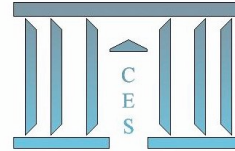




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## Option Pricing under GARCH models with Generalized Hyperbolic innovations (I) : Methodology

Christophe CHORRO, Dominique GUEGAN, Florian IELPO

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# Option Pricing under GARCH models with Generalized Hyperbolic innovations (I): Methodology

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## Abstract

In this paper, we present an alternative to the Black Scholes model for a discrete time economy using GARCH-type models for the underlying asset returns with Generalized Hyperbolic (GH) innovations that are potentially skewed and leptokurtic. Assuming that the stochastic discount factor is an exponential affine function of the states variables, we show that this class of distributions is stable under the Risk neutral change of probability.

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# 1 Introduction

After the celebrated Black-Scholes formula for pricing call options under constant volatility, researchers have paid attention on more general models to explain some well known mispricing phenomena. It is now generally admitted that returns exhibit a time varying conditional variance, leading to the building of models consistent with this stylized fact, either in a continuous (Heston (1993)) or discrete time setting (Engle (1982) and Bollerslev (1986)). Unfortunately, discrete time and a continuum of states of nature give rise to incompleteness: the multiplicity of equivalent martingale measures involves a continuum of equilibrium prices. Thus, the question of selecting the best one naturally arises.

Through an equilibrium argument Duan (1995) gave an economically consistent approach to price options in related GARCH models with Gaussian innovations. Following this methodology, Heston and Nandi (2000) considered a new conditionally Gaussian GARCH model able to cope with the skews in option prices. They derived an almost closed form expression for call option prices and empirically demonstrated its pricing performance. The model being conditionally Gaussian, it usually fails to capture the short term behavior of equity options smiles: Christoffersen et al. (2006) thus extended the Heston Nandi (2000) model by using the Inverse Gaussian distribution to increase the skewness effect and empirically assessed the higher performances of their approach of option pricing. However, it is not the only way around the skewness effect.

In this paper we present a GARCH-type model with Generalized Hyperbolic innovations. This distribution introduced by Barndorff-Nielsen (1977) is known to fit financial dataset remarkably: for example, figures 1, 2 and 3 show its ability to handle the particular tail behavior found in equity indexes returns (CAC 40, FTSE, DAX, SP&500). More, the distribution clearly passes the usual adequation tests, as presented in table 1. This family of distribution has already been used with empirical successes to model the dynamic of several stock markets (see Barndorff-Nielsen (1995), Eberlein and Keller (1995), Jensen and Lunde (2001), Eberlein and Prause (2002), Fergusson and Platen (2006), Guégan and Zhang (2007)) and even to price options in continuous time Lévy type markets (Eberlein and Keller (1995), Eberlein and Prause (2002)).

In the latter article, the authors use the Esscher transform method introduced for option pricing by Gerber and Shiu (1994 a) to select a particular equivalent martingale measure with an interesting economic interpretation. For discrete time models this powerful tool has been extended in a general way by Bühlmann et al. (1998) (see also Siu et al. (2004) and Gouriéroux and Monfort (2007) for an equivalent formulation in terms of the exponential affine parametrization of the stochastic discount factor). Note that, although this elegant approach provides an unique martingale measure, there may be other equivalent martingale measures (see e.g Elliot and Madan (1998)). Nevertheless, from a practical point of view, one of the main feature of this method with respect to the others is that the conditional distribution of the returns of the risky asset is in general stable under the historical and the risk neutral probabilities (Bertholon et al. (2003), Siu et al. (2004), Gouriéroux and Monfort (2006), Christoffersen et al.

(2006)).

The main novelty of this paper is to show how to apply this methodology for GARCH-type models with Generalized Hyperbolic innovations. In particular, the conditional dynamic of the log-returns under the chosen equivalent martingale measure is also of the Generalized Hyperbolic type. Provided that the equivalent martingale measure has been properly selected, no more calibration exercise is required and option prices can be simply computed using Monte Carlo simulations. Since several classical distributions e.g. Normal, Skewed Laplace, Gamma or Inverse Gaussian may be obtained as a limiting case of the Generalized Hyperbolic one, many classical dynamics (see Duan (1995), Heston and Nandi (2000), Gouriéroux and Monfort (2006), Siu et al. (2004), Christoffersen et al. (2006)) are recovered from our approach.

In a forthcoming paper (Chorro, Guégan and Ielpo (2008)) the associated empirical results corresponding to this new methodology will be presented. A study based on four world indexes (CAC 40, FTSE, DAX, SP&500) is ongoing to compare, in particular, the pricing performances of our model with respect to its natural competitors. More generally the idea is to test the ability of parametric innovations to reproduce market prices in the context of the exponential affine specification of the stochastic discount factor.

The remainder of the paper is organized as follows. In Section 2 the definition and the main features of the Generalized Hyperbolic distribution are recalled and we present the GARCH-type models that will be used in the sequel. In Section 3, the basis of the exponential affine approach for the specification of the stochastic discount factor are presented and we show how to apply it in the Generalized Hyperbolic framework. The conclusions and related future empirical research are presented in Section 4.

## 2 GARCH-type models with Generalized Hyperbolic innovations

This section is devoted to the presentation of a new asset pricing model. We first review the main features of the Generalized Hyperbolic distribution. We then present our economy modeling under the historical measure.

### 2.1 The Generalized Hyperbolic distribution

It is well known that for lower sampling frequencies (e.g. monthly) asset returns empirical distributions are closer to the Gaussian case. Unfortunately, these kind of dataset ignore too much information to be considered especially for pricing purposes. On the contrary, looking at daily data leads to the excess kurtosis phenomena first described by Mandelbrot (1963). Several families of distributions have already been used to reproduce this stylized fact, such as the Paretian distributions or the double Weibull distribution for instance. However, the question of conditional distribution modeling

remains an opened one.

The recently introduced Generalized Hyperbolic (GH) distributions (Barndorff-Nielsen (1977)) have been suggested as a model for financial price processes. Their exponentially decreasing tails seem to fit the statistical behavior of asset returns (Barndorff-Nielsen (1995), Eberlein and Prause (2002)).

For  $(\lambda, \alpha, \beta, \delta, \mu) \in \mathbb{R}^5$  with  $\delta > 0$  and  $\alpha > |\beta| > 0$ , the one dimensional  $GH(\lambda, \alpha, \beta, \delta, \mu)$  distribution is defined by the following density function

$$d_{GH}(x, \lambda, \alpha, \beta, \delta, \mu) = \frac{(\sqrt{\alpha^2 - \beta^2}/\delta)^\lambda}{\sqrt{2\pi}K_\lambda(\delta\sqrt{\alpha^2 - \beta^2})} e^{\beta(x-\mu)} \frac{K_{\lambda-1/2}\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right)}{\left(\sqrt{\delta^2 + (x-\mu)^2}/\alpha\right)^{1/2-\lambda}} \quad (1)$$

where  $K_\lambda$  is the modified Bessel function of the third kind. We may remark that for  $\beta = 0$  this distribution is symmetric.

For  $\lambda \in \frac{1}{2}\mathbb{Z}$ , the basic properties of the Bessel function (see Abramowitz and Stegun (1964)) allow to find simpler forms for the density. In particular, for  $\lambda = 1$  we get the Hyperbolic distributions (HYP) which log-density is a hyperbola and for  $\lambda = -\frac{1}{2}$  we obtain the Normal Inverse Gaussian distributions (NIG) which are closed under convolution. More generally, many important distributions can be found either by constraining the distribution parameters or as a limiting case, e.g Gaussian distribution, Student's t-distribution or the Laplace-distribution (Barndorff-Nielsen and Blaesild (1981)).

Contrary to Paretian distributions, the moment generating function of a  $GH$  distribution exists and is given by

$$\mathbb{G}_{GH}(u) = e^{\mu u} \left( \frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + u)^2} \right)^{\frac{\lambda}{2}} \frac{K_\lambda(\delta\sqrt{\alpha^2 - (\beta + u)^2})}{K_\lambda(\delta\sqrt{\alpha^2 - \beta^2})}, \quad |\beta + u| < \alpha. \quad (2)$$

In particular, moments of all orders are finite (e.g exact values of the skewness and kurtosis are provided in Barndorff-Nielsen and Blaesild (1981)) allowing to apply Central Limit Theorem arguments to ensure the convergence of long time horizon returns towards Normal distributions. Finally, this family is also stable under affine transforms. This property is interesting because in the GARCH setting we will be able to deduce the conditional distribution of the log-returns from the innovations' ones. More precisely,

**PROPOSITION 1** *Let  $(M, \Sigma) \in \mathbb{R}^2$ . If  $X$  follows a  $GH(\lambda, \alpha, \beta, \delta, \mu)$  then  $M + \Sigma X$  follows a  $GH\left(\lambda, \frac{\alpha}{|\Sigma|}, \frac{\beta}{\Sigma}, \delta \mid \Sigma, M + \mu\Sigma\right)$ .*

**Proof:** See Blaesild (1981).□

In particular if we define  $\alpha^* = \alpha\delta$  and  $\beta^* = \beta\delta$  and if  $X \hookrightarrow GH(\lambda, \alpha^*, \beta^*, \delta, \mu)$  then  $\frac{X-\mu}{\delta} \hookrightarrow GH(\lambda, \alpha^*, \beta^*, 1, 0)$ : the parameters  $\mu$  and  $\delta$  respectively describe the location and the scale.

## 2.2 Description of the economy under the historical probability $\mathbb{P}$

It is now well documented that empirical evidence suggest that equity return volatility is stochastic and mean reverting. More, the response of volatility to positive or negative returns is asymmetric (see e.g. Ghysels et al. (1996)). In the discrete time setting the stochastic volatility is often captured using extensions of the autoregressive conditional heteroscedasticity model (ARCH); see for instance Bollerslev (1986). This kind of specification will be classically used in the sequel.

Let  $(z_t)_{t \in \{0,1,\dots,T\}}$  be i.i.d random variables defined on the sample space  $(\Omega, \mathcal{A}, \mathbb{P})$  and  $(\mathcal{F}_t = \sigma(z_u; 0 \leq u \leq t))_{t \in \{0,1,\dots,T\}}$  the associated information structure. Then, we assume that under the historical probability  $\mathbb{P}$ , the dynamics of the bond price process  $(B_t)_{t \in \{0,1,\dots,T\}}$  and the stock price process  $(S_t)_{t \in \{0,1,\dots,T\}}$  are given by

$$B_t = B_{t-1}e^r, \quad B_0 = 1, \quad (3)$$

where  $r$  is the corresponding risk free rate expressed on a daily basis and supposed to be constant and

$$Y_t = \log\left(\frac{S_t}{S_{t-1}}\right) = \underbrace{r + m_t}_{M_t} + \underbrace{\sqrt{h_t}z_t}_{\varepsilon_t}, \quad S_0 = s, \quad (4)$$

where  $z_t \hookrightarrow f(0, 1)$  ( $f$  being an arbitrary distribution with mean 0 and variance 1) and

$$\begin{cases} m_t = F(h_t, \lambda_0); \quad \lambda_0 \in \mathbb{R}, \\ \Sigma_t^2 = h_t = G(h_{t-1}, \varepsilon_{t-1}). \end{cases} \quad (5)$$

In (4) we consider a general time varying excess of return  $m_t$  that depends on the constant unit risk premium  $\lambda_0$ . In practice, it will be fixed for the empirical study as in the Duan model (1995):

$$m_t = \lambda_0\sqrt{h_t} - \frac{1}{2}h_t. \quad (6)$$

Since the vast majority of papers find very few advantages to work with high order GARCH we will consider only the first order case. Moreover to be able to capture the leverage effect we will favor the two following asymmetric GARCH specifications: the GJR-GARCH model (see Glosten et al. (1993))

$$h_t = a_0 + a_1\varepsilon_{t-1}^2 - \gamma\varepsilon_{t-1}\max(0, -\varepsilon_{t-1}) + b_1h_{t-1} \quad (7)$$

with nonnegative coefficients and the EGARCH model that ensures positivity without restrictions on the coefficients (see Nelson (1991))

$$\log(h_t) = a_0 + a_1\left(\left|\frac{\varepsilon_{t-1}}{\sqrt{h_{t-1}}}\right| + \gamma\frac{\varepsilon_{t-1}}{\sqrt{h_{t-1}}}\right) + b_1\log(h_{t-1}). \quad (8)$$

In this model we allow for non Gaussian innovations in order to model extreme returns behavior. Several distributions have been already used to reproduce excess skewness

or kurtosis and outperform the Black-Scholes pricing e.g. Generalized Exponential distribution (Nelson (1991)), Gamma distribution (Siu et al. (2004)), Inverse Gaussian distribution (Christoffersen et al. (2006)) or mixture of Normal distributions (Bertholon et al. (2003)). Here we focus our attention on the GH distribution presented in the preceding section.

According to Proposition 1, if  $z_t$  follows a  $GH(\lambda, \alpha, \beta, \delta, \mu)$  distribution then, given  $\mathcal{F}_{t-1}$ ,

$$Y_t \hookrightarrow GH\left(\lambda, \frac{\alpha}{\Sigma_t}, \frac{\beta}{\Sigma_t}, \delta\Sigma_t, M_t + \mu\Sigma_t\right). \quad (9)$$

Thus, for the estimation of the GARCH model under the historical probability, we may adopt a two-stages procedure. At the first stage, the Quasi Maximum Likelihood Estimation (QMLE) (see e.g. Franses and van Dijk (2000)) is used to determine the parameters  $(\lambda_0, a_0, a_1, b_1, \gamma)$ . This is an approximation for the exact Maximum Likelihood Estimation obtained by replacing in (4) the specific distribution  $f$  by a standard normal one. It is well known that under mild technical conditions this method provides efficient parameter estimates. At the second stage, since we exactly know the form of the density function (1) we adopt a classical maximum likelihood approach to estimate the unknown remaining parameters  $(\lambda, \alpha, \beta, \delta, \mu)$ .

Now, our model is entirely specified under  $\mathbb{P}$ . Since we want to use it to price contingent claims, we need to postulate an explicit risk premium to perform the change in distribution. This is the aim of the next part.

### 3 The stochastic discount factor

We consider the preceding economy with time horizon  $T$  consisting of two assets namely a risk-free bond and a risky stock. Remind that we denote by  $(B_t)_{t \leq T}$  and  $(S_t)_{t \leq T}$  the dynamics of the bond and the stock price processes under the historical probability  $\mathbb{P}$ .

Classically, in a discrete time dynamic equilibrium model (or in an arbitrage free continuous one), the price of any asset equals the expected present value of its future payoffs under an equivalent martingale measure  $\mathbb{Q}$ . For example, the price  $P_t$  at time  $t$  of an European asset paying  $\Phi_T$  at  $T$  ( $\Phi_T$  being  $\mathcal{F}_T$  measurable) is given by

$$P_t = E_{\mathbb{Q}}[\Phi_T e^{-r(T-t)} \mid \mathcal{F}_t] \quad (10)$$

or equivalently

$$P_t = E_{\mathbb{P}}[\Phi_T M_{t,T} \mid \mathcal{F}_t]. \quad (11)$$

The  $\mathcal{F}_{t+1}$  measurable random variable  $M_{t,t+1}$  is the so called stochastic discount factor (SDF) (the quantity  $M_{t,t+1}e^r$  is also known as the pricing kernel). In general, the stochastic discount factor depends on several state variables of the economy (e.g aggregate consumption in a Lucas (1978) economy or past consumptions and equity market returns in Rosenberg and Engle (2002)). Nevertheless, following Rubinstein

(1976), Gibbons (1985) or Cochrane (2001) we suppose that equity market returns are the only variables to deal with for pricing purpose or equivalently that we may project the original stochastic discount factor onto the sigma-algebra generated by the payoffs of the risky asset.

We recall a well known example: in the Black-Scholes economy, the dynamic of the risky asset under the unique equivalent martingale measure  $\mathbb{Q}$  given by the Girsanov theorem is

$$dS_t = rS_t dt + \sigma S_t dW_t,$$

where  $W$  is a standard Brownian motion. If we denote by  $\mu$  the drift coefficient under  $\mathbb{P}$  we obviously have in this case that

$$M_{t,T} = e^{\frac{\mu-r}{\sigma^2} \left[ \log\left(\frac{S_T}{S_t}\right) + \frac{(\sigma^2-r-\mu)(T-t)}{2} \right]}. \quad (12)$$

Thus, in this case, the unique SDF is an exponential affine function of the log-returns and pricing may easily be done by closed form expressions or numerical tools.

In discrete time, it is well known that markets are in general incomplete (see e.g Elliot and Madan (1998)) thus the martingale measure  $\mathbb{Q}$  is not unique and there exists a multiplicity of SDF that are compatible with the previous pricing formulas.

Once the dynamics under the historical probability have been specified throughout statistical modelings, we may overcome this problem adopting one of the two following equivalent point of view:

- impose some constraints on the form of the SDF,
- choose a particular martingale measure that fulfills some economic or risk criteria (e.g the minimal martingale measure in the sense of Föllmer and Schweizer that minimizes the variance of the hedging loss (1991)).

When this choice has been made, if we know the dynamic of the risky asset under the new probability, it is possible to price contingent claims from (10) or (11) using Monte Carlo simulations.

Following the first approach, Rosenberg and Engle (2002) and Gouriéroux and Monfort (2007) consider several parametric specifications for the SDF as power functions or as exponentials of polynomials of the returns. In particular, Gouriéroux and Monfort (2007) show that the power functions case appears naturally in many classical situations. However, their estimation strategies are quite different.

Rosenberg and Engle (2002) is based on a GARCH-type model with empirical innovations. They minimize the classical mean square error criterion between the obtained Monte Carlo prices and the option market quotes in order to estimate the SDF. Contrary to this semi-parametric approach, the knowledge of the distribution of the log-returns is needed in the Gouriéroux and Monfort (2007) framework: conditional Laplace transforms have to be computed to determine the SDF as explained in the



next section. A very similar approach using a dynamic Gerber-Shiu's argument can be found in Siu et al. (2004) (see also the seminal paper of Bühlmann et al. (1998)): in this framework, the conditional Esscher transform is used in order to select a particular martingale measure in a discrete time setting.

The objective of this paper is to apply this second point of view for GARCH-type models with Generalized Hyperbolic innovations. This choice may be interesting because Esscher transform has been successfully applied in continuous time to price derivatives when the underlying follows an exponential Generalized Hyperbolic Lévy motion (see Eberlein and Keller (1995), Eberlein and Prause (2002)).

### 3.1 Pricing options with exponential affine SDF

The methodology unfolds as follows. We assume for the SDF a particular parametric form:  $\forall t \in \{0, \dots, T-1\}$

$$M_{t,t+1} = e^{\theta_{t+1}Y_{t+1} + \xi_{t+1}} \quad (13)$$

where  $Y_{t+1} = \log\left(\frac{S_{t+1}}{S_t}\right)$  and where  $\theta_{t+1}$  and  $\xi_{t+1}$  are  $\mathcal{F}_t$  measurable random variables.

Recall that in a discrete time version of the Black-Scholes economy, the corresponding SDF is given by  $\theta_{t+1} = \frac{\mu-r}{\sigma^2}$  and  $\xi_{t+1} = \frac{(\sigma^2-r-\mu)(\mu-r)}{2\sigma^2}$  that are independent of  $t$  (see (12)). In particular, the parameter  $\theta$  corresponds to a constant risk aversion. Here, the specification (13) allows for time variation in risk aversion.

We need to compute explicitly  $(\theta_{t+1}, \xi_{t+1})$ . Considering the bond and the risky asset, the pricing relation (11) gives for  $T = t+1$  the following restrictions for the SDF

$$\begin{cases} E_{\mathbb{P}}[e^r M_{t,t+1} | \mathcal{F}_t] = 1 \\ E_{\mathbb{P}}[e^{Y_{t+1}} M_{t,t+1} | \mathcal{F}_t] = 1. \end{cases} \quad (14)$$

For all  $t \in \{0, \dots, T-1\}$ , we denote by  $\mathbb{G}_t$  the conditional moment generating function of  $Y_{t+1}$  given  $\mathcal{F}_t$  defined on a convex set  $\mathcal{D}_{\mathbb{G}_t}$  that is not reduced to  $\{0\}$  and by  $\Theta_t$  the parameter set  $\{\theta \in \mathbb{R}; \theta \text{ and } 1+\theta \in \mathcal{D}_{\mathbb{G}_t}\}$ . We now introduce the mapping  $\Phi_t : \Theta_t \rightarrow \mathbb{R}$  such that

$$\Phi_t(\theta) = \log\left(\frac{\mathbb{G}_t(1+\theta)}{\mathbb{G}_t(\theta)}\right).$$

Thus, the preceding system is equivalent to

$$\begin{cases} \mathbb{G}_t(\theta_{t+1}) = e^{-(r+\xi_{t+1})} \\ \mathbb{G}_t(\theta_{t+1} + 1) = e^{-\xi_{t+1}} \end{cases} \quad (15)$$

and, with our notations, we have to solve

$$\begin{cases} \Phi_t(\theta_{t+1}) = r \\ \mathbb{G}_t(\theta_{t+1} + 1) = e^{-\xi_{t+1}}. \end{cases} \quad (16)$$

The next proposition shows that, under the pricing constraints (14), there is no ambiguity in the choice of SDF (13).

**PROPOSITION 2** *Suppose that  $\mathbb{G}_t$  is twice differentiable. If there exists a solution to the equation  $\Phi_t(\theta) = r$ , it is unique.*

**Proof:** See Gerber and Shiu (1994 b).□

In the remaining, we suppose that (16) leads for each  $t \in \{0, \dots, T-1\}$  to a unique solution denoted by  $(\theta_{t+1}^q, \xi_{t+1}^q)$  (see the next section for the proof of the existence in the case of the GH distribution). The SDF

$$M_{t,t+1} = e^{\theta_{t+1}^q Y_{t+1} + \xi_{t+1}^q} \quad (17)$$

being explicit known, we may deduce easily the form of the associated equivalent martingale measure  $\mathbb{Q}$ . In fact, remarking that  $\forall k \in \{0, \dots, T-1\}$

$$\frac{M_{k,k+1}}{E_{\mathbb{P}}[M_{k,k+1} | \mathcal{F}_k]} = \frac{e^{\theta_{k+1}^q Y_{k+1}}}{\mathbb{G}_k(\theta_{k+1}^q)},$$

we define the stochastic process

$$\left( L_t = \prod_{k=1}^t \frac{e^{\theta_k^q Y_k}}{\mathbb{G}_{k-1}(\theta_k^q)} \right)_{t \in \{1, \dots, T\}} \quad (18)$$

that is obviously a martingale under  $\mathbb{P}$ . Then, we have the following proposition:

**PROPOSITION 3** *Let  $\mathbb{Q}$  be the probability owning the density  $L_T$  with respect to  $\mathbb{P}$ , then,*

*a)  $\mathbb{Q}$  is the unique probability associated to the exponential affine SDF (17), in particular, the discounted stock price process  $(e^{-rt} S_t)_{t \in \{0, \dots, T\}}$  is a martingale under  $\mathbb{Q}$  and the price  $P_t$  at time  $t$  of a European asset paying  $\Phi_T$  at  $T$  is given by*

$$P_t = E_{\mathbb{Q}}[\Phi_T e^{-r(T-t)} | \mathcal{F}_t]. \quad (19)$$

*b) Under  $\mathbb{Q}$ , the moment generating function of  $Y_t$  given  $\mathcal{F}_{t-1}$  is given by*

$$E_{\mathbb{Q}}[e^{uY_t} | \mathcal{F}_{t-1}] = E_{\mathbb{P}}[e^{uY_t} \frac{e^{\theta_t^q Y_t}}{\mathbb{G}_{t-1}(\theta_t^q)} | \mathcal{F}_{t-1}] = \frac{\mathbb{G}_{t-1}(\theta_t^q + u)}{\mathbb{G}_{t-1}(\theta_t^q)}. \quad (20)$$

**Proof:** First, when  $s \leq T$ , remark that for a  $\mathcal{F}_s$  measurable and non negative random variable  $Z$  we may deduce from the martingale property of  $(L_t)$  that

$$E_{\mathbb{Q}}[Z] = E_{\mathbb{P}}[L_s Z].$$

Moreover, for  $T \geq t \geq s > 0$ , if  $Z$  (resp.  $X$ ) is  $\mathcal{F}_s$  (resp.  $\mathcal{F}_t$ ) measurable and non negative then

$$E_{\mathbb{Q}}[XZ] = E_{\mathbb{P}}[L_T XZ] = E_{\mathbb{P}}[E_{\mathbb{P}}[X L_T | \mathcal{F}_s] Z] = E_{\mathbb{P}}[E_{\mathbb{P}}[X E_{\mathbb{P}}[L_T | \mathcal{F}_t] | \mathcal{F}_s] Z]$$

thus

$$E_{\mathbb{Q}}[XZ] = E_{\mathbb{P}}[E_{\mathbb{P}}[L_t X | \mathcal{F}_s] Z] = E_{\mathbb{P}} \left[ \frac{L_s}{L_s} E_{\mathbb{P}}[X L_t | \mathcal{F}_s] Z \right] = E_{\mathbb{Q}} \left[ \frac{1}{L_s} E_{\mathbb{P}}[X L_t | \mathcal{F}_s] Z \right].$$

Hence,

$$E_{\mathbb{Q}}[X | \mathcal{F}_s] = \frac{1}{L_s} E_{\mathbb{P}}[X L_t | \mathcal{F}_s], \quad (21)$$

a) and b) easily follow.  $\square$

Under  $\mathbb{Q}$ , the conditional distribution of  $Y_t$  given  $\mathcal{F}_{t-1}$  is none other than the conditional Esscher transform of parameter  $\theta_t^q$  (in the sense of Siu et al. (2004)) of the distribution of  $Y_t$  given  $\mathcal{F}_{t-1}$  under  $\mathbb{P}$ . Moreover, for pricing purposes, relation (20) is fundamental because it gives explicitly the conditional distribution of the log-returns under  $\mathbb{Q}$  and allows for Monte Carlo simulation methods. Furthermore, as underlined in the next proposition, under mild assumptions, this conditional distribution under  $\mathbb{Q}$  belongs to the same family than under the historical one.

**PROPOSITION 4** *For all  $t \in \{1, \dots, T\}$ , if the conditional distribution of  $Y_t$  given  $\mathcal{F}_{t-1}$  is under  $\mathbb{P}$  infinitely divisible and if  $\mathbb{G}_{t-1}$  is twice differentiable, then, the conditional distribution of  $Y_t$  given  $\mathcal{F}_{t-1}$  is also infinitely divisible under  $\mathbb{Q}$  with finite moment of order 2.*

**Proof:** From the Kolmogorov representation theorem (see e.g Mainardi and Rogosin (2006) for an interesting historical approach of this result), we have for all  $u \in \mathcal{D}_{\mathbb{G}_{t-1}}$ ,

$$\log(\mathbb{G}_{t-1}(u)) = \gamma_t u + \int_{-\infty}^{+\infty} (e^{zu} - 1 - zu) \frac{dK_t(z)}{z^2}$$

where  $\gamma_t$  is a  $\mathcal{F}_{t-1}$  measurable real valued random variable and  $K_t$  a  $\mathcal{F}_{t-1}$  measurable random variable with values in the space of the non-decreasing and bounded functions with limit zero in  $-\infty$ . Thus, from Proposition 3,  $\forall u \in \{\theta \in \mathbb{R}; \theta + \theta_t^q \in \mathcal{D}_{\mathbb{G}_{t-1}}\}$ ,

$$\log(E_{\mathbb{Q}}[e^{uY_t} | \mathcal{F}_{t-1}]) = \gamma_t u + \int_{-\infty}^{+\infty} (e^{z(u+\theta_t^q)} - e^{\theta_t^q z} - zu) \frac{dK_t(z)}{z^2}$$

thus

$$\log(E_{\mathbb{Q}}[e^{uY_t} | \mathcal{F}_{t-1}]) = \tilde{\gamma}_t u + \int_{-\infty}^{+\infty} (e^{zu} - 1 - uz) \frac{e^{\theta_t^q z} dK_t(z)}{z^2} \quad (22)$$

where

$$\tilde{\gamma}_t = \gamma_t + \int_{-\infty}^{+\infty} (e^{\theta_t^q z} - 1) z \frac{dK_t(z)}{z^2}.$$

Since  $\mathbb{G}_{t-1}$  is twice differentiable, we have in particular that

$$\int_{-\infty}^{+\infty} e^{\theta_t^q z} dK_t(z) < \infty$$

and we may define  $\forall x \in \mathbb{R}$ ,

$$\tilde{K}_t(x) = \int_{-\infty}^x e^{\theta_t^q z} dK_t(z)$$

that is a  $\mathcal{F}_{t-1}$  measurable random variable with values in the space of the non-decreasing and bounded functions with limit zero in  $-\infty$ . The conclusion follows from (22) and from the Kolmogorov representation theorem.  $\square$

The preceding result is not so surprising because several authors have already remarked that it is true for particular subclasses of distributions (see Gouriéroux and Monfort (2006), Siu et al. (2004), Christoffersen et al. (2006)). This point is one of the main features of the exponential affine specification of the pricing kernel that is not fulfilled, for example, in the framework of Elliot and Madan (1998). For the GH distributions the stability is proved in the next section.

### 3.2 GH-GARCH option pricing model

Here, we apply the methodology of the preceding section using the GH setting of Section 2.2. Thus, we have to identify (if there exists) the unique exponential affine SDF and describe explicitly the dynamic of the log-returns under the associated equivalent martingale measure.

First, we obtain a result that ensures, under mild conditions, the existence of a solution  $(\theta_{t+1}^q, \xi_{t+1}^q)$  of (16) for all  $t \in \{0, \dots, T-1\}$ .

**PROPOSITION 5** *For a GH( $\lambda, \alpha, \beta, \delta, \mu$ ) distribution with  $\alpha > \frac{1}{2}$ , then,*

a) *If  $\lambda \geq 0$ , the equation  $\log\left(\frac{\mathbb{G}_{GH}(1+\theta)}{\mathbb{G}_{GH}(\theta)}\right) = r$  has a unique solution,*

b) *If  $\lambda < 0$ , the equation  $\log\left(\frac{\mathbb{G}_{GH}(1+\theta)}{\mathbb{G}_{GH}(\theta)}\right) = r$  has a unique solution if and only if  $\mu - C < r < \mu + C$  where*

$$C = \log\left(\frac{\Gamma[-\lambda]}{2^{\lambda+1}}\right) - \log\left(\frac{K_\lambda(\delta\sqrt{\alpha^2 - (\alpha-1)^2})}{\delta\sqrt{\alpha^2 - (\alpha-1)^2}}\right).$$

**Proof:** For  $|\beta + u| < \alpha$ ,

$$\mathbb{G}_{GH}(u) = e^{\mu u} \left(\frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + u)^2}\right)^{\frac{\lambda}{2}} \frac{K_\lambda(\delta\sqrt{\alpha^2 - (\beta + u)^2})}{K_\lambda(\delta\sqrt{\alpha^2 - \beta^2})},$$

hence  $\mathbb{G}_{GH}$  is twice differentiable. Moreover,  $\Phi(\theta) = \log\left(\frac{\mathbb{G}_{GH}(1+\theta)}{\mathbb{G}_{GH}(\theta)}\right)$  is defined on the interval  $]-\alpha + \beta, \alpha - \beta - 1[$  that is not empty because  $\alpha > \frac{1}{2}$ . Thus we may apply

Proposition 2 and the unicity holds. It remains to prove the existence.

a) For  $x > 0$ , we define  $\Psi(x) = \log\left(\frac{K_\lambda(x)}{x^\lambda}\right)$ . Thus,

$$\Phi(\theta) = \mu + \Psi(\delta\sqrt{\alpha^2 - (\beta + 1 + \theta)^2}) - \Psi(\delta\sqrt{\alpha^2 - (\beta + \theta)^2}).$$

For the properties of the Bessel function used in the sequel we refer the reader to Abramowitz and Stegun (1964).

If  $\lambda > 0$ ,

$$\frac{K_\lambda(x)}{x^\lambda} \sim_{x \rightarrow 0+} \frac{\Gamma[\lambda]2^{\lambda-1}}{x^{2\lambda}}.$$

So we have  $\lim_{\theta \rightarrow \alpha - \beta - 1} \Phi(\theta) = +\infty$  and  $\lim_{\theta \rightarrow -(\alpha + \beta)} \Phi(\theta) = -\infty$ . The conclusion follows from the intermediate value theorem.

When  $\lambda = 0$ , we may conclude as before, remembering that  $K_0(x) \sim_{x \rightarrow 0+} -\log(x/2) - \gamma$  where  $\gamma$  is the Euler-Mascheroni constant.

b) When  $\lambda < 0$  using the relation  $K_\lambda(x) = K_{-\lambda}(x)$  we obtain that  $\frac{K_\lambda(x)}{x^\lambda} \sim_{x \rightarrow 0+} \Gamma[-\lambda]2^{-\lambda-1}$ . Thus,  $\lim_{\theta \rightarrow \alpha - \beta - 1} \Phi(\theta) = \mu + C$  and  $\lim_{\theta \rightarrow -(\alpha + \beta)} \Phi(\theta) = \mu - C$  and we conclude applying again the intermediate value theorem.  $\square$

**Remark:** The constant  $C$  is strictly positive because  $\frac{d}{dx} \frac{K_\lambda(x)}{x^\lambda} = -\frac{K_{\lambda+1}(x)}{x^\lambda} < 0$ .

Even if we are not able to obtain a closed form formula for the solution of (16), we may apply the risk neutral valuation presented in Section 3.1. For practical cases  $(\theta_{t+1}^q, \xi_{t+1}^q)$  may be computed efficiently using e.g. a refined bracketing method.

The next proposition describes the dynamic of the risky asset under the chosen equivalent martingale measure.

**PROPOSITION 6** *Under  $\mathbb{Q}$ , the distribution of  $Y_t$  given  $\mathcal{F}_{t-1}$  is a*

$$GH\left(\lambda, \frac{\alpha}{\Sigma_t}, \frac{\beta}{\Sigma_t} + \theta_t^q, \delta\Sigma_t, M_t + \mu\Sigma_t\right) \quad (23)$$

where  $M_t = r + m_t$  and  $\Sigma_t = \sqrt{h_t}$ .

**Proof:** It is a direct consequence of the Proposition 3.b).  $\square$

It is interesting to notice that the parameter  $\xi_t^q$  in the SDF (13) does not appear in the stock prices dynamic under the equivalent martingale measure  $\mathbb{Q}$  (23). Moreover, the appearance of  $\theta_t^q$  in (23) induces not only a shift in the skewness of the GH distribution but also an excess kurtosis (exact values of the skewness and kurtosis are provided in Barndorff-Nielsen and Blaesild (1981)).

We deduce from the preceding result that, under  $\mathbb{Q}$ ,

$$Y_t = r + m_t + \underbrace{\sqrt{h_t} z_t}_{\varepsilon_t}, \quad S_0 = s, \quad (24)$$

where the  $z_t$  are  $\mathcal{F}_t$  measurable random variables such that, conditionally to  $\mathcal{F}_{t-1}$ ,

$$z_t \hookrightarrow GH(\lambda, \alpha, \beta + \sqrt{h_t} \theta_t^q, \delta, \mu). \quad (25)$$

In particular, the GH distribution is stable under the change of measure allowing us to simulate easily the sample paths of the risky asset.

Under  $\mathbb{Q}$ , conditionally to  $\mathcal{F}_{t-1}$ ,  $\varepsilon_t$  is no more centered and its variance is not  $h_t$  but

$$\text{var}(\varepsilon_t) = h_t \left( \frac{\delta K_{\lambda+1}(\delta\gamma)}{\gamma K_{\lambda}(\delta\gamma)} + \frac{(\beta + \Sigma_t \theta_t^q)^2 \delta^2}{\gamma_t^2} \left( \frac{K_{\lambda+2}(\delta\gamma_t)}{K_{\lambda}(\delta\gamma_t)} - \frac{K_{\lambda+1}^2(\delta\gamma_t)}{K_{\lambda}^2(\delta\gamma_t)} \right) \right),$$

where  $\gamma_t = \sqrt{\alpha^2 - (\beta + \Sigma_t \theta_t^q)^2}$ . Thus the GARCH structure of the volatility is modified in a nonlinear way from  $\mathbb{P}$  to  $\mathbb{Q}$ .

Several classical distributions e.g. Normal, Skewed Laplace, Gamma or Inverse Gaussian may be obtained as a limiting case (in the sense of the convergence in distribution or for the Wasserstein distance) of the GH distribution (see Eberlein and Hammerstein (2004) for details). Thus, it may be seen easily that the risk neutral dynamics of Duan (1995), Heston and Nandi (2000), Gouriéroux and Monfort (2006), Siu et al. (2004), Christoffersen et al. (2006) may be recovered with an appropriate choice of  $F$  and  $G$  in (5). In particular, as remarked in Siu et al. (2004), the exponential affine specification of the SDF give rise in the normal case to the same results than the ones obtained using the classical locally risk-neutral valuation relationship of Duan (1995). Moreover in all these cases we have explicit solutions for (16) due to the special forms of the considered densities

## 4 Conclusion and forthcoming empirical results

In this article we present a GARCH-type model with Generalized Hyperbolic innovations in order to price contingent claims. Supposing an exponential affine parametrization for the stochastic discount factor, we show that, under the risk neutral probability, the conditional distribution of the log returns remains a Generalized Hyperbolic one with an explicit form. An extensive Monte Carlo study on four world indexes is ongoing to compare this model to its natural competitors in particular to test its efficiency to cope with the skewness effect. More generally, our empirical investigations will analyze the ability of parametric innovations to reproduce market prices in the context of the exponential affine specification of the stochastic discount factor. The results will be presented in a forthcoming paper (Chorro, Guégan and Ielpo (2008)).

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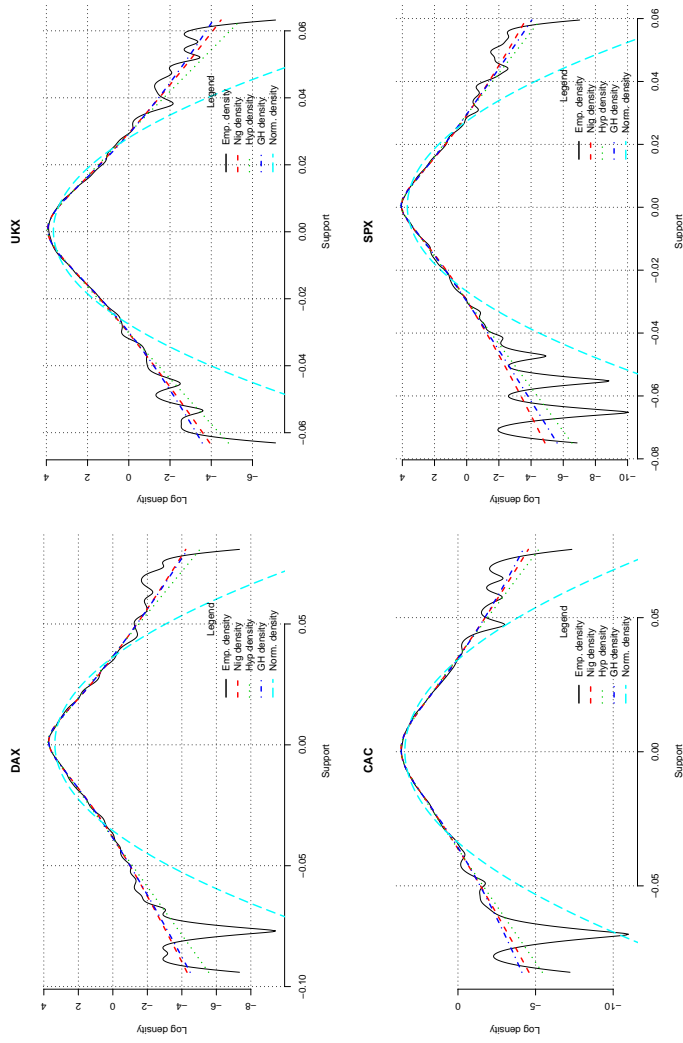


Figure 1: Log density

This figure presents the empirical log density (plain black line) vs. the estimated log density obtained with the NIG (red), Hyperbolic (green), Generalized Hyperbolic (dark blue) and Gaussian (light blue) distributions over different indexes. The estimation has been performed using the daily log returns of four major indexes: the French CAC (bottom left), the German DAX (top left), the US SP500 (bottom right) and the UK FTSE indexes (top right). The sample starts on January, 2nd of 1988 and ends on the December, 31st of 2007.

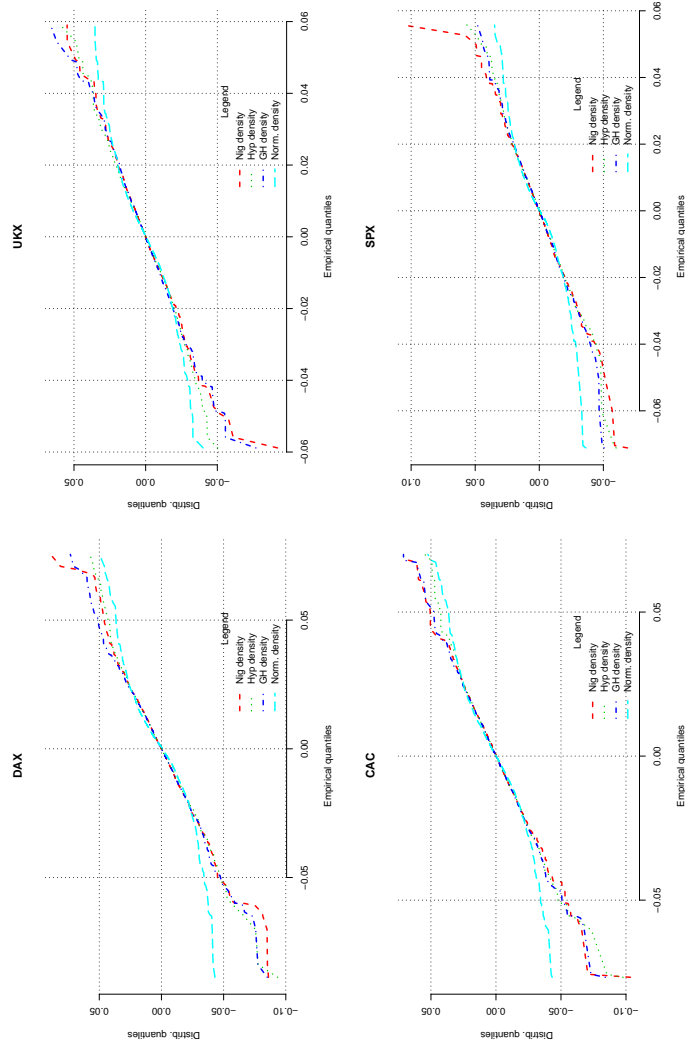


Figure 2: QQ-Plot

This figure presents the qq-plots comparing the empirical quantiles over several indexes vs. the estimated quantiles obtained with the NiG (red), Hyperbolic (green), Generalized Hyperbolic (dark blue) and Gaussian (light blue) distributions. The estimation has been performed using the daily log returns of four major indexes: the French CAC (bottom left), the German DAX (top left), the US SP500 (bottom right) and the UK FTSE indexes (top right). The sample starts on January, 2nd of 1988 and ends on the December, 31st of 2007.

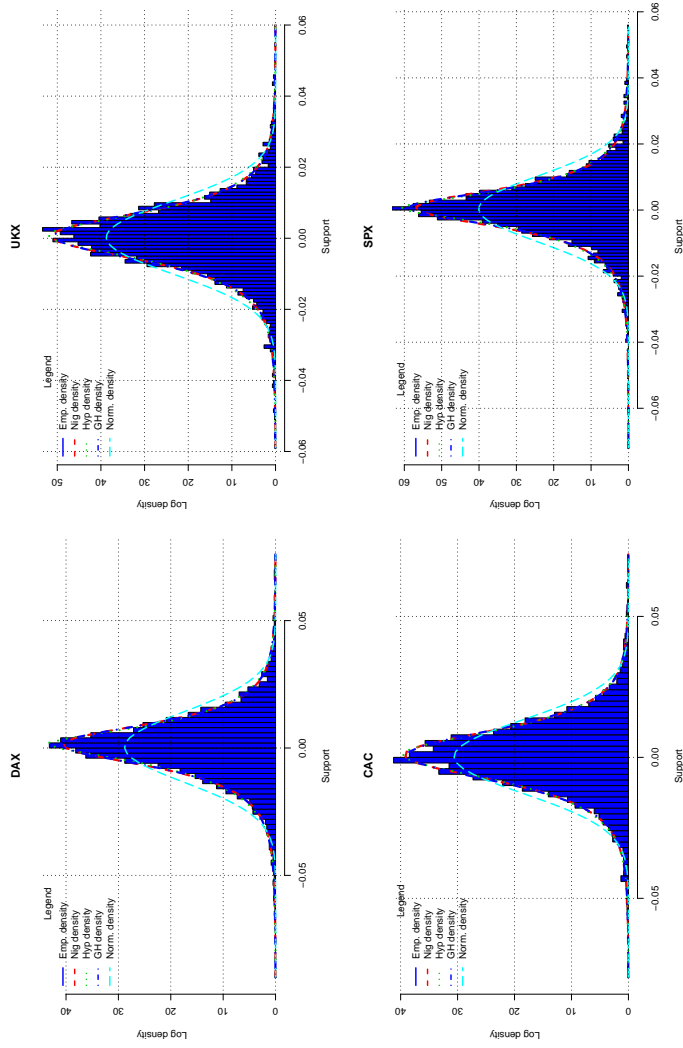


Figure 3: Empirical histogram

This figure presents the empirical density of several indexes' returns vs. the estimated densities obtained with the NIG (red), Hyperbolic (green), Generalized Hyperbolic (dark blue) and Gaussian (light blue) distributions. The estimation has been performed using the daily log returns of four major indexes: the French CAC (bottom left), the German DAX (bottom right), the US SP500 (top left) and the UK FTSE indexes (top right). The sample starts on January, 2nd of 1988 and ends on the December, 31st of 2007.

	<b>DAX</b>	<b>CAC</b>	<b>UKX</b>	<b>SPX</b>
<b>KS p-value for NIG</b>	0,32	0,3	0,83	0,67
<b>KS p-value for HYP</b>	0,95	0,73	0,53	0,69
<b>KS p-value for GH</b>	0,48	0,31	0,69	0,78
<b>KS p-value for Gaussian</b>	0	0	0	0
<b>AD p-value for NIG</b>	0,28	0,45	0,55	0,5
<b>AD p-value for HYP</b>	0,61	0,5	0,48	0,47
<b>AD p-value for GH</b>	0,45	0,23	0,45	0,55
<b>AD p-value for Gaussian</b>	0	0	0	0

Table 1: Kolmogorov-Smirnov and Andersen-Darling adequation tests

This table presents the Kolmogorov- Smirnov and Andersen-Darling adequation tests, testing the ad-equation of the NIG, Hyperbolic, Generalized Hyperbolic and Gaussian distributions to a dataset of the daily log returns of four major indexes: the French CAC, the German DAX, the US SP500 and the UK FTSE indexes. The sample starts on January, 2nd of 1988 and ends on the December, 31st of 2007.