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Equilibrium dynamics in an aggregative model of capital accumulation with heterogeneous agents and elastic labor *

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Abstract

The paper extends the canonical representative agent Ramsey model to include heterogeneous agents and elastic labor supply. The welfare maximization problem is analyzed and shown to be equivalent to a non-stationary reduced form model. An iterative procedure is exploited to prove the supermodularity of the indirect utility function. Supermodularity is subsequently used to establish the convergence of optimal paths.

Keywords: Single-sector growth model, heterogeneous agents, elastic labor supply
JEL Classification: C62, D51, E13

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1 Introduction

Optimal growth theory is useful in qualitatively characterizing simple dynamical systems and in providing constructive methods for the quantitative analysis of the solutions to more complex ones. The usefulness is, for some purposes, enhanced because of the intimate connections between optimal growth theories and their equilibrium counterparts. In a decentralized economy, we seek knowledge about the time paths of the various prices for goods and production factors as well as the distribution of income and wealth. Dynamic optimization techniques used extensively in growth theory facilitate the study of the evolution of those economic aggregates.

A major concern in the area of optimal growth has been the analysis of the short-run and asymptotic behavior of optimal solutions. At issue are questions concerning the existence and asymptotic stability of optimal programs with respect to the stationary optimal stock (turnpike results) as well as the possibility of cyclical or even chaotic behavior.

One-sector representative agent models, in which utility is derived solely from consumption have been studied extensively in the literature under a variety of different technological specifications. A well known property of these models is the monotonicity of the optimal capital path. This property is persistent even when technology has increasing returns (see Dechert and Nishimura (1983)). Thus, it is often suggested that one-sector models exhibit simple dynamics.

Becker and Foias (1987) show that agents' heterogeneity plays a crucial role to the appearance of nonmonotonic dynamics in a single-sector model. Studying a specific economy with incomplete markets as represented by borrowing constraints, they demonstrate that deterministic cycles with period 2 may occur. In Becker and Foias (1994) they discuss in more detail the issue of equilibrium cycles and their construction using bifurcation analysis. Their work has been further elaborated by Sorger (1994).

In a complete market model, Le Van and Vailakis (2003) have also shown that the monotonicity property does not carry over if one permits many consumers, each with a different discount factor. The model does not exhibit cyclical behavior. The convergence of the optimal capital sequence to a particular stock k^s is still true, but that stock is not itself a steady state. This result implies that the optimal capital sequence initiated at $k_0 = k^s$ converges to it in the long-run, but it is not a constant sequence. Hence, the resulting optimal capital sequence cannot be monotonic. The model exhibits the *twisted turnpike property* (see Mitra (1979), Becker (2005)): the optimal capital accumulation paths starting from different initial capital stocks converge to each other, or come together, in the limit but this limit is not itself an optimal stationary program. This is a fundamental property of the heterogeneous agent model and it shows one way in which this model differs significantly from its representative agent counterpart.

In this paper we examine whether and under which conditions similar properties can be established when the heterogeneous-agents Ramsey model studied in Le Van-Vailakis is extended to include an endogenous non-reproducible factor such as labor.

The analysis in Le Van and Vailakis (2003) is carried out by exploiting the so called reduced form model associated with the welfare maximization problem. The presence of heterogeneous discount factors turns out the reduced-form problem to be nonstationary, making the issue of convergence of optimal paths a nontrivial one. Their argument exploits the fact that the indirect utility function V_t associated with the reduced form model is C^2 in the interior of a set D describing feasible activities in period t. This allows them to show that V_t is supermodular. The supermodularity of V_t then implies that the stationary problem involving the agents with a discount factor equal to the maximum one, has a unique stable steady state k^s. Exploiting additional properties of optimal paths, they subsequently show that the optimal capital sequence associated with the initial problem converges to k^s.

Several complications arise by applying a similar method of proof in the presence of elastic labor supply. The problems arise largely from the fact that one cannot exclude the existence of corner solutions in the welfare maximization problem. More precisely, one cannot ensure that all consumers supply labor at any period. As a consequence, the indirect utility function V_t associated with the reduced form model is not necessarily C^2 in the interior of D. Hence, one cannot use the differentiable characterization of supermodilarity. To overcome the problem and establish the supermodularity of V_t, we employ an alternative argument based on a iterative procedure in which a sequence of functions, $(V_t^n)_n$, are shown to be supermodular and to be converging to the function V_t.

Other issues are associated with the properties of optimal paths. Several proofs in Le Van-Vailakis (2003) cannot be carried out due to the presence of elastic labor supply. New and general arguments are given to establish the validity of those properties.

The outline of the paper is as follows: Section 2 describes the model. In section 3 we present its reduced-form counterpart and establish some preliminary results. Section 4 contains our basic results.

2 The model

We consider an intertemporal one-sector model with $m \geq 1$ consumers and one firm. At each period, individuals consume a quantity $c_{i,t}$, and decide how to divide the available time, normalized at 1, between leisure activities $l_{i,t}$, and work $L_{i,t}$. Preferences are represented by a functional that takes the usual additively separable form:

$$\sum_{t=0}^{\infty} \beta_i^t u^i(c_{i,t}, l_{i,t}),$$

where u^i denotes the instantaneous utility function and $\beta_i \in (0, 1)$ is the discount factor.

The initial endowment of capital, the single reproducible factor in the economy, is denoted by $k_0 \geq 0$. Technology is described by a gross production function F. Capital evolves according to:

$$k_{t+1} = (1 - \delta)k_t + I_t,$$

where I_t is gross investment and $\delta \in (0, 1)$ is the rate of depreciation for capital.

At each period, the economy's resource constraints, restricting the allocation of income and time, are:

$$\sum_{i=1}^{m} c_{i,t} + I_t \leq F(k_t, L_t) + (1 - \delta)k_t, \ \sum_{i=1}^{m} L_{i,t} = L_t. \tag{1}$$

We next specify a first set of assumptions imposed on preferences and production technology. The assumptions on period utility function $u^i : \mathbb{R}^2_+ \to \mathbb{R}$ are as follows:

Assumption U1: *u^i is continuous, strictly concave, increasing in $\mathbb{R}^2_+$ and strictly increasing in $\mathbb{R}^2_{++}$.*

Assumption U2: $u^i(0, 0) = 0$.

Assumption U3: *u^i is twice continuously differentiable in $\mathbb{R}^2_{++}$ with partial derivatives that satisfy the Inada conditions:* $\lim_{c\to 0} u^i_c(c, l) = +\infty$, $\forall l > 0$ and $\lim_{l\to 0} u^i_l(c, l) = +\infty$, $\forall c > 0$.

Assumption U4: *u^i_{cl} has a constant sign and the second partial derivatives satisfy the following condition:*

$$\frac{u^i_{cc}}{u^i_c} \leq \frac{u^i_{cl}}{u^i_l}.$$

The assumptions on the production function $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ are as follows:

Assumption F1: *F is continuous, concave, increasing in $\mathbb{R}^2_+$ and strictly increasing in $\mathbb{R}^2_{++}$.*

Assumption F2: $F(0, L) = F(k, 0) = 0$.

Assumption F3: *F is twice continuously differentiable in $\mathbb{R}^2_{++}$ with partial derivatives that satisfy:* $\lim_{k\to 0} F_k(k, 1) \geq \frac{1}{\min_i \beta_i} - (1-\delta)$, $\lim_{k\to +\infty} F_k(k, m) = 0$ and $\lim_{L\to 0} F_L(k, L) = +\infty$, $\forall k > 0$.

Assumption F4: F_{kL} *is nonnegative.*

We conclude this section by introducing some notation.
Let $f(k_t, L_t) = F(k_t, L_t) + (1-\delta)k_t$. Observe that under the previous assumptions, $\lim_{k\to 0} f_k(k,1) \geq \frac{1}{\min_i \beta_i}$, $\lim_{k\to+\infty} f_k(k,m) < 1$ and $\lim_{L\to 0} f_L(k,L) = +\infty$. Consider the set of feasible capital sequences:

$$\Pi(k_0) = \{\mathbf{k} \in (\mathbb{R}_+)^\infty \colon 0 \leq k_{t+1} \leq f(k_t, m),\ \forall t\}.$$

Let $c_t = (c_{1,t}, c_{2,t}, ..., c_{m,t})$ and $l_t = (l_{1,t}, l_{2,t}, ..., l_{m,t})$ denote the $m-$vectors of consumption-leisure allocations at date t. The nonnegative consumption, leisure, labor sequences $(\mathbf{c}, \mathbf{l}, \mathbf{L}) = (c_t, l_t, L_t)_{t=0}^{\infty}$ is said to be feasible from $k_0 \geq 0$, if there exists a sequence $\mathbf{k} \in \Pi(k_0)$ such that $(\mathbf{c}, \mathbf{l}, \mathbf{L}, \mathbf{k})$ satisfy the economy's resource constraint (1) together with the individual time constraint $l_{i,t}+L_{i,t} \leq 1$. The set of feasible from k_0 consumption, leisure-labor allocations is denoted by $\Sigma(k_0)$.

3 Planner's Problem

The planner's welfare function is taken to be a weighted function of the underlying households' intertemporal functions. Let $\Delta = \left\{\lambda \in \mathbb{R}^m_+ \mid \sum_{i=1}^{m} \lambda_i = 1\right\}$. Given nonnegative welfare weights $\lambda \in \Delta$ the social planner maximizes:

$$\begin{aligned}
\max \quad & \sum_{i=1}^{m} \lambda_i \sum_{t=0}^{\infty} \beta_i^t u^i(c_{i,t}, l_{i,t}) \qquad (P)\\
s.t. \quad & \sum_{i=1}^{m} c_{i,t} + k_{t+1} \leq f(k_t, \sum_{i=1}^{m} L_{i,t}), \forall t \\
& c_{i,t} \geq 0,\ l_{i,t} \geq 0,\ L_{i,t} \geq 0,\ l_{i,t} + L_{i,t} \leq 1,\ \forall i, \forall t \\
& k_t \geq 0,\ \forall t \text{ and } k_0 \text{ given.}
\end{aligned}$$

It is well known that any Pareto-efficient allocation can be represented as the solution to problem (P). In other words, by varying the welfare weights it is possible to trace the economy's utility possibility frontier. This procedure can also be used to prove the existence of a price system that support Pareto-optima and characterize competitive equilibria as a set of welfare weights such that the associated transfer payments are zero (Negishi's approach).

3.1 Preliminary results

Since u and F are assumed to be strictly increasing, $\mathbf{L}$ can be dropped from the list of planner's choices. Consider the technology set D:

$$D = \left\{(k, y) \in \mathbb{R}^2_+ : 0 \leq y \leq f(k, m)\right\},$$

and define the correspondence Γ :

$$(k,y) \in D \to \left\{ (c_i, l_i)_i : \sum_{i=1}^{m} c_i + y \leq f(k, m - \sum_{i=1}^{m} l_i),\ c_i \geq 0,\ l_i \in [0,1],\ \forall i \right\}.$$

Given $\lambda \in \Delta$, let $I = (i \mid \lambda_i > 0\}$, $\beta = \max\{\beta_i \mid i \in I\}$, $I_1 = \{i \in I \mid \beta_i = \beta\}$ and $I_2 = \{i \in I \mid \beta_i < \beta\}$. Let $\zeta = (\zeta_i)_{i \in I}$ where $\zeta_i = 1$, $\forall i \in I_1$ and $\zeta_i \in [0,1]$, $\forall i \in I_2$. Given $(k,y) \in D$, let

$$\begin{aligned} V(\lambda, \zeta, k, y) &= \max \left[\sum_{i \in I_1} \lambda_i u^i(c_i, l_i) + \sum_{i \in I_2} \lambda_i \zeta_i u^i(c_i, l_i) \right] \\ & \quad s.t. \ \sum_{i \in I} c_i + y \leq f(k, m - \sum_{i \in I} l_i) \\ & \quad c_i \geq 0,\ l_i \in [0,1],\ \forall i \in I \end{aligned}$$

Let also

$$(c_i(\zeta,k,y), l_i(\zeta,k,y))_{i \in I} = \arg\max \left\{ \sum_{i \in I} \lambda_i \zeta_i u^i(c_i, l_i),\ (c_i, l_i)_{i \in I} \in \Gamma(k,y) \right\}$$

Let $\zeta_i = 1$, $\forall i \in I_1$ and $\zeta_i = \frac{\beta_i}{\beta}$, $\forall i \in I_2$. For simplicity assume that $I_1 = \{1, ..., \#I_1\}$. In this case, for any $t \geq 0$, we use the notation $\zeta^t = (1, ..., 1, (\zeta_i^t)_{i \in I_2})$. We subsequently introduce the time-dependent function V_t:

$$\begin{aligned} V_t(\lambda, k, y) &= V(\lambda, \zeta^t, k, y) \\ &= \max \left[\sum_{i \in I_1} \lambda_i u^i(c_i, l_i) + \sum_{i \in I_2} \lambda_i \zeta_i^t u^i(c_i, l_i) \right] \\ & \quad s.t. \ \sum_{i \in I} c_i + y \leq f(k, m - \sum_{i \in I} l_i) \\ & \quad c_i \geq 0,\ l_i \in [0,1],\ \forall i \in I \end{aligned}$$

Let $(c_i^*, l_i^*)_{i \in I} = (c_i(\zeta^t, k, y), l_i(\zeta^t, k, y))_{i \in I}$ denote the solution to this problem.

Consider the following intertemporal problem:

$$\begin{aligned} \max \quad & \sum_{t=0}^{\infty} \beta^t V(\lambda, \zeta^t, k_t, k_{t+1}) \qquad (Q) \\ s.t. \quad & 0 \leq k_{t+1} \leq f(k_t, m),\ \forall t \\ & k_0 \geq 0 \text{ is given.} \end{aligned}$$

The following proposition shows that problems (P) and (Q) are equivalent. More precisely we have:

Proposition 1 *Let $k_0 \geq 0$ be given. Under assumptions **U1**, **F1**:*

i) If $((\mathbf{c}_i^*, \mathbf{l}_i^*)_i, \mathbf{k}^*)$ is a solution to problem (P), then $\mathbf{k}^*$ is a solution to problem (Q).

ii) If $\mathbf{k}^*$ is a solution to problem (Q), then there exists $(\mathbf{c}_i^*, \mathbf{l}_i^*)_i$ such that $((\mathbf{c}_i^*, \mathbf{l}_i^*)_i, \mathbf{k}^*)$ is a solution to problem (P).

Proof: It is easy. ■

Lemma 1 *Under assumptions **U1-U2**, **F1-F2**, Γ is upper hemicontinuous on D and continuous at any $(k, y) \in D$ with $k > 0$.*

Proof: It is a direct consequence of Lemma 4 proven below (refer to Remark 2). ■

Remark 1 *To see why lower hemicontinuity fails when $k = 0$ observe that, under assumption **F2**, for $k = 0$ we have $D = \{(0, 0)\}$ and*

$$\Gamma(0, 0) = \{(c_i, l_i)_i : c_i = 0 \text{ and } l_i \in [0, 1],\ \forall i\}.$$

Choose $(c_i, l_i)_i \in \Gamma(0, 0)$ such that $c_i = 0$ and $l_i > 0$ for some i. Consider next a sequence (k^n, y^n) such that $y^n = f(k^n, m)$ and $(k^n, y^n) \to (0, 0)$. Note that there is no sequence $(c_i^n, l_i^n)_i$ such that $(c_i^n, l_i^n)_i \in \Gamma(k^n, y^n)$ and $(c_i^n, l_i^n)_i \to (c_i, l_i)_i$.

Proposition 2 *Assume **U1-U3**, **F1-U3**. Then $V : (\lambda, \zeta, k, y) \in \Delta \times [0, 1]^{\#I_2} \times D \to \mathbb{R}_+$ is:*

i) increasing in k, decreasing in y and strictly concave in (k, y)

ii) upper semicontinuous and continuous at any $(\zeta, k, y) \in [0, 1]^{\#I_2} \times D$ with $k > 0$.

iii) The functions $c_i : [0, 1]^{\#I_2} \times intD \to \mathbb{R}_+$ and $l_i : [0, 1]^{\#I_2} \times intD \to \mathbb{R}_+$ are continuous.

Let $(k, y) \in intD$ and $c^* = (c_i(\zeta^t, k, y))_{i \in I}$, $l^* = (l_i(\zeta^t, k, y))_{i \in I}$ denote the solution to the static maximization problem.

iv) If $\lambda_i = 0$, then $c_i^* = 0$ and $l_i^* = 0$. If $i \in I$, then $c_i^* > 0$ and $l_i^* > 0$. In addition, there exists $i \in I$ such that $l_i^* < 1$.

v) V_t is differentiable at any $(k, y) \in intD$ with partial derivatives given by:

$$\frac{\partial V_t(\lambda, k, y)}{\partial k} = \mu_t f_k(k, m - \sum_{i \in I} l_i^*)$$

$$\frac{\partial V_t(\lambda, k, y)}{\partial y} = -\mu_t$$

where $\mu_t = \lambda_i \left(\frac{\beta_i}{\beta}\right)^t u_c^i(c_i^*, l_i^*)$, $\forall i$.

Proof: (i) is standard. (ii) and (iii) follow from the Maximum Theorem.

(iv) It is obvious that $\lambda_i = 0$ implies $c_i^* = 0$, $l_i^* = 0$. Since $(k, y) \in intD$, there exists $\varepsilon > 0$ such that $0 < y + \varepsilon < f(k, m - \varepsilon)$. By letting $c_i = \frac{\varepsilon}{\#I}$, $l_i = \frac{\varepsilon}{\#I}$, $\forall i \in I$, the Slater condition is satisfied. Hence, there exists Lagrange multipliers $\mu_t(\zeta^t, k, y) \in \mathbb{R}_+$ associated with the constraint $\sum_i c_i + y \leq f(k, m - \sum_{i \in I} l_i)$ and $\eta_{i,t}(\zeta^t, k, y) \in \mathbb{R}_+$ associated with the constraints $l_i \leq 1$ such that $(c^*, l^*, \mu_t, (\eta_{i,t})_{i \in I})$ maximizes the associated Lagrangian :

$$\mathcal{L} = \sum_{i \in I} \lambda_i \left(\frac{\beta_i}{\beta}\right)^t u^i(c_i, l_i) - \mu_t \left[\sum_{i \in I} c_i + y - f(k, m - \sum_{i \in I} l_i)\right] - \sum_{i \in I} \eta_{i,t}(l_i - 1).$$

That $c_i^* > 0$ and $l_i^* > 0$, $\forall i \in I$ is a consequence of the Inada conditions imposed on period utilities. The existence of some $i \in I$ with $l_i^* < 1$ is a consequence of the limiting conditions imposed on technology. (v) follows from Corollary 7.3.1 in Florenzano, Le Van and Gourdel (2001). ■

Since $f_k(\infty, m) < 1$, there exists some $\overline{k}$ such that $f(\overline{k}, m) = \overline{k}$. It is easy to show that $\mathbf{k} \in \Pi(k_0)$ implies $k_t \leq A(k_0) = \max\{k_0, \overline{k}\}$. This in turn implies that $\Pi(k_0)$ is included in a compact set for the product topology. Since f is continuous, the set $\Pi(k_0)$ is closed for the product topology, and therefore, is compact in this topology. Define next the function $U : \mathbb{R}_+ \times \Pi(k_0) \to \mathbb{R}_+$:

$$U(k_0, \mathbf{k}) = \sum_{t=0}^{\infty} \beta^t V(\lambda, \zeta^t, k_t, k_{t+1}).$$

We have the following result.

Lemma 2 *i) The correspondence* Π *is continuous for the product topology.*
ii) $U(k_0, \cdot)$ *is upper semicontinuous on* $\Pi(k_0)$ *for the product topology.*

Proof: Refer to Le Van and Morhaim (2002, Lemma 2, Proposition 2). ■

It follows that problem (Q) is equivalent to the maximization of an upper semicontinuous function over a compact set, and therefore it admits a solution. Observe also that the strict concavity of V_t implies that the solution is unique.

Proposition 3 *For all* $k_0 \geq 0$, *there is a unique optimal accumulation path.*

3.2 Value function-Bellman equation-Optimal policy

One way to make any further analysis easier is to work with the value function. Let $\zeta = (\zeta_i)_{i \in I}$ where $\zeta_i = 1$, $\forall i \in I_1$ and $\zeta_i \in [0, 1]$, $\forall i \in I_2$. Given $T \geq 0$, define the function $W_T : (\zeta, k_0) \in [0,1]^{\# I_2} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$:

$$\begin{aligned} W_T(\zeta, k_0) &= W(\zeta^T, k_0) \\ &= \max \sum_{t=0}^{\infty} \beta^t V(\lambda, \zeta^{T+t}, k_t, k_{t+1}) \\ &\quad s.t.\ 0 \leq k_{t+1} \leq f(k_t, m),\ \forall t \\ &\qquad k_0 \geq 0 \text{ is given.} \end{aligned}$$

It is obvious that when $\zeta_i = 1,\ \forall i \in I_1$ and $\zeta_i = \frac{\beta_i}{\beta},\ \forall i \in I_2$, $W_0(\zeta, k_0)$ is the value function associated with problem (Q). In infinite-horizon problems with time-invariant period return functions (stationary problems) the value function is a function of the initial state alone. In the above problem the period return function is time-dependent, so the problem is a nonstationary one. In this case, as the time index on W indicates, the value function is time-dependent.

Proposition 4 *The value function $W : (\zeta, k) \in [0,1]^{\# I_2} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is strictly increasing, strictly concave, nonnegative with $W(\zeta, k) > 0$ for $k > 0$, upper semicontinuous and continuous when $k > 0$.*

Proof: It follows from Proposition 2 and Lemma 2. ■

The next proposition states formally what is known as the Principle of Optimality. It will help us characterize basic properties of optimal paths.

Proposition 5 *The value function solves the Bellman equation, i.e.*

$$\forall k_0 \geq 0,\ W(\zeta^0, k_0) = \max\{V(\lambda, \zeta^0, k_0, k_1) + \beta W(\zeta, k_1) : 0 \leq k_1 \leq f(k_0, m)\}$$

and for all $k_0 \geq 0$, a feasible path $\mathbf{k}$ is optimal, if and only if,

$$W(\zeta^t, k_t) = V(\lambda, \zeta^t, k_t, k_{t+1}) + \beta W(\zeta^{t+1}, k_{t+1})$$

holds for all t.

We next define the optimal policy function $\varphi_T : (\zeta, k) \in [0,1]^{\# I_2} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$:

$$\varphi_T(\zeta, k) = \varphi(\zeta^T, k) = \arg\max \left\{V(\lambda, \zeta^T, k, y) + \beta W(\zeta^{T+1}, y) : 0 \leq y \leq f(k, m)\right\}.$$

When $\zeta_i = 1,\ \forall i \in I_1$ and $\zeta_i = \frac{\beta_i}{\beta},\ \forall i \in I_2$, it follows that $\forall t \geq 1$:

$$k_t^* = \varphi(\zeta^{t-1}, k_{t-1}^*) = \varphi(\zeta^{t-1}, \varphi(\zeta^{t-2}, k_{t-2}^*)) = \varphi(\zeta^{t-1}, \varphi(\zeta^{t-2}, ..., \varphi(\zeta^0, k_0)...)).$$

For simplicity we write $k_t^* = \varphi^t(\zeta^0, k_0)$.

Lemma 3 *Under assumptions **U1-U2**, **F1-F2**, φ is continuous in $(\zeta, k) \in [0,1]^{\# I_2} \times \mathbb{R}_+$, and therefore uniformly continuous in $(\zeta, k) \in [0,1]^{\# I_2} \times [0, A(k_0)]$.*

Proof: When $k > 0$, the continuity of φ follows from the Maximum Theorem. When $k = 0$, feasibility implies that $\varphi(\zeta, k) = 0$. Since $f(0, m) = 0$, for any sequence $(\zeta_n, k_n) \in [0,1]^{\# I_2} \times \mathbb{R}_+$ such that $0 \leq \varphi(\zeta_n, k_n) \leq f(k_n, m)$ and $k_n \to 0$, it follows that the sequence of maximizers converges to 0 as $n \to +\infty$. ■

Consider next the problem involving the agents having a discount factor equal to the maximum one. Let $\widetilde{0} = (1, ..., 1, 0, ..., 0)$. Define the time invariant function $\widehat{V} : \Delta \times D \to \mathbb{R}_+$:

$$\begin{aligned} V(\lambda, \widetilde{0}, k, y) &= \widehat{V}(\lambda, k, y) \\ &= \max \sum_{i \in I_1} \lambda_i u^i(c_i, l_i) \\ &\quad s.t. \sum_{i \in I_1} c_i + y \leq f(k, m - \sum_{i \in I_1} l_i) \\ &\quad\quad c_i \geq 0,\ l_i \in [0,1],\ \forall i \in I_1 \end{aligned}$$

The intertemporal problem associated with $\widehat{V}$ is now stationary, i.e.:

$$\begin{aligned} W(\widetilde{0}, k_0) &= \widehat{W}(k_0) \\ &= \max \sum_{t=0}^{\infty} \beta^t \widehat{V}(\lambda, k_t, k_{t+1}) \\ &\quad s.t.\ 0 \leq k_{t+1} \leq f(k_t, m),\ \forall t \\ &\quad\quad k_0 \geq 0,\ \text{is given.} \end{aligned} \qquad (\widehat{Q})$$

The optimal policy function for this problem is given by:

$$\varphi(\widetilde{0}, k_0) = \arg\max \left\{ \widehat{V}(\lambda, k_0, k_1) + \beta \widehat{W}(k_1) : 0 \leq k_1 \leq f(k_0, m) \right\}.$$

In this case, we write $k_t^* = \varphi(\widetilde{0}, k_{t-1}^*) = \varphi^t(\widetilde{0}, k_0)$.

Given $k_0 \geq 0$, for any feasible allocation $((\mathbf{c}_i, \mathbf{l}_i)_i, \mathbf{k})$ from k_0, we have:

$$u^i(c_{i,t}, l_{i,t}) \leq u^i(f(A(k_0), m), 1) = B(k_0),\ \forall i.$$

This in turn implies that $\forall t$ and $\forall (k, y) \in D$:

$$\widehat{V}(\lambda, k, y) \leq V_t(\lambda, k, y) \leq \left(\frac{\max_{i \in I_2} \beta_i}{\beta} \right)^t C(k_0) + \widehat{V}(\lambda, k, y),$$

where $C(k_0) = \sum_{i \in I_2} B(k_0)$. Let $\varepsilon > 0$. Since $\left(\frac{\max_{i \in I_2} \beta_i}{\beta}\right) < 1$, there exists T independent of k_0 such that:

$$\widehat{V}(\lambda, k, y) \leq V_{T+t}(\lambda, k, y) \leq \varepsilon C(k_0) + \widehat{V}(\lambda, k, y), \ \forall t.$$

It follows that, $\forall k_0 \geq 0$, $\forall T' \geq T$:

$$\widehat{W}(k_0) \leq W_{T'}(\zeta, k_0) \leq \varepsilon C(k_0) \frac{1}{1-\beta} + \widehat{W}(k_0).$$

For any sequence $(k_t)_{t=0}^{\infty}$ feasible from k_0, we have $C(k_t) \leq C(k_0)$. Hence, the following inequalities also hold:

$$\widehat{V}(\lambda, k_t, k_{t+1}) \leq V_t(\lambda, k_t, k_{t+1}) \leq \varepsilon C(k_0) + \widehat{V}(\lambda, k_t, k_{t+1}), \ \forall t \geq T,$$

$$\widehat{W}(k_t) \leq W_t(\zeta, k_t) \leq \varepsilon C(k_0) \frac{1}{1-\beta} + \widehat{W}(k_t), \ \forall t \geq T.$$

Consider now a feasible capital sequence $(k_t)_{t=0}^{\infty}$ starting from some $k_0 \geq 0$. Using the previous results, for any subsequence $(t_n)_{n=1}^{\infty}$ such that $k_{t_n} \to k \geq 0$ and $k_{t_n+1} \to k' \geq 0$, we have:

$$\lim_{n \to \infty} V_{t_n}(\lambda, k_{t_n}, k_{t_n+1}) = \widehat{V}(\lambda, k, k') \text{ and } \lim_{n \to \infty} W_{t_n}(\zeta, k_{t_n}) = \widehat{W}(k).$$

4 Dynamic Equilibrium Properties

Claim (iv) in Proposition 2 shows that we cannot ensure that all consumers supply labor at any period. As a consequence, we cannot exclude the existence of corner solutions in the welfare maximization problem. This in turn implies that we cannot use the differentiable characterization of supermodilarity for the indirect utility function V_t. To overcome the problem and establish the supermodularity of V_t, we employ an alternative argument based on a iterative procedure in which a sequence of functions are shown to be supermodular and to be converging to the function V_t. In what follows we deal with the construction of such a sequence.

Let $\alpha \in (0,1)$ and $\nu \in [0,1]$. Consider the production function $\widetilde{f}$ defined as follows:

$$\widetilde{f}\,(\nu, k, (l_i)_i) = f(k, m - \sum_{i=1}^{m} l_i) + \nu \sum_{i=1}^{m} (1 - l_i)^{\alpha}$$

Observe that $\lim_{l_i \to 1} \widetilde{f}_{l_i}\,(\nu, k, (l_i)_i) = +\infty$. Let $\widetilde{D}$ denote the technology set:

$$\widetilde{D} = \left\{ (k, y) \in \mathbb{R}_+^2 : 0 \leq y \leq \widetilde{f}(\nu, k, m) \right\},$$

and define the correspondence $\widetilde{\Gamma}$:

$$(\nu,k,y)\in[0,1]\times\widetilde{D}\to\left\{(c_i,l_i)_i:\sum_{i=1}^{m}c_i+y\leq\widetilde{f}\left(\nu,k,(l_i)_i\right),\ c_i\geq 0,l_i\in[0,1],\forall i\right\}.$$

Lemma 4 *Under assumptions **U1-U2**, **F1-F2**, $\widetilde{\Gamma}$ is upper hemicontinuous on $[0,1]\times\widetilde{D}$ and continuous at any $(\nu,k,y)\in[0,1]\times\widetilde{D}$ with $k>0$.*

Proof: Let $(\nu^n,k^n,y^n)\in D$ be a sequence that converges to some $(\nu,k,y)\in[0,1]\times\widetilde{D}$. Let also $(c_i^n,l_i^n)_i$ be a sequence such that, $(c_i^n,l_i^n)_i\in\widetilde{\Gamma}(\nu^n,k^n,y^n)$, $\forall n$. Since $l_i^n\in[0,m]$, $c_i^n\leq\widetilde{f}(\nu^n,k^n,m)$ and $k^n\to k$, there exists a subsequence $(c_i^n,l_i^n)_i$ that converges to some $(c_i,l_i)_i$. Since $\sum_{i=1}^m c_i^n+y^n\leq\widetilde{f}\left(\nu^n,k^n,(l_i^n)_i\right)$, $\forall n$, we have $(c_i,l_i)_i\in\widetilde{\Gamma}(\nu,k,y)$.

We next show that $\widetilde{\Gamma}$ is lower hemicontinuous at any (ν,k,y) with $k>0$. Let $(\nu^n,k^n,y^n)\in D$ be a sequence that converges to some (ν,k,y) with $k>0$. Take $(c_i,l_i)_i\in\widetilde{\Gamma}(\nu,k,y)$. We will show that there exists $N\geq 1$ and a sequence $(c_i^n,l_i^n)_i$ such that $(c_i^n,l_i^n)_i\in\Gamma(\nu^n,k^n,y^n)$, $\forall n\geq N$ and $(c_i^n,l_i^n)_i\to(c_i,l_i)_i$. We consider three cases:

Case 1: $\sum_{i=1}^m c_i+y<\widetilde{f}\left(\nu,k,(l_i)_i\right)$.
Observe that for n large enough, $\sum_{i=1}^m c_i+y^n<\widetilde{f}\left(\nu^n,k^n,(l_i)_i\right)$. In this case, for any n, let $c_i^n=c_i$ and $l_i^n=l_i$, $\forall i$.

Case 2: $\sum_{i=1}^m c_i+y=\widetilde{f}\left(\nu,k,(l_i)_i\right)$ and $c_i>0$ for $i=\{1,...,J\}$.
Observe that there exists N such that $y^n<\widetilde{f}\left(\nu^n,k^n,(l_i)_i\right)$, $\forall n\geq N$. Let $\xi=\widetilde{f}\left(\nu,k,(l_i)_i\right)-y$ and $\xi^n=\widetilde{f}\left(\nu^n,k^n,(l_i)_i\right)-y^n$, $\forall n\geq N$. Since $\xi^n\to\xi$, N can be chosen such that $\xi^n-\xi$ is sufficiently small. For any $n\geq N$, let $c_i^n=c_i+\frac{\xi^n-\xi}{J}$ when $i=1,...,J$, $c_i^n=0$ when $i>J$ and $l_i^n=l_i$, $\forall i$.

Case 3: $\sum_{i=1}^m c_i+y=\widetilde{f}\left(\nu,k,(l_i)_i\right)$ and $c_i=0$, $\forall i$.
We have $y=\widetilde{f}\left(\nu,k,(l_i)_i\right)$. We consider three subcases.

1) Assume that $l_i=0$, $\forall i$
In this case, $y=\widetilde{f}\left(\nu,k,m\right)$. Let $c_1^n=\widetilde{f}\left(\nu^n,k^n,m\right)-y^n$, $c_i^n=0$, $\forall i\neq 1$ and $l_i^n=0$, $\forall i$.

2) Assume that $l_i\in(0,1)$ for $i=\{1,...,J\}$.
In this case, $y=\widetilde{f}\left(\nu,k,(l_i)_i\right)<\widetilde{f}\left(\nu,k,m\right)$. Assume first that there exists a subsequence (ν^n,k^n,y^n) such that $y^n\leq\widetilde{f}\left(\nu^n,k^n,(l_i)_i\right)$. In this case, let $c_1^n=\widetilde{f}\left(\nu^n,k^n,(l_i)_i\right)-y^n$, $c_i^n=0$, $\forall i\neq 1$ and $l_i^n=l_i$, $\forall i$. Assume next that there exists a subsequence (ν^n,k^n,y^n) such that $y^n>\widetilde{f}\left(\nu^n,k^n,(l_i)_i\right)$. Choose $j\in\{1,...,J\}$. Define the functions $\psi:[0,1]\to\mathbb{R}$ and, for any n, $\psi_n:[0,1]\to\mathbb{R}$:

$$\begin{aligned}\psi(\xi) &= f(k,(1-\xi)+m-1-\sum_{i\neq j}l_i)+\nu((1-\xi)^\alpha+\sum_{i\neq j}(1-l_i)^\alpha)-y\\ \psi_n(\xi) &= f(k^n,(1-\xi)+m-1-\sum_{i\neq j}l_i)+\nu^n((1-\xi)^\alpha+\sum_{i\neq j}(1-l_i)^\alpha)-y^n.\end{aligned}$$

$\psi_n(\xi)$ are decreasing in ξ. Observe that there exists N large enough such that $\psi_n(0) > 0$, $\forall n \geq N$. Note also that $\psi_n(l_j) = \widetilde{f}\,(\nu^n, k^n, (l_i)_i) - y^n < 0$, $\forall n \geq N$. By the Intermediate Value Theorem, for any $n \geq N$, there exists $\xi^n \in (0, l_j)$ such that $\psi_n(\xi^n) = 0$. For any $n \geq N$, let $l_j^n = \xi^n$ $l_i^n = l^i$, $\forall i \neq j$ and $c_i^n = 0$, $\forall i$. Since the function $\psi(\xi)$ is decreasing in ξ, it follows that $l_j^n \to l_j$.

3) Assume that $l_i = 1$, $\forall i$.
In this case, $y = (1-\delta)k$. Assume first that there exits a subsequence (ν^n, k^n, y^n) such that $y^n \leq (1-\delta)k^n$. In this case, let $c_1^n = (1-\delta)k^n - y^n$, $c_i^n = 0$, $\forall i \neq 1$ and $l_i^n = 1$, $\forall i$. Assume next that there exists a subsequence (k^n, y^n) such that $y^n > (1-\delta)k^n$. Choose $j \in \{1, ..., J\}$. Define the functions $\psi : [0,1] \to \mathbb{R}$ and, for any n, $\psi_n : [0,1] \to \mathbb{R}$:

$$\begin{aligned}
\psi(\xi) &= f\,(k, (1-\xi)) + \nu(1-\xi)^\alpha - y \\
\psi_n(\xi) &= f\,(k^n, (1-\xi)) + \nu^n(1-\xi)^\alpha - y^n.
\end{aligned}$$

$\psi_n(\xi)$ is decreasing in ξ. Observe that there exists N large enough such that $\psi_n(0) > 0$, $\forall n \geq N$. Note also that $\psi_n(1) = (1-\delta)k^n - y^n < 0$, $\forall n \geq N$. By the Intermediate Value Theorem, for any $n \geq N$, there exists $\xi_n \in (0,1)$ such that $\psi_n(\xi^n) = 0$. For any $n \geq N$, let $l_j^n = \xi^n$ $l_i^n = 1$, $\forall i \neq j$ and $c_i^n = 0$, $\forall i$. Since the function $\psi(\xi)$ is decreasing in ξ, it follows that $l_j^n \to 1$. ■

Remark 2 *When* $\nu = 0$, $\widetilde{f}\,(0, k, (l_i)_i) = f\,(k, m - \sum_{i=1}^m l_i)$ *and consequently* $\widetilde{D} = D$ *and* $\widetilde{\Gamma} = \Gamma$. *Therefore, Lemma 1 is a direct consequence of Lemma 4. When* $\nu \in (0,1]$, $D \subset \widetilde{D}$ *and* $\widetilde{\Gamma}(D) \subset \widetilde{\Gamma}(\widetilde{D})$.

Consider the indirect utility function $\widetilde{V}_t$ associated with the production function $\widetilde{f}$.

$$\begin{aligned}
\widetilde{V}_t(\nu, \lambda, k, y) &= \widetilde{V}(\nu, \lambda, \zeta^t, k, y) \\
&= \max \sum_{i \in I} \lambda_i \zeta^t u^i(c_i, l_i) \\
&\quad s.t. \sum_{i \in I} c_i + y \leq \widetilde{f}\,(\nu, k, (l_i)_{i \in I}) \\
&\quad\quad c_i \geq 0,\ 0 \leq l_i \leq 1,\ \forall i \in I
\end{aligned}$$

We have the following result:

Lemma 5 $\widetilde{V}_t : (\nu, \lambda, k, y) \in [0,1] \times \Delta \times \widetilde{D} \to \mathbb{R}_+$ *is:*

i) continuous at any $(\nu, k, y) \in [0,1] \times \widetilde{D}$ *with* $k > 0$.

ii) $\forall (\lambda, k, y) \in \Delta \times D$ *with* $k > 0$*:*

$$\lim_{\nu \to 0} \widetilde{V}_t(\nu, \lambda, k, y) = V_t(\lambda, k, y)$$

iii) $\forall(\nu,\lambda,k,y)\in(0,1]\times\Delta\times int\widetilde{D}$:

$$\frac{\partial\widetilde{V}_t(\nu,\lambda,k,y)}{\partial k\partial y}>0.$$

That is, $\widetilde{V}_t$ is supermodular in the interior of $\widetilde{D}$.

Proof: (i) and (ii) follow from Lemma 4 and the Maximum Theorem. The proof of (iii) follows in several steps. We need the following two results.

Claim 1 *Let*

$$V=\begin{pmatrix} a_1+b & b & \cdots & b & b\\ b & a_2+b & \cdots & \cdots & b\\ \cdots & \cdots & \vdots\ddots & \cdots & \vdots\\ b & b & \cdots & a_{q-1}+b & b\\ b & b & \cdots & b & a_q+b \end{pmatrix}$$

with $b<0$, $a_i<0$, $\forall i=1,...,q$. Then V is invertible and $V^{-1}\mathbf{1}<\mathbf{0}$.

Proof*: It is easy to show that V is negative definite. Hence, V is invertible. Let $\mathbf{x}=V^{-1}\mathbf{1}$. Since $V\mathbf{x}=\mathbf{1}$, it follows that*

$$a_ix_i+b\sum_{i=1}^{q}x_i=1,\ \forall i.$$

This in turn implies that $x_1=\frac{a_i}{a_1}x_i$, $\forall i\neq 1$. If $x_1\geq 0$, then $x_i\geq 0$, $\forall i\neq 1$ while $a_ix_i+b\sum_{i=1}^{q}x_i\leq 0$: a contradiction. Hence, $x_i<0$, $\forall i=1,...,q$. ■

Claim 2 *Let A be a $N\times N$ symmetric matrix. Given $r=1,...,N$, denote by ${}_rA$ the $r\times N$ submatrix where only the r rows are retained and by ${}_rA_r$ the $r\times r$ submatrix where only the first $r\leq N$ rows and $r\leq N$ columns are retained. Let B be an $N\times S$ submatrix with $S\leq N$ and rank equal to S. A is negative definite on $\{\mathbf{z}\in\mathbb{R}^N:B\mathbf{z}=0\}$ (i.e. $\mathbf{z}^TA\mathbf{z}<0$, $\forall\mathbf{z}\in\mathbb{R}^N$ with $B\mathbf{z}=0$ and $\mathbf{z}\neq 0$) if and only if*

$$(-1)^r\begin{vmatrix} {}_rA_r & {}_rB\\ ({}_rB)^T & 0\end{vmatrix}>0$$

for $r=S+1,...,N$.

Proof*: See Mas-Colell et al. (1995) Theorem M.D.3.* ■

Let $c^*=(c_i(\nu,\zeta^t,k,y))_{i\in I}$, $l^*=(l_i(\nu,\zeta^t,k,y))_{i\in I}$ denote a solution for the maximization problem. Since $(k,y)\in int\widetilde{D}$, there exists $\varepsilon>0$ such that $0<y+\varepsilon<\widetilde{f}(\nu,k,m-\varepsilon)$. By letting $c_i=\frac{\varepsilon}{\#I}$, $l_i=\frac{\varepsilon}{\#I}$, $\forall i\in I$, the Slater condition is satisfied. Hence there exists Lagrange multipliers $\mu_t(\nu, \zeta^t, k, y) \in \mathbb{R}$ associated with the constraint $\sum_i c_i + y \leq \widetilde{f}\,(\nu, k, (l_i)_{i\in I})$ and $\eta_{i,t}(\nu, \zeta^t, k, y) \in \mathbb{R}$ associated with the constraints $l_i \leq 1$ such that $(c^*, l^*, \mu_t, (\eta_{i,t})_{i\in I})$ maximizes the associated Lagrangian :

$$\mathcal{L} = \sum_{i\in I} \lambda_i \left(\frac{\beta_i}{\beta}\right)^t u^i(c_i, l_i) - \mu_t \left[\sum_{i\in I} c_i + y - \widetilde{f}\,(\nu, k, (l_i)_i)\right] - \sum_{i\in I} \eta_{i,t}(l_i - 1).$$

From the The Kuhn-Tucker first-order conditions we get:

$$\lambda_i \left(\frac{\beta_i}{\beta}\right)^t u^i_c(c^*_i, l^*_i) - \mu_t = 0, \forall i \in I$$

$$\lambda_i \left(\frac{\beta_i}{\beta}\right)^t u^i_l(c^*_i, l^*_i) - \mu_t \left[f_L(k, m - \sum_{i\in I} l^*_i) + \nu\alpha(1 - l^*_i)^{\alpha-1}\right] - \eta_{i,t} = 0,\ \forall i \in I$$

$$\mu_t \geq 0,\ \mu_t \left[\sum_{i\in I} c^*_i + y - f(k, m - \sum_{i\in I} l^*_i) - \nu \sum_{i=1}^{m} (1 - l^*_i)^\alpha\right] = 0$$

$$\eta_{i,t} \geq 0,\ \eta_{i,t}(l^*_i - 1) = 0,\ \forall i \in I.$$

Since u^i is strictly increasing, $u^i(0, 0) = 0$ and u^i satisfies the Inada conditions, it follows that $c^*_i > 0$ and $l^*_i > 0$, $\forall i \in I$. Therefore, $\mu_t > 0$. Moreover, $l^*_i < 1$, $\eta_{i,t} = 0$, $\forall i \in I$, since $l^*_j = 1$ for some $j \in I$ implies $\eta_{j,t} = +\infty$. The first-order conditions become:

$$\lambda_i \left(\frac{\beta_i}{\beta}\right)^t u^i_c(c^*_i, l^*_i) - \mu_t = 0,\ \forall i \in I$$

$$\lambda_i \left(\frac{\beta_i}{\beta}\right)^t u^i_l(c^*_i, l^*_i) - \mu_t \left[f_L(k, m - \sum_{i\in I} l^*_i) + \nu\alpha(1 - l^*_i)^{\alpha-1}\right] = 0,\ \forall i \in I$$

$$\sum_{i\in I} c^*_i + y - f(k, m - \sum_{i\in I} l^*_i) - \nu \sum_{i=1}^{m} (1 - l^*_i)^\alpha = 0$$

Differentiating the above equations and rearranging we get:

$$\lambda_i \left(\frac{\beta_i}{\beta}\right)^t u^i_{cc} dc^*_i + \lambda_i \left(\frac{\beta_i}{\beta}\right)^t u^i_{cl} dl^*_i - d\mu_t = 0,\ \forall i \in I$$

$$\lambda_i \left(\frac{\beta_i}{\beta}\right)^t u^i_{cl} dc^*_i + \left[\lambda_i \left(\frac{\beta_i}{\beta}\right)^t u^i_{ll} + \mu_t f_{LL} - \frac{\nu\alpha(1-\alpha)}{(1 - l^*_i)^{2-\alpha}}\right] dl^*_i$$

$$-\mu_t \left[f_{kL} dk - f_{LL} \sum_{j\in I, j\neq i} dl^*_j\right] - \left[f_L + \frac{\nu\alpha}{(1 - l^*_i)^{1-a}}\right] d\mu_t = 0,\ \forall i \in I$$

$$\sum_{i\in I} dc^*_i + dy - f_k dk + f_L \sum_{i\in I} dl^*_i + \sum_{i\in I} \frac{\nu\alpha}{(1 - l^*_i)^{1-\alpha}} dl^*_i = 0.$$

Let

$$p = \mu_t f_{LL},\ p_{1i} = \frac{\nu\alpha}{(1-l_i^*)^{1-\alpha}},\ p_{2i} = -\mu_t \frac{\nu\alpha(1-\alpha)}{(1-l_i^*)^{2-\alpha}},\ \forall i \in I.$$

and

$$a_i = \lambda_i \left(\frac{\beta_i}{\beta}\right)^t u^i_{cc},\ b_i = \lambda_i \left(\frac{\beta_i}{\beta}\right)^t u^i_{cl},\ c_i = \lambda_i \left(\frac{\beta_i}{\beta}\right)^t u^i_{ll} + p_{2i}.$$

With this notation the first order conditions can be written as follows:

$$a_i dc_i^* + b_i dl_i^* - d\mu_t = 0,\ \forall i \in I \tag{1}$$

$$b_i dc_i^* + [c_i + p] dl_i^* + p \sum_{j\in I, j\neq i} dl_j^* - (f_L + p_{1i}) d\lambda_t = \mu_t f_{Lk} dk,\ \forall i \in I. \tag{2}$$

$$\sum_{i\in I} dc_i^* + \sum_{i\in I} (f_L + p_{1i}) dl_i^* = f_k dk - dy. \tag{3}$$

Denote $q = \#I$. We can alternatively write these equations in a matrix form, $AX = X_0$, where:

$$A = \begin{pmatrix} a_1 & 0\cdots & 0 & b_1 & 0\cdots & 0 & -1 \\ 0 & \vdots\ddots & \vdots & 0 & \vdots\ddots & \vdots & -1 \\ 0 & 0\cdots & a_q & 0 & 0\cdots & b_q & -1 \\ b_1 & 0\cdots & 0 & c_1 + p & p\cdots & p & -f_L - p_{11} \\ 0 & \vdots\ddots & 0 & \vdots & \vdots\ddots & \vdots & \vdots \\ 0 & 0\cdots & b_q & p & p\cdots & c_q + p & -f_L - p_{1q} \\ 1 & 1\cdots & 1 & f_L + p_{11} & \cdots & f_L + p_{1q} & 0 \end{pmatrix},$$

$$X = \begin{pmatrix} dc_1^* \\ \vdots \\ dc_q^* \\ dl_1^* \\ \vdots \\ dl_q^* \\ d\mu_t \end{pmatrix},\ X_0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mu_t f_{kL} dk \\ \vdots \\ \mu_t f_{kL} dk \\ f_k dk - dy \end{pmatrix}.$$

In particular,

$$A = \begin{pmatrix} M & -\mathbf{d}^T \\ \mathbf{d} & 0 \end{pmatrix},$$

where:

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},\ \mathbf{d} = (1, ..., 1, f_L + p_{11}, ..., f_L + p_{1q})$$

and

$$M_{11} = \begin{pmatrix} a_1 & 0\cdots & 0 \\ \vdots & \vdots\ddots & \vdots \\ 0 & 0 & a_q \end{pmatrix}, \ M_{12} = M_{21} = \begin{pmatrix} b_1 & 0\cdots & 0 \\ \vdots & \vdots\ddots & \vdots \\ 0 & 0\cdots & b_q \end{pmatrix}$$

$$M_{22} = \begin{pmatrix} c_1 + p & p\cdots & p \\ \vdots & \vdots\ddots & \vdots \\ p & p\cdots & c_q + p \end{pmatrix}.$$

We show that A is invertible. Let $\widehat{A}$ be the matrix obtained from A by changing the sign of the last column, i.e.

$$\widehat{A} = \begin{pmatrix} M & \mathbf{d}^T \\ \mathbf{d} & 0 \end{pmatrix}.$$

Observe that $M_{22} = N_1 + N_2$, where

$$N_1 = \begin{pmatrix} c_1 & 0\cdots & 0 \\ \vdots & \vdots\ddots & \vdots \\ 0 & 0\cdots & c_q \end{pmatrix}, \ N_2 = \begin{pmatrix} p & p\cdots & p \\ \vdots & \vdots\ddots & \vdots \\ p & p\cdots & p \end{pmatrix}.$$

Let $\mathbf{z} =\{z_1, ..., z_q, \varsigma_1, ..., \varsigma_q)$ where $\mathbf{z} \neq \mathbf{0}$. Since $p_{2i} < 0$ and $a_i c_i - b_i^2 \geq 0$ (since u is concave), it follows that

$$\begin{aligned} \mathbf{z}^T M \mathbf{y} &= \mathbf{z}^T \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & N_1 \end{pmatrix} \mathbf{z} + \mathbf{z}^T \begin{pmatrix} 0 & 0 \\ 0 & N_2 \end{pmatrix} \mathbf{z} \\ &= \sum_{i \in I} \left[a_i \left(z_i + \frac{b_i}{a_i} \varsigma_i \right)^2 + \frac{a_i c_i - b_i^2}{a_i} \varsigma_i^2 \right] + p \left(\sum_{i \in I} \varsigma_i \right)^2 < 0, \end{aligned}$$

Therefore, M is negative definite. Let

$$B = \begin{pmatrix} \mathbf{d}^T \\ 0 \end{pmatrix}.$$

Since M is negative definite on $\{\mathbf{z} \in \mathbb{R}^{2q+1} : B\mathbf{z} = 0\}$, (i.e. $\mathbf{z}^T M \mathbf{z} < 0$, $\forall \mathbf{z} \in \mathbb{R}^{2q+1}$ with $B\mathbf{z} = 0$ and $\mathbf{z} \neq 0$), it follows from claim 2 that:

$$(-1)^{2q} \det \begin{pmatrix} {}_{2q}M_{2q} & {}_{2q}B \\ ({}_{2q}B)^T & 0 \end{pmatrix} = (-1)^{2q} |\widehat{A}| > 0 \text{ or } |\widehat{A}| > 0.$$

Since $|A| = -|\widehat{A}| < 0$ the matrix A is invertible. By the Implicit Function Theorem, $c_i(\nu, \zeta^t, k, y), l_i(\nu, \zeta^t, k, y)$ and $\mu_t(\nu, \zeta^t, k, y)$ are C^1 in a neighborhood of $(k, y) \in int\widetilde{D}$. The Envelope Theorem then implies:

$$\begin{aligned} \frac{\partial \widetilde{V}_t(\nu, \lambda, k, y)}{\partial k} &= \mu_t f_k(k, m - \sum_{i \in I} l_i^*) \\ \frac{\partial \widetilde{V}_t(\nu, \lambda, k, y)}{\partial y} &= -\mu_t. \end{aligned}$$

By equations (1),(2),(3) we obtain that:

$$\left[\left(f_L + p_{11} - \frac{b_1}{a_1}, .., f_L + p_{1q} - \frac{b_q}{a_q} \right) V^{-1} \begin{pmatrix} f_L + p_{11} - \frac{b_1}{a_1} \\ \vdots \\ f_L + p_{1q} - \frac{b_q}{a_q} \end{pmatrix} \right] + \sum_{i=1}^{q} \frac{1}{a_i} d\mu_t$$

$$= \left[-\mu_t f_{Lk} \left(f_L + p_{11} - \frac{b_1}{a_1}, .., f_L + p_{1q} - \frac{b_q}{a_q} \right) V^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} + f_k \right] dk - dy,$$

with

$$V = \begin{pmatrix} c_1 - \frac{b_1^2}{a_1} + p & \cdots & p \\ \vdots & \vdots \ddots & \vdots \\ p & \cdots & c_q - \frac{b_q^2}{a_q} + p \end{pmatrix}.$$

Observe that the matrix V satisfies the hypothesis of claim 1. Therefore, V^{-1} exists and

$$V^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} < \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

From the first-order conditions we have:

$$f_L + p_{1i} = \frac{u_l^i(c_i^*, l_i^*)}{u_c^i(c_i^*, l_i^*)}, \ \forall i.$$

Given assumption **U4** we also have:

$$\begin{aligned} f_L + p_{1i} - \frac{b_i}{a_i} &= \frac{u_l^i}{u_c^i} - \frac{u_{cl}^i}{u_{cc}^i} \\ &= \frac{u_l^i}{u_{cc}^i} \left(\frac{u_{cc}^i}{u_c^i} - \frac{u_{cl}^i}{u_l^i} \right) \geq 0. \end{aligned}$$

It follows that:

$$\frac{\partial \mu_t}{\partial k} < 0,$$

and in particular

$$\frac{\partial^2 \widetilde{V}_t(\nu, \lambda, k, y)}{\partial k \partial y} = -\frac{\partial \mu_t}{\partial k} > 0, \ \forall (k, y) \in int\widetilde{D}.$$

■

Equipped with the result established in the last proposition we return to the study of problem (Q). We have the following result.

Proposition 6 *i) V_t is supermodular in the interior of D, i.e.*
$\forall (k,y),(k',y') \in intD$:

$$V_t(\lambda,(k,y)\vee(k',y')) + V_t(\lambda,(k,y)\wedge(k',y')) \geq V_t(\lambda,k,y) + V_t(\lambda,k',y').$$

ii) The policy function $\varphi(\widetilde{0},k)$ is non-decreasing in k. As a consequence, the optimal capital path associated with problem $\widehat{Q}$ is monotonic. Moreover, if $k_0 \leq k_0'$ and $\mathbf{k}^*, \mathbf{k}'$ are the optimal paths starting respectively from k_0 and k_0', then $k_t^* \leq k_t'$, $\forall t$.

Proof: i) Note that $(k,y) \in intD$ implies that $0 < y < \widetilde{f}(\nu,k,m)$, $\forall \nu \in [0,1]$. From Lemma 5, it follows that $\forall (k,y),(k',y') \in intD$:

$$\widetilde{V}_t(\nu,\lambda,(k,y)\vee(k',y')) + \widetilde{V}_t(\nu,\lambda,(k,y)\wedge(k',y')) \geq \widetilde{V}_t(\nu,\lambda,k,y) + \widetilde{V}_t(\nu,\lambda,k',y').$$

Letting $\nu \to 0$ and taking the limits of both sides we get:

$$V_t(\lambda,(k,y)\vee(k',y')) + V_t(\lambda,(k,y)\wedge(k',y')) \geq V_t(\lambda,k,y) + V_t(\lambda,k',y').$$

That is, V_t is supermodular on $intD$.

ii) Recall that problem $\widehat{Q}$ is stationary. Assume that $k_0 < k_0'$. If $k_0 = 0$, then $\varphi(\widetilde{0},k_0) = 0$, while $\varphi(\widetilde{0},k_0') > 0$. Let $k_0 > 0$, but assume the contrary, i.e. $\varphi(\widetilde{0},k_0) > \varphi(\widetilde{0},k_0')$. We consider two cases:

Case 1: Assume that $\varphi(\widetilde{0},k_0) < fk_0, m)$.
Observe that

$$\begin{aligned} 0 &< \varphi(\widetilde{0},k_0') < f(k_0,m) \\ 0 &< \varphi(\widetilde{0},k_0) < f(k_0',m). \end{aligned}$$

Therefore, $\varphi(\widetilde{0},k_0')$ is feasible from k_0 and $\varphi(\widetilde{0},k_0)$ is feasible from k_0'. Since $\widehat{V}$ is supermodular in the interior of D, the claim follows directly (see Majumdar et al. (2000), Chap. 2, Proposition 5.2).

Case 2: Assume that $\varphi(\widetilde{0},k_0) = f(k_0,m)$.
We have that $0 < \varphi(\widetilde{0},k_0') < f(k_0',m)$. The continuity of φ implies that for any $k < k_0'$ with k sufficiently close to k_0', we have $0 < \varphi(\widetilde{0},k) < f(k,m)$. Let $k^1 \in [k_0,k_0')$ be the first element, such that, $\varphi(\widetilde{0},k^1) = f(k^1,m)$. For any $k \in (k^1,k_0']$, we have $0 < \varphi(\widetilde{0},k) < f(k,m)$ and $\varphi(\widetilde{0},k) \leq \varphi(\widetilde{0},k_0')$ (recall case 1). It follows that

$$\varphi(\widetilde{0},k_0') \geq \varphi(\widetilde{0},k^1) = f(k^1,m) \geq f(k_0,m) = \varphi(\widetilde{0},k_0).$$

We conclude the proof. ■

Lemma 6 *Let* $\mathbf{k}^*$ *denote the solution to problem* (Q)*. Under assumptions* ***U1-U4***, ***F1-F4***, *if* $k_0 > 0$, *then* $k_t^* > 0$, $\forall t$.

Proof: See Appendix. ■

We need to impose some additional structure on preferences and production technology.

Assumption U5: For any period utility function u^i that satisfies i) $u^i(c,0) = u^i(0,l) = 0$, $\forall c \geq 0$, $l \geq 0$ and ii) $u^i_{cl}(c,l) > 0$, $\forall (c,l) \in \mathbb{R}^2_{++}$, we additionally require $u^i_c(x,x)$ and $u^i_l(x,x)$ to be non-increasing in x.

Assumption F5: F is assumed to be homogeneous of degree $a \in (0,1]$, i.e. $F(k,L) = L^a F\left(\frac{k}{L},1\right)$.

As the following lemma shows, these additional assumptions are sufficient to exclude convergence to zero.

Lemma 7 *Let* $k_0 > 0$*. If* $\mathbf{k}^*$ *denotes the optimal path starting from* k_0*, then* k_t^* *cannot converge to zero.*

Proof: See Appendix. ■

Consider next the problem involving the agents with a discount factor equal to the maximum one (i.e. the agents that belong to the set I_1).

Proposition 7 *Let* $k_0 > 0$ *and* $((\mathbf{c}_i^*, \mathbf{l}_i^*)_{i\in I_1}, \mathbf{k}^*)$ *denote the solution to the Pareto-optimum problem involving only agents in* I_1*. There exists* $((c_i^s, l_i^s)_{i\in I_1}, k^s)$ *such that* $\sum_{i\in I_1} c_i^s + k^s = f\left(k^s, m - \sum_{i\in I_1} l_i^s\right)$, $f_k\left(k^s, m - \sum_{i\in I_1} l_i^s\right) = \frac{1}{\beta}$, *and* $k_t^* \to k^s$, $c_{i,t}^* \to c_i^s$, $l_{i,t}^* \to l_i^s$, $\forall i \in I_1$.

Proof: From Proposition 6(ii) we know that the optimal capital sequence $\mathbf{k}^*$ is bounded and monotonic. In addition Lemma 7 implies that $k_t^* \to k^s > 0$. By the principle of optimality we have:

$$\widehat{W}(k_t^*) = \widehat{V}(\lambda, k_t^*, k_{t+1}^*) + \beta \widehat{W}(k_0)(k_{t+1}^*), \forall t \geq 0.$$

Taking the limits as $t \to +\infty$ we obtain:

$$\widehat{W}(k^s) = \widehat{V}(\lambda, k^s, k^s) + \beta \widehat{W}(k^s).$$

It follows that k^s is a steady state. By Proposition 1 there exists $(c_i^s, l_i^s)_{i\in I_1}$ associated with k^s that solve:

$$\begin{aligned} \widehat{V}(\lambda, k^s, k^s) \quad &= \quad \max \sum_{i\in I_1} \lambda_i u_i(c_i, l_i) \\ s.t. \quad & \sum_{i\in I_1} c^i + k^s \leq f(k^s, m - \sum_{i\in I_1} l^i) \\ & c^i \geq 0,\ l^i \in [0,1],\ i \in I_1 \end{aligned}$$

Observe that $k^s < f\left(k^s, m\right)$. If not, then:

$$\sum_{i \in I_1} c_i^s = 0 \text{ and } \sum_{i \in I_1} l_i^s = 1.$$

In this case, $\widehat{V}(\lambda, k^s, k^s) = 0$: a contradiction since by Proposition 4 $k^s > 0$ implies $\widehat{W}(k^s) > 0$.

Hence, $0 < k^s < f(k^s, m)$. From Proposition 2(iv) $c_i(k^s, k^s) > 0$ and $l_i(k^s, k^s) > 0$, $\forall i \in I_1$. Moreover, since $c_i(k_t^*, k_{t+1}^*)$ and $l_i(k_t^*, k_{t+1}^*)$ are continuous functions and $k_t^* \to k^s$ we have:

$$c_i(k_t^*, k_{t+1}^*) \to c_i(k^s, k^s),\ l_i(k_t^*, k_{t+1}^*) \to l_i^s(k^s, k^s).$$

Since $k_t^* \to k^s > 0$, there exists T such that $0 < k_{t+1}^* < f(k_t^*, m)$ $\forall t \geq T$.

Thus, for any $t \geq T$ Euler equation holds, i.e.

$$u_c^i(c_{i,t}^*, l_{i,t}^*) = \beta u_c^i(c_{i,t+1}^*, l_{i,t+1}^*) f_k(k_{t+1}^*, m - \sum_{i \in I_1} l_{i,t+1}^*)$$

Taking the limits we get:

$$\beta f_k(k^s, m - \sum_{i \in I_1} l_i^s) = 1.$$

■

Lemma 8 *Let* $a \in (0, \overline{k}]$. *Then,* $\forall \varepsilon > 0$, $\exists T(a, \varepsilon)$, *such that,* $\forall k \geq a$, $\forall t \geq T(a, \varepsilon)$:

$$\left| \varphi^t(\widetilde{0}, k) - k^s \right| < \varepsilon.$$

Proof: Let $\varepsilon > 0$ and $k \geq a$. Given Proposition 7, $\exists T(k, \varepsilon)$ such that $\forall t \geq T(k, \varepsilon)$:

$$\left| \varphi^t(\widetilde{0}, k) - k^s \right| < \varepsilon.$$

Since φ^t is continuous, there exists a neighborhood $v(k)$ of k, such that, $\forall k' \in v(k)$ we have:

$$\left| \varphi^{T(k,\varepsilon)}(\widetilde{0}, k') - k^s \right| < \varepsilon.$$

Assume that $k < k^s$. Since $(\varphi^t(0, k))_t$ is non-decreasing, $\forall k' \in v(k)$, $\forall t \geq T(k, \varepsilon)$:

$$\begin{aligned} \left| \varphi^t(\widetilde{0}, k') - k^s \right| &= k^s - \varphi^t(\widetilde{0}, k') \\ &\leq k^s - \varphi^{T(k,\varepsilon)}(\widetilde{0}, k') \\ &< \varepsilon. \end{aligned}$$

When $k \geq k^s$ the argument is similar. Now consider a finite covering $(v(k_j))_{j=1}^n$ of $[a, \overline{k}]$ and let $T(a, \varepsilon) = \max_j \{T(k^j, \varepsilon)\}$. ■

We now return to the initial problem involving all agents. The following Proposition shows that the optimal capital sequence is bounded away from zero.

Proposition 8 *For any $k_0 > 0$ and $\mathbf{k}^*$ optimal from k_0, there exists $\gamma > 0$ such that $k_t^* \geq \gamma$, $\forall t$.*

Proof: Lemma 7 implies that there exists $a \in (0, k^s)$ and a subsequence $(k^*_{T_n})_{n \in \mathbb{N}}$ such that $k^*_{T_n} \geq a$, $\forall n \in \mathbb{N}$. Choose $\varepsilon > 0$ such that $a - \varepsilon > 0$ and $k^s - 2\varepsilon > a$. Let $T(a, \varepsilon)$ be as in Lemma 8. It follows that:

$$\left| \varphi^{T(a,\varepsilon)}(\widetilde{0}, k^*_{T_n}) - k^s \right| < \varepsilon, \ \forall n \in \mathbb{N}.$$

Since φ is uniformly continuous, $\exists n$ large enough, such that:

$$\left| \varphi(\zeta^{T_n}, k^*_{T_n}) - \varphi(\widetilde{0}, k^*_{T_n}) \right| < \varepsilon.$$

In particular uniform continuity of φ^t implies that $\forall t = 1, ..., T(a, \varepsilon)$:

$$\begin{aligned} & \left| \varphi^t(\zeta^{T_n}, k^*_{T_n}) - \varphi^t(\widetilde{0}, k^*_{T_n}) \right| \\ = & \left| \varphi(\zeta^{T_n+t-1}, \varphi(\zeta^{T_n+t-2}, ..., \varphi(\zeta^{T_n}, k^*_{T_n})...)) - \varphi(\widetilde{0}, k^*_{T_n+t}) \right| < \varepsilon. \end{aligned}$$

Observe that $k^*_{T_k} \geq a$ implies $\varphi^t(\widetilde{0}, k^*_{T_n}) \geq \varphi^t(\widetilde{0}, a) \geq a$, $\forall t = 1, ..., T(a, \varepsilon)$. The above inequalities imply that $\forall t = 1, ..., T(a, \varepsilon)$:

$$\begin{aligned} \varphi^t(\zeta^{T_n}, k^*_{T_n}) &> a - \varepsilon \\ \varphi^{T(a,\varepsilon)}(\zeta^{T_n}, k^*_{T_n}) &> k^s - 2\varepsilon > a. \end{aligned}$$

By definition:

$$\varphi^{T(a,\varepsilon)}(\zeta^{T_n}, k^*_{T_n}) = k^*_{T_n+T(a,\varepsilon)} > a.$$

By Lemma 8 we have:

$$\left| \varphi^{T(a,\varepsilon)}(\widetilde{0}, k^*_{T_n+T(a,\varepsilon)}) - k^s \right| < \varepsilon.$$

The uniform continuity of φ implies that:

$$\left| \varphi(\zeta^{T_n+T(a,\varepsilon)}, k^*_{T_n+T(a,\varepsilon)}) - \varphi(\widetilde{0}, k^*_{T_n+T(a,\varepsilon)}) \right| < \varepsilon.$$

As before we have that $\forall t = 1, ..., T(a, \varepsilon)$

$$\begin{aligned} \varphi^t(\zeta^{T_n+T(a,\varepsilon)}, k^*_{T_n+T(a,\varepsilon)}) &> a - \varepsilon \\ \varphi^{T(a,\varepsilon)}(\zeta^{T_n+T(a,\varepsilon)}, k^*_{T_n+T(a,\varepsilon)}) &> k^s - 2\varepsilon > a. \end{aligned}$$

Observe that

$$\varphi^t(\zeta^{T_n+T(a,\varepsilon)}, k^*_{T_n+T(a,\varepsilon)}) = \varphi^{T(a,\varepsilon)+t}(\zeta^{T_n}, k^*_{T_n}) = k^*_{T_n+T(a,\varepsilon)+t},$$

in which case we have:

$$\varphi^t(\zeta^{T_n}, k^*_{T_n}) > a - \varepsilon,\ \forall t = 1, ..., 2T(a, \varepsilon).$$

Repeating the above argument one can establish that :

$$\varphi^t(\zeta^{T_n}, k^*_{T_n}) > a - \varepsilon,\ \forall t = 1, ...\infty.$$

The claim is true for

$$\gamma = \min\{k^*_1, ..., k^*_{T_n}, a - \varepsilon\}$$

■

Proposition 9 *Let $k_0 > 0$ and $((\mathbf{c}^*_i, \mathbf{l}^*_i)_i, \mathbf{k}^*)$ be the solution to problem (P). Let $((c^s_i, l^s_i)_{i\in I_1}, k^s)$ denote the steady state associated with problem $(\widehat{Q})$. Then, i) $k^*_t \to k^s$, ii) $c^*_{i,t} \to 0$ and $l^*_{i,t} \to 0$, $\forall i \in I_2$, iii) $c^*_{i,t} \to c^s_i$ and $l^*_{i,t} \to l^s_i$, $\forall i \in I_1$.*

Proof: Given Propositions 6,7 and 8, the proof of (i) parallels the one presented in Le Van-Vailakis (2003, Proposition 4). Since $k^*_t \to k^s$ there exists some T such that $(k^*_t, k^*_{t+1}) \in intD$, $\forall t \geq T$. We know that for any $i \in I$, $c_i(\zeta^t, k^*_t, k^*_{t+1})$ and $l_i(\zeta^t, k^*_t, k^*_{t+1})$ are continuous functions in $[0,1]^{\#I_2} \times intD$ and that $V(\lambda, \zeta^t, k^*_t, k^*_{t+1}) \to \widehat{V}(\lambda, k^s, k^s)$. This proves claims (ii) and (iii). ■

Remark 3 *The last proposition shows that the equilibrium paths associated with problem (P) converge to a limit point. This limit point is the steady state associated with the planner's problem involving only the most patient consumers. The model exhibits the well known property of "the emergence of a dominant consumer" found in the seminal papers of Becker (1980) and Bewley (1982). After all, one can ask if the convergence point is itself a steady state. It is easy to show that this is not true.*

Let $k_0 = k^s$ and for any $\forall t \geq 1$ assume that $k^*_t = k^s$ and $c^*_{i,t} = c^s_i$, $l^*_{i,t} = l^s_i$, $\forall i \in I_1$, $c^*_{i,t} = 0$, $l^*_{i,t} = 0$, $\forall i \in I_2$. Assume that there exists $j \in I$ with $\beta_j < \beta$. Since $0 < k^s < f(k^s, m)$, from Proposition 2(iiib) we have that $c^*_{j,t} > 0$, $l^*_{j,t} > 0$, $\forall t \geq 0$. This in turn contradicts the optimality of $k^*_t = k^s$ and $c^*_{i,t} = c^s_i$, $l^*_{i,t} = l^s_i$, $\forall i \in I_1$.

It follows that in case where agents have different discount factors and the economy starts at $k_0 = k^s$ any optimal path (k^*_t) converges to k^s with $k^*_1 \neq k_s$. As a result, the optimal path may exhibit fluctuations at least for the beginning periods.

5 Appendix

Proof of Lemma 6: Let $k_0 > 0$ but assume that $k_1^* = 0$. Feasibility implies that $c_{i,t}^* = 0$, $l_{i,t}^* \in [0,1]$, $\forall i, \forall t \geq 1$, $k_t^* = 0$, $\forall t \geq 1$. Since $f(k_0, L_0^*) > 0$ there exists some $j \in I$ such that $c_{j,0}^* > 0$. We distinguish two cases:

Case 1: Assume that $l_{j,1}^* = 1$.

Choose $\varepsilon > 0$ such that $c_{j,0}^* > \varepsilon$. Consider the alternative feasible path $(\mathbf{c}, \mathbf{l}, \mathbf{k})$, defined as follows:

$$\begin{aligned}
\text{i) } c_{j,0} &= c_{j,0}^* - \varepsilon, \ c_{j,1} = f(\varepsilon, L_1), \ c_{j,t} = c_{j,t}^*, \ \forall t \geq 2, \ l_{j,t} = l_{j,t}^*, \ \forall t \geq 0 \\
\text{ii) } c_{i,t} &= c_{i,t}^* \text{ and } l_{i,t} = l_{i,t}^*, \ \forall i \neq j, \forall t \\
\text{iii) } k_1 &= \varepsilon, \ k_t = k_t^*, \ \forall t \geq 2
\end{aligned}$$

Define:

$$\Delta_\varepsilon = \sum_{i=1}^{m} \lambda_i \sum_{t=0}^{\infty} \beta_i^t u^i(c_{i,t}, l_{i,t}) - \sum_{i=1}^{m} \lambda_i \sum_{t=0}^{\infty} \beta_i^t u^i(c_{i,t}^*, l_{i,t}^*).$$

The concavity of u and f implies that

$$\begin{aligned}
\Delta_\varepsilon &= u^j(c_{j,0}, l_{j,0}) - u^j(c_{j,0}^*, l_{j,0}^*) + \beta \left[u^j(c_{j,1}, l_{j,1}) - u^j(c_{j,1}^*, l_{j,1}^*) \right] \\
&\geq u_c^j(c_{j,0}, l_{j,0})(c_{j,0} - c_{j,0}^*) + \beta u_c^j(c_{j,1}, 1) c_{j,1} \\
&= -u_c^j(c_{j,0}, l_{j,0}) \varepsilon + \beta u_c^j(c_{j,1}, 1) f(\varepsilon, L_1) \\
&\geq \varepsilon \left[\beta u_c^j(c_{j,1}, 1)(1 - \delta) - u_c^j(c_{j,0}, l_{j,0}) \right].
\end{aligned}$$

As $\varepsilon \to 0$, $c_{j,1} \to 0$ and $u_c^j(c_{j,1}, 1) \to +\infty$, while $u_c(c_{j,0}, l_{j,0}) \to u_c(c_{j,0}^*, l_{j,0}^*) < +\infty$. Hence, for $\varepsilon > 0$ small enough, $\Delta_\varepsilon > 0$: a contradiction. It follows that $k_1^* > 0$.

Case 2: Assume that $l_{j,1}^* < 1$.

Consider the alternative feasible path $(\mathbf{c}, \mathbf{l}, \mathbf{k})$ described in case 1 with the only difference that $l_{j,1} \in (l_{j,1}^*, 1]$. Following the same argument we obtain that $k_1^* > 0$.

An induction argument proves the claim. ■

Proof of Lemma 7: Assume the contrary: $k_0 > 0$ and $\mathbf{k}^*$ is optimal with $k_t^* \to 0$. Observe that feasibility implies that $c_{i,t}^* \to 0$, $\forall i$. Since $f_k(0,1) > 1$ for k small enough, we have that $f(k, m) > k$. This implies that there exists a date T, such that, $0 < k_{t+1}^* < f(k_t^*, m)$, $\forall t \geq T$. We know that with any optimal solution $\mathbf{k}^*$ of problem (Q), there exist associated sequences $(\mathbf{c}_i^*, \mathbf{l}_i^*)_i$ for consumption and leisure, such that, $((\mathbf{c}_i^*, \mathbf{l}_i^*)_i, \mathbf{k}^*)$ is a solution to problem (P). For any $t \geq T$ there exists $i \in I$ such that $c_{i,t}^* > 0$, $l_{i,t}^* \in (0,1)$. Observe that $L_t^* = m - \sum_{i \in I} l_{i,t}^*$. The proof follows in two steps:

Step 1: We claim that the sequence $\left(\frac{k_t^*}{L_t^*} \right)$ converges to zero.

Let $\left(\frac{k^*_{t_n}}{L^*_{t_n}}\right)$ be a subsequence such that:

$$\limsup_t \frac{k^*_t}{L^*_t} = \lim_n \frac{k^*_{t_n}}{L^*_{t_n}}.$$

Without loss of generality assume that $0 < k^*_{t_n+1} < f(k^*_{t_n}, m)$, $\forall n$. One can find a sequence of consumers denoted by $(i_n)_n$, such that, $c^*_{i_n,t_n} > 0$, $l^*_{i_n,t_n} \in (0,1)$ and $\lim_n c^*_{i_n,t_n} = 0$, $\lim_n l^*_{i_n,t_n} = l \in [0,1]$.

Assume that $(i_n)_n$ is such that $\lim_n l^*_{i_n,t_n} = l \in [0,1)$. This implies that there exists some agent $i \in I$ such that $\lim l^*_{i,t_n} = l_i \in [0,1)$. In this case, $L^*_{t_n} \to L > 0$ which proves the claim.

Consider next the case where for any sequence $(i_n)_n$ we have $\lim_n l^*_{i_n,t_n} = 1$. Observe that in this case, $\lim_n l^*_{i,t_n} = 1$, $\forall i \in I$ and $L^*_{t_n} \to 0$.
The first order conditions for problem (P) imply that $\forall i \in I$, $\forall n$:

$$F_L\left(k^*_{t_n}, L^*_{t_n}\right) \leq \frac{u^i_l(c^*_{i,t_n}, l^*_{i,t_n})}{u^i_c(c^*_{i,t_n}, l^*_{i,t_n})}.$$

Define $\xi(x) = F(x,1) - xF_k(x,1)$. Since f is homogeneous of degree $a \in (0,1]$ we have:

$$\begin{aligned} F_L k^*_{t_n}, L^*_{t_n}) &= a(L^*_{t_n})^{a-1}\xi\left(\frac{k^*_{t_n}}{L^*_{t_n}}\right) \\ &\leq \frac{u^i_l(c^*_{i,t_n}, l^*_{i,t_n})}{u^i_c(c^*_{i,t_n}, l^*_{i,t_n})}. \end{aligned}$$

Taking the limits on both sides as $n \to +\infty$ we get:

$$\lim_n F_L\left(k^*_{t_n}, L^*_{t_n}\right) \leq \frac{u^i_l(0,1)}{u^i_c(0,1)} = 0.$$

For this to be true we must have:

$$\xi\left(\lim_n \frac{k^*_{t_n}}{L^*_{t_n}}\right) = 0.$$

Since ξ is increasing, there exists $M > 0$, such that, $\frac{k^*_{t_n}}{L^*_{t_n}} \leq M$, $\forall n$. If $\frac{k^*_{t_n}}{L^*_{t_n}} \to z > 0$, from the definition of ξ and the strict concavity of F it follows that $\xi(z) = F(z,1) - zF_k(z,1) > 0$: a contradiction. Therefore, $\lim_n \frac{k^*_{t_n}}{L^*_{t_n}} = 0$.

Step 2: Choose some $\varepsilon > 0$, such that, $f_k(\varepsilon, 1) > \frac{1}{\min_i \beta_i}$. Since $k^*_t \to 0$ and $\frac{k^*_t}{L^*_{t_n}} \to 0$, there exists some date T', such that, $k^*_t \leq \varepsilon$ and $\frac{k^*_t}{L^*_t} \leq \varepsilon$, $\forall t \geq T'$.

For any $t \geq T_1 = \max\{T, T'\}$ Euler's equations hold, i.e.

$$u^i_c(c^*_{i,t}, l^*_{i,t}) = \beta_i u^i_c(c^*_{i,t+1}, l^*_{i,t+1}) f_k\left(k^*_{t+1}, L^*_{t+1}\right)$$

Observe that $f_k\left(k_{t+1}^*, L_{t+1}^*\right) \geq f_k\left(\varepsilon, 1\right),\ t \geq T_1$. It follows that there exists $T_2 \geq T_1$ such that for any $t \geq T_2$:

$$\begin{aligned}
+\infty > u_c^i(c_{i,T_2}^*, l_{i,T_2}^*) &\geq u_c^i(c_{i,t}^*, l_{i,t}^*) \prod_{\tau=1}^{t-T_2} \left[\left(\min_i \beta_i\right) f_k\left(k_{t+\tau}^*, L_{t+\tau}^*\right)\right] \\
&\geq u_c^i(c_{i,t}^*, l_{i,t}^*) \prod_{\tau=1}^{t-T_2} \left[\left(\min_i \beta_i\right) f_k\left(\varepsilon, 1\right)\right] \\
&\geq A^{t-T_2} u_c^i(c_{i,t}^*, l_{i,t}^*).
\end{aligned}$$

with $A = (\min_i \beta_i) f_k(\varepsilon, 1) > 1$. Fix some $i \in I$. We distinguish two cases:

Case 1: Assume that $u_{cl}^i \leq 0$.

In this case, $\forall t > T_2$ we have:

$$+\infty > u_c^i(c_{i,T_2}^*, l_{i,T_2}^*) \geq A^{t-T_2} u_c^i(c_{i,t}^*, l_{i,t}^*) \geq A^{t-T_2} u_c^i(c_{i,t}^*, 1).$$

Since $\lim_t A^{t-T_2} = +\infty$, $c_{i,t}^* \to 0$ and u^i satisfies the Inada conditions, we obtain that $A^{t-T_2} u_c^i(c_{i,t}^*, 1) \to +\infty$: a contradiction.

Case 2: Assume that $u_{cl}^i > 0$.

In this case, $\forall t > T_2$ we have:

$$+\infty > u_c^i(c_{i,T_2}^*, l_{i,T_2}^*) \geq A^{t-T_2} u_c^i(c_{i,t}^*, l_{i,t}^*)$$

Since $\lim_t A^{t-T_2} = +\infty$, it follows that $u_c^i(c_{i,t}^*, l_{i,t}^*) \to 0$. Given that $c_{i,t}^* \to 0$ and u^i satisfies the Inada conditions, we have that $l_{i,t}^* \to 0$. Observe also that $\forall t > T_2$ we have:

$$\begin{aligned}
u_l^i(c_{i,t}^*, l_{i,t}^*) &= u_c^i(c_{i,t}^*, l_{i,t}^*) f_L(k_t^*, L_t^*) \\
&\leq u_c^i(c_{i,t}^*, l_{i,t}^*) f\left(\frac{k_t^*}{L_t^*}, 1\right) \\
&\leq u_c^i(c_{i,t}^*, l_{i,t}^*) f\left(\varepsilon, 1\right)
\end{aligned}$$

This implies that $u_l^i(c_{i,t}^*, l_{i,t}^*) \to 0$.

Since by assumption $u^i(0,0) = 0$, we have to distinguish three subcases.

1) Assume first that there exists $\widetilde{c} > 0$ such that $u^i(\widetilde{c}, 0) > 0$.

In this case, there exists $\underline{c} > 0$ such that $u_c^i(\underline{c}, 0) > 0$ (if not, then $u_c^i(c, 0) = 0$, $\forall c > 0$ and $u^i(\widetilde{c}, 0) > 0$: a contradiction). Since $c_{i,t}^* \to 0$, there exists $T_3 \geq T_2$, such that, $\forall t > T_3$ we have $c_{i,t}^* < \underline{c}$:

$$+\infty > u_c^i(c_{i,T_2}^*, l_{i,T_2}^*) \geq A^{t-T_3} u_c^i(c_{i,t}^*, 0) \geq A^{t-T_2} u_c^i(\underline{c}, 0)$$

Since $\lim_t A^{t-T_2} = +\infty$, taking the limits on both sides we obtain a contradiction.

2) Consider next the case where there exists $\widetilde{l} > 0$ such that $u^i(0, \widetilde{l}) > 0$.

A similar argument implies that there exists $\underline{l}$ such that $u_l^i(0,\underline{l}) > 0$. Since $l_{i,t}^* \to 0$, there exists $T_3 \geq T_2$ such that $\forall t > T_3$ we have $l_{i,t}^* < \underline{l}$ and

$$u_l^i(c_{i,t}^*, l_{i,t}^*) \geq u_l^i(0, l_{i,t}^*) \geq u_l^i(0,\underline{l}) > 0.$$

Taking the limits on both sides as $t \to +\infty$ we obtain a contradiction.

3) Consider finally the case where $u^i(c,0) = u^i(0,l) = 0$, $\forall c, \forall l$.

We know that $c_{i,t}^* \to 0$ and $l_{i,t}^* \to 0$. Observe that for any subsequence of $(c_{i,t}^*, l_{i,t}^*)_t$ such that $c_{i,t}^* < l_{i,t}^*$, assumption **U5** implies that $u_c^i(c_{i,t}^*, l_{i,t}^*) \geq u_c^i(l_{i,t}^*, l_{i,t}^*) \geq u_c^i(1,1) > 0$: a contradiction since we know that $u_c^i(l_{i,t}^*, l_{i,t}^*) \to 0$. In a similar way, for any subsequence of $(c_{i,t}^*, l_{i,t}^*)_t$ such that $c_{i,t}^* > l_{i,t}^*$, assumption **U5** implies that $u_l^i(c_{i,t}^*, l_{i,t}^*) \geq u_l^i(c_{i,t}^*, c_{i,t}^*) \geq u_l^i(A(k_0), A(k_0)) > 0$: a contradiction since we know that $u_l^i(l_{i,t}^*, l_{i,t}^*) \to 0$. ■

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