (x^i - e^i)^-) = \widetilde{V}\theta^i + w^i + \xi^i \widetilde{t}\}$$

We set $\widehat{\Theta} = \prod_{i \in I} \widehat{\Theta}^i$.

Notice here that the attainable portfolio sets depend now on transfers $t \in \widehat{T}$, thus compactness of $\widehat{\Theta}$ will crucially depend on the compactness of $\widehat{T}$.

Lemma 4.1. *Let $\mathcal{E}^c$ commodity market and $\mathcal{E}^f$ a financial market. Then*

(a) *There exists $h > 0$ such that*

$$\forall i \in I,\ \widehat{X}^i \subset \operatorname{int} B(\Sigma \times L, h), \quad \widehat{\Theta}^i \subset \operatorname{int} B(J, h) \quad \textit{and} \quad \widehat{T} \subset B(\Sigma, h) \tag{4.1}$$

(b) *If $h > 0$ is sufficiently large such that 4.1 is satisfied, then for each money market $\mathcal{E}^m$, any monetary equilibrium of the truncated economy $(\mathcal{E}^c_h, \mathcal{E}^f_h, \mathcal{E}^m)$ is a monetary equilibrium of the initial economy $(\mathcal{E}^c, \mathcal{E}^f, \mathcal{E}^m)$.*

The proof of this lemma is referred in appendix (A4). We can now fix $h > 0$. Following Lemma 4.1, we can suppose without any loss of generality that for each $i \in I$, the sets X^i and Θ^i are compact. For convenience, we set $T = T_h$.

4.2. **Modifying budget sets.** We will begin by defining the adequate price level that one should consider. Let an agent $i \in I$. According to assumption (C3), for every agent $i \in I$, there exists a consumption plan $x^i \in X^i$ such that $x^i - e^i + g^i \ll 0$. Thus, for every date-event $\sigma \in \Sigma$, there exists $\chi^i(\sigma) > 0$ such that $x^i - e^i + g^i \ll -\chi^i 1\!\!1_\Sigma$.

We will need the following notation: $\underline{g}(\sigma) = \operatorname{Inf}\{g(\sigma, l),\ l \in L\}$. Let

$$d^\star(0) \quad := \quad \operatorname{Max}\left\{ 0 : \frac{\left(w(0) - \min\limits_{i \in I}\{\frac{w^i(0)}{\xi^i}\} \right)}{\min\{1, \underline{g}(0)\}} (1 + r(0)) \right\} \tag{4.2}$$

$$d^\star(s) \quad := \quad \frac{R(s)\theta}{\underline{g}(s) + \min\limits_{i \in I}\{\frac{\chi^i(s)}{\xi^i}\}}, \quad \forall s \in S \tag{4.3}$$

Consider price levels $d \gg 0$, $d \geqslant d^\star$, and let the mapping γ, from T into T,[7] be defined by:

$$\gamma(t) := \gamma((t, \sigma),\ \sigma \in \Sigma) \quad \text{where} \quad \gamma(t, \sigma) = \max\{\widetilde{t}(\sigma), \widetilde{K}(\sigma)\}, \tag{4.4}$$

given that

$$K(0) := d(0)\operatorname{Min}\{1, \underline{g}(0)\} - w(0)(1 + r(0)), \quad K(s) := d(s)\underline{g}(s) - R(s)\theta, \quad \forall s \in S. \tag{4.5}$$

For each $i \in I$, for each $(p, q, t) \in \Pi_d \times T$, we define the following correspondances:

$B^i_d(p, q, \gamma(t)) := \{(x^i, \theta^i) \in X^i \times \Theta^i \ :\ p \square (x^i - e^i + \widetilde{g}^i) + \widetilde{r} \circ \left(p \square (x^i - e^i)^-\right) \leqslant \widetilde{V}\theta^i + w^i + \xi^i \gamma(t)\}$.

$\beta^i_d(p, q, \gamma(t)) := \{(x^i, \theta^i) \in X^i \times \Theta^i \ :\ p \square (x^i - e^i + \widetilde{g}^i) + \widetilde{r} \circ \left(p \square (x^i - e^i)^-\right) \ll \widetilde{V}\theta^i + w^i + \xi^i \gamma(t)\}$.

$d^i_d(p, q, \gamma(t)) := \{(x^i, \theta^i) \in X^i \times \Theta^i \ :\ (x^i, \theta^i) \in B^i_d(p, q, \gamma(t)) \quad \text{and} \quad [P^i(x^i) \times \Theta^i] \cap B^i_d(p, q, \gamma(t)) := \emptyset\}$

As we shall see in the following lemma, the constructed price level $d^\star \in \mathbb{R}^\Sigma_+$ leads to the non-emptyness of β^i_d on $\Pi_d \times T$.

Lemma 4.2. *For every agent $i \in I$, the correspondence β^i_d has non-empty values on $\Pi_d \times T$.*

For the proof of Lemma 4.2, refer to appendix (A5). We have the following properties for the modified correspondences:

Claim 4.1. *For each agent $i \in I$,*

(i) *the correspondence B^i_d is u.s.c. on $\Pi_d \times T$ with compact convex values.*

[7] Recall that $T = T_h$, and given the definition of K, one can always choose $h > 0$ big enough in order for γ to be defined from T into T.

(ii) *the correspondence* B_d^i *is l.s.c. on* $\Pi_d \times T$.
(iii) *the demand correspondence is u.s.c. on* $\Pi_d \times T$ *with non-empty compact, convex values.*

The constructed correspondences are well behaved. We may now apply a fixed point theorem that will lead us to the existence of a monetary equilibrium.

4.3. **Applying Kakutani's fixed point theorem.** Let us define the correspondence

$$F_d \;:\; \Pi_d \times \prod_{i\in I} X^i \times \prod_{i\in I} \Theta^i \times T \longrightarrow \Pi_d \times \prod_{i\in I} X^i \times \prod_{i\in I} \Theta^i \times T$$

such that:

$$F_d(p,q,x^\bullet,\theta^\bullet,t) = \Phi_d(x^\bullet,\theta^\bullet) \times \prod_{i\in I} d_d^i(p,q,\gamma(t)) \times \Gamma_d(p,q,x^\bullet,\theta^\bullet)$$

where

$$\Phi_d(x^\bullet,\theta^\bullet) = \left\{(p,q) \in \Pi_d \colon \forall (p',q') \in \Pi_d, (p-p')\cdot \sum_{i\in I}(\overline{x}^i - e^i) + (\widetilde{q}-\widetilde{q}')\cdot (\sum_{i\in I}\theta^i - \theta) \geqslant 0)\right\}$$

and

$$\Gamma_d(p,q,x^\bullet) = \left\{ t\in T \;:\; \widetilde{t} = \widetilde{r}\circ \left(p\Box \sum_{i\in I}\left(x^i - e^i\right)^-\right) + p\Box\widetilde{g} - \widetilde{V}\theta - w\right\}$$

The correspondence F_d is u.s.c. with non-empty convex compact values. Applying Kakutani's fixed point theorem, there exists $(\overline{p},\overline{q},\overline{x}^\bullet,\overline{\theta}^\bullet,\overline{t}\,) \in \Pi_d \times \prod_{i\in I} X^i \times \prod_{i\in I}\Theta^i \times T$ such that:

$$\forall i\in I, (\overline{x}^i,\overline{\theta}^i) \in d_d^i(\overline{p},\overline{q},\gamma(\overline{t})), \tag{4.6}$$

$$\forall (p,q)\in\Pi_d,\;\; (p-\overline{p})\cdot\sum_{i\in I}(\overline{x}^i - e^i) + (\widetilde{q}-\widetilde{\overline{q}})\cdot(\sum_{i\in I}\overline{\theta}^i - \theta) \leqslant 0, \tag{4.7}$$

$$\widetilde{\overline{t}} = \widetilde{r}\circ\left(\overline{p}\Box\sum_{i\in I}\left(\overline{x}^i - e^i\right)^-\right) + \overline{p}\Box\widetilde{g} - \widetilde{V}\theta - w. \tag{4.8}$$

Notice here that the fixed point obtained satisfies $(\overline{x}^i,\overline{\theta}^i) \in d^i(\overline{p},\overline{q},\gamma(t))$. According to the authority's constraints (4.8) and the definition of γ (refer to 4.4), one has $\gamma(\overline{t}) = \widetilde{\overline{t}}$. Thus, $(\overline{x}^i,\overline{\theta}^i) \in d_d^i(\overline{p},\overline{q},\overline{t})$. Finally, in order to prove that the obtained fixed point is achieved as a monetary equilibrium, we need only to show that commodity and asset markets clear. These proofs are very similar to the case of no transfers, one needs only to replace conditions (3.5) and (3.6) by the new authority's constraint (4.8).

4.4. **Application: The case of positive transfers.** This section is devoted to the proof of Theorem 2.3: In choosing a higher price level, we get to positive transfers.

Claim 4.2. *Under assumptions (M1)–(PR) and (PP), there exists a price level* $c^\star \in \mathbb{R}_+^\Sigma$ *above which transfers are positive elements of* $\mathbb{R}^\Sigma$, *for any consumption allocation* $x^\bullet \in \mathbb{R}^{\Sigma LI}$.

Proof. Let $c^\star \in \mathbb{R}_{++}^\Sigma$, $c^\star = (c^\star(\sigma),\ \sigma\in\Sigma)$. Consider a state $s\in S$ and let

$$c^\star(0) \geqslant \frac{w(0)}{\text{Min}\,\left\{1,\underline{g}(0)\right\}}(1+r(0)) \quad\text{and}\quad c^\star(s) \geqslant \frac{R(s)\theta}{\underline{g}(s)}, \forall s\in S \tag{4.9}$$

Following (PR), $c^\star(\sigma)$ is well-defined. Note that constant K defined in the previous section (4.5) is nul, for all price level $c^\star$. Applying the results of sections 4.2 to 4.3 we obtain $\gamma(\bar{t}) = \text{Max}\,\{\widetilde{K}, \widetilde{t}\}$, i.e. $\widetilde{\bar{t}} \geqslant 0$. $\square$

Appendix

A1. **Proof of Lemma 3.1:**

Proof of Part (a) :
This part follows from the compactness of the sets $\widehat{X}$, and $\widehat{\Theta}$. Indeed, the compactness of $\widehat{X}$ follows from Assumptions (C1). Following Assumptions (F1) and (F2), for each $i \in I$, the set $\widehat{\Theta}^i$ is closed and bounded: Indeed, let us consider (x^i_ν, p_ν, q_ν) be a sequence in $\widehat{X}^i \times \Pi$ and (θ^i_ν) a sequence in $\widehat{\Theta}^i$. Then, for each $\nu \in \mathbb{N}$, we have

$$p_\nu \,\square\, (x^i_\nu - e^i + \widetilde{g}^i) + \widetilde{r} \circ (p_\nu \,\square\, (x^i_\nu - e^i)^-) = \widetilde{V}\theta^i_\nu + w^i \tag{4.10}$$

By a classical compactness argument, we may suppose that the sequence (x^i_ν, p_ν, q_ν) converges to (x^i, p, q). If the sequence (θ^i_ν) is not bounded, then, passing to a subsequence if necessary, we can suppose that $\lim_n \left\|\theta^i_\nu\right\| = +\infty$. Multiplying (4.12) by $1/\left\|\theta^i_\nu\right\|$ and passing to the limit, there exists $\kappa \in \mathbb{R}^J$ with $\widetilde{V}\kappa = 0$ where $\|\kappa\| = 1$. Assumption (F2) implies that if $\widetilde{V}\kappa = 0$ then $\kappa = 0$: a contradiction. It follows that the sequence (θ^i_ν) is bounded, and passing to a subsequence if necessary, we can suppose that there exists $\theta^i \in \mathbb{R}^J$ such that (θ^i_ν) converges to θ^i and $\theta^i \in \widehat{\Theta}^i$.

Proof of Part (b) :
Let $(\overline{x}^\bullet, \overline{\theta}^\bullet, \overline{p}, \overline{q})$ be a monetary equilibrium with no-transfers of $\mathcal{E} = (\mathcal{E}^c_k, \mathcal{E}^f_k, \mathcal{E}^m)$. Suppose that it is not a monetary equilibrium of $\mathcal{E}$. Then for some i, there exists $(x^i, \theta^i) \in X^i \times \Theta^i$ such that $u^i(x^i) > u^i(\overline{x}^i)$ and (x^i, θ^i) is budget feasible. Since $(\overline{x}^i, \overline{\theta}^i)$ belongs to $\text{int}\, B(\Sigma \times L, k)$ then, it is easy to find $0 < \lambda \leqslant 1$ such that

$$(\overline{x}^i + \lambda(x^i - \overline{x}^i)) \in X^i_k, \quad \text{and} \quad (\overline{\theta}^i + \lambda(\theta^i - \overline{\theta}^i)) \in \Theta^i_k$$

Moreover, $(\overline{x}^i + \lambda(x^i - \overline{x}^i), \overline{\theta}^i + \lambda(\theta^i - \overline{\theta}^i))$ is budget feasible. Indeed, we see in the following that budget sets are convex: for this, we need only to recall that,

$$\forall a,\ \overline{a} \in \mathbb{R}^{\Sigma L},\ \forall \lambda \in [0,1],\ (\mu a + (1-\lambda)\overline{a})^- \leqslant \lambda a_- + (1-\lambda)\overline{a}_-$$

Finally, from Assumption C.2, we also have

$$u^i(\overline{x}^i + \lambda(x^i - \overline{x}^i)) > u^i(\overline{x}^i),$$

which yields a contradiction. $\square$

A2. **Proof of Lemma 3.2:**

Proof. Let $(p, q, \mu) \in \Pi \times M$. Let an agent $i \in I$. According to (C3), we can choose a consumption plan $x^i \in X^i$ such that $x^i - e^i + g^i \ll 0$. Since $g^i \geqslant 0$ and $x^i - e^i \ll -g^i \leqslant 0$, we get $(x^i - e^i)^+ = 0$ and $(x^i - e^i)^- = -(x^i - e^i) \gg 0$. Thus, for every state $s \in S$, one has:

$$\begin{aligned} p(s)\cdot(x^i(s) - e^i(s) + \widetilde{g}^i(s)) + \widetilde{r}(s)p(s)\cdot(x^i(s) - e^i(s))^- &= (1 - \widetilde{r}(s))p(s)\cdot(x^i(s) - e^i(s)) + p(s)\cdot\widetilde{g}^i(s) \\ &= \frac{p(s)}{1 + r(s)}\cdot\left(x^i(s) - e^i(s) + g^i(s)\right) < 0. \end{aligned}$$

If $p(0) \neq 0$, similarly, one has:

$$\begin{array}{rcl} p(0)\cdot(x^i(0)-e^i(0)+\widetilde{g}^i(0))+\widetilde{r}(0)p(0)\cdot(x^i(0)-e^i(0))^- & = & (1-\widetilde{r}(0))p(0)\cdot(x^i(0)-e^i(0))+p(0)\cdot\widetilde{g}^i(0) \\ & = & \dfrac{p(0)}{1+r(0)}\cdot\big(x^i(0)-e^i(0)+g^i(0)\big)<0. \end{array}$$

and $(x^i, 0)$ belongs to $\beta'^i(p, q, \mu)$.
If $p(0) = 0$, one has $q \neq 0$. Since $0 \in \operatorname{int}\Theta^i$ and for all $\sigma \in S$,

$$p(\sigma)\cdot(x^i(\sigma)-e^i(\sigma)+\widetilde{g}^i(\sigma))+\widetilde{r}(\sigma)\left(p(\sigma)\cdot(x^i(\sigma)-e^i(\sigma))^-\right)<0.$$

By a continuity argument, one can choose a portfolio $\theta^i \in \Theta^i$ such that

$$\left\{\begin{array}{rcl} \widetilde{q}\cdot\theta^i & < & 0 \\ p(s)\cdot(x^i(s)-e^i(s)+\widetilde{g}^i(s))+\widetilde{r}(s)\left(p(s)\cdot(x^i(s)-e^i(s))^-\right) & < & \mu(s)R(s)\theta^i, \quad \forall s\in S \end{array}\right.$$

which means that $(x^i, \theta^i) \in \beta^i(p, \widetilde{q}, \mu)$. □

A3. **Proof of Claim 3.2:**

Proof. : Let us begin by showing property (i): Let $i \in I$ and $(x_n, \theta_n, p_n, q_n, \mu_n)$ be a sequence in $X^i \times \Theta^i \times \Pi \times M$. Following standard compactness argument, one can assume that the sequence $(x_n, \theta_n, p_n, q_n, \mu_n)$ converges to (x, θ, p, q, μ) and such that $(x_n, \theta_n) \in \beta^i(p_n, q_n, \mu_n)$. For each $n \in \mathbb{N}$,

$$\begin{array}{rcl} p_n(0)\cdot(x_n(0)-e^i(0)+\widetilde{g}^i(0))+\widetilde{r}(0)\left(p_n(0)\cdot(x_n(0)-e^i(0))^-\right) & \leqslant & -\widetilde{q}\cdot\theta_n+\mu_n(0)w^i(0) \\ p_n(s)\cdot(x_n(s)-e^i(s)+\widetilde{g}^i(s))+\widetilde{r}(s)\left(p_n(s)\cdot(x_n(s)-e^i(s))^-\right) & \leqslant & \mu_n(s)(\widetilde{R}\theta_n)(s), \qquad \forall s\in S \end{array}$$

Passing to the limit, we get (x, θ) belongs to $\beta^i(p, q, \mu)$.

Let us now show that β^i is l.s.c. on $\Pi \times M$: Let $(p, q, \mu) \in \Pi \times M$. Since $\beta'^i(p, q, \mu)$ has non-empty, convex values (refer to lemma 3.1), we have $\beta^i(p, q, \mu) = \operatorname{cl}\beta'^i(p, q, \mu)$. Finally, the claim follows from the fact that $\beta'^i(p, q, \mu)$ has an open graph.

Finally, the u.s.c. follows from the continuity of the utility functions. Indeed, $\delta^i(p, q, \mu)$ is the argmax of u^i on $\beta^i(p, q, \mu)$. Since u^i is continuous and β^i is continuous on $\Pi \times M$, it follows from Berge's Maximum theorem that δ^i is u.s.c. on $\Pi \times M$ with non-empty values. The convexity of $\delta^i(p, q, \mu)$ follows from the quasi-concavity of u^i. □

A4. **Proof of Lemma 4.1:**

Proof of Part (a) :
This part follows from the compactness of the sets $\widehat{X}$, $\widehat{T}$ and $\widehat{\Theta}$. Indeed, the compactness of $\widehat{X}$ follows from Assumptions (C1). Following Assumptions (F1) and (F2), for each $i \in I$, the set $\widehat{\Theta}^i$ is bounded. Indeed, for this end, let us begin by showing that $\widehat{T}$ is a closed and bounded subset of $\mathbb{R}^\Sigma$. Let $(x^\bullet_\nu, p_\nu, q_\nu)$ be a sequence in $\widehat{T} \times \widehat{X} \times \Pi_d$. Following standard compactness argument, we may assume that the sequence $(x^\bullet_\nu, p_\nu, q_\nu)$ converges to $(x^\bullet, p, q)$. Let (t_ν) a sequence in $\widehat{T}$. For each $\nu \in \mathbb{N}$, we thus have

$$t_\nu = \widetilde{r}\circ\left(p_\nu \,\Box \sum_{i\in I}\left(x^i - e^i\right)^-\right) + p_\nu \,\Box\, \widetilde{g} - \widetilde{V}\theta - w \tag{4.11}$$

Passing to a subsequence if necessary, we can suppose that there exists $t \in \mathbb{R}^{\Sigma}$ such that (t_ν) converges to t and $t \in \widehat{T}$. Let us consider an agent $i \in I$. We now show that $\widehat{\Theta}^i$ is a closed and bounded subset of $\mathbb{R}^J$. Let us consider $(x^i_\nu, p_\nu, q_\nu, t_\nu)$ be a sequence in $\widehat{X}^i \times \Pi_d \times \widehat{T}$. By a classical compactness argument, we may assume that the sequence $(x^i_\nu, p_\nu, q_\nu, t_\nu)$ converges to (x^i, p, q, t). Let (θ^i_ν) be a sequence in $\widehat{\Theta}^i$. Then, for each $\nu \in \mathbb{N}$, we thus have

$$p_\nu \square (x^i_\nu - e^i + \widetilde{g}^i) + \widetilde{r} \circ (p_\nu \square (x^i_\nu - e^i)^-) = \widetilde{V}\theta^i_\nu + w^i + \xi^i \widetilde{t}_\nu \tag{4.12}$$

If the sequence (θ^i_ν) is not bounded, then, passing to a subsequence if necessary, we can suppose that $\lim_n \left\|\theta^i_\nu\right\| = +\infty$. Multiplying (4.12) by $1/\left\|\theta^i_\nu\right\|$ and passing to the limit, there exists $\kappa \in \mathbb{R}^J$ with $\kappa = 0$ where $\|\kappa\| = 1$: a contradiction. It follows that the sequence (θ^i_ν) is bounded, and passing to a subsequence if necessary, we can suppose that there exists $\theta^i \in \mathbb{R}^J$ such that (θ^i_ν) converges to θ^i and $\theta^i \in \widehat{\Theta}^i$.

Proof of Part (b) :
Let $(\overline{x}^\bullet, \overline{\theta}^\bullet, \overline{p}, \overline{q}, \overline{t})$ be a monetary equilibrium of $\mathcal{E}_h = (\mathcal{E}^c_h, \mathcal{E}^f_h, \mathcal{E}^m)$. Suppose that it is not a monetary equilibrium of $\mathcal{E}$. Then for some i, there exists $(x^i, \theta^i) \in X^i \times \Theta^i$ such that $u^i(x^i) > u^i(\overline{x}^i)$ and (x^i, θ^i) is budget feasible. Since $(\overline{x}^i, \overline{\theta}^i)$ belongs to $\operatorname{int} B(\Sigma \times L, h) \times \operatorname{int} B(J, h)$ then, it is easy to find $0 < \gamma \leqslant 1$ such that

$$(\overline{x}^i + \gamma(x^i - \overline{x}^i)) \in X^i_k, \quad \text{and} \quad (\overline{\theta}^i + \gamma(\theta^i - \overline{\theta}^i)) \in \Theta^i_k$$

Moreover, $(\overline{x}^i + \gamma(x^i - \overline{x}^i), \overline{\theta}^i + \gamma(\theta^i - \overline{\theta}^i))$ is budget feasible. Indeed, we see in the following that budget sets are convex: for this, we need only to recall that,

$$\forall a,\ \overline{a} \in \mathbb{R}^{\Sigma L},\ \forall \gamma \in [0,1],\ (\gamma a + (1-\gamma)\overline{a})^- \leqslant \gamma a_- + (1-\gamma)\overline{a}_-$$

Finally, from Assumption C.2, we also have

$$u^i(\overline{x}^i + \gamma(x^i - \overline{x}^i)) > u^i(\overline{x}^i),$$

which yields a contradiction. □

A.5 **Proof of Lemma 4.2**

Proof. Let an agent $i \in I$. According to (C3), we can choose a consumption plan $x^i \in X^i$ such that $x^i - e^i + g^i \ll -\chi^i 1\!\!1_\Sigma$. Since $g^i \geqslant 0$ and $x^i - e^i \ll -g^i \leqslant 0$, we get $(x^i - e^i)^+ = 0$ and $(x^i - e^i)^- = -(x^i - e^i) \gg 0$. Thus, for every state $s \in S$, one has:

$$\begin{aligned} p(s) \cdot (x^i(s) - e^i(s) + \widetilde{g}^i(s)) + \widetilde{r}(s) p(s) \cdot (x^i(s) - e^i(s))^- &= (1 - \widetilde{r}(s)) p(s) \cdot (x^i(s) - e^i(s)) + p(s) \cdot \widetilde{g}^i(s) \\ &= \frac{p(s)}{1 + r(s)} \cdot \left(x^i(s) - e^i(s) + g^i(s)\right) \end{aligned}$$

Note that $p(s) \cdot (x^i(s) - e^i(s) + g^i(s)) < -\chi^i(s)d(s)$. But remark that by construction of $d(s)$ (refer to 4.3), one has $-\chi^i(s)d(s) \leqslant \xi^i K(s)$. Indeed,

$$\begin{aligned}
d(s) &\geqslant \frac{R(s)\theta}{\underline{g}(s) + \min_{i\in I}\left\{\frac{\chi^i(s)}{\xi^i}\right\}} \\
d(s)\underline{g}(s) + d(s)\min_{i\in I}\left\{\frac{\chi^i(s)}{\xi^i}\right\} &\geqslant R(s)\theta \\
d(s)\underline{g}(s) + d(s)\frac{\chi^i(s)}{\xi^i} &\geqslant R(s)\theta \\
-\frac{\chi^i(s)}{\xi^i}d(s) &\leqslant d(s)\underline{g}(s) - R(s)\theta \\
-\chi^i(s)d(s) &\leqslant \xi^i\left(d(s)\underline{g}(s) - R(s)\theta\right)
\end{aligned}$$

Recalling the definition of $K(s)$, one has $-\chi^i(s)d(s) \leqslant \xi^i K(s)$, thus $\frac{p(s)}{1+r(s)} \cdot (x^i(s) - e^i(s) + g^i(s)) < \xi^i\gamma(t,s)$.
Moreover, if $p(0) \neq 0$, similarly to the case $s \in S$, one has:

$$p(0)\cdot(x^i(0) - e^i(0) + \widetilde{g}^i(0)) + \widetilde{r}(0)p(0)\cdot(x^i(0) - e^i(0))^- = \frac{p(0)}{1+r(0)}\cdot\left(x^i(0) - e^i(0) + g^i(0)\right) < 0$$

At date-event $\sigma = 0$, one has $w^i(0) + \xi^i\gamma(t,0) \geqslant 0$. Indeed, by construction (refer to 4.2), one has

$$\begin{aligned}
d(0) &\geqslant \frac{\left(w(0) - \min_{i\in I}\{\frac{w^i(0)}{\xi^i}\}\right)(1+r(0))}{\mathrm{Min}\,\{1, \underline{g}(0)\}} \\
d(0)\mathrm{Min}\,\{1,\underline{g}(0)\} + (1+r(0))\min_{i\in I}\left\{\frac{w^i(0)}{\xi^i}\right\} &\geqslant (1+r(0))w(0) \\
(1+r(0))w^i(0) + \xi^i\left(\mathrm{Min}\,\{1,\underline{g}(0)\}d(0) - (1+r(0))w(0)\right) &\geqslant 0 \\
(1+r(0))w^i(0) + \xi^i K(0) &\geqslant 0.
\end{aligned}$$

Since $\gamma(t,0) = \mathrm{Max}\,\{\widetilde{t}(0), \widetilde{K}(0)\}$, one has $w^i(0) + \xi^i\gamma(t,0) \geqslant 0$.
Hence, $(x^i, 0)$ belongs to $\beta^i(p, \tilde{q}, t)$.
If $p(0) = 0$, one has $q \neq 0$. Since $0 \in \mathrm{int}\,\Theta^i$ and for all $s \in S$,

$$p(s)\cdot(x^i(s) - e^i(s) + \widetilde{g}^i(s)) + \widetilde{r}(s)\left(p(s)\cdot(x^i(s) - e^i(s))^-\right) < \xi^i\gamma(t,s),$$

by a continuity argument, one can choose a portfolio $\theta^i \in \Theta^i$ such that $\widetilde{q}\cdot\theta^i < 0 \leq w^i(0) + \xi^i\gamma(t,0)$ and

$$p(s)\cdot(x^i(s) - e^i(s) + \widetilde{g}^i(s)) + \widetilde{r}(s)\left(p(s)\cdot(x^i(s) - e^i(s))^-\right) < \widetilde{R}(s)\theta^i + \xi^i\gamma(t,s), \quad \forall s \in S$$

and (x^i, θ^i) belongs to $\beta^i(p, q, \gamma(t))$. □

A6. **Proof of Claim 4.1:**

Proof. : Let us begin by showing property (i): Let $i \in I$ and $(x_n, \theta_n, p_n, q_n, \gamma(t_n))$ be a sequence in $X^i \times \Theta^i \times \Pi_d \times T$. Following classical compactness argument, one can assume that the sequence

$(x_n, \theta_n, p_n, q_n, \gamma(t_n))$ converges to $(x, \theta, p, q, \gamma(t))$ and such that $(x_n, \theta_n) \in B^i(p_n, q_n, \gamma(t_n))$. For each $n \in \mathbb{N}$,

$$p_n \,\Box\, (x_n - e^i + \widetilde{g}^i) + \widetilde{r} \circ \left(p_n \,\Box\, (x_n - e^i)^-\right) \;\leqslant\; \widetilde{V}\theta_n + w^i + \xi^i \gamma(t_n) \tag{4.13}$$

Passing to the limit, we get (x, θ) belongs to $B^i_d(p, q, \gamma(t))$.

Let us now show that B^i is l.s.c. on Π_d: Let $(p, q) \in \Pi_d$. Since $\beta^i_d(p, q, \gamma(t))$ has non-empty, convex values (refer to lemma 4.1), we have $B^i_d(p, q, \gamma(t)) = \mathrm{cl}\, \beta^i_d(p, q, \gamma(t))$. Finally, the claim follows from the fact that $\beta^i_d(p, q, \gamma(t)$ has an open graph.

Finally, the u.s.c. follows from the continuity of the utility functions. Indeed, $d^i(p, q, \gamma(t))$ is the argmax of u^i on $B^i_d(p, q, \gamma(t))$. Since u^i is continuous and B^i_d is continuous on Π_d, it follows from Berge's Maximum theorem that d^i is u.s.c. on Π_d with non-empty values. The convexity of $d^i(p, q, \gamma(t))$ follows from the quasi-concavity of u^i. □

A.7. **Proof of Corollary 2.1 and Corollary 2.1**

Corollaries 2.1 and 2.2 follow from this result:

Lemma 4.3. *There exists $(\widehat{R}, \widehat{\theta})$ such that*

$$\widehat{R} \gg 0, \quad \mathrm{Span}\, \widehat{R} = \mathrm{Span}\, R, \quad \widehat{R}\widehat{\theta} = R\theta \quad \textit{and} \quad \widehat{\theta} = 1\!\!1_{\mathbb{R}^J}.$$

Proof. The proof will be one in two steps. We begin by showing that there exists $(\widetilde{R}, \widetilde{\theta})$ such that

$$\widetilde{R}\widetilde{\theta} = R\theta, \quad \mathrm{Span}\, \widetilde{R} = \mathrm{Span}\, R \quad \text{and} \quad \widetilde{\theta} = e_k.$$[8]

Indeed, according to assumption (NRA), $R\theta \gg 0$. This implies that $\theta \neq 0$, i.e. there exits an asset $k \in J$ such that $\theta(k) \neq 0$. We posit

$$\begin{array}{lcl} \widetilde{\theta}(k) = 1 & \text{and} & \widetilde{\theta}(j) = 0, \quad \forall j \neq k \\ \widetilde{R}(k) = R\theta & \text{and} & \widetilde{R}(j) = R(j) \quad \forall j \neq k. \end{array} \tag{4.14}$$

It is evident that $\widetilde{R}\widetilde{\theta} = R\theta$. We now show that $\mathrm{Span}\, \widetilde{R} = \mathrm{Span}\, R$. By construction, $\mathrm{Span}\, \widetilde{R} \subset \mathrm{Span}\, R$. Reciprocally, in order to show that $\mathrm{Span}\, R \subset \mathrm{Span}\, \widetilde{R}$, one only needs to show that $R(k) \subset \mathrm{Span}\, \widetilde{R}$. This follows from the fact that

$$R(k) = \frac{1}{\theta(k)} \left(R\theta - \sum_{j \neq k} R(j)\theta \right) = \frac{1}{\theta(k)} \left(\widetilde{R}(k) - \sum_{j \neq k} \widetilde{R}(j)\theta \right) \quad \subset \quad \mathrm{Span}\, \widetilde{R}$$

We now show that there exists $(\widehat{R}, \widehat{\theta})$ such that

$$\widehat{R}\widehat{\theta} = \widetilde{R}\widetilde{\theta}, \quad \mathrm{Span}\, \widehat{R} = \mathrm{Span}\, \widetilde{R} \quad \text{and} \quad \widehat{\theta} = 1\!\!1_{R^J}$$

Let $\epsilon > 0$ and define:

$$\begin{array}{rcl} \widehat{R}(j) & = & \dfrac{1}{J}\widetilde{R}(k) + \epsilon \widetilde{R}(j), \quad \forall j \neq k \\ \widehat{R}(j) & = & \dfrac{1}{J}\widetilde{R}(k) - \epsilon \displaystyle\sum_{j \neq k} \widetilde{R}(j) \end{array}$$

[8]The vector e_k designates the vector in $\mathbb{R}^J$ with all its components equal to 0 except for the k_{th} one.

Notice that, for $\epsilon > 0$ small enough, referring to (4.14), we have $\widetilde{R}(k) = R\theta \gg 0$, one has $\widehat{R} \gg 0$. Let us check that $\operatorname{Span}\widehat{R} = \operatorname{Span}\widetilde{R}$. By construction, $\operatorname{Span}\widehat{R} \subset \operatorname{Span}\widetilde{R}$. Reciprocally, since $\sum_{j=1}^{J} \widehat{R}(j) = \widetilde{R}(k)$, one has $\widetilde{R}(k) \subset \operatorname{Span}\widehat{R}$. Finally, for all $j \in J,\ j \neq k$, one has

$$\widetilde{R}(j) = \frac{1}{\epsilon}\left(\widehat{R}(j) - \frac{1}{J}\widetilde{R}(k)\right) = \frac{1}{\epsilon}\left(\widehat{R}(j) - \frac{1}{J}\sum_{j=1}^{J}\widehat{R}(j)\right) \subset \operatorname{Span}\widehat{R}$$

□

The proof of Corollaries 2.1 and 2.2 follow:

Proof. Consider a standard economy $\mathcal{E} := (\mathcal{E}^c, \mathcal{E}^f, \mathcal{E}^m)$ satisfying assumption (NRA), i.e. $R\theta \gg 0$. According to Lemma 4.3, there exists $(\widehat{R}, \widehat{\theta})$ such that

$$\widehat{R} \gg 0, \quad \operatorname{Span}\widehat{R} = \operatorname{Span}R, \quad \widehat{R}\widehat{\theta} = R\theta \quad \text{and} \quad \widehat{\theta} = 1\!\!I_{\mathbb{R}^J}.$$

If we set $\widehat{\mathcal{E}} := (\mathcal{E}^c, \widehat{\mathcal{E}}^f, \mathcal{E}^m)$ where we modify the financial market $\mathcal{E}^f = (R, \Theta^\bullet, \theta)$ by $\widehat{\mathcal{E}}^f = (\widehat{R}, \Theta^\bullet, \widehat{\theta})$. The auxiliary economy $\widehat{\mathcal{E}}^f$ satisfies assumption (PP), (F1) and (F2). Thus, according to Theorem 2.3, there exists a monetary equilibrium $(p, \widehat{q}, x^\bullet, \widehat{\theta}^\bullet)$ of $\widehat{\mathcal{E}}$.
Since $\operatorname{Span}R \subset \operatorname{Span}\widehat{R}$, we have the following property:

$$\forall k \in J, \quad \exists \gamma \in \mathbb{R}^J \,:\, R(k) = \sum_{j \in J} \gamma(j)\widehat{R}(j)$$

For every asset $k \in J$, define $q(k) = \sum_{j \in J} \gamma(j)\widehat{q}(j)$. If $(p, \widehat{q}, x^\bullet, \widehat{\theta}^\bullet)$ is a monetary equilibrium of $\widehat{\mathcal{E}}$, then $(p, q, x^\bullet, \theta^\bullet)$ is a monetary equilibrium of $\mathcal{E}$, with

$$\forall i \in I, \quad \forall k \in J, \quad \theta^i(k) = \sum_{j \in J} \gamma(j)\widehat{\theta}^i(j).$$

Recalling the price levels properties for $(p, \widehat{q})$, we have $\|p(s)\| = c(s)$, whereas we do not know anything on first period price levels $\|p(0)\| + \|q\|$. □

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