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Decomposing Berge graphs and detecting balanced skew partitions

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Abstract

A hole in a graph is an induced cycle on at least four vertices. A graph is Berge if it has no odd hole and if its complement has no odd hole. In 2002, Chudnovsky, Robertson, Seymour and Thomas proved a decomposition theorem for Berge graphs saying that every Berge graph either is in a well understood basic class or has some kind of decomposition. Then, Chudnovsky proved stronger theorems. One of them restricts the allowed decompositions to 2-joins and balanced skew partitions.

We prove that the problem of deciding whether a graph has a balanced skew partition is NP-hard. We give an $O(n^9)$-time algorithm for the same problem restricted to Berge graphs. Our algorithm is not constructive: it certifies that a graph has a balanced skew partition if it has one. It relies on a new decomposition theorem for Berge graphs, that is more precise than the previously known theorems and implies them easily. Our theorem also implies that every Berge graph can be decomposed in a first step by using only balanced skew partitions, and in a second step by using only 2-joins. Our proof of this new theorem uses at an essential step one of the theorems of Chudnovsky.

AMS Mathematics Subject Classification: 05C17, 05C75
Key words: perfect graph, Berge graph, 2-join, balanced skew partition, decomposition, detection, recognition.

1 Introduction

In this paper graphs are simple and finite. A *hole* in a graph is an induced cycle of length at least 4. An *antihole* is the complement of a hole. A graph is said to be Berge if it has no odd hole and no odd antihole. A graph G is said to be *perfect* if for every induced subgraph G' the chromatic number of G' is equal to the maximum size of a clique of G'. In 1961, Berge [2] conjectured that every Berge graph is

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perfect. This was known as the *Strong Perfect Graph Conjecture*, was the object of much research and was finally proved by Chudnovsky, Robertson, Seymour and Thomas in 2002 [7]. In fact, they proved a stronger result: a decomposition theorem, first conjectured by Conforti, Cornuéjols and Vušković [11], stating that every Berge graph is either in a well understood basic class of perfect graph, or has a structural fault that cannot occur in a minimum counter-example to Strong Perfect Graph Conjecture. Before stating this decomposition theorem, we need some definitions.

We call *path* any connected graph with at least a vertex of degree 1 and no vertex of degree greater than 2. A path has at most two vertices of degree 1 that are the *ends* of the path. If a, b are the ends of a path P we say that P is *from a to b*. The other vertices are the *interior* vertices of the path. We denote by $v_1 - \cdots - v_n$ the path whose edge set is $\{v_1v_2, \ldots, v_{n-1}v_n\}$. When P is a path, we say that P is *a path of* G if P is an induced subgraph of G. If P is a path and if a, b are two vertices of P then we denote by $a-P-b$ the only induced subgraph of P that is path from a to b. The *length* of a path is the number of its edges. An *antipath* is the complement of a path. Let G be a graph and let A and B be two subsets of $V(G)$. A path of G is said to be *outgoing from A to B* if it has an end in A, an end in B, length at least 2, and no interior vertex in $A \cup B$.

If $X, Y \subset V(G)$ are disjoint, we say that X is *complete* to Y if every vertex in X is adjacent to every vertex in Y. We also say that (X, Y) is a *complete pair*. We say that X is *anticomplete* to Y if there are no edges between X and Y. We also say that (X, Y) is an *anticomplete pair*. We say that a graph G is anticonnected if its complement $\overline{G}$ is connected.

Skew partitions were first introduced by Chvátal [8]. A *skew partition* of a graph $G = (V, E)$ is a partition of V into two sets A and B such that A induces a graph that is not connected, and B induces a graph that is not anticonnected. When A_1, A_2, B_1, B_2 are non-empty sets such that (A_1, A_2) partitions A, (A_1, A_2) is an anticomplete pair, (B_1, B_2) partitions B, and (B_1, B_2) is a complete pair, we say that (A_1, A_2, B_1, B_2) is a *split* of the skew partition (A, B). A *balanced skew partition* (first defined in [7]) is a skew partition (A, B) with the additional property that every induced path of length at least 2 with ends in B, interior in A and every antipath of length at least 2 with ends in A, interior in B have even length. If (A, B) is a skew partition, we say that B is a *skew cutset*. If (A, B) is balanced we say that the skew cutset B is *balanced*. Note that Chudnovsky et al. [7] proved that no minimum counter example to the strong perfect graph conjecture has a balanced skew partition.

We call *double split graph* (first defined in [7]) any graph G that may be constructed as follows. Let $m, n \geq 2$ be integers. Let $A = \{a_1, \ldots, a_m\}$, $B = \{b_1, \ldots, b_m\}$, $C = \{c_1, \ldots, c_n\}$, $D = \{d_1, \ldots, d_n\}$ be four disjoint sets. Let G have vertex set $A \cup B \cup C \cup D$ and edges in such a way that:

- a_i is adjacent to b_i for $1 \leq i \leq m$. There are no edges between $\{a_i, b_i\}$ and $\{a_{i'}, b_{i'}\}$ for $1 \leq i < i' \leq m$;
- c_j is non-adjacent to d_j for $1 \leq j \leq n$. There are all four edges between $\{c_j, d_j\}$ and $\{c_{j'}, b_{j'}\}$ for $1 \leq j < j' \leq n$;

- there are exactly two edges between $\{a_i, b_i\}$ and $\{c_j, d_j\}$ for $1 \leq i \leq m$, $1 \leq j \leq n$ and these two edges are disjoint.

Note that $C \cup D$ is a non-balanced skew cutset of G and that $\overline{G}$ is a double split graph. Note that in a double split graph, vertices in $A \cup B$ all have degree $n+1$ and vertices in $C \cup D$ all have degree $2n+m-2$. Since $n \geq 2, m \geq 2$ implies $2n-2+m > 1+n$, it is clear that given a double split graph it is relevant to consider the *matching edges*, that have an end in A and an end in B, independently of the choice of the sets A, B, C, D.

A graph is said to be *basic* if one of $G, \overline{G}$ is either a bipartite graph, the line-graph of a bipartite graph or a double split graph.

The 2-join was first defined by Cornuéjols and Cunningham [13]. We say that a partition (X_1, X_2) of the vertex set is a *2-join* when there exist disjoint non-empty $A_i, B_i \subseteq X_i$ $(i = 1, 2)$ satisfying:

- every vertex of A_1 is adjacent to every vertex of A_2 and every vertex of B_1 is adjacent to every vertex of B_2;
- there are no other edges between X_1 and X_2.

The sets X_1, X_2 are the two *sides* of the 2-join. When sets A_i's B_i's are like in the definition we say that $(X_1, X_2, A_1, B_1, A_2, B_2)$ is a *split* of (X_1, X_2). Implicitly, for $i = 1, 2$, we will denote by C_i the set $X_i \setminus (A_i \cup B_i)$.

A 2-join (X_1, X_2) in a graph G is said to be *connected* when for $i = 1, 2$, every component of $G[X_i]$ meets both A_i and B_i. A 2-join (X_1, X_2) is said to be *substantial* when for $i = 1, 2$, $|X_i| \geq 3$ and X_i is not a path of length 2 with an end in A_i, an end in B_i and its unique interior vertex in C_i. A 2-join (X_1, X_2) in a graph G is said to be *proper* when it is connected and substantial.

A 2-join is said to be a *path 2-join* if it has a split $(X_1, X_2, A_1, B_1, A_2, B_2)$ such that $G[X_1]$ is a path with an end in A_1, an end in B_1 and interior in C_1. Implicitly we will then denote by a_1 the unique vertex in A_1 and by b_1 the unique vertex in B_1. We say that X_1 is the *path-side* of the 2-join. Note that when G is not a hole then this path-side is unique. A *non-path 2-join* is a 2-join that is not a path 2-join.

The homogeneous pair was first defined by Chvátal and Sbihi [9]. The definition that we give here is a slight variation used in [7]. An *homogeneous pair* is a partition of $V(G)$ into six non-empty sets (A, B, C, D, E, F) such that:

- every vertex in A has a neighbor in B and a non-neighbor in B, and vice versa;
- the pairs (C, A), (A, F), (F, B), (B, D) are complete;
- the pairs (D, A), (A, E), (E, B), (B, C) are anticomplete.

A graph G is path-cobipartite if it is a Berge graph obtained by subdividing an edge between the two cliques that partitions a cobipartite graph. More accurately, a graph is *path-cobipartite* if its vertex set can be partitioned into three sets A, B, P where A and B are non-empty cliques and P consist of vertices of degree 2, each of which belongs to the interior of a unique path of odd length with one end a

in A, the other one b in B. Moreover, a has neighbors only in $A \cup P$ and b has neighbors only in $B \cup P$. Note that a path-cobipartite graph such that P is empty is the complement of bipartite graph. Note that our path-cobipartite graphs are simply the complement of the *path-bipartite* graphs defined by Chudnovsky in [5]. For convenience, we prefer to think about them in the complement as we do.

A *cutset* is a graph G is a set $C \subset V(G)$ such that $G \setminus C$ is disconnected ($G \setminus C$ means $G[V(G) \setminus C]$). A *double star* in a graph is a subset D of the vertices such that there is an edge ab in $G[D]$ satisfying: $D \subset N(a) \cup N(b)$.

Now we can state the known decomposition theorems of Berge graphs. The first decomposition theorem for Berge graph ever proved is the following:

Theorem 1.1 (Conforti, Cornuéjols and Vušković, 2001, [12]) *Every graph with no odd hole is either basic or has a proper 2-join or has a double star cutset.*

It could be thought that this theorem is useless to prove the Strong Perfect Graph Theorem since there are minimal imperfect graphs that have double star cutsets: the odd antiholes of length at least 7. However, by the Strong Perfect Graph Theorem, we know that the following fact is true: for any minimal non-perfect graph G, one of $G, \overline{G}$ has no double star cutset. A direct proof of this — of which we have no idea — would yield together with Theorem 1.1 a new proof of the Strong Perfect Graph Theorem.

The following theorem was first conjectured in a slightly different form by Conforti, Cornuéjols and Vušković, who proved it in the particular case of square-free graphs [11]. A corollary of it is the Strong Perfect Graph Theorem.

Theorem 1.2 (Chudnovsky, Robertson, Seymour and Thomas, 2002, [7]) *Let G be a Berge graph. Then either G is basic or G has an homogeneous pair, or G has a balanced skew partition or one of $G, \overline{G}$ has a proper 2-join.*

The two theorems that we state now are due to Chudnovsky who proved them from scratch, that is without assuming Theorem 1.2. Her proof uses the notion of *trigraph*. The first theorem shows that homogeneous pairs are not necessary to decompose Berge graphs. Thus it is a result stronger than Theorem 1.2. The second one shows that path 2-joins are not necessary to decompose Berge graphs, but at the price of extending balanced skew partitions to general skew partitions and introducing a new basic class. Note that a third theorem can be obtained by viewing the second one in the complement of G.

Theorem 1.3 (Chudnovsky, 2003, [4, 5]) *Let G be a Berge graph. Then either G is basic, or one of $G, \overline{G}$ has a proper 2-join or G has a balanced skew partition.*

Theorem 1.4 (Chudnovsky, 2003, [5]) *Let G be a Berge graph. Then either G is basic, or one of $G, \overline{G}$ is path-bipartite, or G has a proper non-path 2-join, or $\overline{G}$ has a proper 2-join or G has a skew partition.*

Main results and Motivation

Our main result is Theorem 2.1, a new decomposition for Berge graph that is a generalisation of Theorems 1.2, 1.3 and 1.4. Note that our proof of Theorem 2.1 is not a new proof of the previously known decomposition theorems for Berge graphs, since it uses at an essential step Theorem 1.3. We also give algorithmic applications. Figueiredo, Klein, Kohayakawa and Reed devised an algorithm that given a graph G computes in polynomial time a skew partition if G has one [14]. But the problem of detecting *balanced* skew partitions has not been studied so far. Let us call BSPD the decision problem whose input is a graph and whose answer is YES if the graph has a balanced skew partition and NO otherwise. Using a construction due to Bienstock [3], we prove that BSPD is NP-hard (we are not able to prove that BSPD in NP or in CoNP). Using Theorem 2.1 we give an $O(n^9)$-time algorithm for BSPD restricted to Berge graphs.

In 2002, Chudnovsky, Cornuéjols, Liu, Seymour and Vušković [6] gave an algorithm that recognizes Berge graphs in time $O(n^9)$. This algorithm may be used to prove that, when restricted to Berge graphs, BSPD is in NP. Indeed, a balanced skew partition is a good certificate for BSPD: given a Berge graph and a partition (A, B) of its vertices, one can easily check that (A, B) is a skew partition; to check that it is balanced, it suffices to add a vertex adjacent to every vertex of B, to no vertex of A, and to check that this new graph is still Berge.

Proving that BSPD is in fact in P by a decomposition theorem uses a classical idea, used for instance in [10] to check whether a given graph has or not an even hole. First, solve BSPD for each class of basic graph. This is done in Section 5 in time $O(n^5)$. Note that bipartite graphs are the most difficult to handle efficiently. For them, we use an algorithm due to Reed [18]. For a graph G such that one of $G, \overline{G}$ has a 2-join, try to break G into smaller blocks in such a way that G has a balanced skew partition if and only if one of the blocks has one, allowing us to recurse. And when a graph is not basic and has no 2-join, simply answer "the graph has a balanced skew partition", the correct answer because of the decomposition theorem. This blind use of decomposition is not safe from criticism, but this will be discussed later.

Unfortunately, with the usual notions of 2-join and blocks, this approach does not work. Building the blocks of a 2-join preserves existing balanced skew partitions, but some 2-joins can create balanced skew partitions when building the blocks. In the graph depicted Fig. 1 on the left, we have to simplify somehow the left part of the obvious 2-join to build one of the blocks. The most reasonable way to do so seems to be replacing X_1 by a path of length 1. But this creates a skew cutset: the black vertices on the right. Of course, this graph is bipartite but one can imagine more complicated examples based on the same template and another template exists. These bad 2-joins will be described in more details in Section 2 and called *cutting 2-joins*. All of them are path 2-joins.

Theorem 2.1 shows that cutting 2-joins are not necessary to decompose Berge graphs. A more general statement is proved, that makes use of a new basic class and of a new kind of decomposition that are quite long to describe. But an interesting corollary can be stated with no new notions. By *contracting a path P* that is the

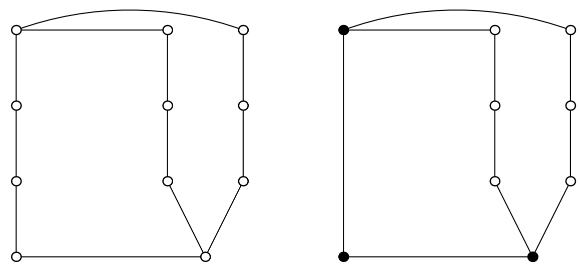

Figure 1: Contracting a path creates a skew cutset

side of a proper path 2-join of a graph we mean delete the interior vertices of P, and link the ends of P with a path of length 1 or 2 according to the original parity of the length of P.

Theorem 1.5 *Let G be a Berge graph. Then either:*

- *G is basic;*
- *one of $G, \overline{G}$ has a non-path proper 2-join;*
- *G has no balanced skew partition and exactly one of $G, \overline{G}$ (say G) has at least a proper path 2-join. Moreover, for every proper path 2-join of G, the graph obtained by contracting its path-side has no balanced skew partition;*
- *G has a balanced skew partition.*

The algorithm for detecting balanced skew partitions is now easy to sketch. Since the balanced skew partition is a self-complementary notion, we may switch from the graph to its complement as often as needed. First check whether the input graph is basic, and if so look directly for a balanced skew partition. Else, try to decompose along non-path 2-joins (they preserve the existence of balanced skew partitions). If there are none of them, try to decompose along path 2-joins (possibly, this creates balanced skew partitions but do not destroy them). At the end of this process, one of the leaf of the decomposition tree has a balanced skew partition if and only if the root has one. Note that a balanced skew partition in a leaf may have been created by the contraction of a cutting 2-join since such 2-joins do exist (we are not able to recognize all of them, it seems to be a difficult task). But Theorem 1.5 shows that when such a bad contraction occurs, the graph has anyway a balanced skew cutset somewhere. The proof of correctness and complexity analysis are given in Section 5.

Theorem 1.5 gives a structural description of Berge graphs that have no balanced skew partitions: these graphs can be decomposed along 2-joins till reaching basic graphs. This could be used to solve algorithmic problems for the class of Berge graphs with no balanced skew partitions (together with the Berge graphs recognition algorithm [6], our work solves the recognition in $O(n^9)$). Note that this class has an unusual feature in the field of perfect graphs: it is not closed under taking induced subgraph. Theorem 1.5 also gives a structural information on every Berge graph: it can be decomposed in a first step by using only balanced skew partitions, and in a second step by using only 2-joins, possibly in the complement.

Let us come back to the weak point of our recognition algorithm: when it answers "the graph has a balanced skew-partition" using blindly some decomposition theorem. This weakness is the reason why we are not able to find explicitly a balanced skew partition when there is one. However, our results shows that an explicit algorithm might exist. The proof of Theorem 1.2 or Theorem 1.3 might contain its main steps and ideas. However, we would like to point out that if someone manage to read algorithmically the proof of Theorem 1.2 or of Theorem 1.3, (s)he will probably end up with an algorithm that given a graph, either finds an odd hole/antihole, or certifies that the graph is basic, or finds some decomposition. If the decomposition found is not a balanced skew partition, the algorithm will probably not certify that there is no balanced skew partition in the graph, and thus BSPD will not be solved entirely. To solve it, one will still have to think about the detection of balanced skew partitions in basic graphs, and in graphs having a 2-join: this is what we are doing here. Thus an effective algorithm might have to use much of the present work.

This paper answers in some respect questions asked by several authors, for instance the problem of how 2-joins and balanced skew partitions interact in Berge graphs. See [1] where a section is devoted to open problems about skew partitions. One of them is the fast detection of general skew partitions in Berge graphs. This has been solved for basic graphs by Reed [18], so a decomposition based approach might work. Moreover, at first glance, general skew partitions seem easier than balanced skew partitions: in general graphs the first ones are polynomial [14] to detect while the second ones are NP-hard. However, at Subsection 4.3 we explain why our work does not improve the general skew partition detection in Berge graphs, why we are not able to prove Theorem 1.5 with "skew partition" instead of "balanced skew partition". Rather than a failure, we consider this as a further indication that balanced skew partition is the relevant decomposition for Berge graphs.

Section 2 gives the new definitions necessary to state properly Theorem 2.1, states it, and explain why it is a generalisation of the previously known decomposition theorems for Berge graphs. Section 3 gives some useful technical lemmas and study how 2-joins and balanced skew partitions can overlap in a Berge graph. Section 4 gives the proof of Theorems 2.1 and of its corollary 1.5. Section 5 describes the algorithms announced above. Section 6 proves that BSPD is NP-hard.

2 The decomposition theorem

We call *flat path of a graph G* any path whose interior vertices all have degree 2 in G and whose ends have no common neighbors outside of the path.

We call path-double split graph any graph obtained from a double split graph G by subdividing matching edges of G into paths of odd length. Note that a double split graph is a path-double split graph. More accurately, a *path-double split graph* is any graph G that may be constructed as follows. Let $m, n \geq 2$ be integers. Let $A = \{a_1, \dots, a_m\}$, $B = \{b_1, \dots, b_m\}$, $C = \{c_1, \dots, c_n\}$, $D = \{d_1, \dots, d_n\}$ be four disjoint sets. Let E be another possibly empty set disjoint from A, B, C, D. Let G have vertex set $A \cup B \cup C \cup D \cup E$ and edges in such a way that:

- for every vertex v in E, v has degree 2 and there exists $i \in \{1, \dots m\}$ such that

v lies on a path of odd length from a_i to b_i;

- for $1 \leq i \leq m$, there is a unique path of odd length (possibly 1) between a_i and b_i whose interior is in E. There are no edges between $\{a_i, b_i\}$ and $\{a_{i'}, b_{i'}\}$ for $1 \leq i < i' \leq m$;

- c_j is non-adjacent to d_j for $1 \leq j \leq n$. There are all four edges between $\{c_j, d_j\}$ and $\{c_{j'}, b_{j'}\}$ for $1 \leq j < j' \leq n$;

- there are exactly two edges between $\{a_i, b_i\}$ and $\{c_j, d_j\}$ for $1 \leq i \leq m$, $1 \leq j \leq n$ and these two edges are disjoint.

Now we turn our attention to types of 2-join whose contraction may create balanced skew partitions:

- A 2-join is said to be *cutting of type 1* if it has a split (X_1, X_2, A_1, B_1, A_2, B_2) such that:

  1. (X_1, X_2) is a path 2-join with path-side X_1;
  2. $G[X_2 \setminus A_2]$ is disconnected.

- A 2-join is said to be *cutting of type 2* if it has a split (X_1, X_2, A_1, B_1, A_2, B_2) such that there exist sets A_3, B_3 satisfying:

  1. (X_1, X_2) is a path 2-join with path-side X_1;
  2. $A_3 \neq \emptyset$, $B_3 \neq \emptyset$, $A_3 \subset A_2$, $B_3 \subset B_2$;
  3. A_3 is complete to B_3;
  4. every outgoing path from $B_3 \cup \{a_1\}$ to $B_3 \cup \{a_1\}$ (resp. from $A_3 \cup \{b_1\}$ to $A_3 \cup \{b_1\}$) has even length;
  5. every antipath with its ends outside of $B_3 \cup \{a_1\}$ (resp. $A_3 \cup \{b_1\}$) and its interior in $B_3 \cup \{a_1\}$ (resp. $A_3 \cup \{b_1\}$) has even length;
  6. $G \setminus (X_1 \cup A_3 \cup B_3)$ is disconnected.

- A 2-join is said to be *cutting* if it is either cutting of type 1 or cutting of type 2.

An *homogeneous 2-join* is a partition of $V(G)$ into six non-empty sets (A, B, C, D, E, F) such that:

- (A, B, C, D, E, F) is an homogeneous pair;

- every vertex in E has degree 2 and belongs to a flat path of odd length with an end in C, an end in D and whose interior is in E;

- every flat path outgoing from C to D and whose interior is in E is the path-side of a non-cutting proper 2-join of G.

Our main result is the following:

Theorem 2.1 *Let G be a Berge graph. Then either G is basic, or one of $G, \overline{G}$ is a path-cobipartite graph, or one of $G, \overline{G}$ is a path-double split graph, or one of $G, \overline{G}$ has an homogeneous 2-join, or one of $G, \overline{G}$ has a non-path proper 2-join, or G has a balanced skew partition.*

This theorem generalises Theorems 1.2, 1.3 and 1.4: path-cobipartite graphs may be seen either as graphs having a proper path 2-join (Theorems 1.2 and 1.3) or as a new basic class (Theorem 1.4). Path-double split graphs may be seen as graphs having a proper path 2-join (Theorems 1.2 and 1.3) or as graphs having a non-balanced skew partition (Theorem 1.4). And graphs having an homogeneous 2-join may be seen as graphs having an homogeneous pair (Theorems 1.4 and perhaps 1.2) or as graphs having a proper path 2-join (Theorems 1.3 and perhaps 1.2). Formally all these remarks are not always true: it may happen in special cases that path-cobipartite graphs and path-double split graphs have no proper 2-join. But such graphs are established in Lemma 3.4 to be basic or to have a balanced skew partition.

3 Lemmas

The following is a useful characterization of line-graphs of bipartite graphs:

Theorem 3.1 (Harary and Holzmann [16]) *G is the line-graph of a bipartite graph if and only if G contains no odd hole, no claw and no diamond as induced subgraphs.*

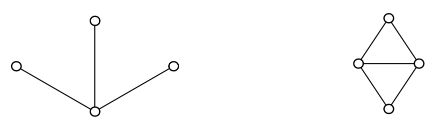

Figure 2: A claw and a diamond

The following fact is clear and useful:

Lemma 3.2 *If (A, B) is a balanced skew partition of a graph G then (B, A) is a balanced skew partition of $\overline{G}$. In particular, a graph G has a balanced skew partition if and only if $\overline{G}$ has a balanced skew partition.*

A *star* in a graph is a set of vertices B such that there is a vertex x in B, called a *center* of the star, seeing every vertex of $B \setminus x$. Note that a star cutset of size at least 2 is a skew cutset.

Lemma 3.3 *Let G be a Berge graph of size at least 4, with at least an edge and that is not the complement of C_4. If G has a star cutset then G has a balanced skew partition*

PROOF — Let B be a star cutset of G. Let us suppose $|B|$ being maximum with that property. Let A_1, A_2 be such that A_1, A_2, B are pairwise disjoint, there are no edges between A_1, A_2, and $A_1 \cup A_2 \cup B = V(G)$.

Suppose first that B has size 1. Thus up to a symmetry $|A_1| \geq 2$ since G has at least 4 vertices. There is no edge between B and A_1 for otherwise such an edge would be a cutset contradicting $|B|$ being maximum. There is no edge in A_2 since such an edge would be a cutset of G. If there is no edge in A_1, any edge of G is a cutset of G. So, there is an edge e in A_1. So, $|A_1| = 2$ and B is complete to A_2 for otherwise, e is a cutset of G. So, $|A_2| = 1$ for otherwise, any edge between B and A_2 is a cutset edge of G. Now, we observe that G is the complement of C_4.

If B has size at least 2 then B is a skew cutset of G. Let x be a center of B. By maximality of B, every component of $G \setminus B$ has either size 1 or contains no neighbor of x. Thus, if P is a path that makes the skew cutset B non-balanced, then $P \cup x$ induces an odd hole of G. If Q is an antipath that makes the skew cutset B non-balanced, then $Q \cup x$ induces an odd antihole of G. □

The following lemma is useful to establish formally that Theorem 2.1 really implies Theorems 1.2, 1.3 and 1.4. But we also need it at several places in the next section.

Lemma 3.4 *Let G be a Berge graph. Then:*

- *If G has a flat path P of length at least 3 then either G is bipartite, or G has a balanced skew partition or P is the path-side of a proper path 2-join of G.*
- *If G is a path-cobipartite graph, a path-double split graph or has an homogeneous 2-join, then either G has a proper 2-join or G has a balanced skew partition or G is a bipartite graph, the complement of a bipartite graph, or a double split graph.*

PROOF — Let us prove the first item. Let P be a flat path of G of length at least 3. So $(P, V(G) \setminus P)$ is a path 2-join of G. Let $(P, X_2, \{a_1\}, \{b_1\}, A_2, B_2)$ be a split of this 2-join. If (P, X_2) is not proper, then either there is a component of X_2 that does not meet one of A_2, B_2, or X_2 induces a path of length 1 or 2. In the last case, G is bipartite, and in the first one, we may assume that there is a component C of X_2 that does not meet B_2. But then, $\{a_1\} \cup (A_2 \setminus C)$ is a star cutset of G that separates C from B_2, and so by Lemma 3.3, G has a balanced skew partition.

The second item follows easily: if G is a path-cobipartite graph, then we may assume that G is not the complement of a bipartite graph. If G is a path-double split graph then we may assume that G is not a double split graph. In both cases, G has a flat path of length at least 3. If G has an homogeneous 2-join then it also has a flat path of length at least 3. In every case, the conclusion follows from the first item. □

The following is well known for double split graphs (mentioned in [7]):

Lemma 3.5 *A path-double split graph G has exactly one skew partition and this skew partition is not balanced.*

PROOF — Let $V(G)$ be partitioned into sets A, B, C, D, E like in the definition of path-double split graphs. Obviously, $(A \cup B \cup E, C \cup D)$ is a non-balanced skew partition of G. Every vertex of $A \cup B \cup E$ has a non neighbor in every anticomponent of $C \cup D$. Hence, every subset of $V(G)$ strictly containing $C \cup D$ is anticonnected. So, if $X \neq C \cup D$ is a skew cutset of G, we may assume that X does not contain c_1. So, c_1 is in a component of $G \setminus X$, and there is a vertex y of G that is in another component. Up to a symmetry, we have two cases to consider:

First case: $y = d_1$. Hence, every vertex of $C \cup D \setminus \{c_1, d_1\}$ must be in X. Every vertex in $A \cup B \cup E$ has a non neighbor in every anticomponent of $C \cup D \setminus \{c_1, d_1\}$. So, since X is not anticonnected, we have $X = C \cup D \setminus \{c_1, d_1\}$. This contradicts $G \setminus X$ being disconnected.

Second case: y is on a path P from a_1 to b_1 whose interior is in E. Since P has a vertex adjacent to c_1, at least a vertex of P must be in X. If this vertex u is in E then we may assume up to a symmetry $b_1 \in X$ since u and c_1 must have a common neighbor in X because X is not anticonnected. Else we may also assume $b_1 \in X$. Note that $a_1 \notin X$, because either a_1 and b_1 are not adjacent, and then cannot be both in X because they have no common neighbor; or a_1 and b_1 are adjacent and then $y = a_1$ is the only possibility left for y. Hence, X is a skew cutset that separates a_1 from c_1. Now, for every $2 \leq j \leq n$, one of c_j, d_j is a common neighbor of a_1, c_1. Hence, up to a symmetry, we may assume $\{c_2, \dots, c_n\} \subset X$. Every vertex of $V(G) \setminus \{b_1, c_2, \dots, c_n\}$ has a non neighbor in the unique anticomponent of $\{b_1, c_2, \dots, c_n\}$. Hence, $X = \{b_1, c_2, \dots, c_n\}$. So, X is anticonnected. This contradicts X being a skew cutset. □

The following is needed twice in the proof of Theorem 2.1:

Lemma 3.6 *Let G be a Berge graph. Suppose that G has a vertex u of degree 3 whose neighborhood induces a stable set. Moreover, G has a stable set $\{x, y, z\}$ such that x, y, z all have degree at least 3. Then G is not a path-cobipartite graph, not a path-double split graph and G has no non-degenerate homogeneous 2-join.*

PROOF — In a path-cobipartite graph the vertices of degree at least 3 partition into 2 cliques. Since $\{x, y, z\}$ contradicts this property, G is not a path-cobipartite graph.

In a path-double split graph, every vertex of degree exactly 3 must have an edge in his neighborhood. Since u contradicts this property, G is not a path-double split graph.

If G has a non-degenerate homogeneous 2-join (A, B, C, D, E, F), then every vertex in F has degree at least 4. Every vertex in A, B has an edge in his neighborhood. Every vertex in C has a neighbor in C or F for otherwise, (A, B, C, D, E, F) is degenerate. Thus, every vertex in C, and by the same way every vertex in D, has an edge in his neighborhood. Every vertex in E has degree 2. Hence, u is in none of A, B, C, D, E, F, a contradiction. □

3.1 Paths and antipaths overlapping 2-joins

Here, we state easy facts about 2-joins. Some of them are well known but they need to be stated and proved clearly, especially because most of them are needed for possibly non-proper 2-joins.

Lemma 3.7 *Let G be a Berge graph with a connected 2-join (X_1, X_2). Then all the paths with an end A_1, an end in B_1, no interior vertex in $A_1 \cup B_1$, and all the paths with an end A_2, an end in B_2, no interior vertex in $A_2 \cup B_2$ have same parity.*

PROOF — Note that since (X_1, X_2) is connected there actually exists in $G[X_1]$ a path P_1 with an end in A_1, an end in B_1 and interior in C_1. There exists a similar path in $G[X_2]$ from A_2 to B_2. The paths P_1, P_2 have same parity because $P_1 \cup P_2$ induces a hole. Let P be a path from A_1 to B_1 with no interior vertex in $A_1 \cup B_1$ (the proof is the same for an a path from A_2 to B_2). Let P^* be the interior of P. Then one of $P \cup P_2$, $P^* \cup P_1$ induces a hole. Hence, P, P_1, P_2 have same parity. □

Lemma 3.8 *Let G be a Berge graph with a 2-join (X_1, X_2). Let i be in $\{1, 2\}$. Then every outgoing path from A_i to A_i (resp. from B_i to B_i) has even length. Every antipath of length at least 2 whose interior is in A_i (resp. B_i) and whose ends are outside A_i (resp. B_i) has even length.*

PROOF — Note that we do not suppose (X_1, X_2) being connected, so Lemma 3.7 does not apply. Let P be an outgoing path from A_1 to A_1 (the other cases are similar). If P has a vertex in A_2, then P has length 2. Else, P must lie entirely in X_1 except possibly for one vertex in B_2. If P lies entirely in X_1, then $P \cup \{a_2\}$ where a_2 is any vertex in A_2 induces a hole, so P has even length. If P has a vertex $b_2 \in B_2$, then we must have $P = a \!-\! \cdots \!-\! b \!-\! b_2 \!-\! b' \!-\! \cdots \!-\! a'$ where $a \!-\! P \!-\! b$ and $b' \!-\! P \!-\! a'$ are paths with an end in A_1, an end in B_1 and interior in C_1. Suppose that P has odd length. Let a_2 be a vertex of A_2. Then $V(P) \cup \{a_2\}$ induces an odd cycle of G whose only chord is a_2b_2. So one of $V(a \!-\! P \!-\! b_2) \cup \{a_2\}$, $V(a' \!-\! P \!-\! b_2) \cup \{a_2\}$ induces an odd hole of G, a contradiction.

Let Q be an antipath of length at least 2 whose interior is in A_1 and whose ends are outside A_1 (the other cases are similar). If Q has length at least 3, then the ends of Q must have a neighbor in A_1 and a non-neighbor in A_1. Hence these ends are in X_1. Thus, $Q \cup \{a\}$, where a is any vertex of A_2 is an antihole of G. Thus, Q has even length. □

Lemma 3.9 *Let G be a graph with a 2-join (X_1, X_2). Let P be a path of G whose end-vertices are in X_2. Then either:*

1. *There are vertices $a \in A_1$, $b \in B_1$ such that $V(P) \subseteq X_2 \cup \{a, b\}$. Moreover, if a, b are both in $V(P)$, then they are non-adjacent.*

2. *$P = c \!-\! \cdots \!-\! a_2 \!-\! a \!-\! \cdots \!-\! b \!-\! b_2 \!-\! \cdots \!-\! c'$ where: $a \in A_1$, $b \in B_1$, $a_2 \in A_2$, $b_2 \in B_2$. Moreover $V(c \!-\! P \!-\! a_2) \subset X_2$, $V(b_2 \!-\! P \!-\! c') \subset X_2$, $V(a \!-\! P \!-\! b) \subset X_1$.*

PROOF — If P has no vertex in X_1, then for any $a \in A_1, b \in B_1$, the first outcome holds. Else let c, c' be the end-vertices of P. Starting from c, we may assume that first vertex of P in X_1 is $a \in A_1$. Note that a is the only vertex of P in A_1. If a has its two neighbors on P in X_2, then P has no other vertex in X_1, except possibly a single vertex $b \in B_1$ and the first outcome holds. If a has only one neighbor on P in X_2, then let a_2 be this neighbor. Note that P must have a single vertex b in B_1.

Let b_2 be the neighbor of b in X_2 along P. Vertices a_2, a, b, b_2 show that the second outcome holds. □

Lemma 3.10 *Let G be a Berge graph with a 2-join (X_1, X_2). Let P be a path of G whose end-vertices are in $A_1 \cup X_2$ (resp. $B_1 \cup X_2$) and whose interior vertices are not in A_1 (resp. B_1). Then either:*

1. *P has even length.*
2. *There are vertices $a \in A_1$, $b \in B_1$ such that $V(P) \subseteq X_2 \cup \{a, b\}$. Moreover, if a, b are both in $V(P)$, then they are non-adjacent.*
3. *$P = a - \cdots - b - b_2 - \cdots - c$ where: $a \in A_1$, $b \in B_1$, $b_2 \in B_2$, $c \in X_2$.*

   Moreover $V(a - P - b) \subset X_1$ and $V(b_2 - P - c) \subset X_2$.

   (resp. $P = b - \cdots - a - a_2 - \cdots - c$ where: $b \in B_1$, $a \in A_1$, $a_2 \in A_2$, $c \in X_2$.

   Moreover $V(b - P - a) \subset X_1$ and $V(a_2 - P - c) \subset X_2$.)

PROOF — Note that we do not suppose (X_1, X_2) being proper. Suppose that the end-vertices of P are in $A_1 \cup X_2$ (the case when the end-vertices of P are all in $B_1 \cup X_2$ is similar).

If P has its two end-vertices in A_1, then by Lemma 3.8, P has even length and Output 1 of the lemma holds.

If P has exactly one end-vertex in A_1, let a be this vertex. Let $c \in X_2$ be the other end-vertex of P. Let a' be the neighbor of a along P. If a' is in A_2, then we may apply Lemma 3.9 to $a' - P - c$: Outcome 2 is impossible and Outcome 1 yields Outcome 2 of the lemma we are proving now since P has exactly one vertex in A_1. If a' is not in A_2, then let b be the last vertex of X_1 along P and b_2 the first vertex of X_2 along P. Outcome 3 of the lemma holds.

If P has no end-vertex in A_1 then Lemma 3.9 applies to P. The second outcome is impossible. The first outcome implies that there is a vertex $b \in B_1$ such that $V(P) \subseteq X_2 \cup \{b\}$ since no interior vertex of P is in A_1. So, Outcome 2 of the lemma we are proving now holds. □

Lemma 3.11 *Let G be a graph with a 2-join (X_1, X_2). Let Q be an antipath of G of length at least 4 whose interior vertices are all in X_2. Then there is a vertex a in $A_1 \cup B_1$ such that $V(Q) \subseteq X_2 \cup \{a\}$.*

PROOF — Let c, c' be the end-vertices of Q. Note that $N(c) \cap N(c') \cap X_2$ have to be non-empty and that $N(c) \cap X_2$ must be different of $N(c') \cap X_2$, because c, c' are the end-vertices of an antipath of length at least 4. No pair of vertices in X_1 satisfies these two properties, so at most one of c, c' is in $V(Q) \cap X_1$. If none of c, c' are in X_1, then let a be any vertex in A_1, else let a be the unique vertex in X_1 among c, c'. Since c, c' must have a neighbor in X_2, $a \in A_1 \cup B_1$ and clearly $V(Q) \subseteq X_2 \cup \{a\}$. □

Lemma 3.12 *Let G be a Berge graph with a 2-join (X_1, X_2). Let Q be an antipath of G of length at least 5 whose interior vertices are all in $A_1 \cup X_2$ (resp. $B_1 \cup X_2$) and whose end-vertices are not in A_1 (resp. B_1). Then either:*

1. *Q has even length.*
2. *There is a vertex $a \in A_1 \cup B_1$ such that $V(Q) \subseteq X_2 \cup \{a\}$.*

PROOF — We suppose that the interior vertices of Q are all in $A_1 \cup X_2$. The case when the interior vertices of Q are all in $B_1 \cup X_2$ is similar.

If Q has at least 2 vertices in A_1, then let $a \neq a'$ be two of these vertices. Since the end-vertices of Q are not in A_1, a, a' may be chosen in such a way that there are vertices $c, c' \notin A_1$ such that $\overline{c - a - \overline{Q} - a' - c'}$ is an antipath of G. Since c must miss a while seeing a', c must be in $X_1 \setminus A_1$, and so is c'. But the interior vertices of Q cannot be in $X_1 \setminus A_1$, so c, c' are in fact the end-vertices of Q. Also, every interior vertex of Q must be adjacent to at least one of c, c'. If all the interior vertices of Q are in A_1 then by Lemma 3.8, Q has even length. Else, Q must have at least an interior vertex $b \in X_2$. Since b must see at least one of c, c' we have $b \in B_2$, so b misses both a, a'. Hence $\overline{a - b - a'}$ is an induced subgraph of Q and b must see both c, c', so $c, c' \in B_1$. Now we observe that $Q = \overline{c - a - b - a' - c'}$, contradicting Q having length at least 5.

If Q has exactly one vertex a in A_1 then by assumption, a is an interior vertex of Q. Let c, c' be the ends of Q. Suppose $c \in X_1$. Since Q has length at least 5, c must have a neighbor in the interior Q that is different of a, hence $c \in B_1$. Since Q has length at least 5, a and c must have a common neighbor, that must be c' since it must be in X_1. Hence $c' \in X_1$, implying $c' \in B_1$ since c' must have a neighbor in X_2. Now the non-neighbor of c' along Q is not a, so it must be a vertex of X_2 while seeing c and missing c', a contradiction. We proved $c \in X_2$, and similarly $c' \in X_2$. Hence $V(Q) \subset X_2 \cup \{a\}$.

If Q has no vertex in A_1 then Lemma 3.11 applies: there is a vertex $a \in A_1 \cup B_1$ such that $V(Q) \subseteq X_2 \cup \{a\}$. □

3.2 Balanced skew partitions overlapping 2-joins

Let G be a Berge graph and $(X_1, X_2, A_1, B_1, A_2, B_2)$ be a split of a proper 2-join of G. The *blocks* of G with respect to (X_1, X_2) are the two graphs G_1, G_2 that we describe now. We obtain G_1 by replacing X_2 by a flat path P_2 from a vertex a_2 complete to A_1, to a vertex b_2 complete to B_1. This path has the same parity than a path from A_1 to B_1 whose interior is in C_1. There is such a path since (X_1, X_2) is proper and all such paths have same parity by Lemma 3.7. The length of P is decided as follow: if (X_1, X_2) is a path 2-join with path-side X_2 then P has length 1 or 2, else it has length 3 or 4. The block G_2 is obtained similarly by replacing X_1 by a flat path.

It is convenient to consider a degenerated kind of 2-join that implies the existence of a balanced skew partition. A 2-join (X_1, X_2) is said to be *degenerate* if either:

- there exists $i \in \{1, 2\}$ and a vertex v in A_i (resp. B_i) that has no neighbor in $X_i \setminus A_i$ (resp. in $X_i \setminus B_i$);

- one of $A_1 \cup A_2$, $B_1 \cup B_2$ is a skew cutset of G;
- the 2-join (X_1, X_2) is not connected (ie, there exists $i \in \{1, 2\}$ and a component of X_i that does not meet both A_i, B_i);
- there exists $i \in \{1, 2\}$ and a vertex in A_i that is complete to B_i or a vertex in B_i that is complete to A_i;
- there exists $i \in \{1, 2\}$ and a vertex in C_i that is complete to $A_i \cup B_i$.

Lemma 3.13 *Let G be a Berge graph and (X_1, X_2) be a degenerate substantial 2-join of G. Then G has a balanced skew partition and at least one of the blocks G_1, G_2 of G has a balanced skew partition.*

PROOF — Let us look at the possible reasons why (X_1, X_2) is degenerate.

If there is a vertex v in A_1 that has no neighbor in $X_1 \setminus A_1$ then suppose first $|A_1| > 1$. So $(A_1 \setminus \{v\}) \cup A_2$ is a skew cutset separating v from the rest of the graph. Hence, in $\overline{G}$ there is a star cutset of center v, and by Lemmas 3.3 and 3.2, G has a balanced skew partition. Hence we may assume $A_1 = \{v\}$. Since (X_1, X_2) is substantial, $|X_1| \geq 3$. Thus, for any $b \in B_1$, $\{b\} \cup B_2$ is a star cutset that separates v from $X_1 \setminus \{b, v\}$ and G has a balanced skew partition by Lemma 3.3. By the same way, the block G_1 has a balanced skew partition. The cases with A_2, B_1, B_2 are similar.

If $A_1 \cup A_2$ is a skew cutset of G then let us check that this skew cutset is balanced (the case when $B_1 \cup B_2$ is a skew cutset is similar). Since A_1 is complete to A_2, any outgoing path from $A_1 \cup A_2$ to $A_1 \cup A_2$ is either outgoing from A_1 to A_1 or outgoing from A_2 to A_2. Thus, such a path has even length by Lemma 3.8. If there is an antipath Q of length at least 5 with its interior in $A_1 \cup A_2$ and its ends in the rest of the graph, then it must lie entirely in X_1 or X_2, say X_1 up to symmetry. Thus, such an antipath has even length by Lemma 3.8. By the same way $A_1 \cup \{a_2\}$, where a_2 is the vertex of G_1 that represents A_2, is a balanced skew cutset of G_1

If (X_1, X_2) is not connected, then let for instance Y be a component of X_1 that does not meet B_1. If $Y \cap C_1 \neq \emptyset$ then $A_1 \cup A_2$ is a skew cutset of G that separates $Y \cap C_1$ from B_1. So, by the preceding paragraph, G and G_1 have a balanced skew partition and we may assume that $Y \subset A_1$. Hence, every vertex in Y has no neighbor in $X_1 \setminus A_1$. So, by the penultimate paragraph, G and G_1 have a balanced skew partition.

If there is a vertex $a \in A_1$ that is complete to B_1 (the other cases are symmetric) then suppose first $|A_1| > 1$. Consider $a' \neq a$ in A_1. Hence $(\{a\} \cup N(a)) \setminus a'$ is a star cutset of G separating a' from B_2. So, by Lemma 3.3, we may assume $A_1 = \{a\}$. If $|B_1| > 1$, consider $b \neq b'$ in B_1. Hence, $(\{b\} \cup N(b)) \setminus b'$ is a star cutset of G separating b' from A_2. So we may assume $B_1 = \{b\}$. Since (X_1, X_2) is substantial, $|X_1| \geq 3$, and there is a vertex c in $V(G) \setminus (A_1 \cup B_1)$. Now, $\{a, b\}$ is a star cutset separating c from X_2. By the same way, G_1 has a balanced skew partition.

If there is a vertex c complete to $A_i \cup B_i$ then we may assume $C_i = \{c\}$ for otherwise there is another vertex c' in C_i and $\{c\} \cup A_i \cup B_i$ is a star cutset separating c' from the rest of the graph. By the preceding paragraph, we may assume that there is a vertex $a \in A_1$ and a vertex $b \in B_1$ missing a. Then a–c–b is an outgoing path of even length from A_i to B_i. By the penultimate paragraph, we may assume (X_1, X_2) connected. Thus by Lemma 3.7, there is no edge between A_i and B_i. If there are two vertices $a \neq a' \in A_i$ then $\{a\} \cup N(a) \setminus \{a'\}$ is a star cutset of G separating a' from B_{3-i}. Thus may assume $|A_i| = 1$, and similarly $|B_i| = 1$. Thus, X_i is an outgoing path of length 2 from A_i to B_i contradicting (X_1, X_2) being substantial. By the same way, G_1 has a balanced skew partition. □

Lemma 3.14 *Let G be a graph with a non-degenerate 2-join (X_1, X_2). Let i be in $\{1, 2\}$. Then for every vertex $v \in X_i$ there is a path $P_a = a - \cdots - v$ and a path $P_b = b - \cdots - v$ such that:*

- *$a \in A_i$, $b \in B_i$;*
- *Every interior vertex of P_a, P_b is in $X_i \setminus (A_i \cup B_i)$.*

PROOF — Note that (X_1, X_2) is connected since it is not degenerate. Suppose first $v \in X_i \setminus (A_i \cup B_i)$. By the definition of connected 2-joins, the connected component X_v of v in $G[X_i]$ meets both A_i, B_i and there is at least one path from v to a vertex of B_i in $G[X_i]$. If every such path of $G[X_i]$ from v to B_i goes through A_i, then A_i is a cutset of $G[X_i]$ that separates v from B_i. Thus $A_1 \cup A_2$ is a skew cutset of G, so (X_1, X_2) is degenerate, a contradiction. So there is a path P_b as desired, and by the same way, P_a exists.

If $v \in A_i$, then P_a exists and have length 0: put $P_a = v$. The vertex v has a neighbor w in $X_i \setminus A_i$ otherwise (X_1, X_2) is degenerate. By the preceding paragraph, there is a path Q from w to $b \in B_i$ whose interior vertices lie in $X_i \setminus (A_i \cup B_i)$. So P_b exists: consider a shortest path from v to b in $G[V(Q) \cup \{b\}]$. □

Lemma 3.15 *Let G be a Berge graph with a non-degenerate 2-join (X_1, X_2). Let F be a balanced skew cutset of G. Then for some $i \in \{1, 2\}$ either:*

- *$F \subsetneq X_i$;*
- *$F \cap X_i \subsetneq X_i$ and one of $(F \cap X_i) \cup A_{3-i}$, $(F \cap X_i) \cup B_{3-i}$ is a balanced skew cutset of G.*

PROOF — We consider three cases:
Case 1: $F \cap A_1$, $F \cap A_2$, $F \cap B_1$, $F \cap B_2$ are all non-empty.

If there is a vertex $a \in A_1 \cap F$ non-adjacent to a vertex $b \in B_1 \cap F$ then there is an antipath of length at most 3 between any vertex of F and a, contradicting $\overline{G}[F]$ being disconnected. Thus $A_1 \cap F$ is complete to $B_1 \cap F$, and similarly $A_2 \cap F$ is complete to $B_2 \cap F$. Similarly, we prove $F \cap C_1 = F \cap C_2 = \emptyset$. If $A_1 \subset F$ then there is a vertex in B_1 that is complete to A_1, contradicting (X_1, X_2) being non-degenerate. Thus $A_1 \setminus F \neq \emptyset$, and similarly $A_2 \setminus F \neq \emptyset$, $B_1 \setminus F \neq \emptyset$, $B_2 \setminus F \neq \emptyset$.

Let E_1 be the component of $G \setminus F$ that contains $(A_1 \setminus F) \cup (A_2 \setminus F)$. Let E_2 be another component of $G \setminus F$. Up to a symmetry we assume $E_2 \cap X_2 \neq \emptyset$. We claim that $F' = (F \cap X_2) \cup A_1$ is a skew cutset of G that separates $E_1 \cap X_2$ from $E_2 \cap X_2$. For suppose not. This means that there is a path P of $G \setminus F'$ with an end in $E_1 \cap X_2$

and an end in $E_2 \cap X_2$. If P has no vertex in X_1 then $P \subset G \setminus F$ and P contradicts E_1, E_2 being components of $G \setminus F$. If P has a vertex in X_1 then this vertex b is unique and is in B_1 because $A_1 \subset F'$. By replacing b by any vertex of $B_1 \setminus F$, we obtain again a path that contradicts E_1, E_2 being components of $G \setminus F$. Thus F' is a skew cutset of G. Note that this skew cutset is included in $A_1 \cup A_2 \cup B_2$. Let us prove that this skew cutset is balanced.

Let P be an outgoing path from F' to F'. Let us apply Lemma 3.10 to P. If Outcome 1 of the lemma holds then P has even length. If Outcome 2 of the lemma holds then $V(P) \subset X_2 \cup \{a, b\}$. Let a_1 be a vertex of $A_1 \cap F$ and b_1 be a vertex of $B_1 \setminus F$ such that a_1 misses b_1. Note that b_1 exists for otherwise (X_1, X_2) is a degenerate 2-join of G. After possibly replacing a by a_1 and b by b_1, we obtain an outgoing path from F to F that has same length than P. Thus, P has even length since F is a balanced skew cutset. If Outcome 3 of the lemma holds then P has one end in A_1 and one end in B_2 and P is a path from A_1 to B_1 whose interior is in C_1, plus one edge. Note that there is an edge between A_2 and B_2 so by Lemma 3.7 every path from A_1 to B_1 whose interior is in C_1 has odd length. Hence in every case P has even length.

Let Q be an antipath with both ends in $G \setminus F'$ and interior in F'. If Q has length 3 then Q may be seen as an outgoing path from F' to F', so we may assume that Q has length at least 5. By Lemma 3.12 applied to Q, either Q has even length or $V(Q) \subset X_2 \cup \{a\}$. If $a \in A_1$ let us replace a by a vertex of $F \cap A_1$ and if $a \in B_1$ let us replace a by a vertex of $B_1 \setminus F$. We obtain an antipath that have same length than Q, that has both ends outside of F and interior in F. Thus Q has even length because F is a balanced skew cutset.

Case 2: one of $F \cap A_1$, $F \cap A_2$, $F \cap B_1$, $F \cap B_2$ is empty and $F \cap X_1$, $F \cap X_2$ are both non-empty.

We assume up to a symmetry that one of $B_1 \cap F$, $B_2 \cap F$ is empty. Since $F \cap X_1$ and $F \cap X_2$ are both non-empty, there is a least an edge between $F \cap X_1$ and $F \cap X_2$ because $\overline{G}[F]$ is disconnected. Thus we know that $F \cap A_1$ and $F \cap A_2$ are both non-empty. If $(F \cap X_1) \setminus A_1$ and $(F \cap X_2) \setminus A_2$ are both non-empty then there is a vertex of F in one of C_1, C_2 since one of $B_1 \cap F$, $B_2 \cap F$ is empty. Up to a symmetry, suppose $C_1 \cap F \neq \emptyset$. Then $\overline{G}[F]$ is connected since every vertex in it can be linked to a vertex of C_1 by an antipath of length at most 2, a contradiction. Hence one of $(F \cap X_1) \setminus A_1$ and $(F \cap X_2) \setminus A_2$ is empty. Thus we may assume $F \subset X_2 \cup A_1$. Suppose $B_2 \subset F$. Then B_2 and $F \cap A_1$ are in the same component of $\overline{G}[F]$, thus there must be a vertex v in F that is complete to $B_2 \cup (F \cap A_1)$. So, v is in A_2, and v is complete to B_2, contradicting (X_1, X_2) being non-degenerate. We proved that there is at least a vertex u in $B_2 \setminus F$. In particular, $F \cap X_2 \subsetneq X_2$. By Lemma 3.14 there is a path from every vertex of $X_1 \setminus F$ to u whose interior is in $X_1 \setminus A_1$, thus there is a component E_1 of $G \setminus F$ that contains $X_1 \setminus F$ and u. There is another component E_2 included in X_2. Thus $(F \cap X_2) \cup A_1$ is a skew cutset of G that separates B_1 from E_2. We still have to prove that the skew cutset $(F \cap X_2) \cup A_1$ is balanced.

Let P be an outgoing path from $(F \cap X_2) \cup A_1$ to $(F \cap X_2) \cup A_1$. Let us apply Lemma 3.10 to P. If Outcome 1 of the lemma holds then P has even length. If Outcome 2 of the lemma holds then $V(P) \subset X_2 \cup \{a, b\}$. Let a_1 be a vertex of $A_1 \cap F$ and b_1 be a vertex of B_1 such that a_1 misses b_1. Note that b_1 exists for otherwise

(X_1, X_2) is a degenerate 2-join of G. After possibly replacing in P a by a_1 and b by b_1, we obtain an outgoing path from F to F that has the same length than P. Thus, P has even length since F is a balanced skew cutset. If Outcome 3 of the lemma holds then $P = a-\cdots-b-b_2-\cdots-c$. Let a_1 be in $A_1 \cap F$. By Lemma 3.14 there is a path P_1 of $G[X_1]$ from a_1 to a vertex $b_1 \in B_1$. Moreover, P_1 has an end in A_1, an end in B_1 and interior in C_1. Note that by Lemma 3.7, P_1 and $a-P-b$ have same parity. Thus $a_1-P_1-b_1-b_2-P-c$ is an outgoing path from F to F that has the same parity that P. Thus P has even length.

If Q is an antipath with both ends in $G \setminus ((F \cap X_2) \cup A_1)$ and its interior in $(F \cap X_2) \cup A_1$, we prove that Q has even length like in Case 1.

Case 3: One of $F \cap X_1, F \cap X_2$ is empty.

Since $F \subsetneq X_2$ is an output of the lemma, we may assume up to a symmetry $F = X_2$ an look for a contradiction. If there is a path of odd length from A_2 to B_2 whose interior is in C_2, then there is by Lemma 3.7 a similar path P from A_1 to B_1 of odd length. Hence A_2 is complete to B_2 because a pair of non-adjacent vertices yields together with P an outgoing path of odd length from F to F, contradicting F being a balanced skew cutset. In particular, there is a vertex of A_2 that is complete to B_2, implying (X_1, X_2) being degenerate, a contradiction. If there is a path of even length from A_2 to B_2 whose interior is in C_2 then by Lemma 3.7 there are no edges between A_2 and B_2. Since $X_2 = F$ is not anticonnected, there is a vertex in C_2 that is complete to $A_2 \cup B_2$, implying again (X_1, X_2) being degenerate, a contradiction. □

Lemma 3.16 *Let G be a Berge graph and (X_1, X_2) be a proper 2-join of G. If G has a balanced skew partition then at least one of the blocks of G has a balanced skew partition.*

PROOF — If (X_1, X_2) is degenerate, then the conclusion holds by Lemma 3.13. From now on, we assume that (X_1, X_2) is non-degenerate. Suppose that G has a balanced skew partition (E, F). By Lemma 3.15 and up to a symmetry either $F \subsetneq X_2$, or $(F \cap X_2) \subsetneq X_2$ and $A_1 \subset F$, after possibly replacing F by $(F \cap X_2) \cup A_1$.

If $F \subsetneq X_2$ then we claim that F is a balanced skew cutset of G_2. Note that there is at least a component E of $G \setminus F$ that has some vertex in X_2 but no vertex in $A_2 \cup B_2$. Else every component of $G \setminus F$ has neighbors in A_1 or B_1, and therefore contains $A_1 \cup B_1$ because (X_1, X_2) is connected. This implies $G \setminus F$ being connected, a contradiction. Thus, F is a skew cutset of G_2 that separates E from $V(G_2) \setminus X_2$. Let P be an outgoing path of G_2 from F to F. Note that G_2 has an obvious 2-join, $(V(G_2) \setminus X_2, X_2)$, possibly non-substantial. Let us apply Lemma 3.9 to P. If Outcome 1 of the Lemma holds then after possibly replacing a by any $a_1 \in A_1$ and b by any $b_1 \in B_1$ non-adjacent to a_1, P may be viewed as an outgoing of G from F to F, thus P has even length. Note that b_1 may be chosen non-adjacent to a_1 because (X_1, X_2) is non-degenerate. If Outcome 2 of the lemma holds, then $P = c-\cdots-a_2-a_1-\cdots-b_1-b_2-\cdots-c'$. Let P' be any path from A_1 to B_1 whose interior is in C_1. Then $c-\cdots-a_2-P'-b_2-\cdots-c'$ is an outgoing path of G from F to F that has same parity than P by Lemma 3.7. Thus P has even length. Let Q be an antipath of G_2 with its ends out of F and its interior in F. Let us apply

Lemma 3.11 to Q: $V(Q) \subseteq X_2 \cup \{a\}$. Thus, after possibly replacing a by a vertex in $A_1 \cup B_1$, Q may be seen as an antipath of G that has same length than Q. Thus Q has even length.

If $(F \cap X_2) \subsetneq X_2$ and $A_1 \subset F$ then we put $F' = (F \cap X_2) \cup \{a_1\}$. We claim that F' is a balanced skew cutset of G_2. Exactly as above, we prove that F' is a skew cutset of G_2 that separates b_1 from a component of $G \setminus F$ that has vertices in X_2 but no vertex in B_2. Let P be an outgoing path from F' to F'. As above we prove that P has even length by Lemma 3.10. Let Q be an antipath of G_2 with its ends out of F' and its interior in F'. As above, we prove that Q has even length by Lemma 3.12. □

Lemma 3.17 *Let G be a Berge graph and (X_1, X_2) be a non-cutting substantial 2-join of G. Then G has a balanced skew partition if and only if one of the blocks of G has a balanced skew partition.*

PROOF — If G has a balanced skew partition then by Lemma 3.16 one of the blocks of G has a balanced skew partition. If (X_1, X_2) is degenerate, then the conclusion holds by Lemma 3.13. From now on, we assume that (X_1, X_2) is non-degenerate. In particular, it is connected and proper. Let us suppose that one of G_1, G_2 (say G_2 up to a symmetry) has a balanced skew cutset F'. We denote by $P_1 = a_1 - \cdots - b_1$ the path induced by $V(G_2) \setminus X_2$. Note that G_2 has an obvious connected path 2-join: (P_1, X_2), possibly non-substantial.

(*1) Either:*

- *$F' \subsetneq X_2$;*
- *$F' \cap X_2 \subsetneq X_2$ and one of $(F' \cap X_2) \cup \{a_1\}$, $(F \cap X_2) \cup \{b_1\}$ is a balanced skew cutset of G_2.*

If P_1 has length 3 or 4, then (P_1, X_2) is proper. It is non-degenerate because (X_1, X_2) is non-degenerate. Let us apply Lemma 3.15. The conclusion $F' \subsetneq X_1$, is impossible since then by Lemma 3.14, $G_2 \setminus F'$ is connected. Also $(F' \cap P_1) \cup A_2$ and $(F' \cap P_1) \cup B_2$ cannot be skew cutsets of G_2, because a_1, b_1 cannot be both in a skew cutset of G_2 since they are non adjacent with no common neighbors. Hence, Lemma 3.14 proves that $(F' \cap P_1) \cup A_2$ and $(F' \cap P_1) \cup B_2$ are not cutsets of G_2. Thus (1) is simply the only possible conclusion of Lemma 3.15.

If P_1 has length 2 then $P_1 = a_1 - c_1 - b_1$. If a_1, b_1 are both in F', then $F' = \{a_1, c_1, b_1\}$ because c_1 is the only common neighbor of a_1, b_1 in G_2. This means that $G_2[X_2] = G[X_2]$ is disconnected, implying that (X_1, X_2) is a cutting 2-join of type 1, a contradiction. By Lemma 3.14 applied to $G_2[X_2] = G[X_2]$, none of a_1, b_1 can be the center of a star cutset of G. Hence, $c_1 \notin F'$. Thus, $F \cap X_2 \subsetneq X_2$ because any induced subgraph of P_1 containing c_1 is connected. We proved (1) when P_1 has length 2.

We are left with the case when $P_1 = a_1 - b_1$. If a_1, b_1 are both in F' then $F' \subset \{a_1, b_1\} \cup A_2 \cup B_2$. If $F' \cap A_2 \neq \emptyset$ and $F' \cap B_2 \neq \emptyset$ then putting $A_3 = F' \cap A_2$ and $B_3 = F' \cap B_2$ we see that (X_1, X_2) is a cutting 2-join of type 2 of G. Indeed, A_3

is complete to B_3 for otherwise, F' is anticonnected. The requirements on the parity of paths and antipaths are satisfied because F' is a balanced skew cutset. If at least one of $F' \cap A_2$ and $F' \cap B_2$ is empty then we see that (X_1, X_2) is a cutting 2-join of type 1. Both cases contradict (X_1, X_2) being non-cutting. Thus we know that at most one of a_1, b_1 is in F. Also $F' \cap X_2 \subsetneq X_2$ because every induced subgraph of P_1 is connected. This proves (1).

By (1), we may assume that not both a_1, b_1 are in F'. Up to a symmetry, we assume $b_1 \notin F'$. If $a_1 \in F'$, put $A'_1 = A_1$, else put $A'_1 = \emptyset$. Now $F = (F' \cap X_2) \cup A'_1$ is a skew cutset of G that separates a vertex of X_2 from $X_1 \setminus A'_1$. The proof that F is a balanced skew cutset of G is entirely similar to the similar proofs above: we consider an outgoing path of G from F to F. Lemma 3.9 or Lemma 3.10 shows that P has the same parity than an outgoing path of G_2 from F' to F'. We consider an antipath Q of G of length at least 2 with all its interior vertices in F and with its end-vertices outside of F. Lemma 3.11 or Lemma 3.12 shows that Q has the same parity than a similar antipath with respect to F' in G_2. □

Lemma 3.18 *Let G be a Berge graph and (X_1, X_2) be a non-path proper 2-join of G. Then G has a balanced skew partition if and only if one of the blocks of G has a balanced skew partition.*

PROOF — Clear by Lemma 3.17 since a non-path 2-join is a non-cutting 2-join. □

3.3 Balanced skew partitions overlapping homogeneous 2-joins

An homogeneous 2-join (A, B, C, D, E, F) is said to be *degenerate* if either:

- there is a vertex $x \in C$ with no neighbor in $E \cup D$ or a vertex $y \in D$ with no neighbor in $E \cup C$;
- there is a vertex $x \in C$ such that $N(x) \subset A \cup D \cup E$ or a vertex $y \in D$ such that $N(y) \subset B \cup C \cup E$.

Lemma 3.19 *Let G be a Berge graph with a degenerate homogeneous 2-join. Then G has a balanced skew partition.*

PROOF — Suppose first that there exists a vertex $x \in C$ with no neighbor in $E \cup D$ (the case with $y \in D$ is similar). Then, $(A \cup C \cup F) \setminus \{x\}$ is a skew cutset that separates x from the rest of the graph. Thus, $\overline{G}$ has a star cutset centered at x. By Lemma 3.3, $\overline{G}$ has a balanced skew partition and by Lemma 3.2 so is G.

Suppose now that there exists $x \in C$ such that $N(x) \subset A \cup D \cup E$ (the case with $y \in D$ is similar). Let D_x be the set of those vertices of D that are the ends of a path from C to D whose interior is in E and starting from x. Note that all such paths have odd length (possibly 1). If a vertex $f \in F$ misses $d \in D_x$, then consider a pair $a \in A, b \in B$ of non-adjacent vertices. Then $\{a, b, f\} \cup P$, where P is a path from x to d whose interior is in E, induces an odd hole. Thus F is complete to D_x. Thus, for any $f \in F$, $\{f\} \cup N(F) \setminus B$ is a star cutset of G that separates x from B. Thus, by Lemma 3.3, G has a balanced skew partition. □

4 Proof of Theorems 2.1 and 1.5

4.1 Proof of Theorem 2.1

For any graph G, let $f(G)$ be the number of maximal flat paths of length at least 3 in G. Let us consider G, a counter-example to Theorem 2.1 such that $f(G) + f(\overline{G})$ is minimal. Since G is a counter-example and since G is Berge, by Theorem 1.3 and up to a complementation of G, we may assume that:

a. G is not basic, none of $G, \overline{G}$ is a path-cobipartite graph, none of $G, \overline{G}$ is a path-double split graph, G has no balanced skew partition, none of $G, \overline{G}$ has a non-path proper 2-join, none of $G, \overline{G}$ has an homogeneous 2-join;

b. G has a path proper 2-join.

Since G has a path proper 2-join, G has flat path of length at least 3, implying $f(G) \geq 1$. We choose such a flat path X_1 inclusion-wise maximal. Note that by Lemma 3.4, $(X_1, V(G) \setminus X_1)$ is a proper 2-join of G since G is not basic and has no balanced skew partition. Let us consider $(X_1, X_2, A_1, B_1, A_2, B_2)$ a split of this 2-join. Note that $G[X_2]$ is not a path since G is not bipartite. We denote by a_1 the only vertex in A_1 and by b_1 the only vertex in B_1. We put $C_1 = X_1 \setminus \{a_1, b_1\}$, and $C_2 = X_2 \setminus (A_2 \cup B_2)$.
If one of G, $\overline{G}$ has a degenerate proper 2-join, a degenerate homogeneous 2-join or a star cutset then one of $G, \overline{G}$ has a balanced skew partition by Lemma 3.13, Lemma 3.19 or Lemma 3.3. So G has a balanced skew partition by Lemma 3.2. This contradicts G being a counter-example. Thus:

c. G and $\overline{G}$ have no degenerate proper 2-join, no degenerate homogeneous 2-join and no star cutset.

Suppose that a_1 has degree 2 in G. Since X_1 is the path-side of a path 2-join, this means that the unique neighbor a of a_1 in X_2 sees at least a neighbor $b \in X_2$ of b_1. Otherwise, $X_1 \cup \{a\}$ is flat path contradicting X_1 being maximal. Hence, b is a vertex of B_2 complete to $A_2 = \{a\}$, implying (X_1, X_2) being degenerate, a contradiction. Hence:

d. a_1, b_1 both have degree at least 3 in G.

Let us study the connectivity of G. If $G[X_2]$ is disconnected, then let X_2' be any component of $G[X_2]$. Since (X_1, X_2) is proper, the sets $A_2 \cap X_2'$ and $B_2 \cap X_2'$ are not empty. So $(V(G) \setminus X_2', X_2')$ is a 2-join of G. Let us suppose that X_2' is not a path of length 1 or 2 from A_2 to B_2 whose interior is in C_2. This implies that $(V(G) \setminus X_2', X_2')$ is a proper 2-join. So since G is a counter example, we know that $(V(G) \setminus X_2', X_2')$ is a path 2-join of G. Since X_1 is a maximal flat path of G, $V(G) \setminus X_2'$ cannot be the path side of this 2-join. Thus $G[X_2']$ is the path side of this 2-join. Hence we know that every component of X_2 is a path from A_2 to B_2 whose interior is in C_2. This implies that G is bipartite contradicting G being a counter example. Hence:

e. $G[X_2]$ is connected.

Since by Property c, (X_1, X_2) is non-degenerate, the following is a direct consequence of Lemma 3.14:

f. In $G[X_2]$, there exists a path from A_2 to B_2 whose interior is in C_2. Moreover, for every $A'_2 \subseteq A_2$, $B'_2 \subseteq B_2$ the graphs $G[A'_2 \cup C_2 \cup B_2 \cup \{b_1\}]$ and $G[B'_2 \cup C_2 \cup A_2 \cup \{a_1\}]$ are connected.

The six properties listed above will be referred as the *properties of* G in the rest of proof. We denote by $\varepsilon \in \{0, 1\}$ the parity of the length of the path X_1. We now consider three cases according to the properties of (X_1, X_2). In each case, we will consider a graph G' obtained from G by destroying the path 2-join (X_1, X_2), and we will show that G' is a counter-example that contradicts $f(G) + f(\overline{G})$ being minimal.

Case 1: X_1 may be chosen in such a way that (X_1, X_2) is cutting of type 1.

Up to a symmetry we assume that $G[X_2 \setminus A_2]$ is disconnected. Let X be a component of $G[X_2 \setminus A_2]$. If X is disjoint from B_2 then $\{a_1\} \cup A_2$ is a star cutset of G separating X from $X_2 \setminus X$, contradicting the properties of G. Thus X intersects B_2, and by the same proof so is any component of $X_2 \setminus X$. Hence, there are two non-empty sets $B_3 = B_2 \cap X$ and $B_4 = B_2 \setminus X$. Also we put $C_3 = C_2 \cap X$, $C_4 = C_2 \setminus X$. Possibly, C_3, C_4 are empty. There are no edges between $B_3 \cup C_3$ and $B_4 \cup C_4$.

We consider the graph G' obtained from G by deleting $X_1 \setminus \{a_1, b_1\}$. Moreover, we add new vertices: c_1, c_2, b_3, b_4. Then we add every possible edge between b_3 and B_3, between b_4 and B_4. We also add edges a_1c_1, c_2b_3, c_2b_4. If $\varepsilon = 0$, we consider for convenience $c_1 = c_2$, so that c_1 is always a vertex of G'. Else we consider $c_1 \neq c_2$ and we add an edge between c_1 and c_2. Note that in G', $N(b_1) = B_2$. Here are seven claims about the parity of various kinds of paths and antipaths in G'.

(1) Every path of G' from B_2 to A_2 with no interior vertex in $A_2 \cup B_2$ has length of parity ε.

If such a path contains one of a_1, b_3, b_4, c_1, c_2 then it has length $4 + \varepsilon$. Else such a path may be viewed as a path of G from B_2 to A_2. By Lemma 3.7 it has parity ε. This proves (1).

(2) Every outgoing path of G' from B_2 to B_2 has even length.

For suppose there is such a path $P = b - \cdots - b'$, $b, b' \in B_2$. If P goes through b_1 then it has length 2. If P goes through b_3 and b_4 it has length 4. If P goes through only one of b_3, b_4 then either P has length 2 or we may assume up to a symmetry that $P = b - b_3 - c_2 - c_1 - a_1 - a - \cdots - b'$ where $a \in A_2$. So, $a - P - b'$ is a path from A_2 to B_2 whose interior is in C_2 and by (1) it has parity ε. So, P has even length. If P goes through c_2 or c_1 then it must goes through at least one of b_3, b_4, and by the discussion above it must have even length. So we may assume that P goes through none of c_1, c_2, b_1, b_3, b_4. Hence P may be viewed as a path of G. Thus, P has even length by Lemma 3.8. In every case, P has even length. This proves (2).

(3) Every outgoing path of G' from A_2 to A_2 has even length.

For suppose there is such a path $P = a - \cdots - a'$, where $a, a' \in A_2$. If P goes through a_1 then it has length 2. So we may assume that P does not go through a_1. Note that if $c_1 \neq c_2$ then P does not go through c_1.

If P goes through c_2 or through both b_3, b_4 then we may assume $P = a - \cdots - b - b_3 - c_2 - b_4 - b' - \cdots - a'$ where $b \in B_3$ and $b' \in B_4$. By (1) $b - P - a$ and $a' - P - b'$ have both parity ε. Thus, P has even length. If P goes through B_3, b_1 and B_4 then we prove that it has even length by the same way. So we may assume that P neither goes through c_2 nor through both b_3, b_4 nor through B_3, b_1 and B_4.

If P goes through exactly one of b_3, b_4, say b_3 up to a symmetry, then just like above $P = a - \cdots - b - b_3 - b' - \cdots - a'$, where both $b - P - a$ and $a' - P - b'$ are paths from B_2 to A_2. So by (1), they both have parity ε. Thus, P has even length. If P goes through b_1 and exactly one of B_3, B_4, then we prove that it has even length by the same way. So we may assume that P goes though none of b_1, b_3, b_4.

Now P goes through none of $a_1, c_1, c_2, b_1, b_3, b_4$, so P may be viewed as an outgoing path of G from A_2 to A_2. It has even length by Lemma 3.8.

In every case, P has even length. This proves (3).

(4) Every outgoing path of G' from B_3 to B_3 (resp. B_4 to B_4) has even length.

Suppose that there is an outgoing path $P = b - \cdots - b'$ from B_3 to B_3 (the case with B_4 is similar). Note that P may have interior vertices in B_4, so (2) does not apply to P. If P goes through b_1 or b_3 it has length 2. So we may assume that P does not go through $\{b_1, b_3\}$. If P has no vertex in A_2, then P has no interior vertices in B_4 since B_3 and B_4 are in distinct components of $G \setminus (\{b_1, b_3\} \cup A_2)$. So (2) applies and P has even length.

So we may assume that P has at least a vertex in A_2. Let us then call *B-segment of P* every subpath of P whose end vertices are in B_2 and whose interior vertices are not in B_2. Note that P is edgewise partitioned into its B-segment. Similarly, let us call *A-segment of P* every subpath of P whose end-vertices are in A_2 and whose interior vertices are not in A_2. By (3), every A-segment has even length or has length 1. An A-segment of length 1 is called an *A-edge*. Suppose that P has odd length. Let $b, b' \in B_2$ be the end-vertices of P. Along P from b to b', let us call a the first vertex in A_2 after b, and a' the last vertex in A_2 before b'. So $b - P - a$ and $a' - P - b'$ are both paths from B_2 to A_2, and by (1) they have same parity. So $a - P - a'$ is a path of odd length that is edgewise partitioned into its A-segment, and that contains all the A-segments of P. Thus P has an odd number of A-edges. Since P is edgewise partitioned into into its B-segments, there is a B-segment P' of P with an odd number of A-edges. Let β, β' be the end-vertices of P'. Along P' from β to β', let us call α the first vertex in A_2 after β, and α' the last vertex in A_2 before β'. So $P'' = \alpha - P' - \alpha'$ is a path that is edgewise partitioned into its A-segment with an odd number of A-edge. Thus P'' has odd length. Since $\beta - P - \alpha$ and $\alpha' - P - \beta'$ are both paths from B_2 to A_2, they have same parity by (1). Finally, P' is of odd length, outgoing from B_2 to B_2, and contradicts (2). Thus P has even length. This proves (4).

(5) Every antipath of G' with length at least 2, with its end vertices in $V(G') \setminus A_2$, and all its interior vertices in A_2 has even length.

Let Q be such an antipath. We may assume that Q has length at least 3. So each end-vertex of Q must have a neighbor in A_2 and a non-neighbor in A_2. So none of $a_1, c_1, c_2, b_1, b_3, b_4$ can be an end-vertex of Q, and Q may be viewed as an antipath of G. So Q has even length by Lemma 3.8. This proves (5).

(6) Every antipath of G' with length at least 2, with its end vertices in $V(G') \setminus B_2$, and all its interior vertices in B_2 has even length.

Let Q be such an antipath. We may assume that Q has length at least 3. So each end-vertex of Q must have a neighbor in B_2 and a non-neighbor in B_2. So none of a_1, b_1, c_1, c_2 can be an end-vertex of Q. If b_3 is an end-vertex of Q, then the other end-vertex must be adjacent to b_3 while not being in $B_2 \cup \{a_1, b_1, c_1, c_2\}$, a contradiction. So b_3 is not an end-vertex of Q and by a similar proof, neither is b_4. So none of $a_1, c_1, c_2, b_1, b_3, b_4$ is in Q and Q may be viewed as an antipath of G. So Q has even length by Lemma 3.8. This proves (6).

(7) Every antipath of G' with length at least 2, with its end vertices in $V(G') \setminus B_3$ (resp. $V(G') \setminus B_4$), and all its interior vertices in B_3 (resp. B_4) has even length.

Let Q be such an antipath whose interior is in B_3 (the case with B_4 is similar). We may assume that Q has length at least 3. So each end-vertex of Q must have a neighbor in B_3. So no vertex of B_4 can be an end-vertex of Q. Thus (6) applies and Q has even length. This proves (7).

(8) Let Q be an antipath of G' of length at least 4. Then Q does not go through c_1, c_2. Moreover Q goes through at most one of a_1, b_1, b_3, b_4.

In an antipath of length at least 4, each vertex either is in a square of the antipath or in a triangle of the antipath. So, c_1, c_2 are not in Q since they are not in any triangle or square of G'. In an antipath of length at least 4, for any pair x, y of non-adjacent vertices, there must be a third vertex adjacent to both x, y. Thus, Q goes through at most one vertex among a_1, b_3, b_4. Suppose now that Q also goes through b_1. Then it does not go through a_1 since a_1, b_1 have no common neighbours. So, up to a symmetry we may assume that Q goes through b_3 and b_1. There is no vertex in $G' \setminus c_2$ seeing b_3 and missing b_1. So b_1 is an end of Q. Along Q, after b_1 we meet b_3. The next vertex along Q must be in B_4. The next one, in B_3. The next one must see b_3 and must have a neighbor in B_4, a contradiction. This proves (8).

(9) G' is Berge.

Let H be a hole of G'. Suppose first that H goes through a_1. If H does not go through c_1, then $H \setminus a_1$ is a path of even length by (3), so H has even length. If H goes through c_1 then H goes though exactly one of b_3, b_4, say b_3 up to symmetry, and $H \setminus \{a_1, c_1, c_2, b_3\}$ is a path P. If P does not go through b_1 then it has parity ε by (1). If P goes through b_1, then $P = b - b_1 - b' - \ldots - a$ where $b' - P - a$ is from B_4 to A_2. So, again P has parity ε by (1). So H has even length and we may assume that H does not go through a_1. If $c_1 \neq c_2$ then H does not go through c_1. If H goes through c_2 then the path $H \setminus \{b_3, c_2, b_4\}$ has even length by (2), so H is even. If H goes through b_1 then the path $H \setminus \{b_1\}$ has even length by (2), so H is even. So we may assume that H does not go through b_1, c_2. If H goes through both b_3, b_4 then $H \setminus \{b_3, b_4\}$ is partitioned into two outgoing paths from B_2 to B_2 that both have even length by (2). Thus H has even length. If H goes through b_3 and not through b_4, then $H \setminus b_3$ is an outgoing path from B_3 to B_3. By (4) it has even length, so H

is even. If H goes through b_4 and not through b_3 then H is even by a similar proof. So we may assume that H goes through none of b_3, b_4. Now, H goes through none of $a_1, c_1, c_2, b_1, b_3, b_4$. So H may be viewed as a hole of G, and so it is even. So every hole of G' is even.

Let us now consider an antihole H of G'. Since the antihole on 5 vertices is isomorphic to C_5, we may assume that H has at least 7 vertices. Let v be a vertex of H that is not in $\{a_1, c_1, c_2, b_1, b_3, b_4\}$. By (8) applied to $H \setminus \{v\}$, H does not go through c_1, c_2 and goes through at most one vertex of $\{a_1, b_1, b_3, b_4\}$. If H goes through a_1, the antipath $H \setminus a_1$ has all its interior vertices in A_2 and by (5), $H \setminus a_1$ has even length, thus H is even. If H goes through b_1 then the antipath $H \setminus b_1$ has all its interior vertices in B_2 and by (6), $H \setminus b_1$ has even length, thus H is even. If H goes through one of b_3, b_4, say b_3 up to a symmetry, the antipath $H \setminus b_3$ has all its interior vertices in B_3 and by (7), $H \setminus b_3$ has even length, thus H is even. If H goes through none of $a_1, c_1, c_2, b_1, b_3, b_4$ then H may be viewed as an antihole of G. So every antihole of G' has even length. This proves (9).

(10) G' has no balanced skew partition.

Let (F', E') be a balanced skew partition of G' with a split (E'_1, E'_2, F'_1, F'_2). Starting from F', we shall build a balanced skew cutset F of G which contradicts the properties of G.

Let us first suppose $c_1 \neq c_2$ and $c_1 \in F'$. Then, F' must contains at least a neighbor of c_1. If F' contains a_1 and not c_2, then F' is a star cutset of G' centered at a_1. But this contradicts Property f of G. If F' contains c_2 and not a_1, then F' is a star cutset of G' centered at c_2. But this again contradicts Property f of G. So, F' must contain a_1 and c_2. Since a_1, c_2 have no common neighbors we have $F' = \{a_1, c_1, c_2\}$. This is a contradiction since $G' \setminus \{a_1, c_1, c_2\}$ is connected by Property f of G. So if $c_1 \neq c_2$ then $c_1 \notin F'$.

Suppose $c_2 \in F'$. By Property f of G, no subset of $\{c_2, b_3, b_4\}$ can be a cutset of G. So, F' must be a star cutset centered at one of b_3, b_4. This again contradicts Property f of G. So $c_2 \notin F'$. Not both b_3, b_4 can be in F' since they have no common neighbors in F'. So we assume $b_4 \notin F'$

Up to a symmetry, we may assume $\{c_1, c_2, b_4\} \subset E'_1$. Also, $\{a_1, b_3\} \cap E' \subset E'_1$. We claim that $\{b_1\} \cap E' \subset E'_1$. Else, F' separates b_1 from c_2. Hence we must have $B_4 \subset F'$. Now $b_3 \in F'$ is impossible since there is no vertex seeing b_3 and having a neighbor in B_4. So, $B_3 \subset F'$. Since there is no edge between B_3 and B_4, there must be a vertex in F' that is complete to $B_3 \cup B_4 = B_2$. The only place to find such a vertex is in A_2. But this implies (X_1, X_2) being degenerate, contradicting Property c of G.

We proved $\{c_1, c_2, b_4\} \subset E'_1$ and $\{a_1, b_1, b_3\} \cap E' \subset E'_1$. Let v be any vertex of E'_2. Since $\{a_1, c_1, c_2, b_1, b_3, b_4\} \cap E' \subset E'_1$, we have $v \in X_2$. If b_3 is in F, put $B'_1 = \{b_1\}$, else put $B'_1 = \emptyset$. Now $F = (F' \setminus \{b_3\}) \cup B'_1$ is a skew cutset of G that separates v from the interior vertices of the path induced by X_1. Indeed, either $F = F'$, or F' is obtained by deleting b_3 and adding b_1. Since $N(b_3) \cap X_2 \subset N(b_1) \cap X_2$, F is not anticonnected and is a cutset. It suffices now to prove that F is a balanced skew cutset of G.

Let P be an outgoing path of G from F to F. We shall prove that P has even length.

If $a_1, b_1 \notin F$, then $F \subset X_2$ and the end-vertices of P are both in X_2. So Lemma 3.9 applies to P. Suppose that the first outcome of Lemma 3.9 is satisfied: $V(P) \subseteq X_2 \cup \{a_1, b_1\}$. Note that by the definition of F, $b_1 \notin F$ implies $b_1 \notin F'$. Hence, P may be viewed as an outgoing path from F' to F', so P has even length since F' is a balanced skew cutset of G'. Suppose now that the second outcome of Lemma 3.9 is satisfied: $P = c - \cdots - a_2 - a_1 - X_1 - b_1 - b_2 - \cdots - c'$. Put $i = 3$ if $b_2 \in B_3$ and $i = 4$ if $b_2 \in B_4$. Put $P' = c - P - a_2 - a_1 - c_1 - c_2 - b_i - b_2 - P - c'$. Note that by the definition of F, $b_1 \notin F$ implies $b_3 \notin F'$. The paths P and P' have same parity and P' is an outgoing path of G' from F' to F'. So P' and P has even length since F' is a balanced skew cutset of G'.

If $a_1 \in F$, note that $b_1 \notin F$ since a_1, b_1 are non-adjacent with no common neighbors (in both G, G'). We have $F' = F \subset X_2 \cup \{a_1\}$, the end-vertices of P are both in $X_2 \cup \{a_1\}$ and no interior vertex of P is in $\{a_1\}$ since $a_1 \in F$. So Lemma 3.10 applies. If Outcome 1 of the lemma holds, then P has even length. If Outcome 2 of the lemma holds, then just like in the preceding paragraph, we can build a path P' of G' that is outgoing from F to F and that has a length with the same parity than P. So P has even length. If Outcome 3 of the lemma holds, the proof is again similar to the preceding paragraph.

If $b_1 \in F$ then $a_1 \notin F$, $F \subset X_2 \cup \{b_1\}$, and Lemma 3.10 applies. If Outcome 1 of the lemma holds, then P has even length. If Outcome 2 of the lemma holds, we may assume that b_1 that is in $F \setminus F'$ and that b_1 is an end of P, for otherwise the proof works like in the paragraph above. Then we build a path P' of G' that is outgoing from F' to F' and that has a length with same parity than P, by replacing $\{b_1\}$ by $\{b_3\}$ (if P goes through B_3) or by $\{b_3, c_2, b_4\}$ (if P goes through b_4). So P has even length. If Outcome 3 of the lemma holds then $P = b_1 - X_1 - a_1 - a_2 - \cdots - c$ where $a_2 \in A_2$, $c \in X_2$. Note that one of b_1, b_3 is in F'. If $b_3 \in F'$, then we put $P' = b_3 - c_2 - c_1 - a_1 - a_2 - P - c$. If $b_3 \notin F'$ then up to a symmetry, we assume $V(a_2 - P - c) \subset A_2 \cup C_3$. Note that $b_1 \in F'$. We put $P' = b_1 - b - b_4 - c_2 - c_1 - a_1 - a_2 - P - c$ where b is any vertex in B_4. It may happen that P' is not a path of G' because of the chord $a_2 b$. But then we put $P' = b_1 - b - a_2 - P - c$. In every case, P' is outgoing from F' to F', and has same parity than P. Hence, P has even length.

Now, let Q be an antipath of G of length at least 2 with all its interior vertices in F and with its end-vertices outside of F. We shall prove that Q has even length. Note that we may assume that Q has length at least 5, because if Q has length 3, it may be viewed as an outgoing path from F to F, that have even length by the discussion above on paths.

If both $a_1, b_1 \notin F$, then $F \subset X_2$ and the interior vertices of Q are all in X_2. So Lemma 3.11 applies: $V(Q) \subseteq X_2 \cup \{a\}$ where $a \in \{a_1, b_1\}$. So Q may be viewed as an antipath of G' that has even length because F' is a balanced skew cutset of G'.

If $a_1 \in F$, let us remind that $b_1 \notin F$. We have $F \subset X_2 \cup \{a_1\}$, the interior vertices of Q are in $X_2 \cup \{a_1\}$ and the end-vertices of Q are not in $\{a_1\}$ since $a_1 \in F$. So Lemma 3.12 applies. We may assume that Outcome 2 holds. Once again, Q may be viewed as an outgoing path of G' that has even length because F' is balanced.

If $b_1 \in F$, we have to consider the case when $b_1 \notin F'$ (else the proof is like in the paragraph above). Since $b_1 \notin F'$, we have $b_3 \in F'$. Note that $B_4 \cap F' = B_4 \cap F = \emptyset$ since there are no edges between b_3, B_4 and no vertex seeing b_3 while having a neighbor in B_4. So, if Q is an antipath whose interior is in F, then Q does not go through B_4. Hence, if we replace b_1 by b_3, we obtain an antipath Q' whose interior is in F' and whose ends are not. Hence, Q has even length.

In every case, Q has even length. This proves (10).

(11) G' and $\overline{G'}$ have no degenerate substantial 2-join, no degenerate homogeneous 2-join and no star cutset.

If one of G', $\overline{G'}$ has a degenerate substantial 2-join, a degenerate homogeneous 2-join or a star cutset then G' has a balanced skew partition by Lemma 3.13, 3.19 or 3.3. This contradicts (10). This proves (11).

(12) G' is not basic, not a path-cobipartite graph, not a path-double split graph and has no homogeneous 2-join.

If G' is bipartite then all the vertices of A_2 are of the same color because of a_1. Because of b_1 all the vertices of B_2 have the same color. By Property f of G, there is a path from A_2 to B_2 whose interior is in C_2 that has parity ε by (1). So, the number of colors in $A_2 \cup B_2$ is equal to $1 + \varepsilon$, implying that G is bipartite and contradicting the properties of G. Hence G' is not bipartite.

One of the graphs $G'[c_2, c_1, b_3, b_4]$, $G'[a_1, c_1, b_3, b_4]$ is a claw, so G' is not the line-graph of a bipartite graph by Theorem 3.1. Let us choose $b \in B_3, b' \in B_4$. The graph $\overline{G'}[a_1, c_1, b, b']$ is a diamond, so $\overline{G'}$ is not the line-graph of a bipartite graph by Theorem 3.1.

Note that b, b' both have degree at least 3 in G' because since (X_1, X_2) is not degenerate, b, b' have neighbors in $A_2 \cup C_2$. Also a_1 has degree at least 3 in G' by Property d of G. So, there exist in G' a stable set of size 3 containing vertices of degree at least 3 ($\{a_1, b, b'\}$), and a vertex of degree 3 whose neighborhood induces a stable set (c_1). Hence, by Lemma 3.6, G' is not a path-cobipartite graph (and in particular, it is not the complement of a bipartite graph), not a path-double split graph (and in particular, it is not a double split graph) and G' has no non-degenerate homogeneous 2-join. Hence by (11), G' has no homogeneous 2-join. This proves (12).

We now give five claims describing the proper 2-joins of G'.

(13) There exist no sets Y_1, Z_1, Y_2, Z_2 such that:

- *Y_1, Z_1, Y_2, Z_2 are pairwise disjoint and $Y_1 \cup Z_1 \cup Y_2 \cup Z_2 = X_2$;*
- *there are every possible edges between Y_1 and Y_2, and these edges are the only edges between $Y_1 \cup Z_1$ and $Y_2 \cup Z_2$;*
- *$A_2 \subset Y_1 \cup Z_1$ and $B_2 \subset Y_2 \cup Z_2$.*

Suppose such sets exist. Note that $Y_1 \neq \emptyset$ and $Y_2 \neq \emptyset$ since by Property e of G, $G[X_2]$ is connected. Note that Z_1, Z_2 can be empty. Suppose $Y_2 \cap B_2 \neq \emptyset$ and pick a vertex $b \in Y_2 \cap B_2$. Up to a symmetry we assume $b \in B_3$ and we pick a vertex $b' \in B_4$. Since $B_2 \subset Y_2 \cup Z_2$ we have $b' \in Y_2 \cup Z_2$. Now $\{b\} \cup N(b)$ is a star cutset of G that separates a_1 from b', contradicting the properties of G. Thus $Y_2 \cap B_2 = \emptyset$. Hence $(Y_2 \cup Z_2, V(G) \setminus (Y_2 \cup Z_2))$ is a 2-join of G. This 2-join is proper (the check of connectivity relies on the fact that (X_1, X_2) is connected and on Lemma 3.14). By the properties of G, this 2-join has to be a path 2-join. Since X_1 is a maximal flat path of G, $Y_2 \cup Z_2$ is the path-side of the 2-join. This is impossible because $|B_2| \geq 2$. This proves (13).

Implicitly, when (X'_1, X'_2) is a 2-join, we consider a split $(X'_1, X'_2, A'_1, B'_1, A'_2, B'_2)$. We also put $C'_1 = X'_1 \setminus (A'_1 \cup B'_1)$ and $C'_2 = X'_2 \setminus (A'_2 \cup B'_2)$.

(14) If G' has a proper 2-join (X'_1, X'_2) then either $\{c_1, c_2\} \subset X'_1$ or $\{c_1, c_2\} \subset X'_2$.

Suppose not. We may assume that there is a 2-join (X'_1, X'_2) such that $c_1 \in X'_2$ and $c_2 \in X'_1$. In particular, $c_1 \neq c_2$. Up to a symmetry, we assume $c_1 \in A'_2$ and $c_2 \in A'_1$. Then, $a_1 \in X'_2$ for otherwise c_1 is isolated in X'_2, contradicting (X'_1, X'_2) being proper. Also one of b_3, b_4 must be in X'_1 for otherwise c_2 is isolated in X'_1. Up to a symmetry we assume $b_3 \in X'_1$.

By Property f of G there is a path $P = h_1 - \cdots - h_k$ from A_2 to B_3 whose interior is in C_2, with $h_1 \in A_2$, $h_k \in B_3$. We denote by H the hole induced by $V(P) \cup \{a_1, c_1, c_2, b_3\}$. Note that H has an edge whose ends are both in X'_1 (it is $c_2 b_3$) and an edge whose ends are both in X'_2 (it is $a_1 c_1$). So H is vertex-wise partitioned into a path from A'_1 to B'_1 whose interior is in X'_1 and a path from B'_2 to A'_2 whose interior is in X'_2. Hence, starting from c_1, then going to a_1 and continuing along H, one will first stay in X'_2, will meet a vertex in B'_2, immediately after that, a vertex in B'_1, and after that will stay in X'_1 and reach c_2. We now discuss several cases according to the unique vertex x in $H \cap B'_2$.

If $x = a_1$ then $a_1 \in B'_2$. So $b_3 \in C'_1$. This implies step by step $B_3 \subset X'_1$, $B_3 \subset C'_1$, $b_1 \in X'_1$, $b_1 \in C'_1$, $B_4 \subset X'_1$, $B_4 \subset C'_1$, $b_4 \in X'_1$. Let v a vertex in C_2 (if any). Then by Property f of G there is a path Q from v to B_2 with no vertex in A_2. If $v \in X'_2$, then Q must contain a vertex in $A'_1 \cup B'_1$. This is impossible since no vertex in $C_2 \cup B_2$ sees a_1 or c_1. So, $C_2 \subset C'_1$. Let v be a vertex in A_2. Note that by Property f of G, v must have a neighbor in $C_2 \cup B_2$. So, $v \in X'_1$ since $C_2 \cup B_2 \subset C'_1$. Finally, we proved $X'_2 = \{a_1, c_1\}$. This is impossible since (X'_1, X'_2) is proper.

If $x = h_i$ with $1 \leq i < k$, then $h_i \in B'_2 \cap (A_2 \cup C_2)$ and $h_{i+1} \in B'_1$. Note that $b_3 \in C'_1$ since b_3 misses c_1 and h_1. So, $B_3 \subset X'_1$. By the definition of x, we know that $a_1 \in C'_2$. So, $A_2 \subset X'_2$. We consider now two cases.

First case: $b_4 \in X'_1$. Since there are no edges between $\{b_3, b_4\}$ and $\{c_1, h_i\}$ we know that $\{b_3, b_4\} \subset C'_1$. This implies $B_3 \cup B_4 \subset X'_1$. Also, $b_1 \in X'_1$ for otherwise b_1 is isolated in X'_2. Now, $A'_1 \cup B'_1 \subset (B_2 \cup C_2)$. Let us put: $Y_1 = B'_2$, $Z_1 = (X'_2 \cap X_2) \setminus Y_1$, $Y_2 = B'_1$, $Z_2 = (X'_1 \cap X_2) \setminus Y_2$. These four sets yield a contradiction to (13).

Second case: $b_4 \in X'_2$. Then $b_4 \in A'_2$ and $A'_1 = \{c_2\}$. If there is a vertex v of X'_1 in B_4 then $v \in A'_1$. This is impossible since v misses $c_1 \in A'_2$. So, $B_4 \subset X'_2$. Hence, if $b_1 \in X'_1$ then $b_1 \in A'_1 \cup B'_1$. But this is impossible since b_1 misses c_1 and h_i. So, $b_1 \in X'_2$. Since $B_3 \subset X'_1$, we know $B_3 = B'_1$ and $b_1 \in B'_2$. So b_3 is a vertex of C'_1 complete to $A'_1 \cup B'_1$, implying (X'_1, X'_2) being degenerate, a contradiction.

If $x = h_k$ then $a_1 \in C'_2$ and $A_2 \subset X'_2$. Let v be a vertex of $C_2 \cup B_3 \cup B_4 \cup \{b_1, b_4\}$. By Property f of G there is a path Q from v to A_2 with no interior vertex in $B_3 \cup A_2$. If $v \in X'_1$, then Q must have a vertex $u \neq v$ in $A'_2 \cup B'_2$. Note $u \notin B_3$.This is impossible because u misses c_2 and b_3. So, $v \in X'_2$. Hence, $X'_1 = \{c_2, b_3\}$ contradicting (X'_1, X'_2) being proper. This proves (14).

(15) If G' has a proper 2-join (X'_1, X'_2) then either $\{c_1, c_2, b_3, b_4\} \subset X'_1$ or $\{c_1, c_2, b_3, b_4\} \subset X'_2$.

Suppose not. By (14), we may assume that there is a 2-join (X'_1, X'_2) such that $c_1, c_2 \in X'_1$ and $b_3 \in X'_2$. Up to a symmetry, we assume $c_2 \in A'_1$ and $b_3 \in A'_2$. At least one vertex of B_3 is in X'_2 for otherwise b_3 is isolated in X'_2. So let b be a vertex of $X'_2 \cap B_3$. We claim that there is a hole H that goes through b_3, c_2, c_1, a_1, $h_1 \in A_2$, $\dots h_k = b$, with at least an edge in X'_1 and at least an edge in X'_2. If $c_1 \neq c_2$ then our claim hold trivially: $c_1c_2 \in X'_1$ and $b_3b \in X'_2$. If $c_1 = c_2$, suppose that our claim fails. Then $a_1 \in X'_2$, implying $A'_1 = \{c_2\}$ and $a_1 \in A'_2$. We have $b_4 \in X'_1$ for otherwise c_2 is isolated in X'_1. If $b_4 \in B'_1$ then (X'_1, X'_2) is degenerate since b_4 is complete to A'_1. So, $b_4 \in C'_1$ implying $B_4 \subset X'_1$. If $b_1 \in X'_1$ then $b \in B'_1$ since $b \in X'_2$. So $B'_2 \subset B_3$ and b_3 is a vertex of A'_2 that is complete to B'_2, implying (X'_1, X'_2) being degenerate, a contradiction. So $b_1 \in X'_2$. Hence $B'_1 = B_4$ because no vertex of B'_1 can be in B_3 since $b_3 \in A'_2$. So $b_4 \in C'_1$ is complete to $A'_1 \cup B'_1$, implying (X'_1, X'_2) being degenerate, a contradiction. Thus our claim holds: H has an edge in X'_1 and an edge in X'_2. So there is a unique vertex x in $H \cap B'_2$. We now discuss according to the place of x.

If $x = a_1$ then by the discussion above $c_1 \neq c_2$. Also, $a_1 \in B'_2$ and $c_1 \in B'_1$. Suppose that $X'_1 \cap X_2$ and $X'_2 \cap X_2$ are both non-empty. The vertices of $A'_2 \cup B'_2$ are not in X_2 because they have to see either c_1 or c_2. So there are no edges between $X'_1 \cap X_2$ and $X'_2 \cap X_2$. Hence, $G'[X_2]$ is not connected, contradicting Property e of G. So either $X_2 \subset X'_1$ or $X_2 \subset X'_2$. If $X_2 \subset X'_1$ then $X'_2 \subset \{a_1, b_1, b_3, b_4\}$, so X'_2 is a stable set, contradicting (X'_1, X'_2) being proper. If $X_2 \subset X'_2$ then b_1 is in X'_2 for otherwise it is isolated in X'_1. So, $X'_1 \subset \{c_1, c_2, b_4\}$. This is a contradiction since no subset of $\{c_1, c_2, b_4\}$ can be a side of a proper 2-join of G'.

If $x = h_1$ then $h_1 \in B'_2$ and $a_1 \in B'_1$. If $b_4 \in X'_1$ then $b_4 \in C'_1$ because of b_3 and h_1. So, $B_4 \subset X'_1$. But in fact, by the same way, $B_4 \subset C'_1$, and $b_1 \in C'_1$. So, $B_3 \subset X'_1$, contradicting $h_k \in X'_2$. We proved $b_4 \in X'_2$ implying $A'_1 = \{c_2\}$. If a vertex v of $X_2 \cup \{b_1\}$ is in X'_1, then by Lemma 3.14 applied to (X'_1, X'_2) there is a path of X'_1 from v to $A'_1 = \{c_2\}$ with no interior vertex in B'_1, a contradiction. So $X_2 \cup \{b_1\} \subset X'_2$. We proved $X'_1 = \{a_1, c_1, c_2\}$ contradicting (X'_1, X'_2) being proper.

If $x = h_i$, $2 \leq i \leq k$ then $h_i \in B'_2$, $h_{i-1} \in B'_1$. Since $a_1 \in C'_1$ we have $A_2 \subset X'_1$. If $b_4 \in X'_1$ then $b_4 \in C'_1$ implying $B_4 \subset X'_1$. If $b_1 \in X'_2$ then b_1 must be in $A'_2 \cup B'_2$, a contradiction since b_1 misses c_2 and h_{i-1}. So, $b_1 \in X'_1$. Since $h_k \in X'_2$, we know $b_1 \in B'_1$. Thus $B'_2 \subset B_3$. Hence b_3 is a vertex of A'_2 that is complete to B'_2, implying (X'_1, X'_2) being degenerate, a contradiction. We proved $b_4 \in X'_2$. Now $A'_2 = \{b_3, b_4\}$. Suppose that there is a vertex v of X'_1 in $B_3 \cup B_4$. Then v must be in A'_1 since v sees one of b_3, b_4. But this is a contradiction since v misses one of b_3, b_4. We proved $B_3 \cup B_4 \subset X'_2$. Also, $b_1 \in X'_2$ for otherwise, b_1 is isolated in X'_1. Let us put: $Y_1 = B'_1$, $Z_1 = (X'_1 \cap X_2) \setminus Y_1$, $Y_2 = B'_2$, $Z_2 = (X'_2 \cap X_2) \setminus Y_2$. These four sets yield a contradiction to (13). This proves (15).

(16) If G' has a proper 2-join (X'_1, X'_2) then either $\{c_1, c_2, b_1, b_3, b_4\} \subset X'_1$ or $\{c_1, c_2, b_1, b_3, b_4\} \subset X'_2$.

Suppose not. By (15), we may assume that there is a 2-join (X'_1, X'_2) of G' such that $c_1, c_2, b_3, b_4 \in X'_1$ and $b_1 \in X'_2$. If $\{b_3, b_4\} \cap (A'_1 \cup B'_1) = \emptyset$ then $\{b_3, b_4\} \subset C'_1$, so $B_3 \cup B_4 \subset X'_1$. Hence b_1 is isolated in X'_2, a contradiction.

If $|\{b_3, b_4\} \cap (A'_1 \cup B'_1)| = 1$, then up to a symmetry we may assume $b_3 \in A'_1$ and $b_4 \in C'_1$. Thus $B_4 \subset X'_1$. Since $b_1 \in X'_2$, we have $B_4 \subset A'_1 \cup B'_1$. But no vertex x of B_4 can be in A'_1 because x and b_3 have no common neighbors, so $B_4 \subset B'_1$. Thus $b_1 \in B'_2$. Because of b_3, $A'_2 \subset B_3$. So b_1 is a vertex of B'_2 that is complete to A'_2, implying (X'_1, X'_2) being degenerate, a contradiction. We proved $\{b_3, b_4\} \subset (A'_1 \cup B'_1)$.

Since b_3, b_4 have no common neighbors in X'_2, we may assume up to a symmetry that $b_3 \in A'_1$ and $b_4 \in B'_1$. So b_1 have non-neighbors in both A'_1, B'_1. This implies $b_1 \in C'_2$, and $B_3 \cup B_4 \subset X'_2$. Hence $A'_2 = B_3$ and $B'_2 = B_4$. Now, $b_1 \in C'_2$ is complete to $A'_2 \cup B'_2$, implying (X'_1, X'_2) being degenerate, a contradiction. This proves (16).

(17) G' has no proper non-path 2-join.

Let (X'_1, X'_2) be a proper 2-join of G'. By (16), we may assume $\{c_1, c_2, b_1, b_3, b_4\} \subset X'_2$. If $b_3 \notin C'_2$ and $b_4 \notin C'_2$ then up to a symmetry we may assume $b_3 \in A'_2$, $b_4 \in B'_2$ since b_3, b_4 have no common neighbors in X'_1. So, there is a vertex of A'_1 in B_3 and a vertex of B'_1 in B_4 implying $b_1 \in A'_2 \cap B'_2$, a contradiction. We proved $b_3 \in C'_2$ or $b_4 \in C'_2$. Up to a symmetry we assume $b_3 \in C'_2$, implying $B_3 \subset X'_2$. Note that X'_1 is a subset of $V(G)$. If $A'_1 \cap B_4, B'_1 \cap B_4$ are both non-empty then b_1 must be in $A'_2 \cap B'_2$, a contradiction. Thus we may assume $A'_1 \cap B_4 = \emptyset$. If $a_1 \in X'_1$ and $B'_1 \cap B_4 \neq \emptyset$ then $a_1 \notin B'_1$ since a_1 misses b_1. Thus we may assume $B'_1 \cap \{a_1\} = \emptyset$.

Let us now put: $X''_1 = X'_1$, $X''_2 = V(G) \setminus X''_1$, $A''_1 = A'_1$, $B''_1 = B'_1$, $B''_2 = B'_2 \setminus \{b_4\}$. If $a_1 \in A'_1$ then $A''_2 = (A'_2 \cap X_2) \cup (N_G(a_1) \cap X_1)$ else $A''_2 = A'_2$. Note that $A''_2 \cap B''_2 = \emptyset$. Also, if $b_4 \in B'_2$ then $b_1 \in B'_2$ and $b_1 \in B''_2$. From the definitions it follows that (X''_1, X''_2) is a partition of $V(G)$, that $A''_1, B''_1 \subset X''_1$, $A''_2, B''_2 \subset X''_2$, that A''_1 is complete to A''_2, that B''_1 is complete to B''_2 and that there are no other edges between X''_1 and X''_2. So, (X''_1, X''_2) is a 2-join of G.

We claim that (X''_1, X''_2) is a proper 2-join of G. Note that $G[X''_1]$ is not a path of length 1 or 2 from A''_1 to B''_1 whose interior is in C''_1, because $X''_1 = X'_1$ and because (X'_1, X'_2) is a proper 2-join of G'. Also $G[X''_2]$ is not a path from A''_2 to B''_2 whose interior is in C''_2 because b_1 has at least 2 neighbors in X''_2 (one in X_1, one in B_3) while having degree at least 3 because of B_4. Hence (X''_1, X''_2) is substantial. So it is connected and proper for otherwise it is degenerate contradicting (11). This proves our claim.

Since (X''_1, X''_2) is proper, we know by the properties of G that (X''_1, X''_2) is a path 2-join of G. If X''_2 is the path-side of (X''_1, X''_2) then b_1 is an interior vertex of this path while having degree at least 3, a contradiction. Hence, X''_1 is the path-side of (X''_1, X''_2). Since $X''_1 = X'_1$, (X'_1, X'_2) is a path 2-join of G'. This proves (17).

(18) $\overline{G'}$ has no proper 2-join.

In the proof of (18), the word "neighbor" refers to the neighborhood in $\overline{G'}$. Let (X'_1, X'_2) be a proper 2-join of $\overline{G'}$.

Suppose $c_1 \neq c_2$. In $\overline{G'}$, c_1 has degree $n-3$, so up to a symmetry we may assume $c_1 \in A'_1$. In B'_2 there must be a non-neighbor of c_1. Also, since (X'_1, X'_2) cannot be a degenerate 2-join of $\overline{G'}$, vertex c_1 must have a non-neighbor in B'_1. So we have two cases to consider. Case 1: $a_1 \in B'_1$, $c_2 \in B'_2$. Then c_2 must have a non-neighbor in A'_2 for otherwise (X'_1, X'_2) is degenerate. This non-neighbor must be one of b_3, b_4. But this is impossible since b_3, b_4 both see a_1 in $\overline{G'}$. Case 2: $a_1 \in B'_2$, $c_2 \in B'_1$. Then $A'_2 \subset \{b_3, b_4\}$. So, $a_1 \in B'_2$ is complete to A'_2. Again, (X'_1, X'_2) is degenerate.

Suppose $c_1 = c_2$. Up to a symmetry we assume $c_1 \in X'_1$. If $c_1 \in C'_1$ then the only possible vertices in X'_2 are a_1, b_3, b_4, so $\overline{G'}[X'_2]$ induces a triangle. So, any vertex of A'_2 is complete to B'_2 and (X'_1, X'_2) is degenerate, a contradiction. So, $c_1 \notin C'_1$. Up to a symmetry, we assume $c_1 \in A'_1$. So, $B'_2 \subset \{a_1, b_3, b_4\}$ and at least one of a_1, b_3, b_4 (say x) must be in B'_2. Since (X'_1, X'_2) is not degenerate, c_1 must have a non-neighbor in B'_1. So, one of a_1, b_3, b_4 (say y) must be in B'_1. Since (X'_1, X'_2) is not degenerate, x must have a non-neighbor z in A'_2. But z must also be a non-neighbor of y. This is impossible because in $G' \setminus c_1$, $N(a_1), N(b_3), N(b_4)$ are disjoint. This proves (18).

(19) $\overline{G'}$ is not path-cobipartite, not a path-double split graph, has no homogeneous 2-join and has no flat path of length at least 3.

Else, by Lemma 3.4 there is a contradiction with one of (12), (10) or (18). This proves (19).

(20) $f(G') + f(\overline{G'}) < f(G) + f(\overline{G})$.

Every vertex in $\{a_1\} \cup B_3 \cup B_4$ has degree at least 3 in G'. For a_1, this is the property d of G and for vertices in $B_3 \cup B_4$, this is because (X_1, X_2) is not degenerate. Hence no vertex in $\{a_1\} \cup B_3 \cup B_4$ can be an interior vertex of a flat path of G', and no vertex in $\{c_1, c_2, b_3, b_4, b_1\}$ can be in a maximal flat path of G' of length at least 3. Hence, every maximal flat path of G' of length at least 3 is a maximal flat path of G, implying $f(G') \leq f(G)$. But in fact $f(G') < f(G)$ because X_1 is a flat path of G that is no more a flat path in G'. By (19) we know $0 = f(\overline{G'}) \leq f(\overline{G})$. We add these two inequalities. This proves (20).

Let us now finish the case. By (9), G' is Berge. By (12), G' is not basic, not path-cobipartite, not a path-double split graph, and has no homogeneous 2-join. By (10), G' has no balanced skew partition. By (17), G' has no proper non-path 2-join. By (18) $\overline{G'}$ has no proper 2-join. By (19), $\overline{G'}$ is not a path-cobipartite graph, a path-double split graph and has no homogeneous 2-join. So, G' is a counter-example to the theorem we are proving now. Hence there is a contradiction between the initial choice of G and (20). This completes the proof in Case 1.

Case 2: X_1 may be chosen in such a way that there are sets A_3, B_3 satisfying the items 1–5 of the definition of cutting 2-joins of type 2.

The frame of the proof is very much like in Case 1, but the details differ and are simpler. We consider the graph G' obtained from G by deleting $X_1 \setminus \{a_1, b_1\}$. Moreover, we add new vertices: c_1, c_2, a_3, b_3. Then we add every possible edge between a_3 and A_3, between b_3 and B_3. We also add edges a_1c_1, c_1c_2, c_2b_1, a_3b_3, c_1a_3, c_2b_3. Here are six claims about the parity of various kinds of paths and antipaths in G'.

(21) Every path of G' from B_2 to A_2 with no interior vertex in $A_2 \cup B_2$ has odd length.

If such a path contains one of $a_1, b_1, a_3, b_3, c_1, c_2$ then it has length 3 or 5. Else such a path may be viewed as a path of G from B_2 to A_2. By Lemma 3.7 it has odd length. This proves (21).

(22) Every outgoing path of G' from A_2 to A_2 (resp. B_2 to B_2) has even length.

For suppose there is such a path P from A_2 to A_2 (the case with B_2 is similar). If P goes through a_1 then it has length 2. If P goes through at least one of c_1, c_2, a_3, b_3, b_1 then P is the union of two edge-wise-disjoint paths from A_2 to B_2. Thus P has even length by (21). Else, P may be viewed as an outgoing path of G from A_2 to A_2, that has even length by Lemma 3.8. In every case, P has even length. This proves (22).

(23) Every outgoing path of G' from A_3 to A_3 (resp. B_3 to B_3) has even length.

For suppose there is such a path P from A_3 to A_3 (the case with B_3 is similar). If P goes through a_1, a_3 or B_3 then it has length 2. From now on, we assume that P goes through none of a_1, a_3, B_3. Hence P cannot go through b_3, c_1, c_2.

If P goes through b_1 then P is the edge-wise-disjoint union of two outgoing paths of G from $A_3 \cup \{b_1\}$ to $A_3 \cup \{b_1\}$. Thus P has even length by the definition of cutting 2-joins of type 2. Thus we may assume that P does not go through b_1.

Now P may be viewed as an outgoing path of G from A_3 to A_3, that does not go through b_1. Thus P is outgoing from $A_3 \cup \{b_1\}$ to $A_3 \cup \{b_1\}$, it has even length by the definition of cutting 2-joins of type 2. This proves (23).

(24) Every antipath of G' with length at least 2, with its end vertices in $V(G') \setminus A_2$ (resp. $V(G') \setminus B_2$), and all its interior vertices in A_2 (resp. B_2) has even length.

Let Q be such an antipath whose interior is in A_2 (the case with B_2 is similar). We may assume that Q has length at least 3. So each end-vertex of Q must have a neighbor in A_2 and a non-neighbor in A_2. So none of a_1, c_1, c_2, b_1, b_3 can be an end-vertex of Q. If a_3 is an end of Q then the other end of Q must be a neighbor of a_3, a contradiction. Thus Q may be viewed as an antipath of G. By Lemma 3.8, Q has even length. This proves (24).

(25) Every antipath of G' with length at least 2, with its end vertices in $V(G') \setminus A_3$ (resp. $V(G') \setminus B_3$), and all its interior vertices in A_3 (resp. B_3) has even length.

Let Q be such an antipath whose interior is in A_3 (the case with B_3 is similar). We may assume that Q has length at least 3. So each end-vertex of Q must have a neighbor in A_3 and a non-neighbor in A_3. So none of $a_1, a_3, c_1, c_2, b_1, b_3$ can be an end-vertex of Q. Thus Q may be viewed as an antipath of G. It has even length by the definition of cutting 2-joins of type 2. This proves (25).

(26) Let Q be an antipath of G' of length at least 5. Then Q does not go through c_1, c_2. Moreover one of $V(Q) \cap \{a_1, a_3\}$, $V(Q) \cap \{b_1, b_3\}$ is empty.

Let Q be such an antipath. In an antipath of length at least 5, each vertex is in a triangle of the antipath. So, c_1, c_2 are not in Q since they are not in any triangle of G'.

Suppose $V(Q) \cap \{a_1, a_3\}$, $V(Q) \cap \{b_1, b_3\}$ are both non-empty. In an antipath of length at least 6, for every pair u, v of vertices, there is a vertex x seeing both u, v. Thus Q has length 5 because no vertex of G' have neighbors in both $\{a_1, a_3\}$, $\{b_1, b_3\}$. Let $q_1, \dots, q_6$ be the vertices of Q in there natural order. Since $V(Q) \cap \{a_1, a_3\}$, $V(Q) \cap \{b_1, b_3\}$ are both non-empty there are two vertices of Q that have no common neighbors in G'. These vertices must be q_2 and q_5, and up to a symmetry we must have $q_2 = a_3$, $q_5 = b_3$. Thus q_3 must be a vertex of B_3 and q_4 must be a vertex of A_3. There is a contradiction since by the definition of cutting 2-joins of type 2, A_3 is complete to B_3. This proves (26).

(27) G' is Berge.

Let H be a hole of G'.

If H goes through both c_1, c_2 then H has length 4 or it must contains one of $\{a_1, b_1\}$, $\{a_1, b_3\}$, $\{b_1, a_3\}$. In the first case, H is edge-wise partitioned into two paths from A_2 to B_2. Thus H has even length by (21). In the second case H is edge-wise partitioned into two paths outgoing from $B_3 \cup \{a_1\}$ to $B_3 \cup \{a_1\}$, one of them of length 4, the other one included in $V(G)$. Thus H has even length by the definition of cutting 2-joins of type 2. The third case is similar. From now on, we assume that H goes through none of c_1, c_2. If H goes through both a_1, a_3 then it has length 4. If H goes through a_2 and not through a_3 then H has even length by (22). If H goes through a_3 and not through a_2 then H has even length by (23). Thus, we may assume that H goes through none of a_1, a_3. Similarly, we may assume that H goes through none of b_1, b_3. Now H may be viewed as a hole of G. In every case, H has even length.

Let us now consider an antihole H of G'. We may assume that H has length at least 7. Let v be a vertex of $V(H) \setminus \{a_1, b_1, c_1, c_2, a_3, b_3\}$. By (26) the antipath $V(H) \setminus v$ does not go through c_1, c_2 and we may assume up to a symmetry that $V(Q) \cap \{b_1, b_3\}$ is empty. If H goes through both a_1, a_3 then H must contains a vertex that sees a_3 and misses a_1, a contradiction. If H goes through a_1 and not through a_3 then H has even length by (24). If H goes through a_3 and not through a_1 then H has even length by (25). If H goes through none of a_1, a_3 then H may be viewed as an antihole of G. In every case, H has even length. This proves (27).

(28) G' has no balanced skew partition.

Suppose that G' has a balanced skew partition (E', F') with a split (E'_1, E'_2, F'_1, F'_2). Starting from F', we shall build a balanced skew cutset F of G which contradicts the properties of G.

By Property f of G, F' cannot be a star cutset centered at one of a_1, b_1, c_1, c_2, a_3, b_3. For the same reason, F' cannot be a subset of one of $\{c_1, c_2, a_3, b_3\}$, $\{a_1, c_1, a_3\} \cup A_3$, $\{b_1, c_2, b_3\} \cup B_3$. Thus, $c_1 \notin F'$ and $c_2 \notin F'$. Since a_1, b_1 are non-adjacent with no common neighbors, they are not both in F' and we may assume $b_1 \notin F'$. Up to symmetry we may assume $\{c_1, c_2\} \subset E'_1$, implying $\{a_1,\ a_3,\ c_1,\ c_2,\ b_1,\ b_3\} \cap E' \subset E'_1$. Let v be any vertex of E'_2. Since $\{a_1, a_3, c_1, c_2, b_1, b_3\} \cap E' \subset E'_1$, we have $v \in X_2$.

We claim that $F = F' \setminus \{a_3, b_3\}$ is a skew cutset of G that separates v from the interior vertices of the path induced by X_1. Since F' is not a star cutset centered at one of a_3, b_3, we know that if $a_3 \in F'$ (resp $b_3 \in F'$) then a_3 (resp. b_3) is not the only vertex in its anticomponent of F'. Hence, F is not anticonnected. If P is a path of $G \setminus F$ from v to a vertex u in the interior of X_1 then up to a symmetry, $P = v - \cdots - a_1 - X_1 - u$. Hence $v - P - a_1 - c_1$ is a path of $G' \setminus F'$ contradicting F' being a cutset of G'. We proved our claim. Let us prove that the skew cutset F is balanced.

Let P be an outgoing path of G from F to F. We shall prove that P has even length. If $a_1 \notin F$, then $F \subset X_2$ and the end-vertices of P are both in X_2. So Lemma 3.9 applies to P. Suppose that the first outcome of Lemma 3.9 is satisfied: $V(P) \subseteq X_2 \cup \{a_1, b_1\}$. Hence, P may be viewed as an outgoing path from F' to F', so P has even length since F' is a balanced skew cutset of G'. Suppose now that the second outcome of Lemma 3.9 is satisfied: $P = c - \cdots - a_2 - a_1 - X_1 - b_1 - b_2 - \cdots - c'$. Put $P' = c - P - a_2 - a_1 - c_1 - c_2 - b_1 - b_2 - P - c'$. The paths P and P' have same parity and P' is an outgoing path of G' from F' to F'. So P' and P has even length since F' is a balanced skew cutset of G'. If $a_1 \in F$ then $F \subset X_2 \cup \{a_1\}$ and Lemma 3.10 applies. If Outcome 1 of the lemma holds, then P has even length. If Outcome 2 of the lemma holds then P may be viewed as an outgoing path of G' from F' to F'. Hence P has even length. If Outcome 3 of the lemma holds then $P = a_1 - X_1 - b_1 - b_2 - \cdots - c$ where $b_2 \in B_2$, $c \in X_2$. We put $P' = a_1 - c_1 - c_2 - b_1 - b_2 - P - c$. So P' is outgoing from F' to F' in G' while having same parity than P. In every case P has even length.

Now, let Q be an antipath of G of length at least 5 with all its interior vertices in F and with its end-vertices outside of F. We shall prove that Q has even length. If $a_1 \notin F$, then $F \subset X_2$ and the interior vertices of Q are all in X_2. So Lemma 3.11 applies: $V(Q) \subseteq X_2 \cup \{a\}$ where $a \in \{a_1, b_1\}$. So Q may be viewed as an antipath of G' that has even length because F' is a balanced skew cutset of G'. If $a_1 \in F$, the proof is similar. Hence, Q has even length. This proves (28).

(29) G' and $\overline{G'}$ have no degenerate 2-join, no degenerate homogeneous 2-join and no star cutset.

If one of G', $\overline{G'}$ has a degenerate proper 2-join, a degenerate homogeneous 2-join or a star cutset, then G' has a balanced skew partition by Lemma 3.13, 3.19 or 3.3. This contradicts (28). This proves (29).

(30) G' is not basic, not a path-cobipartite graph, not a path-double split graph and has no homogeneous 2-join.

If G' is bipartite then all the vertices of A_2 are of the same color because of a_1. Because of b_1 all the vertices of B_2 have the same color. By Property f of G, there is a path from A_2 to B_2 that has odd length by (21). Thus G is bipartite, contradicting the properties of G. Hence G' is not bipartite.

The graph $G'[c_2, c_1, a_1, a_3]$ is a claw, so G' is not the line-graph of a bipartite graph. $\overline{G'}[a_1, b_1, a_3, b_3]$ is a diamond, so $\overline{G'}$ is not the line-graph of a bipartite graph.

Note that b_1 has degree at least 3 in G' by Property d of G. So, there exist in G' a stable set of size 3 containing vertices of degree at least 3 ($\{b_1, b_3, c_1\}$), and a vertex of degree 3 whose neighborhood induces a stable set (c_1). Hence, by Lemma 3.6, G' is not a path-cobipartite graph (and in particular, it is not the complement of a bipartite graph), not a path-double split graph (and in particular, it is not a double split graph) and G' has no non-degenerate homogeneous 2-join. Hence by (29), G' has no homogeneous 2-join. This proves (30).

(31) If G' has a proper 2-join (X'_1, X'_2) then either $\{c_1, c_2, a_3, b_3\} \subset X'_1$ or $\{c_1, c_2, a_3, b_3\} \subset X'_2$.

Suppose not. Up to a symmetry, we have five cases to consider according to $X'_1 \cap \{c_1, c_2, a_3, b_3\}$. Each of them leads to a contradiction:

Case $\{c_1\} \subset X'_1$ and $\{c_2, a_3, b_3\} \subset X'_2$:

Up to a symmetry, we assume $c_1 \in A'_1$ and $c_2, a_3 \in A'_2$. Note that $A'_1 = \{c_1\}$ because c_1 is the only vertex in X'_1 that sees both c_2, a_3. Note that a_1 is in X'_1 for otherwise c_1 is isolated in X'_1. Also if a vertex x of A_3 is in X'_1 then x must be in A'_1 since it sees a_3. This is impossible since x misses c_2. Thus $x \in X'_2$. Since x sees $a_1 \in X'_1$, x must be in B'_2 and a_1 must be in B'_1. So, a_1 is a vertex of B'_1 that is complete to A'_1, implying (X'_1, X'_2) being degenerate, contradicting (29).

Case $\{a_3\} \subset X'_1$ and $\{c_1, c_2, b_3\} \subset X'_2$:

This case is like the previous one, we just sketch it. We assume $a_3 \in A'_1$, implying $c_1, b_3 \in A'_2$. Thus $A'_1 = \{a_3\}$. There is a x vertex of X'_1 in A_3. Also, $a_1 \in X'_2$ for otherwise $a_1 \in A'_1$ while missing b_3, a contradiction. Thus $x \in B'_1$, and x is a vertex of B'_1 that is complete to A'_1, a contradiction.

Case $\{c_1, c_2\} \subset X'_1$ and $\{a_3, b_3\} \subset X'_2$:

Up to a symmetry, we assume $c_1 \in A'_1$, $a_3 \in A'_2$, $c_2 \in B'_1$, $b_3 \in B'_2$. Since by (29) (X'_1, X'_2) is not degenerate, a_3 must have a non-neighbor x in B'_2. Since x must see c_2 we have $x = b_1$ and $b_1 \in B'_2$. Similarly, b_3 must have a non-neighbor in A'_2, implying $a_1 \in A'_2$. Now put $Y_1 = X_2 \cap X'_1$ and $Y_2 = X_2 \cap X'_2$. Note that $Y_1 \neq \emptyset$ for otherwise $X'_1 = \{c_1, c_2\}$ and (X'_1, X'_2) is not proper. Also $Y_2 \neq \emptyset$ for otherwise, a_1 is isolated in X'_2. If there is an edge of G' with an end in Y_1 and an end y in Y_2, then y_2 must be in one of A'_2, B'_2. This is a contradiction since y misses both c_1, c_2. Thus there is no edge with an end in Y_1 and an end Y_2. This contradicts $G[X_2]$ being connected (Property e of G).

Case $\{c_1, a_3\} \subset X'_1$ and $\{c_2, b_3\} \subset X'_2$:

Up to a symmetry, we assume $c_1 \in A'_1$, $a_3 \in B'_1$, $c_2 \in A'_2$, $b_3 \in B'_2$. Since by (29) (X'_1, X'_2) is not degenerate, a_3 must have a non-neighbor x in A'_1. Since x must see c_2 we have $x = b_1$ and $b_1 \in A'_1$. Similarly, b_3 must have a non-neighbor in A'_2, implying $a_1 \in A'_2$. So, $b_1 \in A'_1$, $a_1 \in A'_2$ and $a_1 b_1 \notin E(G')$, a contradiction.

Case $\{c_1, b_3\} \subset X'_1$ and $\{c_2, a_3\} \subset X'_2$:

Up to a symmetry, we assume $c_1 \in A'_1$, $a_3 \in A'_2$, $c_2 \in A'_2$, $b_3 \in A'_1$. There is a vertex x of X'_1 in B_3 for otherwise b_3 is isolated in X'_1. Also, $b_1 \in X'_2$ for otherwise c_2

is isolated in X'_2. But b sees x. Since $b_1 \in A'_2$ is impossible because b_1 misses c_1 we have $b_1 \in B'_2$. Similarly, we prove $a_1 \in B'_1$. So, $b_1 \in B'_2$, $a_1 \in B'_1$ and $a_1b_1 \notin E(G')$, a contradiction. This proves (31).

(32) G' has no non-path proper 2-join.

By (31), we may assume $\{c_1, c_2, a_3, b_3\} \subset X'_2$. We claim that at most one of c_1, c_2, a_3, b_3 is in $A'_2 \cup B'_2$. For otherwise, up to a symmetry there are four cases. First case, $a_3 \in A'_2$, $b_3 \in B'_2$, implying $A'_1 \subset A_3$ and $B'_1 \subset B_3$, implying (X'_1, X'_2) being degenerate because any vertex of A'_1 is complete to B'_1, contradicting (29). Second case, $c_1 \in A'_2$, $c_2 \in B'_2$, implying $A'_1 = \{a_1\}$, $B'_1 = \{b_1\}$, $a_3, b_3 \in C'_2$, $A_3 \cup B_3 \subset X'_2$. Hence, $X'_1 \cap X_2 \neq \emptyset$ and $A_3 \cup B_3$ are in different components of $G[X_2]$ contradicting Property e of G. Third case, $a_3 \in A'_2$, $c_1 \in B'_2$ implying $A'_1 \subset A_3$, $a_1 \in B'_1$, implying (X'_1, X'_2) being degenerate because $a_1 \in B'_1$ is to complete to A'_1, contradicting (29). Fourth case, $a_3 \in A'_2$, $c_2 \in B'_2$ implying $b_1 \in B'_1$. Also $b_3 \in C'_2$ because b_3, c_2 (resp. b_3, a_3) have no common neighbors in X'_1. So $B_3 \subset X'_2$ and because of b_1, $B_3 \subset B'_2$. Because of a_3 there is a vertex a of A'_1 in A_3. Hence a is a vertex of A'_1 that has a neighbor in B'_2, a contradiction. All four cases yield a contradiction, so our claim is proved.

Thus up to a symmetry we assume that we are in one of the three cases that we describe below:

- $a_3 \in A'_2$. Moreover, $a_1 \in X'_2$ because $c_1 \in C'_2$. Because of a_3, there is a vertex of X'_1 in A_3, implying $a_1 \in A'_2$ and $B_3 \subset A'_2$.
- $c_1 \in A'_2$. This implies $a_1 \in A'_1$. Since $a_3 \in C'_2$, we have $A_3 \subset X'_2$ and $A_3 \subset A'_2$ because of a_1. Note that $A'_1 = \{a_1\}$ because a_1 is the only neighbor of c_1 in X'_1.
- $a_2 \notin A'_2$ and $c_1 \notin A'_2$. Moreover, $a_1 \in X'_2$ and $A_3 \subset X'_2$.

In every case, $c_2, b_3 \in C'_2$, implying $\{b_1\} \cup B_3 \subset X'_2$. Note that $X'_1 \subset V(G)$. Let us now put: $X''_1 = X'_1$, $X''_2 = V(G) \setminus X''_1$, $A''_1 = A'_1$, $B''_1 = B'_1$, $B''_2 = B'_2$. If $c_1 \in A'_2$ then put $A''_2 = (A'_2 \cap X_2) \cup (N_G(a_1) \cap X_1)$. If $c_1 \notin A'_2$ then put $A''_2 = A'_2 \setminus \{a_3\}$. From the definitions it follows that (X''_1, X''_2) is a partition of $V(G)$, that $A''_1, B''_1 \subset X''_1$, $A''_2, B''_2 \subset X''_2$, that A''_1 is complete to A''_2, that B''_1 is complete to B''_2 and that there are no other edges between X''_1 and X''_2. So, $(X''_1, X''_2) = (X'_1, V(G) \setminus X'_1)$ is a 2-join of G.

Note that $G[X''_1]$ is not a path of length 1 or 2 from A''_1 to B''_1 whose interior is in C''_1, because (X'_1, X'_2) is a proper 2-join of G' and because $X''_1 = X'_1$. Also $G[X''_2]$ is not a outgoing path from A''_2 to B''_2 whose interior is in C''_2 because b_1 has at least 2 neighbors in X''_2 (c_2 and one in B_3) while having degree at least 3 by Property d of G. This proves that (X''_1, X''_2) is substantial. It is connected for otherwise it is degenerate, contradicting (29). So (X''_1, X''_2) is proper and we know by the properties of G that (X''_1, X''_2) is a path 2-join of G. If X''_2 is the path-side of (X''_1, X''_2) then b_1 is an interior vertex of this path while having degree at least 3 by Property d of G, a contradiction. Hence, X''_1 is the path-side of (X''_1, X''_2). Thus (X''_1, X''_2) is a path 2-join of G because $X''_1 = X'_1$. This proves (32).

(33) $\overline{G'}$ has no proper 2-join.

In the proof of (33), the word "neighbor" refers to the neighborhood in $\overline{G'}$. Let (X'_1, X'_2) be a proper 2-join of $\overline{G'}$.

If $c_1 \in C'_1$ then $X'_2 \subset \{a_1, a_3, c_2\}$ implying (X'_1, X'_2) being degenerate or non-proper, contradicting (29). Thus, we may assume $c_1 \in A'_1$. Similarly c_2 must be in one of A'_1, A'_2, B'_1, B'_2. But $c_2 \in A'_2$ is impossible because c_2 is not a neighbor of c_1. Also $c_2 \in A'_1$ is impossible because otherwise $B'_2 = \emptyset$ since no vertex of $\overline{G'}$ can be a non-neighbor of both c_1, c_2. Thus c_2 is in one of B'_1, B'_2.

If $c_2 \in B'_1$ then $A'_2 \subset \{b_1, b_3\}$ because of c_2 and $B'_2 \subset \{a_1, a_3\}$ because of c_1. But b_1 must be in A'_2 because it is a common neighbor of c_1, a_1, a_3. Thus b_1 is a vertex of A'_2 that is complete to B'_2, implying (X'_1, X'_2) being degenerate, contradicting (29).

If $c_2 \in B'_2$ then there is a non-neighbor of c_2 in A'_2 for otherwise (X'_1, X'_2) is degenerate. Thus at least one of b_1, b_3 is in A'_2. Similarly, because of c_1, at least one of a_1, a_3 must be in B'_1. But since there is no edge of $\overline{G'}$ between B'_1, A'_2, we have $a_3 \in B'_1$, $b_3 \in A'_2$. Since a_3, b_3, c_2 are neighbors of a_1, we know $a_1 \in B'_2$. Now b_1 is a neighbor of $c_1 \in A'_1$, $a_3 \in B'_1$, $a_1 \in B'_2$, $b_3 \in A'_2$, a contradiction. This proves (33).

(34) $\overline{G'}$ is not path-cobipartite, not a path-double split graph, has no homogeneous 2-join and has no flat path of length at least 3.

Else, by Lemma 3.4 there is a contradiction with one of (30), (28) or (33). This proves (34).

(35) $f(G') + f(\overline{G'}) < f(G) + f(\overline{G})$.

Every vertex in $\{a_1, b_1\} \cup A_3 \cup B_3$ has degree at least 3 in G'. For a_1, this is the property d of G and for vertices in $A_3 \cup B_3$, this is clear. Hence no vertex in $\{a_1, b_1\} \cup A_3 \cup B_3$ can be an interior vertex of a flat path of G', and no vertex in $\{c_1, c_2, a_3, b_3\}$ can be in a maximal flat path of G' of length at least 3. Hence, every maximal flat path of G' of length at least 3 is a maximal flat path of G, implying $f(G') \leq f(G)$. But in fact $f(G') < f(G)$ because X_1 is a flat path of G that is no more a flat path in G'. By (34) we know $0 = f(\overline{G'}) \leq f(\overline{G})$. We add these two inequalities. This proves (35).

Let us now finish the case. By (27), G' is Berge. By (30), G' is not basic, not path-cobipartite, not a path-double split graph, and has no homogeneous 2-join. By (28), G' has no balanced skew partition. By (32), G' has no proper non-path 2-join. By (33) $\overline{G'}$ has no proper 2-join. By (34), $\overline{G'}$ is not a path-cobipartite graph, a path-double split graph and has no homogeneous 2-join. So, G' is a counter-example to the theorem we are proving now. Hence there is a contradiction between the initial choice of G and (35). This completes the proof in Case 2.

Case 3: We are neither in Case 1 nor in Case 2. In particular :

(36) G has no cutting 2-join.

We consider the graph G' obtained from G by replacing X_1 by a path of length $2 - \varepsilon$ from a_1 to b_1. Possibly, this path has length 2. In this case we denote by c_1 its unique interior vertex. Else, this path has length 1, and for convenience we put $c_1 = a_1$ (thus c_1 is a vertex of G' whatever ε). Note that $(V(G') \setminus X_2, X_2)$ is not a proper 2-join of G since $V(G') \setminus X_2$ is a path of length 1 or 2 from a_1 to b_1. Note that $a_1 - c_1 - b_1$ a flat path of G' (possibly of length 1 when $a_1 = c_1$) because if there is a common neighbor c of a_1, b_1, then (X_1, X_2) is not a 2-join of G. Note that G' is what we call in section 3.2 the block G_2 of G with respect to the 2-join (X_1, X_2).

(37) G' has no balanced skew partition, and none of G, $\overline{G'}$ has a star cutset, a degenerate substantial 2-join or a degenerate homogeneous 2-join.

Since G' is a block of G, and since (X_1, X_2) is not cutting, by Lemma 3.17, if G' has a balanced skew partition then so is G, contradicting the properties of G. By Lemma 3.3, 3.13 and 3.19, $G, \overline{G}$ have no star cutset, no degenerate 2-join and no degenerate homogeneous 2-join. This proves (37).

(38) G' is Berge.

Any hole H' of G' yield a hole of G of the same parity after possibly subdividing the flat path $a_1 - c_1 - b_1$. Also, a_1, b_1 cannot both be in an odd antihole of G' because in an antihole of length at least 7, any pair of vertex have a common neighbor. Also, if $c_1 \neq a_1$ then c_1 does not lie in an antihole of G' of length at least 7 because c_1 has degree 2. Thus, any antihole of G' may be viewed as an antihole of G. Thus, every hole and every antihole in G' are even. This proves (38).

(39) G' has no proper non-path 2-join.

Let $(X'_1, X'_2, A'_1, B'_1, A'_2, B'_2)$ be a split of a proper non-path 2-join of G'. If $a_1 \in X'_1$, $b_1 \in X'_1$ then $c_1 \in X'_1$ since otherwise c_1 is isolated in X'_2. If $c_1 \neq a_1$ then $c_1 \in C'_1$ because c_1 has degree 2. So, by subdividing $a_1 - c_1 - b_1$ we obtain a non-path proper 2-join of G, contradicting the properties of G. Thus, since $a_1 - c_1 - b_1$ is a flat path of G', up to a symmetry, we may assume $c_1 \in B'_1$, $b_1 \in B'_2$.

Suppose $|B'_2| = 1$. Then no vertex of A'_2 has a neighbor in B'_2 for otherwise, (X_1, X_2) is degenerate. Thus, $(X'_1 \cup B'_2, X'_2 \setminus B'_2)$ is a non-path proper 2-join of G', and by subdividing $a_1 - c_1 - b_1$, we obtain a non-path proper 2-join of G, contradicting the properties of G. Thus, $|B'_2| \geq 2$. In particular, $c_1 = a_1$, and similarly $|B'_1| \geq 2$.

In G, a_1 is complete to $B'_2 \setminus \{b_1\}$, and b_1 is complete to $B'_1 \setminus \{a_1\}$. We put $A_3 = B'_2 \setminus \{b_1\}$, $B_3 = B'_1 \setminus \{a_1\}$. In G, X_1 is a flat path from a_1 to b_1, $A_3 \subset A_2$ and $B_3 =\subset B_2$ and A_3 is complete to B_3. We claim that every path of G outgoing from $A_3 \cup \{b_1\}$ to $A_3 \cup \{b_1\}$ has even length. Note that after possibly deleting the interior of X_1, such a path P may be viewed as a path P' of G' that has same parity than P. In G', P' is an outgoing path from B'_1 to B'_1 and by Lemma 3.8, P has even length as claimed. We claim that every antipath of G whose interior is in $A_3 \cup \{b_1\}$ and whose ends are outside of $A_3 \cup \{b_1\}$ has even length. Let Q be such an antipath of length at least 5. Note that the interior vertices of X_1 are not in Q since every vertex in Q have degree at least 3. Thus Q is an antipath of G' whose interior is in B'_1 and whose ends are not in B'_1 and by Lemma 3.8, Q has even length as claimed. The same properties hold with $B_3 \cup \{a_1\}$. Now, A_3, B_3 show that (X_1, X_2) satisfies the items 1–5 of the definition of cutting 2-joins of type 2, contradicting that we are not in Case 2 of the proof of our theorem. This proves (39).

(40) $\overline{G'}$ has no proper 2-join.

Let us consider a proper 2-join of $\overline{G'}$ with a split $(X'_1, X'_2, A'_1, B'_1, A'_2, B'_2)$. If $c_1 \neq a_1$ then c_1 has degree $n-3$ in $\overline{G'}$. Thus, up to a symmetry, we may assume $c_1 \in B'_1$. Since (X'_1, X'_2) is not degenerate, c_1 must have a non-neighbor in A'_1. Thus, up to a symmetry, we may assume $a_1 \in A'_1$, $b_1 \in A'_2$. Now, since (X'_1, X'_2) is not degenerate, there exists a vertex of B'_2 that is a common neighbor of a_1, b_1 in G, contradicting $a_1 - c_1 - b_1$ being a flat path of G. We proved $a_1 = c_1$.

Since a_1, b_1 form a flat edge of G', they must be non-adjacent in $\overline{G'}$ with no common non-neighbor. Thus, up to a symmetry we have to deal with three cases:

Case $a_1 \in C'_1$, $b_1 \in X'_2$:

Since in G' $a_1 b_1$ is flat, in $\overline{G'}$ a_1 is complete to $A'_1 \cup B'_1$ or up to a symmetry $b_1 \in A'_2$ while being complete to B'_2. Thus, (X'_1, X'_2) is a degenerate 2-join, a contradiction.

Case $a_1 \in A'_1$, $b_1 \in B'_2$:

Since in G', $a_1 b_1$ is flat, in $\overline{G'}$, a_1 must be complete to $(A'_1 \cup C'_1) \setminus \{a_1\}$.

Suppose first $C'_1 \neq \emptyset$. There is at least a vertex of C'_1 that has a neighbor in B'_1 for otherwise $A'_1 \cup A'_2$ is a skew cutset of $\overline{G'}$, implying (X'_1, X'_2) being degenerate. If a_1 has a neighbor in B_1 then by Lemma 3.7 every path from A'_1 to B'_1 whose interior is in C'_1 has odd length. Thus, a_1 must see every vertex of B'_1 that has a neighbor in C'_1. This implies that $A'_1 \cup (N(a_1) \cap B'_1)$ is a star cutset of G', centered at a_1 and separating C'_1 from X'_2. Thus, a_1 has no neighbor in B_1. Hence, there is at least an outgoing path of even length from A'_1 to B'_1, implying that no vertex in A'_1 has a neighbor in B'_1. If $|A'_1| \geq 2$ then $\{a_1\} \cup C'_1 \cup B'_2$ is a star cutset centered at a_1 that separates $A'_1 \setminus \{a_1\}$ from B'_2. Thus, $|A_1| = 1$. Since, every path from A'_1 to B'_1 whose interior is in C'_1 has even length, we know that every path from A'_2 to B'_2 whose interior is in C'_2 has even length. Thus, $C'_2 \neq \emptyset$. By the same proof than above, this implies $B'_2 = \{b_1\}$. Note that every vertex in C'_1 has a neighbor in B'_1 because a vertex of C'_1 with no neighbor in B'_1 can be separated from the rest of the graph by a star cutset centered at a_1. Every vertex in C'_1 has a non-neighbor in B'_1 because a vertex of C'_1 complete to B'_1 would imply (X'_1, X'_2) being degenerate. Note also that every vertex in B'_1 has a neighbor in C'_1 for otherwise (X'_1, X'_2) is degenerate. Every vertex in B'_1 has a non-neighbor in C'_1 because if there is a vertex $b \in B'_1$ complete to C'_1 then $|B'_1| \geq 2$ implies that $\{b\} \cup C'_1 \cup B'_2$ is a star cutset separating $B'_1 \setminus \{b\}$ from A'_2, and $|B'_1| = 1$ implies that every vertex in C'_1 is complete to $A'_1 \cup B'_1$, a case already treated. Let us come back to G: in G, X_1 is a path from a_1 to b_1. Let us denote by E its interior. We observe that $(C'_1, B'_1, \{b_1\}, \{a_1\}, E, A'_2 \cup C'_2)$ is an homogeneous 2-join of G (the last condition of the definition of homogeneous 2-joins is satisfied by (36)). This contradicts the properties of G.

We proved $C'_1 = \emptyset$. By the same way, $C'_2 = \emptyset$. Thus, (X'_1, X'_2) is a non-path proper 2-join of G', contradicting (39).

Case $a_1 \in A'_1, b_1 \in B'_1$:

Since $a_1 - b_1$ is a flat edge of G', $C'_2 = \emptyset$. If $C'_1 = \emptyset$, then just like above (X'_1, X'_2) is a non-path proper 2-join of G', contradicting (39). So, $C'_1 \neq \emptyset$. Hence, $(A'_2, B'_2, B'_1, A'_1, X_1 \setminus \{a_1, b_1\}, C'_1)$ is an homogeneous 2-join of G (the last condition of the definition of homogeneous 2-joins is satisfied by (36)). This contradicts the properties of G. This proves (40).

(41) G' is neither a bipartite graph nor the line-graph of a bipartite graph.

Subdividing flat paths of a line-graph of a bipartite graph (resp. of a bipartite graph) into a path of the same parity yields a line-graph of a bipartite graph (resp. a bipartite graph). Thus, if G' is the line-graph of a bipartite graph or a bipartite graph, then so is G, contradicting the properties of G. This proves (41).

(42) $\overline{G'}$ is not the line-graph of a bipartite graph.

Suppose that $\overline{G'}$ is the line-graph of bipartite graph. If $c_1 \neq a_1$ then by the properties of G there exists a path of even length from A_2 to B_2 whose interior is in C_2. Thus, there is a vertex $c \in C_2$. Since (X_1, X_2) is not degenerate, c has at least a non-neighbor b in one of A_2, B_2, say B_2 up to symmetry. Now $\{a_1, c_1, c, b\}$ induces a diamond of $\overline{G'}$, a contradiction. We prove $a_1 = c_1$.

Let B be a bipartite graph such that $G' = \overline{L(B)}$. Let (X, Y) be a bipartition of B. So, a_1, b_1 may be seen as edges of B. Let us suppose $a_1 = a_X a_Y$ and $b_1 = b_X b_Y$ where $a_X, b_X \in X$ and $a_Y, b_Y \in Y$. Note that these four vertices of B are pairwise distinct since in $L(B) = \overline{G'}$, a_1 misses b_1. Since $a_1 b_1$ is flat in G', every edge of B is either adjacent to a_X, a_Y, b_X or b_Y. Thus, the vertices of $L(B) = \overline{G'}$ different of a_1, b_1 partition into six sets:

- A_X, the sets of the edges of B seeing a_X and missing b_Y;
- A_Y, the sets of the edges of B seeing a_Y and missing b_X;
- B_X, the sets of the edges of B seeing b_X and missing a_Y;
- B_Y, the sets of the edges of B seeing b_Y and missing a_X;
- possibly a single vertex c representing the edge $a_X b_Y$;
- possibly a single vertex d representing the edge $a_Y b_X$.

Suppose $|A_X| \geq 2$. Then, $|B_X| \neq \emptyset$ for otherwise one of $\{a_1\}$, $\{a_1, c\}$ is a star cutset of $\overline{G'}$ separating A_X from b_1. We observe that $(A_X \cup B_X, V(G') \setminus (A_X \cup B_X))$ is a 2-join of $\overline{G'}$. This 2-join is substantial since $|A_X| \geq 2$ and by (37) it is non degenerate and therefore proper, contradicting (40). Thus, $|A_X| \leq 1$, and similarly $|B_X| \leq 1$, $|A_X| \leq 1$, $|B_Y| \leq 1$. Note that if $|A_X| = 1$, $|B_X| = 1$ then there is an edge between A_X, B_X for otherwise one of $\{a_1\}$, $\{a_1, c\}$ is a star cutset separating A_X from B_X. Similarly, if $|A_Y| = 1$, $|B_Y| = 1$ then there is an edge between A_Y, B_Y. In the case when $|A_X| = |B_X| = |A_Y| = |B_Y| = 1$ and when c, d are both vertices of G', we observe that $\overline{G'}$ is the self-complementary graph $L(K_{3,3} \setminus e)$ (depicted Figure 3). Hence, G' is an induced subgraph of the line-graph of a bipartite graph, and G' is the line-graph of a bipartite graph, contradicting (41). This proves (42).

(43) G' is not a path-cobipartite graph (and in particular, not a cobipartite graph).

If G' is a path-cobipartite graph then let A, B, P, a, b be like in the definition. Suppose first $P = \emptyset$. If $a_1 \in A, b_1 \in A$, then since $a_1 b_1$ is a flat edge of G' we have $|A| = 2$. If a vertex c of B sees none of a_1, b_1 then $B \setminus c$ is a star-cutset of G' separating c from $a_1 b_1$. Thus $\{a_1\} \cup N(a_1)$ and $\{b_1\} \cup N(b_1)$ are two cliques of G' that partition $V(G')$. Thus, we may always assume that $a_1 \in A$, $b_1 \in B$. So, G is obtained by subdividing a_1b_1 implying G being a path-cobipartite graph, contradicting the properties of G.

Thus $P \neq \emptyset$. Note that $(P \cup \{a, b\}, A \setminus \{a\} \cup B \setminus \{b\})$ is a path 2-join of G'. Also, $G'[(A \cup B) \setminus \{a, b\}]$ is not a single edge, for otherwise G' is a hole, contradicting (41). Thus this 2-join is proper, and so it is not degenerate. In particular, every vertex in $A \setminus \{a\}$ has a neighbor and a non-neighbor in $B \setminus \{b\}$, implying $|A| \geq 3$, $|B| \geq 3$. If at least one a_1, b_1 is on P then the graph G obtained by subdividing a_1b_1 is again a path-cobipartite graph, contradicting the properties of G. Thus since a_1b_1 is a flat edge of G', we may assume $a_1 \in A \setminus \{a\}$, $b_1 \in B \setminus \{b\}$. The graph G is obtained by subdividing a_1b_1 into a path Q. Now $(P \cup Q \cup \{a, b\}, V(G) \setminus (P \cup Q \cup \{a, b\})$ is a 2-join of G. By the properties of G this 2-join must be either a path 2-join or a non-proper 2-join, meaning that $V(G') \setminus (P \cup Q \cup \{a, b\})$ is a single edge. Now we observe that G is the line-graph of a bipartite graph (such graphs are called *prisms* in [7]), contradicting the properties of G. This proves (43).

(44) G' is not a path-double split graph.

Suppose that G' is a path-double split graph. Let $A' = \{a'_1, \dots, a'_m\}$, $B' = \{b'_1, \dots, b'_m\}$, $C' = \{c'_1, \dots, c'_n\}$, $D' = \{d'_1, \dots, d'_n\}$ and E' be sets of vertices of G' that are like in the definition. If $a_1 \in A' \cup E'$ and $b_1 \in B' \cup E'$, then G is obtained from G' by subdividing the flat path $a_1 - c_1 - b_1$. If this yields a path of even length between a vertex a'_i and b'_i, then this path together with a neighbor of a'_i in $C' \cup D'$ and a neighbor of b'_i in $C' \cup D'$ that are adjacent, yields an odd hole of G. Thus every path with an end in A', and end in B' and interior in E has odd length, and G is a path-double split graph contradicting the properties of G. The case when $a_1 \in B' \cup E, b_1 \in A' \cup E$ is symmetric. Since $a_1 - c_1 - b_1$ is a flat path of G', there is only one case left up to a symmetry: $a_1 = c_1$, $|C'| = |D'| = 2$, $a_1 = c'_1$, $b_1 = c'_2$ and for every $i \in \{1, \dots, m\}$, a'_i sees c'_1, d'_2 and b'_i sees d'_1, c'_2. So, G is obtained by subdividing $c'_1c'_2$ into a path P. We see that $(P \cup \{d'_1, d'_2\}, A' \cup B' \cup E')$ is a proper non-path 2-join of G, contradicting the properties of G. This proves (44).

(45) G' has no homogeneous 2-join.

Suppose that G' has an homogeneous 2-join (A, B, C, D, E, F). If $c_1 \neq a_1$ then since c_1 has degree 2, c_1 must be in E. Thus, by subdividing $a_1 - c_1 - b_1$ into a path P we obtain a graph G with an homogeneous 2-join. Note that in G, the path definition If $c_1 = a_1$ then a_1b_1 is a flat edge of G', thus, up to a symmetry, either $a_1 \in C$, $b_1 \in E \cup D$ or $a_1 \in C$, $b_1 \in A$. But the last case is impossible since a_1b_1 being flat implies $N(a_1) \subset A \cup D \cup E$, implying (A, B, C, D, E, F) being degenerate, contradicting (37). Hence, $a_1 \in C$ and $b_1 \in D \cup E$. So, by subdividing a_1b_1 we obtain a graph G that has an homogeneous 2-join. The last condition of the definition of homogeneous 2-joins is satisfied by (36). This proves (45).

(46) $\overline{G'}$ is not a path-cobipartite graph, not a path-double split graph, has no homogeneous 2-join and no flat path of length at least 3.

Else, by Lemma 3.4 either $\overline{G'}$ has a proper 2-join, contradicting (40) or $\overline{G'}$ has a balanced skew partition contradicting (37), or $\overline{G'}$ is bipartite contradicting (43), or G' is bipartite contradicting (41), or $\overline{G'}$ is a double split graph and so is G', contradicting (44). This proves (46).

(47) $f(G') + f(\overline{G'}) < f(G) + f(\overline{G})$.

Every flat path of G' is a flat path of G thus $f(G') \leq f(G)$. But in fact $f(G') < f(G)$ since X_1 is a flat path of G and not of G'. By (46) $0 = f(\overline{G'}) \leq f(\overline{G})$. We add these two inequalities. This proves (47).

Let us now finish the proof. By (38), G' is Berge. By (37), G' has no balanced skew partition. By (39), G' has no proper non-path 2-join. By (40) $\overline{G'}$ has no proper 2-join. By (41, 42), none of $G', \overline{G'}$ is the line-graph of a bipartite graph and G' is not bipartite. By (43) G' is not a path-cobipartite graph. By (44) G' is not a path-double split graph. By (45) G' has no homogeneous 2-join. By (46), $\overline{G'}$ is not a path-cobipartite graph, not a path-double split graph and has no homogeneous 2-join. So, G' is a counter-example to the theorem we are proving now. Hence there is a contradiction between the initial choice of G and (47). This completes the proof of Theorem 2.1.

4.2 Proof of Theorem 1.5

Let G be a Berge graph. Note that it is impossible that both $G, \overline{G}$ have a path proper 2-join because in a graph with a proper path 2-join, no vertex has degree $n-3$, and this should be the degree of an interior vertex of the path side of a 2-join of $\overline{G}$. Let us now apply Theorem 2.1 to G. If one of $G, \overline{G}$ is basic, has a non-path proper 2-join, or a balanced skew partition, we are done. From now on, we assume that G has no balanced skew partition and is not basic. So up to a complementation we have three cases to consider. In each case, we have to check that G has at least a path proper 2-join, and that the contraction of any path proper 2-join leaves the graph balanced skew partition-free.

If G has an homogeneous 2-join (A, B, C, D, E, F) then it is not degenerate since G has no balanced skew cutset. So, every vertex in $A \cup B \cup C \cup D \cup F$ has degree at least 3. So every flat path of length at least 3 in G has an end in C, an end in D and interior in E. Let P be such a flat path. By definition of homogeneous 2-joins, such a path is the path side of a non cutting 2-join that is also proper. Hence, by Lemma 3.17, the graph obtained by contracting P has no balanced skew partition.

If G is path-cobipartite then let A, B, P be three sets that partition $V(G)$ like in the definition. Since G is not basic, P is not empty and is the interior of the unique maximal flat path P' of G with ends $a \in A$ and $b \in B$. Since A and B are cliques, $(P', V(G) \setminus P')$ is not a cutting 2-join of type 1 of G. If $(P', V(G) \setminus P')$ is cutting of type 2, this means that there are non empty sets $A_3 \subset A \setminus \{a\}$ and $B_3 \subset B \setminus \{b\}$, complete to one another and such that $H = G \setminus (P' \cup A_3 \cup B_3)$ is disconnected. But since A, B are cliques, this means that H has exactly two components, say $A' \subset A$, and $B' \subset B$. We observe that $A_3 \cup B_3 \cup \{a\}$ is a star cutset of G, centered at any vertex of A_3, that separates A' from $B' \cup P$. This is a contradiction since G has no balanced skew partition. We proved that the unique proper path 2-join of G is not cutting. Hence, its contraction does not create a balanced skew partition.

If G is a path double split graph then let $V(G)$ be partitioned into sets

A, B, C, D, E like in the definition. Since G is not basic, we know $E \neq \emptyset$. Hence, there is a flat path P in G that is the path side of a proper 2-join of G. The contraction of any such path P yields a graph G' that is also a path-double split graph. By Lemma 3.5 G has no balanced skew partition.

This proves Theorem 1.5.

4.3 On the detection of general skew partitions

As mentioned in the introduction we are not able to prove something like Theorem 1.5 with "skew partition" instead of "balanced skew partition". Following our frame, we would have to give up the conditions on the parity of paths in the definition of cutting 2-joins of type 2. But then we would not be able to prove (23), meaning that the graph G' in Case 2 of the proof of Theorem 2.1 would possibly be non-Berge, making the whole proof collapse.

5 Algorithms

By Lemma 3.2, the balanced skew partition is a self-complementary notion. Thus, for basic graphs, we have to deal only with bipartite graph, line-graphs of bipartite graphs and double-split graphs. When decomposing, we may switch from the graph to its complement as often as needed.

5.1 Balanced skew partitions in basic graphs

Lemma 5.1 *Let G be a bipartite graph. Then (A, B) a skew partition of G if and only if it is a balanced skew partition of G.*

PROOF — A balanced skew partition of G is clearly a skew partition. Let us prove the converse. Since G is bipartite, B is a complete bipartite graph. Every path of length at least 2 with its ends in B and it interior in A has even length, because its ends are in the same side of the bipartition. Since G is triangle-free, every antipath of G has length at most 3. Hence, every antipath of length at least 2, with its ends in A and it interior in B has even length. Because otherwise such an antipath has length 3 and may be viewed as a path. □

By the lemma above, detecting balanced skew partition in bipartite graphs can be performed by running an algorithm for general skew partitions. Such a fast algorithm for bipartite graphs has been given by Reed [18]. It is has complexity $O(n^5)$.

Now, we have to decide if the line-graph of a bipartite graph has a balanced skew partition. Note that every case is possible: line-graphs of bipartite graphs may have balanced skew partitions, skew partitions and no balanced skew partition, or no skew partition at all, see figure 3. By Theorem 3.1 the line-graph of a bipartite graph has no claw and no diamond.

Lemma 5.2 *Let G be the line-graph of a bipartite graph with a skew partition (A, B). Then B is a star or B is a square.*

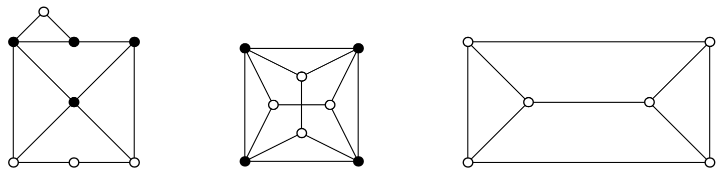

Figure 3: Three line graph of bipartite graph. The second one is $L(K_{3,3} \setminus e)$

PROOF — Suppose that G has a skew partition (A, B) such that B has at least 5 vertices. We may assume that B is not a star, so every anticomponent of B has at least 2 vertices. Let B_0 be such an anticomponent, and let b, b' be non adjacent in B_0 (because B_0 is anticonnected). If B has at least 3 anticomponents say $B_0, B_1, B_2, \dots$, then for $b_1 \in B_1$, $b_2 \in B_2$, $\{b, b', b_1, b_2\}$ induces a diamond, a contradiction. Thus, B has 2 anticomponent B_0, B_1 and we may assume that B_0 has at least 3 vertices. If B_0 has no edge, then we can pick 3 vertices b_1, b_2, b_3 in B_0 and a vertex c in B_1 and $\{c, b_1, b_2, b_3\}$ induces a claw, a contradiction. Thus, B_0 has at least one edge, say bb'. Now consider a non edge c, c' in B_1: $\{b, b', c, c'\}$ induces a diamond, a contradiction. So, we are left with the case where B has at most 4 vertices. The only candidate for a non-star non-anticonnected graph is the square. □

Lemma 5.3 *Let G be the line-graph of a bipartite graph. Suppose that G has at least an edge and size at least 5. Then G has a balanced skew partition if and only if G has a start cutset.*

PROOF — By Lemma 3.3, we know that if G has a star cutset, then it has a balanced skew partition. Let us prove the converse. Suppose that G has a balanced skew partition (A, B). We may assume that B is not a star. So by lemma 5.2, B is a square with vertices say b_1, b_2, b_3, b_4 and edges b_1b_2, b_2b_3, b_3b_4, b_4b_1. Note that Figure 3, the first graph depicted has a square cutset that is a balanced skew cutset. Let X be a connected component of $G \setminus B$. To finish the proof, it suffice to show that one of the star $\{b_1, b_2, b_4\}$, $\{b_2, b_3, b_4\}$ or $\{b_1, b_2, b_3\}$ is a cutset. So, let us suppose for a contradiction that none of these sets is a cutset.

Since $\{b_2, b_3, b_4\}$ is not a cutset, b_1 has a neighbor in X and similarly b_3 has a neighbor in X. Since X is connected, we now that there is a path from b_1 to b_3 that goes through none of b_2, b_4. We may choose this path as short as possible, so it is an induced path, say $P = v_1 - v_2 - \cdots - v_{k-1} - v_k$, with $v_1 = b_1$ and $v_k = b_3$. Since (A, B) is balanced, P has even length. One of b_2, b_4 (say b_2 by symmetry) must see v_2 for otherwise $\{b_1, v_2, b_2, b_4\}$ induces a claw. If P has length 2, then, $\{b_1, b_2, b_3, v_2\}$ induces a diamond, a contradiction. So, P has length at least 4. But then, $v_2 - P - v_k - b_2 - v_2$ is a cycle of odd length ≥ 5, thus it has a chord b_2v_i. But i must equals $k-1$ for otherwise, b_2, b_1, b_3, v_i induces a claw. So $H = b_2 – v_2 – P – v_{k-1} – b_2$ is a hole. We rename its vertices $h_1, \dots, h_l$.

Since $\{b_1, b_2, b_3\}$ is not a cutset, there is a path Q that goes through none of b_1, b_2, b_3, from b_4 to a vertex that has a neighbor in H. Let us choose $Q = b_4 – \cdots – x' – x$ of minimal length. Note that Q has length at least 1, for otherwise, b_4 has a neighbor $v_i \in H$. If $2 < i < k - 1$ then $\{b_4, b_1, b_3, v_i\}$ induces a claw and if $i = 2$ then

$\{b_1, b_2, b_4, v_i\}$ induces a diamond ($i = k-1$ is symmetric). If x sees two non-adjacent vertices y, z in H, then $\{x, x', y, z\}$ induces a claw. If x sees only one vertex h_i in H then $\{h_i, h_{i-1}, h_{i+1}, x\}$ induces a claw. So, x has exactly two adjacent neighbors in H, say h_i, h_{i+1}. Since H is an even hole, the induced paths $b_4–Q–x–h_i–H \setminus h_{i+1}–b_2$ and $b_4–Q–x–h_{i+1}–H \setminus h_i–b_2$ have different parity. So one of them has odd length, contradicting (A, B) being balanced. □

By the previous lemma we know that an algorithm that detects star cutsets is sufficient to decides whether a line-graph of a bipartite graph has or not a balanced skew partition. Chvátal [8] gave such an $O(nm)$-time algorithm. Note that in [18], Reed gives a fairly optimised algorithm for detecting general skew partitions in line graphs with complexity $O(n^2m)$. So, the obvious algorithm for detecting a balanced skew partition in the line-graph of a bipartite graph is faster than the optimised algorithm for general skew partition. This might be general: detecting a skew partition might be harder than a balanced skew partition for perfect graphs.

The detection of balanced skew partitions in double split graph takes constant time by Lemma 3.5: answer "No".

Our main algorithm needs also to recognize basic graphs. This can be done in linear time for bipartite graphs (this is a classical result) and for line-graphs of bipartite graphs (see [17, 19]). For double split graphs, this can be done in linear time by looking at the degrees since vertices of the matching all have degree $1 + n$ and vertices of the anti-matching all have degree $2n - 2 + m$ (these numbers are different since $n \geq 2, m \geq 2$ implies $2n-2+m > 1+n$). Hence, the recognition can be performed as follows: compute the degrees, check whether the vertices of smallest degree induce a matching, that the rest of the graph induces the complement of a matching, and check for every edge xy of the matching and every non-edge $\overline{uv}$ of the antimatching, that $\{x, y, u, v\}$ induces a path on 4 vertices. The computing of degrees takes linear time, and the checking to be done afterward do not take more than $O(m)$ time.

Let us sum up this subsection.

Theorem 5.4 (Several authors) *There is an $O(n + m)$ algorithm that decides whether a given graph is basic. There is an $O(n^5)$ algorithm that given a basic graph G decides whether G has a balanced skew partition or not.*

5.2 2-join decomposition

Let us define a decomposition tree T_G of a Berge graph G. The root of T_G is G itself. If a node F of the tree is a basic graph then it is a leaf marked with label "basic". Else, if F is a graph on at most 10 vertices, then it is a leaf marked with label "small". Else, if none of $F, \overline{F}$ has a substantial 2-join then F is a leaf marked with label "no decomposition". Else, one of $F, \overline{F}$ has a substantial 2-join and has at least 11 vertices. If possible, we choose this substantial 2-join (X_1, X_2) non-path. If (X_1, X_2) is not connected then F is a leaf marked with label "disconnected". Else, up to a complementation, we suppose that the 2-join is in F and we define the children of F to be the blocks of F with respect to this 2-join (these blocks are defined Subsection 3.2).

We claim that T_G has size at most $O(n)$. Indeed, we define for every graph F: $\phi(F) = |V(F)| - 10$. So, in T_G every non-leaf node satisfies $\phi(F) \geq 1$ since it has size at least 11. Furthermore, when F_1, F_2 are the blocks of a graph F with respect to a 2-join (X_1, X_2) of F then $\phi(F_1) + \phi(F_2) \leq \phi(F)$, because for $i = 1, 2$, $|F_i| \leq |X_i| + 5$. Hence, the total number of leaves in T_G is at most $2\phi(G) = O(n)$.

We claim that T_G can be constructed in time $O(n^9)$. Indeed, testing whether G is basic is easy (see Theorem 5.4). In [10], an $O(n^8)$ algorithm, due to Cornuéjols and Cunningham [13], for constructing a substantial non-path 2-join of a an input graph is given. Note that what we call non-path substantial 2-join is simply called 2-join in [10]. Finding substantial path 2-joins is easy in linear time by checking every vertex of degree 2. Testing for the connectivity of a 2-join is easy. By the paragraph above, to construct T_G in the worst case, we will have to run $O(n)$ times the $O(n^8)$ algorithm that detects non-path substantial 2-joins.

We claim that G has a balanced skew partition if and only if one of the leaves of T_G has a balanced skew partition. Indeed, if G has a balanced skew partition then Lemma 3.16 shows by an easy induction that at least one of the leaves of T_G has a balanced skew partition. Conversely, if a leaf F of T_G has a balanced skew partition then suppose for a contradiction that G has no balanced skew partition. Among the nodes of T_G, let H be the graph with no balanced skew partition, closest to F along the unique path of T_G from G to F. The graph H is Berge, has no balanced skew partition, and is not basic. Since it is not a leaf, H has a proper 2-join by definition of T_G. If H has a non-path proper 2-join, then by Lemma 3.18 the children of H in T_G have no balanced skew partitions contradicting the definition of H. Else, by Theorem 1.5, the children of H have no balanced skew partition, a contradiction again.

We claim that we can test whether a leaf L of T_G has a balanced skew partition in $O(n^5)$. If L is marked "basic", this is true by Theorem 5.4. If L is marked "small", this is trivial. If L is marked "no decomposition", this is done in constant time by answering "YES", the correct answer by Theorem 1.5. If L is marked "disconnected", this is done in constant time by answering "YES", the correct answer by Lemma 3.13.

By the claims above, detecting balanced skew partitions in a Berge graph G can be performed as follows: construct T_G and test whether a leaf has or not a balanced skew partition. Note that in the case when G has no balanced skew partition, then the leaves of T_G are all basic.

Theorem 5.5 *There is an $O(n^9)$-time algorithm that decides whether a Berge graph has or not a balanced skew partition.*

6 NP-hardness

We recall here a construction due to Bienstock [3]. Let us call *Bienstock graph* any graph G that can be constructed as follows. Let $n \geq 3$, $m \geq 1$ be two integers. For every $1 \leq i \leq n$ let α_i be the graph depicted Figure 4, with vertex set $\{t_{i,1}, t_{i,2}, t_{i,3}, t_{i,4}, f_{i,1}, f_{i,2}, f_{i,3}, f_{i,4}, c_{i,1}, c_{i,2}, c_{i,3}, c_{i,4}\}$ and with edge set $\{c_{i,1}t_{i,1}, t_{i,1}c_{i,3}, c_{i,1}f_{i,1}, f_{i,1}c_{i,3}, c_{i,2}t_{i,2}, t_{i,2}t_{i,3}, t_{i,3}t_{i,4}, t_{i,4}c_{i,4}, c_{i,2}f_{i,2}, f_{i,2}f_{i,3}, f_{i,3}f_{i,4}, f_{i,4}c_{i,4}, t_{i,1}f_{i,2}, t_{i,1}f_{i,3},$

$f_{i,1}t_{i,2}$, $f_{i,1}t_{i,3}$, $t_{i,3}f_{i,3}\}$. For every $1 \leq j \leq m$, let β_j be the graph depicted Figure 4, with vertex set $\{d_{j,1}, d_{j,2}, d_{j,3}, d_{j,4}, r_j, z_{j,1}, z_{j,2}, z_{j,3}\}$ and edge set $\{d_{j,1}r_j, r_jd_{j,3}, d_{j,2}z_{j,1}, z_{j,1}d_{j,4}, d_{j,2}z_{j,2}, z_{j,2}d_{j,4}, d_{j,2}z_{j,3}, z_{j,3}d_{j,4}\}$.

All the graphs α_i, β_j are pairwise vertex disjoint subgraphs of G that are assembled by adding the following edges: $c_{i,3}c_{i+1,1}$ and $c_{i,4}c_{i+1,2}$ for $1 \leq i < n$, $d_{j,3}d_{j+1,1}$ and $d_{j,4}d_{j+1,2}$ for $1 \leq j < m$. Add a vertex u adjacent to $c_{1,2}$, a vertex w adjacent to $c_{1,1}$, a vertex s adjacent to w and a vertex v adjacent to $d_{m,3}$, $d_{m,4}$. See Figure 4. For every $1 \leq j \leq m$ and every $k \in \{1, 2, 3\}$ we add exactly 2 edges incident to $z_{j,k}$. These edges are either $z_{j,k}f_{i,1}, z_{j,k}f_{i,3}$ for some i, or $z_{j,k}t_{i,1}, z_{j,k}t_{i,3}$ for some i. See Figure 4. Moreover, for every $1 \leq k < k' \leq 3$ and every $1 \leq j \leq m$, $z_{j,k}$ and $z_{j,k'}$ are required to have their neighbors in different α_i's.

By 3-SAT' we mean the usual 3-SAT problem (see [15]) restricted to the sets of clauses on 3 variables such that every clause is on three pairwise distinct variables. Bienstock proved an NP-completeness reduction from 3-SAT that when restricted to 3-SAT' yields:

Theorem 6.1 (Bienstock [3]) *For every instance $\mathcal{I}$ of size x of the NP-complete problem 3-SAT', there is a Bienstock graph $G_{\mathcal{I}}$ of size $O(x)$, obtained from $\mathcal{I}$ by a linear time algorithm and such that the answer to $\mathcal{I}$ is YES if and only if there is a path of $G_{\mathcal{I}}$ of odd length joining u and s.*

Here is why Bienstock's construction is related to the Balanced Skew Partition Problem:

Lemma 6.2 *Let G be a Bienstock graph. Let G' be the graph obtained by adding two vertices: a vertex a seeing both u, s and a vertex b also seeing both u, s. Then G' has a balanced skew partition if and only if there is no path of odd length in G joining u and s.*

PROOF — The graph G' is depicted Figure 4. The sets $\{a, u, s\}$ and $\{b, u, s\}$ are clearly skew cutsets. If there is a path of odd length in G between u and s then these two skew cutset are non-balanced. Else they are clearly both balanced. Hence if suffices to prove that G' has no other skew cutset. Note that G' has no diamonds and no K_4. Hence, every skew cutset of G' is either a star cutset or is a complete bipartite graph. Let us check every star and every square in G'.

We observe that G' has no star cutset centered at: s, u, w, v, $c_{i,k}$; $t_{i,4}$, $f_{i,4}$, $t_{i,2}$, $f_{i,2}$ for $1 \leq i \leq n$; $d_{j,k}$ for $1 \leq j \leq m$, $k \in \{1, 2, 3\}$. Also G' has no star cutset centered at $z_{j,k}$ since $z_{j,k}$ has degree 4 and since for $k' \in \{1, 2, 3\} \setminus k$, $z_{j,k'}$ does not have its neighbors in the same α_i than $z_{j,k}$. A star centered at a vertex x among $t_{i,1}$, $f_{i,1}$, $t_{i,3}$, $f_{i,3}$ is dangerous since x may have large arbitrarily large degree. But this is no enough to disconnect G' since x has at most one neighbor in every β_j.

The square $G'[a, b, s, u]$ is not a skew cutset of G'. Moreover, since s, u (resp. a,b) have no common neighbors in G', no skew cutset can contain $\{a, b, s, u\}$. Similarly, for $1 \leq i \leq n$, no skew cutset of G' can contain $\{c_{i,1}, t_{i,1}, c_{i,3}, f_{i,1}\}$. No skew cutset of G' can contain $\{d_{1,2}, z_{1,1}, d_{1,4}, z_{1,2}\}$ since $z_{1,3}$ is the only possible vertex to be added to the potential skew cutset, and since $z_{1,3}$ has a neighbor in some α_i. By the same way, no skew cutset can be contained in β_j, $1 \leq j \leq m$. The last squares to be checked are those contained in sets consisting of some $t_{i,1}, t_{i,3}$ (resp. $f_{i,1}, f_{i,3}$) plus a collection of $z_{j,k}$'s complete to $\{t_{i,1}, t_{i,3}\}$ (resp. $f_{i,1}, f_{i,3}$). Note that the $z_{j,k}$'s are all in different β_j's. Hence such a set in not a skew cutset. □

Theorem 6.3 *The decision problem whose instance is any graph G and whose question is "does G have a balanced skew partition ?" is NP-hard.*

PROOF — Let $\mathcal{I}$ be an instance of 3-SAT'. By Theorem 6.1, we construct a graph $G_{\mathcal{I}}$. By Lemma 6.2 we construct a graph $G'_{\mathcal{I}}$. By these two results $G'_{\mathcal{I}}$ has a balanced skew partition if and only if the answer to $\mathcal{I}$ is YES. □

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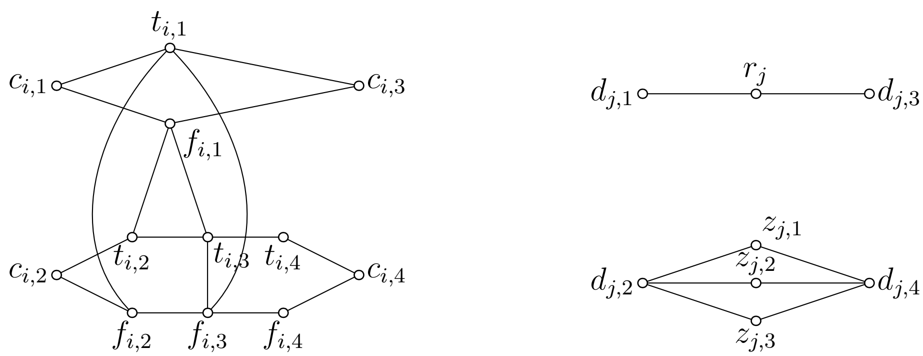
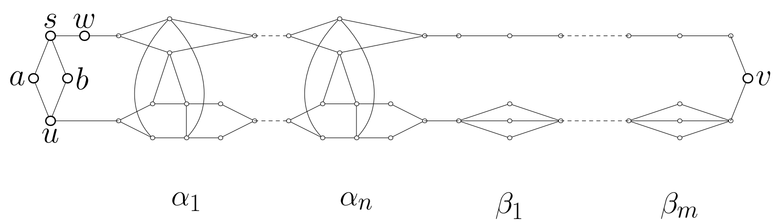
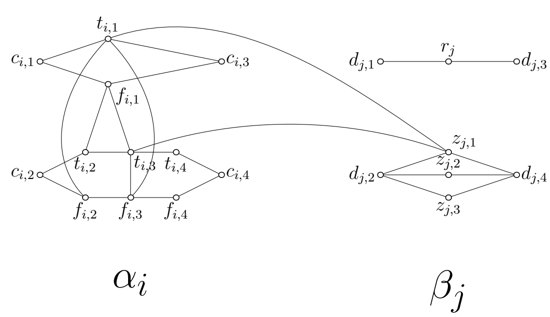


Graph α_i Graph β_i



The whole graph G plus two vertices a, b



The two edges out from $z_{j,1}$, a possibility

Figure 4: Bienstock's construction