On Time Preference, Rational Addiction & Utility Satiation

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Abstract†

A basic consumer problem with a unique good is considered, current consumption of this good influencing in a positive manner consumer intertemporal utility whilst past consumption exerts a negative influence. Moreover, in the line of I. Fisher, a specification of preferences is retained so that the rate of time preference, assumes a long-run value — this means for a stationary consumption-path — that is non monotonic as a function of consumption : impatience increases for low level of consumptions but decreases for higher ones. Such a framework allows for an integrated appraisal of addiction, satiation and the rate of time preference. It is shown that the emergence of an addiction phenomenon in the neighbourhood of an unsatiated long-run position exactly corresponds to letting the rate of time preference be an increasing function of past consumption habits. When addiction becomes sufficiently strong, the unsatiated stationary state becomes unstable and the satiated steady state becomes the only admissible stationary position.

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I – Introduction

In the spirit of Koopmans [11] celebrated contribution, several works have been attempting at understanding intertemporal choices through various assumptions on preferences. Mention should be made in that perspective of the contributions of Epstein & Hynes [8], Shi & Epstein [13], Ryder & Heal [12]. Some authors, *e.g*, Becker & Murphy [2] and Iannaccone [10] have parallely tried to analyse concrete behaviours such as addiction phenomena in the light of utility theory.

The current contribution belongs to both of these bodies of literature with a special emphasis put on the rate of time preference. Along Ryder & Heal [12], Shi & Epstein [13] and Becker & Murphy [2] , it shall be assumed that consumption habits explicitly enter in intertemporal preferences. A basic consumer problem with a unique good is considered, current consumption of this good influencing in a positive manner consumer intertemporal utility whilst past consumption exerts a negative influence. Moreover, in the line of Fisher [9], a specification of preferences is retained so that the rate of time preference assumes a long-run value — for a stationary consumption-path — that is non monotonic as a function of consumption : impatience increases for low level of consumptions but decreases for higher ones.

Such a framework will allow for an inegrated approach of addiction, satiation phenomena, the properties of the rate of time preference and the dynamics of consumption. It is first recalled that satiation describes a long-run stationary level of consumption that maximises intertemporal utility with an unbinding budgetary constraint. Along the contribution of Ryder & Heal [12], satiation emerges as a natural phenomenon in the current model where a rise in contemporary consumption increases intertemporal utility whilst a rise in past consumption has opposite effects. As for addiction phenomena, the exposition shall borrow a definition due to Becker & Murphy [2] : there is *addiction* when past consumption positively influences current consumption levels. More precisely, considering the optimal policy rule for consumption, it depends positively on the inherited level of consumption habits.

The current model then allows for reaching the following list of results. First, the existence of two long-run stationary states is proved, one being interior and unsatiated whilst the outstanding one is satiated. Between these two long-run positions may also exist a third steady state that however happens to be overall unstable. It is then established that the satiated stationary state is always a saddle point whilst the unsatiated interior one can experience various dynamical properties that are directly related to the features of the rate of time preference. In that perspective, it is shown that the emergence of an addiction phenomenon in the neighbourhood of this unsatiated long-run position exactly corresponds to letting the rate of time preference be an increasing function of past consumption habits : the more agents have been consuming in the past and the less patient they will happen to be. Incidently, it is readily checked that this condition remains true for all the previous frameworks of the literature where consumption habits was associated with the occurrence of satiation, namely Becker & Murphy [2] and Shi & Epstein [13].

The emergence of an addiction phenomenon can first lead to a cyclical convergence towards the unsatiated steady state. Furthermore and if addiction happens to be sufficiently strong, this unsatiated interior steady state becomes locally unstable, the precise condition for such an occurrence being that the elasticity of the rate of time preference with respect to past consumption habits is sufficiently high when compared to the inverse of the intertemporal elasticity of substitution in consumption. Such a property is refered to as *strong addiction*. When this latter phenomenon emerges, the unsatiated stationary state becoming unstable, the satiated steady state becomes the only admissible stationary position.

Though the occurrence of strong addiction seems to increase the likelihood of convergence towards a long-run satiated position, the current analysis did not reach a characterisation of the global dynamical properties of the environment under study. It is then unable to state that the satiated stationary state is the only conceivable long-run equilibrium. Indeed, other long-run non stationary equilibria such as cycles may simultaneously exist.

A detailed analysis of the dynamical properties around the unsatiated steady state finally shows that there does exist a selection of parameters of the current framework such that any of the aforementioned configurations, be it addiction, strong addiction or cyclical trajectories, can actually occur.

One may however wonder about the robustness of this line of results obtained through a partial equilibrium approach. The consideration of a productive sector allows to remedy to this line of criticism by endogenising the income of the individuals as well as the rate of interest on the capital market. Within a general equilibrium setting, it is hence established that the whole earlier line on conclusions can be recovered, though they now involve somewhat more stringent conditions. An endogenous determination of the rate of interest renders more difficult the emergence of an unstable unsatiated steady state and the concavity of the production technology exerts a stabilising effect on this long-run unsatiated position. From the sole qualitative standpoint, it should nonetheless be noticed that remains actual scope for any of the configurations reached through a partial equilibrium approach.

The current approach and the associated results articulate as follows with the previous literature. Most of the contributions that have hitherto been concerned with rational addiction refer to *complementarity* between consumptions in preferences. Both Ryder & Heal [12] and Becker & Murphy [2] indeed make their analysis rest upon *adjacent complementarity* whilst Shi & Epstein [13] introduce another understanding of complementarity in intertemporal preferences : complementarity is said to *predominate* between present and future consumptions as soon as the sum of the derivatives of the rate of time preference with respect to future consumptions happens to be positive. As an alternative, the current argument will be organised around the properties of the rate of time preference, *addiction* being understood as a positive dependency with respect to past consumption habits. Within the frameworks used by Ryder & Heal [12] and Becker & Murphy [2], it is readily checked that this characterisation of addiction based upon the rate of time preference would entail a condition similar to the one associated with *adjacent complementarity*. In contradistinction with this, their *adjacent complementarity* property would not provide a suitable characterisation of addiction in the current environment. In the same vein, such an addiction criterion based upon the rate of time preference could have been used in Shi & Epstein [13]. In opposition to this, the aforementioned predominant complementarity notion that is used by these authors reveals as being inappropriate for appraising the current environment : more explicitly, an addiction phenomenon emerges for preferences which would not be complement according to Shi & Epstein [13]. It is hoped that the present study may in this sense have succeeded in introducing a convincing criterion for characterising addiction phenomena in intertemporal preferences.

Another presumably interesting dimension of the current contribution relates to the integrated view of addiction and satiation that it provides. When, as a result of strong addiction, the unsatiated interior steady state happens to become unstable, the satiated steady state emerges as the unique stationary long-run position. In economic terms, this indicates that the occurrence of addiction in preferences may well lead to long-run consumption behaviours that are entirely disconnected from budget constraints. Such a conjunction was oppositely not noticed by earlier contributions of the literature, be it Becker & Murphy [2] or Shi & Epstein [13]. Finally, the current characterisation completes a thorough picture of local dynamics that makes an explicit account of the existence of convergent as well as of divergent cyclical dynamics, their occurrence being further analysed through the properties of the rate of time preference.
More generally, the model under consideration introduces a theory of consumption that leads to two distinct types of behaviours. More explicitly, an individual would consume his whole permanent income if he was to undertake his consumption choice in a close neighbourhood of the unsatiated steady state. In opposition with this somewhat standard result, his consumption behaviour would change as soon as he completes his decision in the neighbourhood of the satiated state : his current consumption becomes unrelated to his permanent income but oppositely fully determined by his past consumption choices. Interestingly, this latter configuration is reminiscent of the alternative approaches raised by J. Duesenberry and T. Brown half a century ago. It is indeed worth recalling that whilst Duesenberry advocated by 1948 a theory where current consumption was determined by a benchmark level of income, namely the maximal one reached by the individual in his lifetime, Brown raised by 1951 an approach where past consumption levels emerge as the main determinant of current consumption behaviours.Though both of those formulations can be compared to numerous features of the current theory of consumption, the latter strongly differs by being based upon fully rational decisions, his main features being further directly understood from the ordinal features of the rate of time preference that was omitted by these early theories.

Section II examines the characteristics of preferences and characterises the features of the rates of time preference. Section III introduces the optimal dynamics of consumption. Section IV provides a characterisation of local dynamics. Section V completes an integrated approach of time preference, addiction and satiation. Section VI is devoted to the characterisation of a general equilibrium framework with an explicit formulated production sector and capital. All proofs are gathered in a final appendix.

II – Inherited Tastes and Time Discounting

II.1 – Preferences

Time is continuous. Let ${}_0\mathcal{C} \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ denote the consumption-path that assigns a value of $c(t)$ at date $t \in \mathbb{R}_+$. Let in turn $z(t)$ denote the time-t value of *consumption habits* acquired by the consumer. It satisfies $\dot{z}(t) = \sigma[c(t) - z(t)]$ for a given $z(0) = z_0$. Being alternatively defined as

$$(1) \qquad z(t) = \sigma \int_0^t \exp\left[\sigma(s-t)\right] c(s)ds + z_0 \exp(-\sigma t),$$

it thus features an average of past consumptions weighted by exponentially decreasing coefficients. It is assumed that, for a given $z(0) = z_0 \geq 0$, $z(t)$ exerts a direct influence on the intertemporal preferences of the representative individual:

$$(2) \qquad U\left({}_0\mathcal{C}; z_0\right) = -\int_{t=0}^{+\infty} \exp\left(-\int_{s=0}^{t} \Theta[c(s), z(s)]ds\right)dt,$$

where it is noted that $U\left({}_0\mathcal{C}; z_0\right)$ is defined on $\mathbb{R}_-$.

This formulation of the intertemporal utility function does not allow for clearly distinguishing between an instantaneous utility function and a discount function. This is a direct outcome of the relaxation of the standard separability assumption. In spite of its widespread use in the literature, and as this was shown by Koopmans [11], the latter does not rest on any firmly established axiomatical foundations.[1]

Remark 1 : A more general specification for the intertemporal utility function would be:

$$U\left({}_0\mathcal{C}; z_0\right) = -\int_{t=0}^{+\infty} u[c(s), z(s)] \exp\left(-\int_{s=0}^{t} \Theta[c(s), z(s)]ds\right)dt,$$

the formulation used by Ryder & Heal [12], Boyer [5] and Becker & Murphy [2] being then recovered when $\Theta(\cdot,\cdot)$ is specialised to a constant value. Shi & Epstein [13], as for themselves, focus on an environment where $u(\cdot,\cdot)$ is univocally determined by c while $\Theta(c,z) = \beta z$, for β a constant. Finally, Epstein & Hynes [8] introduce a formulation that is a special case of (2) when the intertemporal utility function does not explicitly depend upon past consumption habits. ◊

Assumption 1 : *$\Theta \in C^2(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$ and is strictly concave. Further, $\Theta'_c(c,z) > 0$ and $\Theta'_z(c,z) < 0$ for any $c, z > 0$.*

Assumption 1 postulates a negative influence of past consumption levels on intertemporal utility. Such a property makes sense for goods such as cigarettes, alchool and drugs but a

[1] As a matter of fact, even with a separably additive representation, the instantaneous utility function does not have any ordinal meaning. Solely notions such as the ones used in the subsequent arguments, e.g., the rate of time preference or the marginal rate of substitution, make sense from an ordinal standpoint.

positive sign would admittedly be relevant for other addictive goods such as music or cultural goods, a thorough discussion of this topic being completed in Becker & Stigler [3]. While Ryder & Heal [12] assumed a negative influence of past consumption habits, the opposite case is analysed in Boyer [5]. As this shall soon appears, the current analysis being concerned with the possibility of satiation, this phenomenon can only appear through a negative dependence.

Remark 2 : A simple parametric formulation for $\Theta(\cdot,\cdot)$ can be introduced as :

$$\Theta(c,z) = C_0 + Ac^{\alpha} - B(z+b)^{\beta},$$

where C_0, A, B and b denote positive constants, $0<\alpha<1$ and $\beta>1$, b being further such that $C_0 - Bb^{\beta} > 0$. Its properties are carefully examined in the first part of Appendix 14. ◇

Lemma 1. *Under Assumption* 1, $U(\cdot; z_0)$ *is strictly concave.*

Proof : *Vide* Appendix 1. △

Letting ${}_t\mathcal{C}$ denote the tail of the consumption-path that starts at instant $t \geq 0$, consider a variable $\xi(t) := U({}_t\mathcal{C}; z_t)$ that features an intertemporal utility associated with a tail of ${}_t\mathcal{C}$. A new definition follows for $U({}_0\mathcal{C}; z_0)$:

$$U({}_0\mathcal{C}; z_0) = \xi(0) \text{ such that}$$
$$\dot{\xi}(t) = 1 + \Theta[c(t), z(t)]\xi(t),$$
$$(2') \quad \lim_{t\to+\infty} \xi(t)\exp\left(-\int_{s=0}^{t} \Theta\left[c(s), z(s)\right] ds\right) = 0,$$
$$\dot{z}(t) = \sigma[c(t) - z(t)],$$
$$z(0) = z_0 \geq 0 \text{ given.}$$

Lemma 2. *Under Assumption* 1, *consider the intertemporal utility function* $U({}_t\mathcal{C}; z_t)$:

(i) *its dependency with respect to* z_t *is featured by* $\partial U(\mathcal{C}; z_t)/\partial z_t = -\beta(t)$, *for :*

$$(3) \quad \beta(t) = -\int_{x=t}^{+\infty} \exp\left(-\int_{s=t}^{x} \Theta\left[c(s), z(s)\right] ds\right) \times \left[\int_{s=t}^{x} \Theta'_z\left[c(s), z(s)\right] \exp[-\sigma(s-t)] ds\right] dx$$
$$= \int_{x=t}^{+\infty} \exp\left\{-\int_{s=t}^{x} \Theta[c(s), z(s)] ds\right\} \times U({}_x\mathcal{C}, z_x) \times \Theta'_z[c(x), z(x)] \exp[-\sigma(x-t)] dx$$
$$> 0;$$

where $\beta(t)$ *can be understood as minus the present value of the shadow price associated to consumption habits* z_t *at date* $t \geq 0$ *that satisfies the differential equation* $\dot{\beta}(t) = \beta(t)\{\Theta[c(t), z(t)] + \sigma\} - \Theta'_z[c(t), z(t)]\xi(t)$;

(ii) *the Volterra derivative*[2] *at a given date* $\tau \geq t$ *states as :*

[2] *Vide* Ryder & Heal [12] for an intuitive exposition and Volterra [14] for a careful argument.

$$(4)\qquad U'\left({}_0\mathcal{C};z_0;\tau\right) = -\exp\left(-\int_{s=0}^{\tau}\Theta\left[c(s),z(s)\right]ds\right) \times\left\{\Theta'_c\left[c(\tau),z(\tau)\right]U\left({}_\tau C,z_\tau\right)+\sigma\beta(\tau)\right\},$$

(iii) *along an invariant consumption-path for which $c = z$, the stationary values of β and U' state as :*

$$(5a)\qquad \beta = -\frac{\Theta'_z(c,c)}{\Theta(c,c)[\Theta(c,c)+\sigma]};$$

$$(5b)\qquad U'({}_0\mathcal{C};c_0;\tau) = \frac{\exp[-\Theta(c,c)\tau]}{\Theta(c,c)[\Theta(c,c)+\sigma]}\Big\{\Theta'_c(c,c)[\Theta(c,c)+\sigma]+\sigma\Theta'_z(c,c)\Big\}.$$

Proof : *Vide* Appendix 2. △

From Equation (5b), as $\Theta'_z < 0$, the occurrence of $U'(\mathcal{C};\tau) = 0$, *i.e.*, *utility satiation* for a given $\tau \geq 0$, cannot be discarded. A stationary level of c that solves $\Theta'_c(c,c)\,[\Theta(c,c)+\sigma] + \sigma\Theta'_z(c,c) = 0$ would thus be associated with utility satiation.

II.2 – Time Preference & Intertemporal Elasticity of Substitution.

A measure of the impatience properties inherent to $U(\cdot;z_0)$ is then provided by the introduction of the rate of time preference of the individual, the latter being defined as *minus* the rate of growth of marginal utility along a locally constant consumption-path. Let then ϱ feature the rate of time preference :

$$\begin{aligned}\varrho\left[{}_\tau\mathcal{C};z(\tau)\right] &= -\frac{d}{d\tau}\ln U'(\mathcal{C};z_0;\tau)|_{\dot{c}=0}\\ &= \Theta - \frac{\Theta''_{cz}\sigma(c-z)U\left({}_\tau C\right)+\Theta'_c(1+\Theta U\left({}_\tau C\right))+\sigma\left[\beta(\Theta+\sigma)-\Theta'_z U\left({}_\tau C\right)\right]}{\Theta'_c U\left({}_\tau C\right)+\sigma\beta},\\ &= \Theta - \frac{\Theta''_{cz}\sigma(c-z)\xi+\Theta'_c(1+\Theta\xi)+\sigma\left[\beta(\Theta+\sigma)-\Theta'_z\xi\right]}{\Theta'_c\xi+\sigma\beta},\end{aligned}$$

where it is checked that the long-run expression of the rate of time preference, available for $c = z$, $\Theta = -1/\xi$, simplifies to Θ. As this was already pointed out by Epstein & Hynes [8], this is only part of a more general purpose relationship at a given date τ :

$$\begin{aligned}&-\frac{d}{d\tau}\ln U'\left({}_0\mathcal{C};z_0;\tau\right) = \varrho\left[{}_\tau\mathcal{C},z(\tau)\right]+\frac{\dot{c}(\tau)}{c(\tau)}\frac{1}{\Sigma(\tau)},\\ &\text{with}\quad \frac{1}{\Sigma(\tau)} = -\frac{\Theta''_{cc}c}{\Theta'_c+\sigma\beta/\xi},\end{aligned}$$

for $\Sigma(\tau)$ that features the intertemporal elasticity of substitution in consumption.
For future reference, a benchmark characterisation of the properties of time preference will be useful :

Lemma 3. *Along a constant consumption-path, the rate of time preference exhibits the following properties :*

(i) *the derivative with respect to* $z(\tau)$ *is available as :*

$$\frac{\partial\varrho\big({}_\tau\mathcal{C}:z(\tau)\big)}{\partial z(\tau)} = \left\{\frac{\Theta\Theta'_z}{\Theta+\sigma}\left(\Theta'_c + \frac{\Theta'_z}{\Theta+2\sigma}\right) + \frac{\sigma}{\Theta+2\sigma}\left[(\Theta+2\sigma)\Theta''_{cz} + \sigma\Theta''_{zz}\right]\right\} \Big/ \left\{\Theta'_c + \frac{\sigma\Theta'_z}{\Theta+\sigma}\right\},$$

(ii) *the Volterra derivative at a given date* $x \geq \tau$ *is given by :*

$$\frac{\partial\varrho\big({}_\tau\mathcal{C},z(\tau)\big)}{\partial c(x)} = \exp[-\Theta(x-\tau)]\left\{\Theta\big(\Theta'_c + \Theta'_z\big) - \exp[-\sigma(x-\tau)](\Theta+\sigma)\frac{\partial\varrho\big({}_\tau\mathcal{C}:z(\tau)\big)}{\partial z(\tau)}\right\},$$

Proof : *Vide* Appendix 3. △

As a final remark and for future use in the next section, when considered in the course of a dynamical system, this rate of time preference $\varrho\big({}_\tau\mathcal{C},z(\tau)\big)$ will be denoted, in order to avoid any confusion in notations, $\rho\big[c(\tau),\beta(\tau),z(\tau),\xi(\tau)\big]$.

III – On the Dynamics of Consumptions

III.1 – A Consumer Problem

Letting $w(t)$ denote the current revenue of the consumer, it is assumed that he can borrow or lend on a perfect credit market on which prevails a rate of interest of $r(t)$. The level of savings of the consumer is denoted as $a(t)$ and its law of motion is given by:

(6) $\dot{a}(t) = r(t)a(t) + w(t) - c(t),$

with a solvability condition:

(7) $\lim_{t\to\infty} a(t)\exp\left[-\int_{s=0}^{+\infty} r(s)d(s)\right] = 0.$

The consumer problem then formulates along:

(8) Maximise $\xi(0)$ s.t. $(2'), (6), (7),$ for $a(0) = a_0$.

The existence of an optimal solution to recursive programs has first been dealt with by Becker, Boyd & Sung [4] and was later on generalised by Balder [1]. Unfortunately, their class of results does not ecompass the current problem. It shall however be assumed that such a solution exists.
Problem (8) being of the Maier type, it can be solved with the definition of the following Hamiltonian:

(9) $\mathcal{H} = \lambda(ra+w-c) + \mu\left[1+\Theta(c,z)\xi\right] + \nu\sigma(c-z),$

for λ,μ and ν the shadow prices respectively associated to a,ξ and z.

III.2 – Interior Solutions

This configuration being associated with the holding of $\lambda > 0$, the system of first-order conditions leads, for $\beta := \nu/\mu$, to the following autonomous 5-dimensional dynamical system:

$$\begin{aligned} -\Theta''_{cc}(c,z)\xi\dot{c} &= [\beta\sigma + \Theta'_c(c,z)\xi]\,[r - \rho(c,\beta,z,\xi)], \\ \dot{\beta} &= \beta[\Theta(c,z) + \sigma] - \Theta'_z(c,z)\xi, \\ \dot{z} &= \sigma(c - z), \\ \dot{\xi} &= 1 + \Theta(c,z)\xi, \\ \dot{a} &= ra + w - c, \end{aligned} \tag{10}$$

time arguments having been ommitted. Letting w and r assume constant values, an Unsatiated Interior Steady State is then available as a 5-uple $(\bar{c}, \overline{\beta}, \overline{z}, \overline{\xi}, \overline{a})$ that satisfies:

$$\begin{aligned} \Theta\left(\overline{c},\overline{c}\right) &= r, \\ \overline{\xi} &= -1/r, \\ \overline{c} &= \overline{z}, \\ \overline{\beta} &= -\frac{\Theta'_z\left(\overline{c},\overline{c}\right)}{\Theta\left(\overline{c},\overline{c}\right)\left[\Theta\left(\overline{c},\overline{c}\right)+\sigma\right]}, \\ \overline{a} &= \left(\overline{c} - w\right)/r. \end{aligned} \tag{11}$$

III.3 – Satiated Solutions

When $\lambda = 0$, the dimension of the dynamical system shrinks to 4:

$$\begin{aligned} -\Theta''_{cc}(c,z)\xi\dot{c} &= -\xi\left\{\Theta'_c(c,z)\left[\Theta(c,z)+\sigma\right] + \Theta'_z(c,z)\sigma\right\} \\ &\qquad + \Theta'_c(c,z)\left[1 + \Theta(c,z)\xi\right] + \Theta''_{cz}(c,z)\xi\sigma(c-z), \\ \dot{z} &= \sigma(c - z), \\ \dot{\xi} &= 1 + \Theta(c,z)\xi, \\ \dot{a} &= ra + w - c. \end{aligned} \tag{12}$$

Letting w and r assume constant values, a Satiated Steady State is similarly available as a 3-uple $(\tilde{c}, \tilde{z}, \tilde{\xi})$ that solves:

$$\begin{aligned} \Theta'_c\left(\tilde{c},\tilde{c}\right)\left[\sigma + \Theta\left(\tilde{c},\tilde{c}\right)\right] + \sigma\Theta'_z\left(\tilde{c},\tilde{c}\right) &= 0, \\ \tilde{c} &= \tilde{z}, \\ \tilde{\xi} &= -1/\Theta\left(\tilde{c},\tilde{c}\right). \end{aligned} \tag{13}$$

III.4 – Existence of Steady States

The following assumption shall further restrict the set of admissible configurations:

Assumption 2: Consider the function $c \mapsto \Theta(c,c)$. It satisfies:

(i) it is a unimodal function that reaches its maximum at c^*, $\Theta(0,0) > 0$ and $\exists\, c_{\text{Max}} > 0$ such that $\Theta(c_{\text{Max}}, c_{\text{Max}}) = 0$;

(ii) there exist two values $\overline{c}_1$ and $\overline{c}_2$ such that $0 < \overline{c}_1 < c^* < \overline{c}_2 < c_{\text{Max}}$ and $\Theta(\overline{c}_1, \overline{c}_1) = \Theta(\overline{c}_2, \overline{c}_2) = r$;

(iii) there exists a unique $\tilde{c} < c_{\text{Max}}$ that is defined from $\Omega(\tilde{c}) := \Theta'_c(\tilde{c},\tilde{c})\left[\sigma + \Theta(\tilde{c},\tilde{c})\right] + \sigma\Theta'_z(\tilde{c},\tilde{c}) = 0$; moreover $\tilde{c}$ satisfies $\Omega'(\tilde{c}) = [\sigma+\Theta(\tilde{c},\tilde{c})]\Theta''_{cc}(\tilde{c},\tilde{c}) + [2\sigma+\Theta(\tilde{c},\tilde{c})]\Theta''_{cz}(\tilde{c},\tilde{c}) + \sigma\Theta''_{zz}(\tilde{c},\tilde{c}) + \Theta'_c(\tilde{c},\tilde{c})\left[\Theta'_c(\tilde{c},\tilde{c}) + \Theta'_z(\tilde{c},\tilde{c})\right] < 0$.

Remark 2 (Continued) : The nature of the restrictions that are to be superimposed on a basic parametric formulation, namely $\Theta(c,z) = C_0 + Ac^\alpha - B(z+b)^\beta$, to fit with Assumption 2 are detailed in the first part of Appendix 14. ♢

Proposition 1. *Under Assumptions* 1 *and* 2*:*

(i) *there exists a pair of unsatiated steady states, namely* $\overline{c}_1$ *and* $\overline{c}_2$*, together with a satiated steady state, namely* $\tilde{c}$*:*

(ii) *letting* $\widetilde{r} := \Theta(\tilde{c},\tilde{c})$*:*

  a/ *if* $r > \widetilde{r}$*, the unsatiated and satiated steady states rank according to* $\overline{c}_1 < c^* < \overline{c}_2 < \widetilde{c}$*;*

  b/ *if* $r < \tilde{r}$*, the unsatiated and satiated steady states satisfy* $\overline{c}_1 < c^* < \tilde{c} < \overline{c}_2$*.*

Proof : *Vide* Appendix 4. △

— Please insert Figure 1 —

Remark 3 : As $\Omega(c^*) = \Theta(c^*,c^*)\Theta'_c(c^*,c^*) > 0$ and $\Omega(\tilde{c}) = 0$, the satiated steady state is to display $\Omega'(\tilde{c}) \leq 0$. The extra holding of $\Omega'(\tilde{c}) < 0$ has then been retained through Assumption 2(iii) in order to discard any scope for the limit case $\Omega'(\tilde{c}) = 0$. Besides, for $r < \tilde{r}$, the steady state $\overline{c}_2$ is not admissible in that it corresponds to a steady state value for intertemporal utility that is strictly lower than the one associated to the satiated steady state value $\widetilde{c}$. ♢

To sum up and building upon Assumption 2, Proposition 1 finally allows for stating that there exists a unique satiated steady state and one or two unsatiated interior steady states.

IV – A Benchmark Characterization of Local Dynamics Around the Steady States

IV.1 – Appraising Dynamics in the Neighbourhood of Unsatiated Steady States

Lemma 4. *Consider an unsatiated steady state* $(\overline{c},\overline{\beta},\overline{z},\overline{\xi},\overline{a})$*. The eigenvalues associated to the unsatiated dynamical system in the neighbourhood of* $(\overline{c},\overline{\beta},\overline{z},\overline{\xi},\overline{a})$ *are* $r, \lambda_1, r-\lambda_1, \lambda_2, r-\lambda_2$*, for, letting* $\mu_i := \lambda_i(r-\lambda_i), i = 1,2$*,*

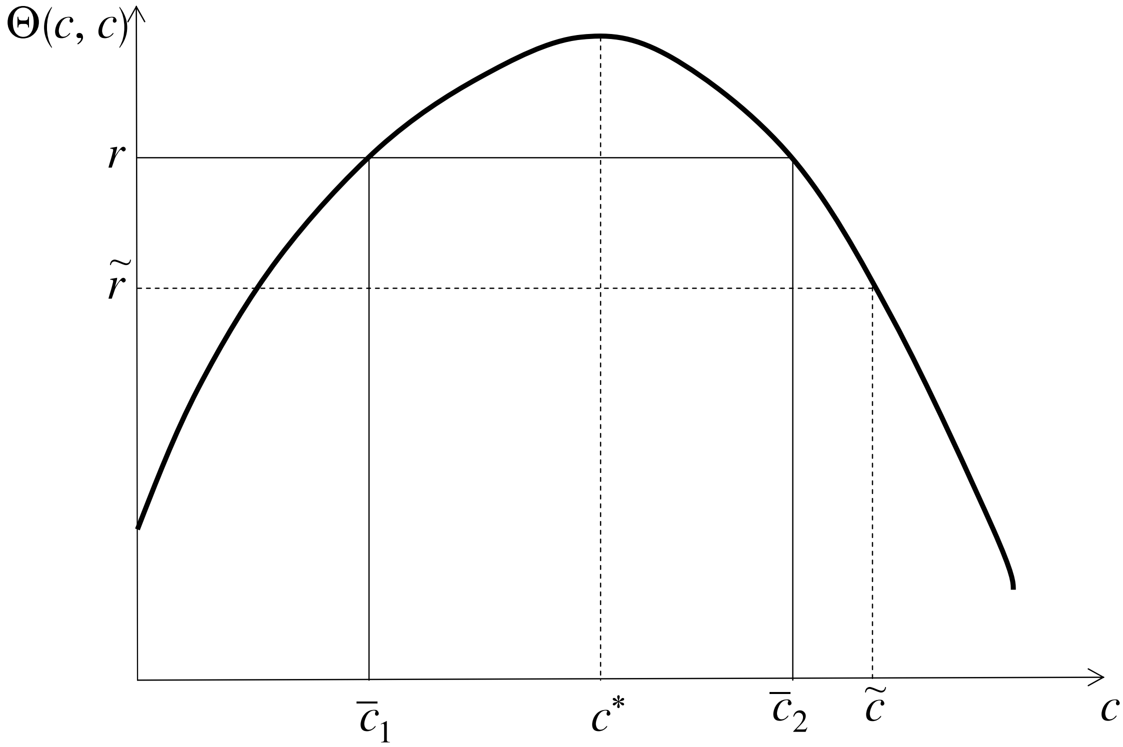

Figure 1.a

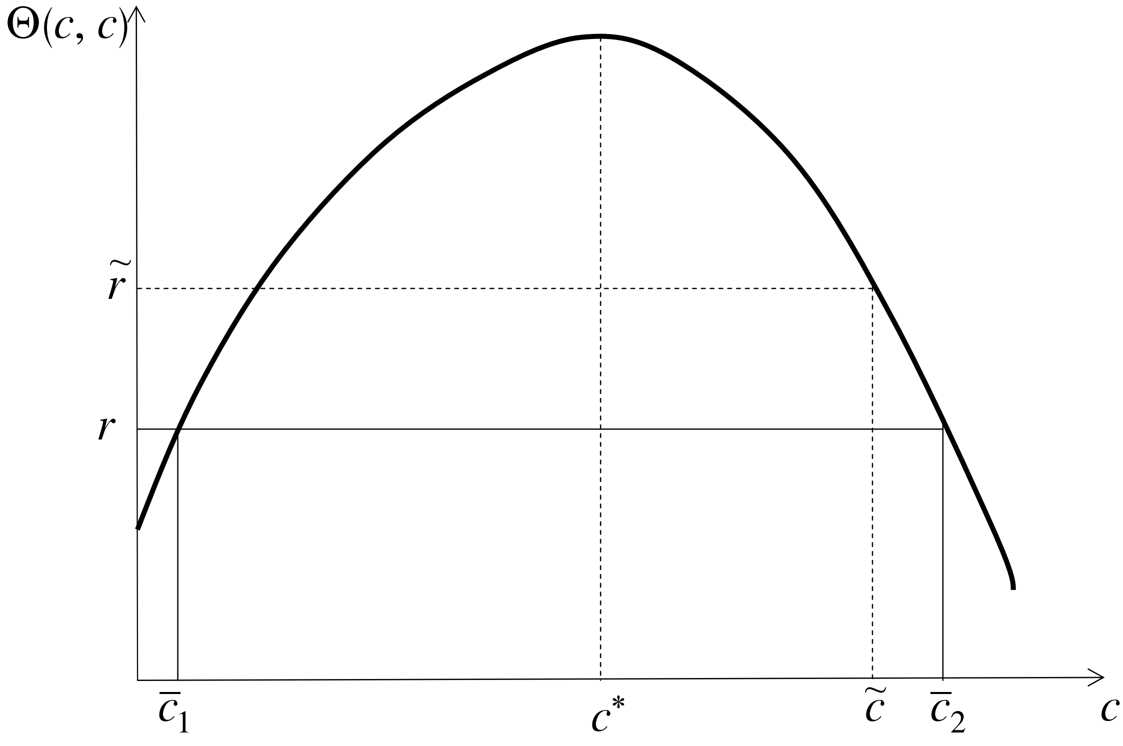

Figure 1.b

$$(14)\quad \mathcal{S} = \mu_1 + \mu_2 = -\frac{\sigma}{\Theta''_{cc}(\overline{c},\overline{c})}\Big[(r+\sigma)\Theta''_{cc}(\overline{c},\overline{c}) + (r+2\sigma)\Theta''_{cz}(\overline{c},\overline{c}) + \sigma\Theta''_{zz}(\overline{c},\overline{c})\Big] + \frac{r\left[\Theta'_c(\overline{c},\overline{c})\right]^2}{\Theta''_{cc}(\overline{c},\overline{c})},$$

$$(15)\quad \mathcal{P} = \mu_1\mu_2 = -\frac{\sigma r}{\Theta''_{cc}(\overline{c},\overline{c})}\left[\Theta'_c(\overline{c},\overline{c}) + \Theta'_z(\overline{c},\overline{c})\right]\left[(r+\sigma)\Theta'_c(\overline{c},\overline{c}) + \sigma\Theta'_z(\overline{c},\overline{c})\right].$$

Proof : *Vide* Appendix 5. △

It is worth recalling that the five dimensional dynamical system (10) includes two backward-looking variables, namely z and a, and three forward-looking ones, namely c, β and ξ. The obtention of a saddlepoint property would hence correspond to the occurrence of two negative eigenvalues — or with a negative real part in the complex case — and three positive eigenvalues — or with a negative real part in the complex case. A detailed appraisal of the stability issue is available in the following statement :

Proposition 2. *Under Assumptions* 1 *and* 2*:*

(i) *the high-level unsatiated steady state* $(\overline{c}_2, \overline{\beta}_2, \overline{z}_2, \overline{\xi}_2, \overline{a}_2)$ *is locally unstable (four eigenvalues with positive real parts and one eigenvalue with a negative real part — Area* I *on Figure* 2 *—);*

(ii) *the low-level unsatiated steady state* $(\overline{c}_1, \overline{\beta}_1, \overline{z}_1, \overline{\xi}_1, \overline{a}_1)$ *can fulfill three distinct configurations:*

a/ *it is a saddlepoint equilibrium with three eigenvalues with positive real parts and two eigenvalues with negative real parts and belongs to Area* II *on Figure* 2 *; Area* II *has two distinct subzonas* $\mathcal{R}_7$ *and* $\mathcal{R}_1$ *on Figure* 3 *: within* $\mathcal{R}_7$*, all of the five eigenvalues are real while in* $\mathcal{R}_1$*, four eigenvalues are complex and only one is real ;*

b/ *it is locally unstable and assumes five eigenvalues with positive real parts ; this corresponds to the Area* III *on Figure* 2 *that encompasses four distinct subzonas* $\mathcal{R}_2$, $\mathcal{R}'_2$, $\mathcal{R}_3$ *and* $\mathcal{R}_4$ *on Figure* 3 *: within* $\mathcal{R}_2$ *and* $\mathcal{R}'_2$*, four eigenvalues are complex while in* $\mathcal{R}_3$*, all of the eigenvalues are real and in* $\mathcal{R}_4$*, two eigenvalues are complex and three eigenvalues are real ;*

c/ *three eigenvalues with positive real parts and two purely imaginary complex eigenvalues— locus* IV *on Figure* 2.

Proof : *Vide* Appendix 6. △

— Please insert Figures 2 and 3 —

Figure 2 portrays the various configurations embedded in the statement of Proposition 2. Area I is hence related to the steady state $(\overline{c}_2, \overline{\beta}_2, \overline{z}_2, \overline{\xi}_2, \overline{a}_2)$ for which $\mu_1\mu_2 < 0$. Zonas II, III and IV are associated with the outstanding steady state $(\overline{c}_1, \overline{\beta}_1, \overline{z}_1, \overline{\xi}_1, \overline{a}_1)$ for which $\mu_1\mu_2 \geq 0$, the locus IV being more specifically associated with the occurrence of a Poincaré-Hopf bifurcation and the emergence of a unique pair of purely imaginary complex eigenvalues. A transition

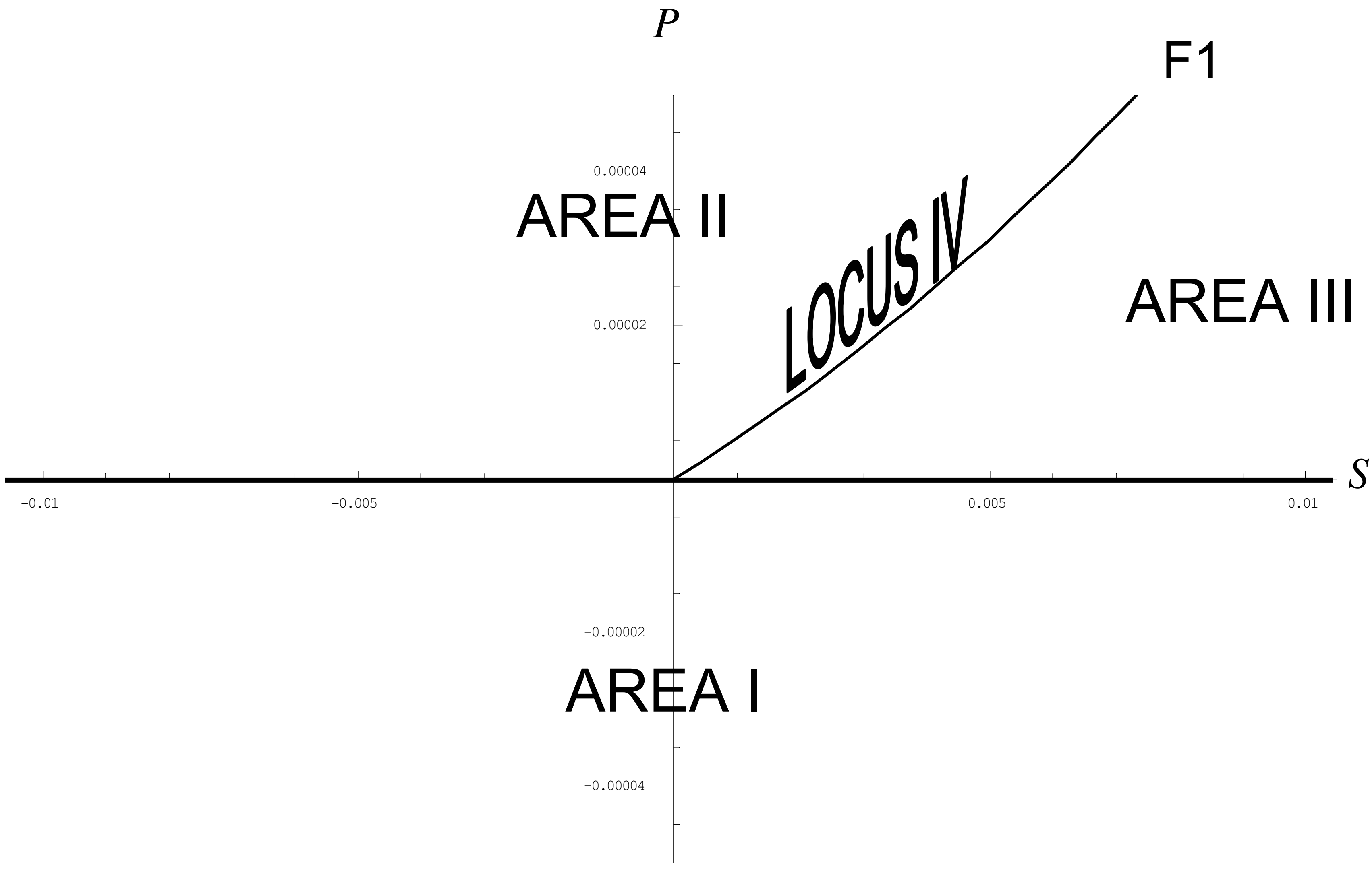
P
F1
0.00004
AREA II
LOCUS IV
AREA III
0.00002
S
-0.01
-0.005
0.005
0.01
-0.00002
AREA I
-0.00004

Figure 2

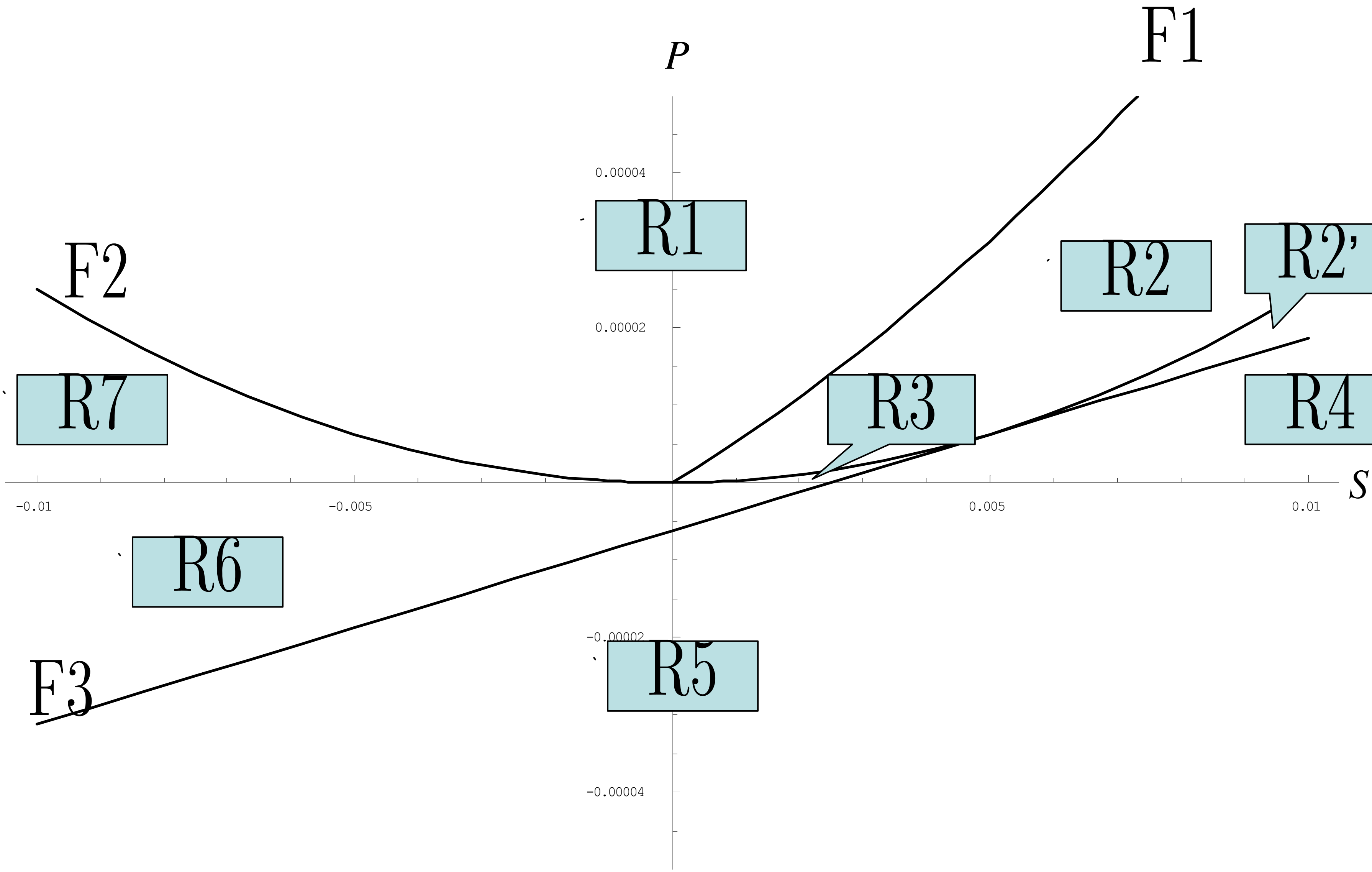
F1
P
0.00004
R1
R2
R2'
F2
0.00002
R7
R3
R4
-0.01
-0.005
0.005
0.01
S
R6
-0.00002
R5
F3
-0.00004

Figure 3

from zona II to zona III would thus correspond to a change in the stability properties of the dynamical system.

From Proposition 2, the stability properties of $(\overline{c}_1, \overline{\beta}_1, \overline{z}_1, \overline{\xi}_1, \overline{a}_1)$ are intimately linked to the sign of $\mu_1 + \mu_2$. As long as this latter expression remains of negative sign, the saddlepoint property is obtained. In opposition to this, a positive sign for $\mu_1 + \mu_2$ is a necessary condition for unstability.

IV.2 – A Characterization of Equilibrium Dynamics in the Neighbourhood of Satiated Steady States

The dynamical system associated with consumption satiation expresses as an autonomous subsystem defined in the space (c, z, ξ):

$$\begin{aligned}
\dot{c} &= -\frac{1}{\Theta''_{cc}(c,z)\xi}\Big\{-\Big[\Theta'_c(c,z)\left[\Theta(c,z)+\sigma\right]+\Theta'_z(c,z)\sigma\Big]\xi \\
&\qquad\qquad + \Theta'_c(c,z)\left(1+\Theta\xi\right)+\Theta''_{cz}\xi\sigma(c-z)\Big\}, \\
\dot{z} &= \sigma(c-z), \\
\dot{\xi} &= 1+\Theta\xi.
\end{aligned} \tag{16}$$

Under Assumption 2, this system assumes a unique steady state $(\widetilde{c}, \widetilde{z}, \widetilde{\xi})$. Furthermore:

Proposition 3. *Under Assumptions 1 and 2:*

(i) *the three eigenvalues $\Theta(\tilde{c}, \tilde{c})$, ν_1 and ν_2 associated to the behavior of the dynamical system in the neighbourhood of the satiated steady state $(\widetilde{c}, \widetilde{z}, \widetilde{\xi})$ are such that:*

$$\nu_1 + \nu_2 = \Theta(\widetilde{c}, \widetilde{c});$$

$$\nu_1\nu_2 = -\sigma\left[\Theta''_{cc}(\Theta+\sigma)+(\Theta+2\sigma)\Theta''_{cz}+\sigma\Theta''_{zz}-\Theta\left(\Theta'_c\right)^2/\sigma\right]/\Theta''_{cc};$$

(ii) *the steady state $(\widetilde{c}, \widetilde{z}, \widetilde{\xi})$ is a saddlepoint of the subsystem associated to the occurrence of consumption satiation.*

Proof : *Vide* Appendix 7. △

To sum up, under Assumption 2 and from Propositions 2 and 3, the long-run properties of the economy list as follows. Firstly , two distinct steady state positions may exist : the first, namely $(\overline{c}_1, \overline{\beta}_1, \overline{z}_1, \overline{\xi}_1, \overline{a}_1)$, that corresponds to an unsatiated intertemporal utility while the second, namely $(\widetilde{c}, \widetilde{z}, \widetilde{\xi})$, is associated to a satiated intertemporal utility. It is to be stressed that this latter satiated long-run position does usually not correspond to a stationary value for the asset a since the individual will not consume the whole amount of his intertemporal wealth.

Two distinct steady states hence emerge as corresponding to a long-run position for the economy, namely a first unsatiated interior steady state and a satiated steady state. According to Proposition 2, the unsatiated steady state may uncover a change to unstability, the satiated then emerging as being the sole conceivable candidate steady state solution for the long run. The forthcoming section shall establish the very possibility of such a conjunction and anchor the whole argument on the uprise of addiction phenomena and the features of the time preference of the individuals.

V – Time Preference, Addiction and Satiation

V.1 – Addiction and time preference

Following Becker & Murphy [2] definition, *addiction* means the property that the optimal consumption policy $c(a,z)$ increases with z. In accordance with the earlier literatures — Becker & Murphy [2], Iannaccone [10] and Shi & Epstein [13] —, this property will be apprehended on a local basis, i.e., in the neighbourhood of a steady state. Considering an initial position located in the neighbourhood of the unsatiated interior steady state $(\bar{c}_1, \bar{\beta}_1, \bar{z}_1, \bar{\xi}_1, \bar{a}_1)$, the holding of an addiction property would correspond to a positive link between c and z in the two-dimensional local stable manifold associated with the interior steady state. In formal terms :

Lemma 5. *Letting λ_1 and λ_2 denote the stable eigenvalues of the characteristic polynomial considered in the neighbourhood of the unsatiated steady state $(\overline{c}_1, \overline{\beta}_1, \overline{z}_1, \overline{\xi}_1, \overline{a}_1)$, the holding of an addiction phenomenon corresponds to the satisfaction of $(\lambda_1/\sigma+1)(\lambda_2/\sigma+1) > 0$.*

Proof : *Vide* Appendix 8. △

Proposition 4. *Consider the unsatiated steady state $(\overline{c}_1, \overline{\beta}_1, \overline{z}_1, \overline{\xi}_1, \overline{a}_1)$:*

(i) *a good is addictive in the neighbourhood of $(\overline{c}_1, \overline{\beta}_1, \overline{z}_1, \overline{\xi}_1, \overline{a}_1)$ if and only if $\partial\varrho/\partial z \geq 0$;*

(ii) *the holding of $\partial\varrho/\partial z = 0$ corresponds to the frontier $\mathcal{F}_4$ of equation $\mathcal{P} + \sigma(r+\sigma)\mathcal{S} + \sigma^2(r+\sigma)^2 = 0$ over the plane $(\mathcal{S}, \mathcal{P})$. This locus divides this plane between two distinct domains A and B. Domain A is an overall non-addiction area where one root is inferior to $-\sigma$ whereas the outstanding one is greater than $-\sigma$. In opposition to this, domain B is an addiction area that turns out to assume three distinct subareas B', B'' and B''' : over B', $\lambda_1 > -\sigma$ and $\lambda_2 > -\sigma$; over B'', $\lambda_1 < -\sigma$ and $\lambda_2 < -\sigma$; finally, over B''', λ_1 and λ_2 are complex conjugate with $(\lambda_1/\sigma+1)(\lambda_2/\sigma+1) > 0$.*

Proof : *Vide* Appendix 9. △

— Please insert Figure 4 —

Incidently, it is readily checked that this *addiction property* — $c(a,z)$ that increases with z — is exactly related to the holding of $\partial\varrho/\partial z \geq 0$ in Becker and Murphy [2] or Shi and Epstein [13]. It is hence ought to figure out a general enough property that is not directly attached to the specification (2) of intertemporal preferences under consideration.

Actually, the north-east move of the straight-line $\mathcal{F}_4$ on Figure 4 is to be understood as the strenghtening of the addiction phenomenon in preferences. Such an increase may lead to the attainment of zona III for which the unsatiated interior steady state loses local stability. Such an occurrence can be viewed as a strenghtened *addiction* from the individual whose consumption threrein becomes *strongly dependent* of his past behaviours. More precisely and in order to portray the occurrence of *strong dependency*, one uses the following property :

$$(17)\quad \mathcal{P} + \sigma(r+\sigma)\mathcal{S} + \sigma^2(r+\sigma)^2 = \frac{\partial\varrho}{\partial z}\left\{\left[\Theta'_c + \frac{\sigma}{r+\sigma}\Theta'_z\right]\sigma(r+\sigma)(r+2\sigma)\right\} \Big/ (-\Theta''_{cc})$$
$$= \left(\frac{\partial\varrho}{\partial z}\frac{z}{\varrho}\right)\Sigma\sigma(r+\sigma)r(r+2\sigma).$$

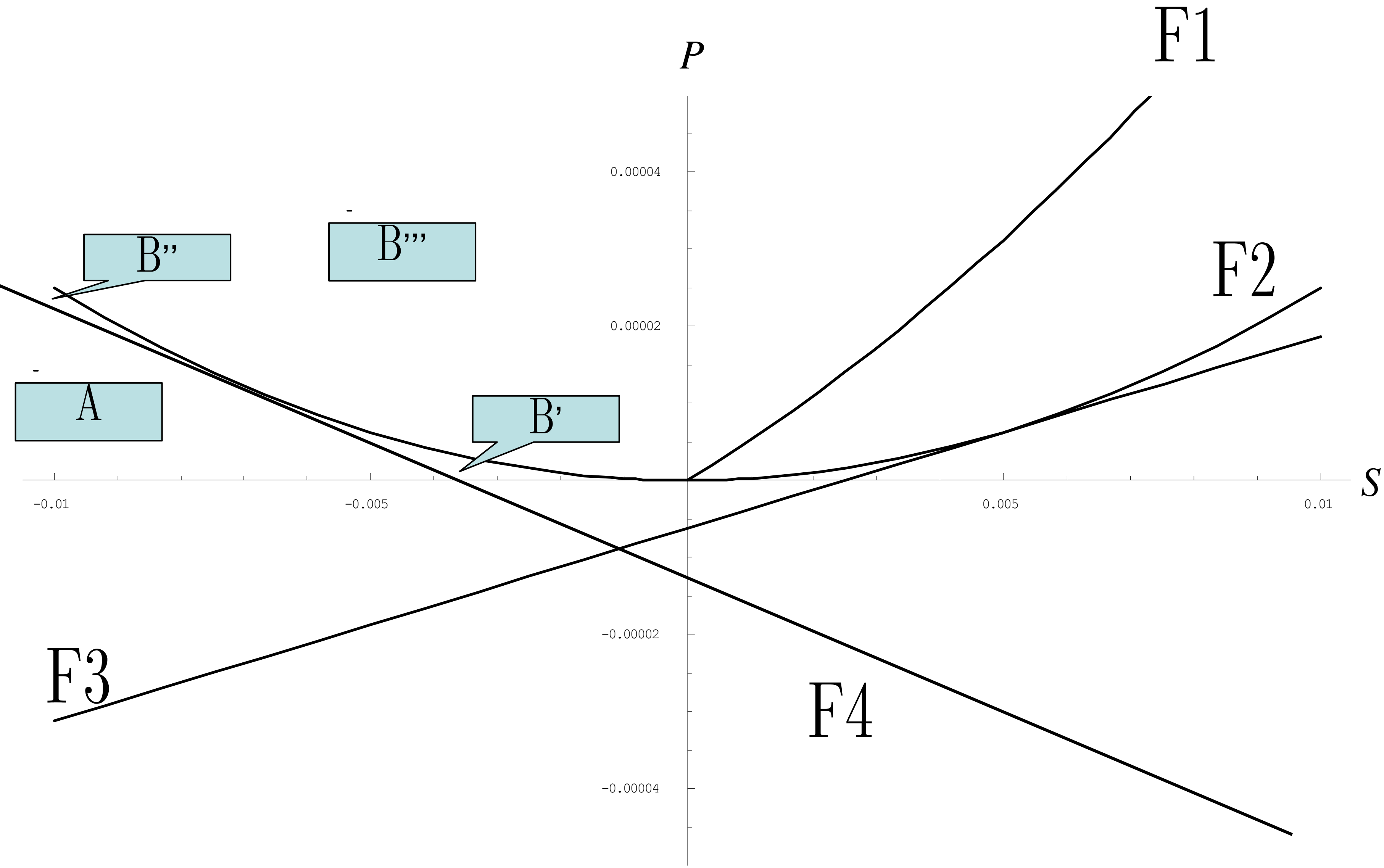
F1
P
0.00004
B'''
B''
F2
0.00002
A
B'
S
-0.01
-0.005
0.005
0.01
-0.00002
F3
F4
-0.00004

Figure 4

The obtention of *strong dependency* then corresponds to a north-east move property according to which at least the point $(\mathcal{S} = 0, \mathcal{P} = 0)$ is reached by $\mathcal{F}_4$ and the associated value of $\partial\varrho/\partial z$. From (17), this property narrows $(\partial\varrho/\partial z)z/\varrho$ so that it satisfies

$$\begin{aligned}\frac{\partial\varrho}{\partial z}\frac{z}{\varrho} &\geq \frac{\sigma(r+\sigma)\big(-\Theta''_{cc}\big)\bar{c}_1}{\big[\Theta'_c + \sigma\Theta'_z/(r+\sigma)\big](r+2\sigma)r} \\ &= \frac{\sigma(r+\sigma)}{r(r+2\sigma)}\frac{1}{\Sigma}.\end{aligned} \tag{18}$$

The satisfaction of such an inequality thus provides a criterion on $(\partial\varrho/\partial z)z/\varrho$ for assessing a *strong dependency* property. Further and from its expression detailed through Lemma 3(i), $\partial\varrho/\partial z$ happens to be an increasing function of Θ''_{cz}. In practical terms, inequality (18) could hence be available from the selection of a value of Θ''_{cz} sufficiently positive so as to fit such a criterion. The subsequent section shall examine the actual scope for such a conjecture. Finally, as soon as strong addiction occurs, the unique satiated steady state[3] emerges as the only stationary state candidate for the long run. The consumption behaviour of an addicted consumer becomes independent of its budget constraint in the long run.

V.2 – Assessing the Relevance of Strong Dependency : A Parametric Approach

This eventual section is intended to prove that both *dependency* and *strong dependency* emerge as actual long-run equilibrium configurations in the current environment. For that intend, the approach shall consider an initial position in a neighbourhood of the unsatiated interior long-run steady state $(\bar{c}_1, \bar{\beta}_1, \bar{z}_1, \bar{\xi}_1, \bar{a}_1)$ and take the values of $(\Theta'_c, \Theta'_z, \Theta''_{cc}, \Theta''_{cz}, \Theta''_{zz})$ as a set of *parameters* constrained so as to satisfy the whole list of previous assumptions. It is going to be established that these parameters can be selected so that *addiction* and *strong addiction* do exist. A direct articulation between the ensued formal conditions and a set of ordinal restrictions shall then be provided.

It is first remarked that, for $\Theta''_{cz} < 0$, one obtains both $\partial\varrho/\partial z < 0$ and $\mathcal{S} < 0$. It is then clear that, under $\Theta''_{cz} < 0$, no addiction happens, the unsatiated interior steady state $(\bar{c}_1, \bar{\beta}_1, \bar{z}_1, \bar{\xi}_1, \bar{a}_1)$ assuming a saddlepoint structure. The coming argument provides conditions such that strong addiction — and standard addiction as a direct corollry — occurs.

In order to make an explicit account of the parameters dependencies of (14) and (15), let $\mu_1\mu_2 := \mathcal{P}\big(\Theta'_z; \Theta'_c, \Theta''_{cc}\big)$, $\mu_1 + \mu_2 := \mathcal{S}\big(\Theta''_{cz}, \Theta''_{zz}; \Theta''_{cc}, \Theta'_c\big)$ and keep unchanged the respectively negative and positive parameters Θ''_{cc} and Θ'_c. The approach will proceed from a variation of the coefficients Θ''_{cz}, Θ''_{zz} and Θ'_z restricted so as to satisfy at $(\bar{c}_1, \bar{\beta}_1, \bar{z}_1, \bar{\xi}_1, \bar{a}_1)$:

$$\begin{aligned}&\Theta''_{cc} < 0 \quad \text{and} \quad \Theta''_{zz} < 0,\\ &\Theta''_{cc}\Theta''_{zz} - \big(\Theta''_{cz}\big)^2 > 0,\\ &\Theta'_c > 0, \Theta'_z < 0 \quad \text{and} \quad \Theta'_c + \Theta'_z > 0.\end{aligned}$$

[3] The existence of a unique satiated steady state is similarly obvious in Becker & Murphy [2] for their benchmark framework that hinges upon a quadratic specification for instantaneous utility.

Noticing however that while $\mathcal{P}$ is univocally related to Θ'_z, $\mathcal{S}$ solely depends on Θ''_{zz} and Θ''_{cz}, this results in a configuration for which a given parameter will be to modify a unique coefficient. From Figure 3 — equivalently, from Proposition 2 —, a requesite for modified dynamical properties states as the simultaneous holding of $\mathcal{S} > 0$ and $\mathcal{P} \leq \mathcal{S}(\mathcal{S}/2 + r^2)/2$ where $\mathcal{P} = \mathcal{S}(\mathcal{S}/2 + r^2)/2$ is the equation of the border $\mathcal{F}_1$ — equivalently locus IV — on Figure 2. In more details :

Lemma 6. *Consider the unsatiated steady state* $(\overline{c}_1, \overline{\beta}_1, \overline{z}_1, \overline{\xi}_1, \overline{a}_1)$:

(i) *when* Θ'_z *varies from* $-\Theta'_c$ *to* 0, $\mu_1\mu_2 = \mathcal{P}(\Theta'_z; \Theta''_{cc}, \Theta'_c)$ *assumes an increase from* 0 *to* $\mathcal{Y}(\Theta'_c; \Theta''_{cc}) := -\sigma r (\Theta'_c)^2 (r+\sigma)/\Theta''_{cc}$;

(ii) *furthermore :*

a/ *when* Θ''_{cz} *varies from* 0 *to* $\sqrt{\Theta''_{cc}\Theta''_{zz}}$, $\mathcal{S}(\Theta''_{cz}, \Theta''_{zz}; \Theta'_c, \Theta''_{cc})$ *assumes an increase from* $-\sigma\{(r+\sigma)\Theta''_{cc} + \sigma\Theta''_{zz} - r(\Theta'_c)^2/\sigma\}/\Theta''_{cc}$ *to* $\mathcal{X}(\Theta''_{zz}; \Theta''_{cc}, \Theta'_c) := -\sigma\{(r+\sigma)\Theta''_{cc} + (2\sigma + r)\sqrt{\Theta''_{cc}\Theta''_{zz}} + \sigma\Theta''_{zz} - r(\Theta'_c)^2/\sigma\}/\Theta''_{cc}$, *with* $\mathcal{X}(\cdot; \cdot, \cdot)$ *the maximum over the range of the admissible values for* Θ''_{cz};

b/ *for* $-\Theta''_{cc} r > 4(\Theta'_c)^2$, *there exists a value of the coefficient* $\Theta''_{zz} < 0$ *such that* $\mathcal{X}(\Theta''_{zz}; \Theta''_{cc}, \Theta'_c) > 0$.

Proof : *Vide* Appendix 10. △

Lemma 6 equips the analysis with a complete characterisation of $\mathcal{S}$ and $\mathcal{P}$ as a function of the parameters Θ'_z, Θ''_{cz} and Θ''_{zz}. Owing to (ii)b/ and for $-\Theta''_{cc} r > 4(\Theta'_c)^2$, it becomes possible to select a value of Θ''_{zz} so that the maximum of $\mathcal{S}$ undergoes a positive value. Then following (ii)a/, there exists a Θ''_{cz} that entails a positive one for $\mathcal{S}$.

A change in the stability properties of the unsatiated steady state $(\overline{c}_1, \overline{\beta}_1, \overline{z}_1, \overline{\xi}_1, \overline{a}_1)$ then occurs when the system moves into the area for which $\mathcal{P} \leq \mathcal{S}(\mathcal{S}/2 + r^2)$. From (i) and for $\mathcal{S} > 0$, it is always conceivable to find out such a Θ'_z, sufficiently close from $-\Theta'_c$. The actual scope for such a conjunction is established in Proposition 5 and finds its graphical justification through Figure 5 :

Proposition 5. *Let the values undergone by* $(\Theta'_c, \Theta'_z, \Theta''_{cc}, \Theta''_{cz}, \Theta''_{zz})$ *at the unsatiated steady state* $(\overline{c}_1, \overline{\beta}_1, \overline{z}_1, \overline{\xi}_1, \overline{a}_1)$ *be considered as a set parameters :*

(i) *for* $-\Theta''_{cc} r > 4(\Theta'_c)^2$, *there exist values of* Θ''_{zz}, Θ''_{cz} *and* Θ'_z *such that* $\mathcal{P} > 0$, $\mathcal{S} > 0$ *and* $\mathcal{P} \leq \mathcal{S}(\mathcal{S}/2 + r^2)/2$ *simultaneously hold ; a Poincaré-Hopf bifurcation then becomes admissible that corresponds to the attainment of the unstability zona* III *by the steady state ;*

(ii) *the attainment of the unstability zona builds upon the satisfaction of two distinct conditions :*

a/ *a sufficiently great value for the elasticity of the rate of time preference with respect to past consumption levels when compared to the inverse of the intertemporal elasticity of substitution :* $(\partial\varrho/\partial z)z/\varrho > \sigma(r+\sigma)/r(r+2\sigma)\Sigma$;

b/ *a sufficiently negative value for* $(\partial\varrho/\partial c)c/\varrho$ *when compared to* $(\partial\varrho/\partial z)z/\varrho$ *so that* $(\partial\varrho/\partial c)c/\varrho + (r+\sigma)(\partial\varrho/\partial z)z/\varrho$ *is arbitrarily small.*

Proof : *Vide* Appendix 11. △

— Please insert Figure 5 —

For $-\Theta''_{cc}r > 4(\Theta'_c)^2$, there exists a value of Θ''_{zz} such that $\mathcal{X} > 0$. The point $\mathcal{B}(\mathcal{X},\mathcal{Y})$ can then be located either above or underneath the locus $\mathcal{F}_1$. The rectangle $(0ABC)$ features the set of reachable points $(\mathcal{S},\mathcal{P})$ subsequent to a modification of the parameters Θ''_{cz} and Θ'_z. Figure 5 focuses on the case for which B is located above the borderline $\mathcal{F}_1$: there then always exists a value of Θ'_z associated with the occurrence of a Poincaré-Hopf bifurcation. Before such a critical value, the stationary state $(\bar{c}_1, \bar{\beta}_1, \bar{z}_1, \bar{\xi}_1, \bar{a}_1)$ becomes unstable. One could then surmise that the satiated steady state emerges as the unique long-run steady state position, a conjunction where addiction would lead to satiation. However, a global picture of equilibrium dynamics is not available and other long-run non stationary equilibria such as limit cycles could also exist.

Remark 2 (Continued) : As it is characterised by $\Theta''_{cz} = 0$, it is clear that the basic parameteric form for $\Theta(\cdot,\cdot)$ is not appropriate for the above line of arguments. It nevertheless becomes suitable for such an analysis through a simple modification. Consider indeed now a function $\Theta(\cdot,\cdot)$ defined by :

$$\Theta(c,z) = C_0 + Ac^{\alpha} - B(z+b)^{\beta} + C\int_{\bar{c}_1-\epsilon}^{c}\int_{\bar{c}_1-\epsilon}^{z} \mathrm{Max}\left[0, 1 - \frac{(x-\bar{c}_1)^2}{\varepsilon^2} - \frac{(y-\bar{c}_1)^2}{\varepsilon^2}\right] dxdy,$$

with $\varepsilon > 0$, C being a positive constant. It is proved in the second part of Appendix 14 that the whole range of properties analysed in the current section then assume an explicit parametric basis. ♢

To sum up, the model under consideration introduces a theory of consumption that leads to two distinct types of behaviours. An individual would consume his whole permanent income if he was to undertake his consumption choice in a close neighbourhood of the unsatiated steady state. His consumption behaviour would change as soon as he completes his decision in the neighbourhood of the satiated state : his current consumption becomes unrelated to his permanent income but oppositely fully determined by his past consumption choices. Interestingly, this latter configuration is reminiscent of the alternative approaches raised by J. Duesenberry and T. Brown half a century ago. It is indeed worth recalling that whilst Duesenberry advocated by 1948 a theory where current consumption was determined by a benchmark level of income, namely the maximal one reached by the individual in his lifetime, Brown raised by 1951 an approach where past consumption levels emerge as the main determinant of current consumption behaviours.

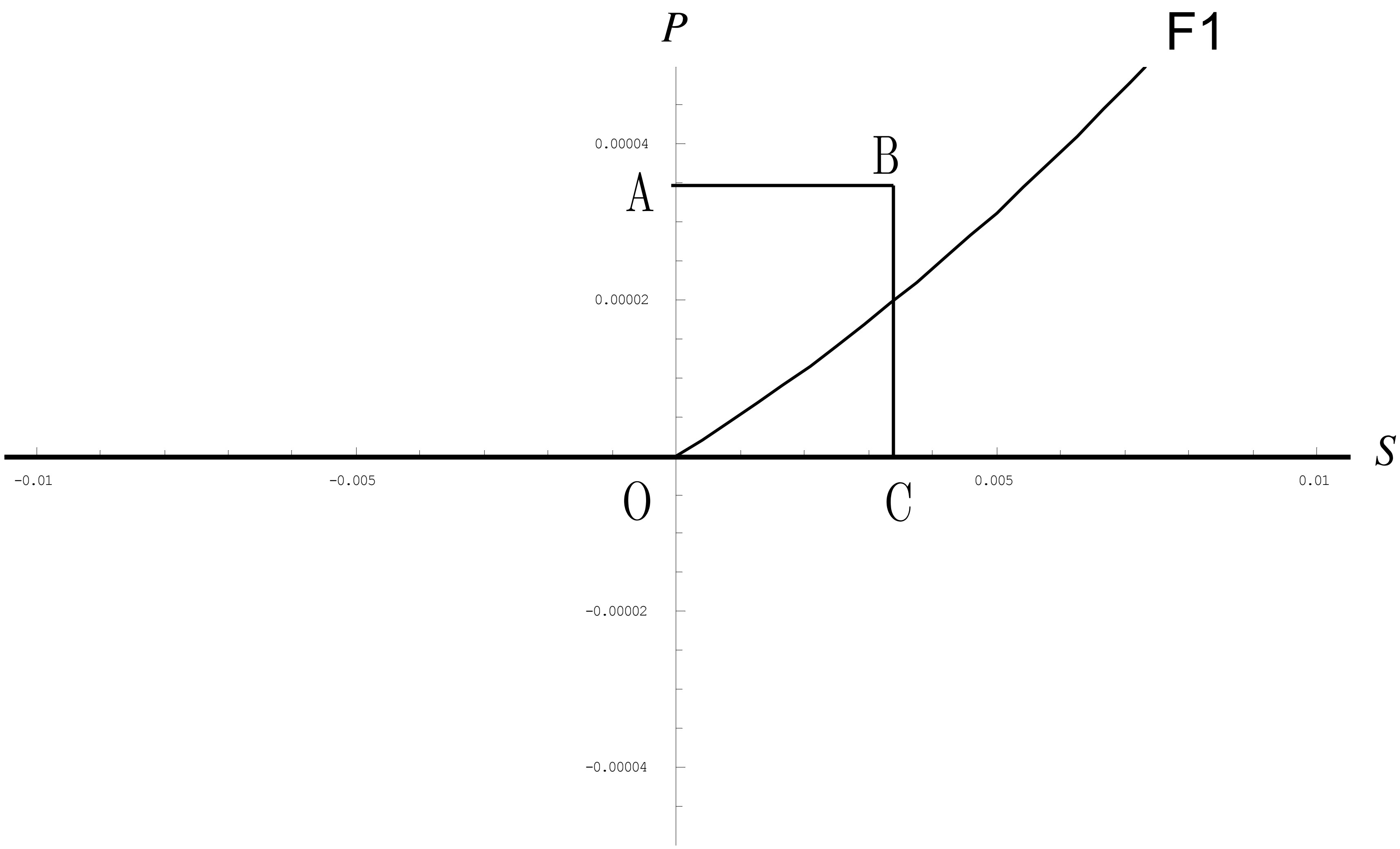

Figure 5

VI – A General Equilibrium Perspective

This section shall be concerned with an extension of the earlier line of arguments to a general equilibrium setting with an explicit productive sector.

VI.1 - A Productive Economy

It shall henceforward be assumed that per head production is available as a production function $y = f(x)$ where x denotes per head capital stock and y features the level of per head production. Further assume that $f(\cdot)$ depicts production net of the depreciation rate of the capital stock.

Assumption 3 : $f : \mathbb{R}_+ \to \mathbb{R}_+$, $f(0) = 0$, $f''_{xx}(x) < 0$ for $k > 0$. $\exists\ \hat{x}$ such that $f'(\hat{x}) = 0$ for $\hat{x} > 0$ or $\hat{x} = +\infty$, $f'_x(x) > 0$ for $x \in\ ]0, \hat{x}[$, $f'_x(x) < 0$ for $x \in\ ]\hat{x}, +\infty)$. Further $\hat{c} := f(\hat{x})$, possibly infinite, satisfies $\hat{c} > c_{\text{Max}}$.

First note that the equations (10) and (12) that described consumption dynamics are let unmodified by this generalisation. The introduction of a productive activity adds the new relations to the definition of an intertemporal equilibrium :

$$\begin{aligned} w(t) &= f(x(t)) - x f'_x(x(t)), \\ r(t) &= f'_x(x(t)), \\ a(t) &= x(t). \end{aligned}$$

Unsatiated consumption dynamics is hence available as :

$$\begin{aligned} -\Theta''_{cc}(c,z)\xi\dot{c} &= [\beta\sigma + \Theta'_c(c,z)\xi]\,[f'_x(x) - \rho(c,\beta,z,\xi)], \\ \dot{\beta} &= \beta[\Theta(c,z) + \sigma] - \Theta'_z(c,z)\xi, \\ \dot{z} &= \sigma(c - z), \\ \dot{\xi} &= 1 + \Theta(c,z)\xi, \\ \dot{x} &= f(x) - c, \end{aligned}$$

Satiated consumption dynamics accordingly reformulate to :

$$\begin{aligned} -\Theta''_{cc}(c,z)\xi\dot{c} &= -\xi\,\{\Theta'_c(c,z)\,[\Theta(c,z) + \sigma] + \Theta'_z(c,z)\sigma\} \\ &\qquad + \Theta'_c(c,z)\,[1 + \Theta(c,z)\xi] + \Theta''_{cz}(c,z)\xi\sigma(c - z), \\ \dot{z} &= \sigma(c - z), \\ \dot{\xi} &= 1 + \Theta(c,z)\xi, \\ \dot{x} &= f(x) - c. \end{aligned}$$

VI.2 – Existence of Steady States

The features of consumption dynamics being independent of the budgetary constraint of the individual when utility statiation occurs, these are let unmodified by the existence of an explicit productive sector : under the earlier range of assumptions, namely Assumption

2(iii), there exists a unique satiated level of consumption $\tilde{c}$ that solves $\Theta'_c(\tilde{c},\tilde{c})\left[\sigma+\Theta(\tilde{c},\tilde{c})\right]+\sigma\Theta'_z(\tilde{c},\tilde{c})=0$. Things are modified when one is concerned with unstatiated consumption dynamics, a steady state $(\bar{c},\bar{\beta},\bar{z},\bar{\xi},\bar{x})$ being now to satisfy :

$$\begin{aligned}
\Theta\left(\overline{c},\overline{c}\right) &= f'_x\left(\bar{x}\right),\\
\overline{\xi} &= -1/r,\\
\overline{c} &= \overline{z},\\
\overline{\beta} &= -\frac{\Theta'_z\left(\overline{c},\overline{c}\right)}{\Theta\left(\overline{c},\overline{c}\right)\left[\Theta\left(\overline{c},\overline{c}\right)+\sigma\right]},\\
f\left(\bar{x}\right) &= \bar{c}.
\end{aligned}$$

The obtention of a configuration qualitatively related to the one raised for the benchmark *exogenous* production economy imposes the retainment of a renewed range of assumptions :

Assumption 4 : The functions $\Theta(\cdot,\cdot)$ and $f(\cdot)$ satisfy :

(i) $\ln f'_x(\cdot)$ is a convex function of $x>0$ and thus satisfies $f'''_{xxx}f'_x-\left(f''_{xx}\right)^2>0$;

(ii) $\Theta(0,0)<f'_x(0)$;

(iii) consider the function $h(c):=\Theta(c,c)-f'\left(f^{-1}(c)\right)$ that is defined over $]0,c_{\mathrm{Max}}[$; $\exists c^{\star\star}$ such that $h(c^{\star\star})>0$ and $h'_c(c^{\star\star})=0$.

The actual status of these assumptions may deserve some clarifications. Assumption 4(i) first noticeably implies the holding of $f'''_{xxx}(x)>0$ for $x>0$ — it is, *e.g.*, satisfied when $f(\cdot)$ is a Cobb-Douglas production function. A remarkable feature of Assumption 4(ii) is that such a condition is filled by any production function $f(\cdot)$ that satisfies the Inada conditions, namely $f'_x(0)=+\infty$. Finally, Assumption 4(iii) allows for the analysis to rest upon at least one unsatiated steady state. More precisely, such an unsatiated steady state $\bar{c}$ being to solve $h(c)=0$, *i.e*, $\Theta(c,c)-f'_x\left(f^{-1}(c)\right)=0$, from the preceding assumption, one can state that : $h(\cdot)$ is a concave function from (i), $h(0)<0$ holds from (ii), $h(c_{\mathrm{Max}})=0-f'\left(f^{-1}(c_{\mathrm{Max}})\right)<0$ and $h(\cdot)$ assumes a maximum at $c^{\star\star}$ and $h(c^{\star\star})>0$ from (iii), There thus exists two unsatiated steady state consumption levels $\bar{c}_1$ and $\bar{c}_2$ such that $0<\bar{c}_1<c^{\star\star}<\bar{c}_2$ and $h(\bar{c}_1)=h(\bar{c}_2)=0$. It is further obtained that $h'_c(\bar{c}_1)>0$ and $h'_c(\bar{c}_2)<0$. An extra eventual assumption is however necessitated for ensuring the existence of $\bar{c}_1$:

Assumption 5 : $\bar{c}_1<\tilde{c}$.

As a matter of fact, a reverse conjunction would imply that the satiated steady state appears to be relevant. Finally, while Assumption 5 ensures that $\bar{c}_1$ is an admissible unsatiated steady state position, the other candidate $\bar{c}_2$ would as well be relevant if $\bar{c}_2<\tilde{c}$ was simultaneously be satisfied. Figure 6 provides an enlightening illustration of this range of considerations :

— Please insert Figure 6 —

VI.3 – Local Dynamics in the Neighbourhood of Steady States

In accordance with the preceding section, the dynamics in the neighbourhood of the satiated steady state position being let unaffected by the consideration of a production economy, the

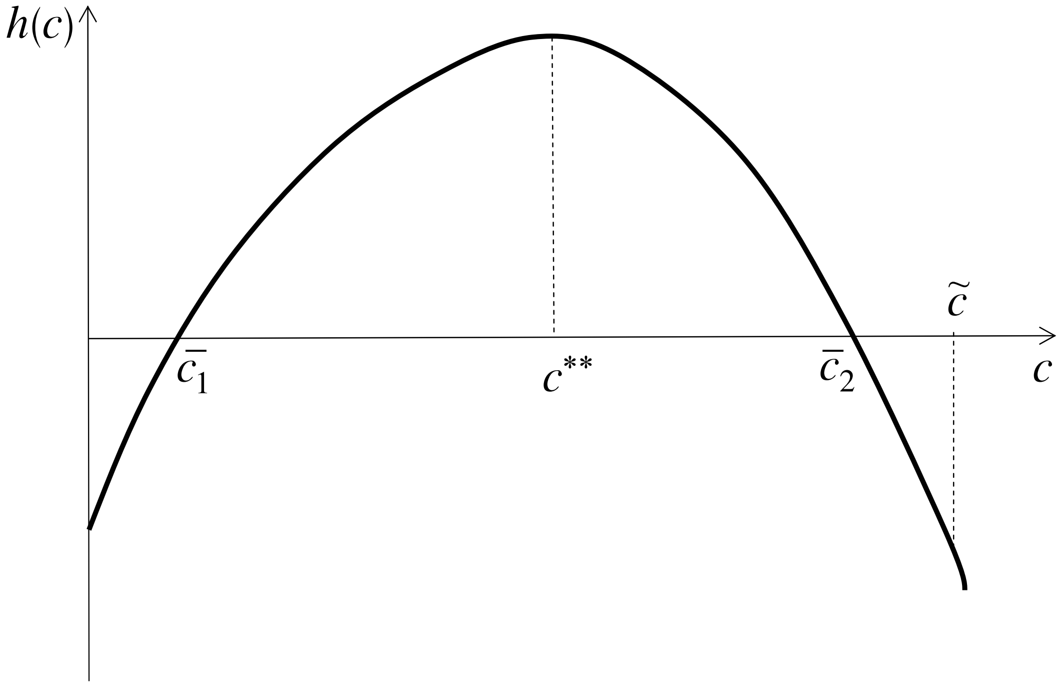

Figure 6.a

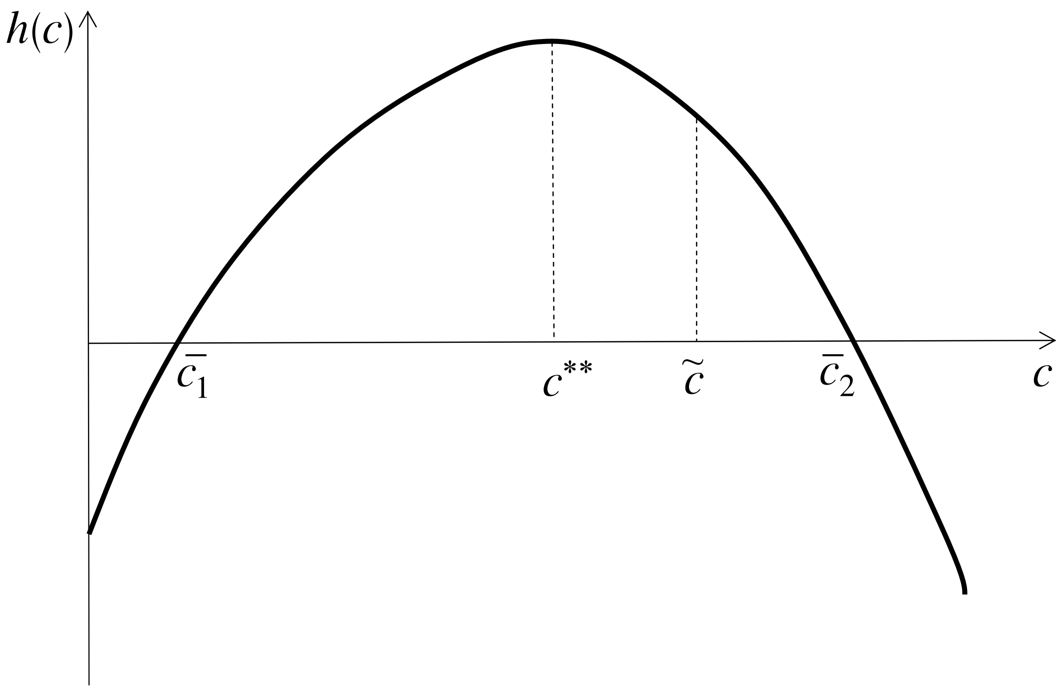

Figure 6.b

subsequent argument shall focus on the properties of the equilibrium dynamical system in the neighbourhood of the unsatiated positions $\bar{c}_1$ and $\bar{c}_2$. From Appendix 12, the characteristic polynomial associated with an unsatiated steady state then writes down along :

$$\mathcal{Z}(\lambda) = (\Theta - \lambda)\{\lambda^4 - 2\Theta\lambda + (\Theta^2 + \mathcal{S})\lambda^2 - \Theta\mathcal{S}\lambda + \mathcal{P}\},$$

$$\mathcal{S} = -\frac{\sigma}{\Theta''_{cc}(\bar{c},\bar{c})}\Big\{[\Theta(\bar{c},\bar{c}) + \sigma]\Theta''_{cc}(\bar{c},\bar{c}) + [\Theta(\bar{c},\bar{c}) + 2\sigma]\Theta''_{cz}(\bar{c},\bar{c}) + \sigma\Theta''_{zz}(\bar{c},\bar{c})\Big\}$$
$$+ \frac{\Theta(\bar{c},\bar{c})[\Theta'_c(\bar{c},\bar{c})]^2}{\Theta''_{cc}(\bar{c},\bar{c})} - \frac{\Theta'_c(\bar{c},\bar{c}) + \sigma\Theta'_z(\bar{c},\bar{c})/[\sigma + \Theta(\bar{c},\bar{c})]}{\Theta''_{cc}(\bar{c},\bar{c})} f''_{xx}(\bar{x}),$$

$$\mathcal{P} = -\frac{\sigma\Theta(\bar{c},\bar{c})}{\Theta''_{cc}(\bar{c},\bar{c})}\left[\Theta'_c(\bar{c},\bar{c}) + \Theta'_z(\bar{c},\bar{c}) - \frac{f''_{xx}(\bar{x})}{f'_x(\bar{x})}\right][(\Theta + \sigma)\Theta'_c(\bar{c},\bar{c}) + \sigma\Theta'_z(\bar{c},\bar{c})],$$

where $\Theta'_c + \Theta'_z - f''_{xx}/f'_x = h'_c$. Besides the obvious root Θ, the roots of the characteristic polynomial again assume a paired root structure : λ_1, $\Theta - \lambda_1$, λ_2, $\Theta - \lambda_2$. Along the approach developed in Section IV, they are analysed through the introduction of $\mu_i = \lambda_i(\Theta - \lambda_i)$, $i = 1, 2$, the two above coefficients then boiling down to $\mathcal{S} = \mu_1 + \mu_2$, $\mathcal{P} = \mu_1\mu_2$. As it was shown in Section VI.2 that $h'_c(\bar{c}_1) > 0$ and $h'_c(\bar{c}_2) < 0$, it is obtained that $\mathcal{P} > 0$ prevails at $\bar{c}_1$ while $\mathcal{P} < 0$ at $\bar{c}_2$. It then follows that the dynamical properties displayed in the neighbourhood of the unsatiated steady states $\bar{c}_1$ and $\bar{c}_2$ replicate the ones raised through Section IV in the benchmark exogenous production economy : $\bar{c}_2$ is locally unstable and is thence not a relevant long-run steady state position. The properties associated with $\bar{c}_1$ depend, following the analysis of the benchmark exogenous production economy, on the relative values assumed by $\mathcal{S}$ and $\mathcal{P}$.

VI.4 – Addiction & Strong Addiction within a Productive Economy

Focusing then on the emergence of addiction and strong addiction within a productive economy, the analysis of Section V can be adaptated to this generalised environment by taking into account the modified expressions of $\mathcal{S}$ and $\mathcal{P}$. More explicitly, it first appears that the expression of the boundary $\mathcal{F}_4$ is let unchanged, the new expressions of $\mathcal{S}$ and $\mathcal{P}$ still satisfying :

$$\mathcal{P} + \sigma(r + \sigma)\mathcal{S} + \sigma^2(\Theta + \sigma)^2 = \frac{\partial\varrho}{\partial z}\left\{\left[\Theta'_c + \frac{\sigma}{\Theta + \sigma}\Theta'_z\right]\sigma(\Theta + \sigma)(\Theta + 2\sigma)\right\} \Big/ (-\Theta''_{cc}).$$

Unsurprising, addiction being a preferences phenomenon whose emergence is unrelated to the embedding environment, the *addiction frontier* still corresponds to the holding of $\partial\varrho/\partial z = 0$ and Figure 4 remains relevant.

The second issue relates to the existence of addiction and strong addiction within a productive economy considered in the neighbourhood of the unsatiated steady state $(\bar{c}_1, \bar{\beta}_1, \bar{z}_1, \bar{\xi}_1, \bar{x}_1)$. The values of $(f''_{xx}/f'_x, \Theta'_c, \Theta'_z, \Theta''_{cc}, \Theta''_{cz}, \Theta'_{zz}, \Theta)$ considered at the steady state are retained as a list of parameters. These are to satisfy[4]

$$\Theta''_{cc} < 0 \quad \text{and} \quad \Theta''_{zz} < 0,$$
$$\Theta''_{cc}\Theta''_{zz} - (\Theta''_{cz})^2 > 0,$$
$$\Theta'_c > 0, \Theta'_z < 0,$$
$$\Theta'_c + \Theta'_z - f''_{xx}/f'_x > 0 \quad \text{and} \quad \Theta\Theta'_c + \sigma f''_{xx}/f'_x > 0.$$

Proposition 5 is finally to be reformulated along the following lines :

Proposition 6. *Let the values undergone by* $(\Theta, \Theta'_c, \Theta'_z, \Theta''_{cc}, \Theta''_{cz}, \Theta''_{zz})$ *and* f''_{xx}/f'_x *at the unsatiated steady state* $(\bar{c}_1, \overline{\beta}_1, \bar{z}_1, \overline{\xi}_1, \bar{x}_1)$ *be considered as a set of parameters :*

(i) *for* $-\Theta''_{cc}\Theta/4 - (\Theta'_c)^2 + (\Theta\Theta'_c + \sigma f''_{xx}/f'_x) f''_{xx} \;/\; f'_x(\Theta+\sigma) > 0$, *there exist values of* Θ''_{zz}, Θ''_{cz} *and* Θ'_z *such that* $\mathcal{P} > 0$, $\mathcal{S} > 0$ *and* $\mathcal{P} \leq \mathcal{S}(\mathcal{S}/2 + \Theta^2)$ *simultaneously hold ; a Poincaré-Hopf bifurcation then becomes admissible that corresponds to the attainment of the unstability zona* III *by the steady state ;*

(ii) *the attainment of the unstability zona builds upon the satisfaction of two distinct conditions :*

a/ *a sufficiently great value for the elasticity of the rate of time preference with respect to consumption habits when compared to the inverse of the intertemporal elasticity of substitution :* $(\partial\varrho/\partial z)z/\varrho > \sigma(\Theta+\sigma)/\Theta(\Theta+2\sigma)\Sigma$ *;*

b/ *a sufficiently small negative value for* $(\partial\varrho/\partial c)c/\varrho$ *so that* $(\partial\varrho/\partial c)c/\varrho + (\Theta + \sigma)(\partial\varrho/\partial z)z/\varrho - f''_{xx}/f'_x$ *assumes arbitrarily small orders.*

Proof : *Vide* Appendix 13. △

Allowing for a general equilibrium argument, it is hence established that the whole earlier line on conclusions can be recovered, though they now involve somewhat more stringent conditions. More explicitly, the endogenous determination of the rate of interest renders more difficult the emergence of an unstable unsatiated steady state and the concavity of the production technology exerts a stabilising effect on this long-run unsatiated position. From the sole qualitative standpoint, it should nonetheless be noticed that remains actual scope for any of the configurations reached through a partial equilibrium approach.

[4]As a matter of fact, the two last constraints are specific to the production economy. Condition $\Theta'_c + \Theta'_z - f''_{xx}/f'_x > 0$ corresponds to the holding of $h'_c(\bar{c}_1) > 0$: as clarified in Section V.2, this is ensured by the retainment of Assumption 4. Condition $\Theta\Theta'_c + \sigma f''_{xx}/f'_x > 0$, as for itself, guarantees that Assumption 5, namely $\bar{c}_1 < \tilde{c}$, continues to prevail when the value of Θ'_z is made to vary. Such an occurrence is indeed equivalent to the one of $\Omega(\bar{c}_1) > \Omega(\tilde{c}) = 0$, that is $\Theta'_c(\bar{c}_1, \bar{c}_1)[\Theta(\bar{c}_1, \bar{c}_1) + \sigma] + \sigma\Theta'_z(\bar{c}_1, \bar{c}_1) > 0$ — *vide* Assumption 2. This last inequality is to continue to hold when $\Theta'_z(\bar{c}_1, \bar{c}_1)$ tends towards its lower admissible bound, *i.e.*, $-\Theta'_c + f''_{xx}/f'_x$. Replacing, it derives that $\Theta\Theta'_c + \sigma f''_{xx}/f'_x > 0$.

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VIII – Proofs

VIII.1 – Proof of Lemma 1: Establishing the Concavity of the Utility Function

Letting $V(c(t); z_0(t))$ denote the utility function, establishing its concavity is equivalent to prove, for distinct $c_1(t)$ and $c_2(t)$ and $\alpha \in]0,1[$, that

$$V\left(\alpha c_1 + (1-\alpha)c_2; z_0\right) \geq \alpha V\left(c_1; z_0\right) + (1-\alpha)V\left(c_2; z_0\right).$$

Considering the two underlying consumption habits paths, their laws of motion are described by:

$$\dot{z}_1 = \sigma(c_1 - z_1),$$
$$\dot{z}_2 = \sigma(c_2 - z_2),$$

that in turn implies:

$$\alpha\dot{z}_1 + (1-\alpha)\dot{z}_2 = \sigma\big\{\alpha c_1 + (1-\alpha)c_2 - [\alpha z_1 + (1-\alpha)z_2]\big\}.$$

Notice also that the concavity of $\Theta(\cdot,\cdot)$ ensures the satisfaction of:

$$\Theta\left(\alpha c_1 + (1-\alpha)c_2; \alpha z_1 + (1-\alpha)z_2\right) \geq \alpha\Theta\left(c_1, z_1\right) + (1-\alpha)\Theta\left(c_2, z_2\right),$$

Parallely using the concavity of the function $x \mapsto -\exp(-x)$ gives

$$-\exp\left[-\left(\alpha y_1(t) + (1-\alpha)y_2(t)\right)\right] \geq -\alpha\exp\left(-y_1(t)\right) - (1-\alpha)\exp\left(-y_2(t)\right).$$

Using the two last inequalities, when $y_1(t)$ and $y_2(t)$ are defined from

$$y_1(t) = \int_{s=0}^{t} \Theta\left[c_1(s), z_1(s)\right] ds,$$
$$y_2(t) = \int_{s=0}^{t} \Theta\left[c_2(s), z_2(s)\right] ds,$$

it is obtained :

$$\begin{aligned} V\big[\alpha c_1 + (1-\alpha)c_2; z_0\big] \\ &= -\int_{t=0}^{+\infty} \exp\Big\{-\int_{s=0}^{t} \Theta\big[\alpha c_1(s) + (1-\alpha)c_2(s); \\ &\qquad \alpha z_1(s) + (1-\alpha)z_2(s)\big] ds\Big\} dt \\ &\geq -\int_{t=0}^{+\infty} \exp\big[-\alpha y_1(t) - (1-\alpha)y_2(t)\big] dt \\ &\geq \alpha V\left(c_1; z_0\right) + (1-\alpha)V\left(c_2; z_0\right), \end{aligned}$$

that establishes the result. △

VIII.2 – Proof of Lemma 2.

(i) Let

$$U\big({}_t\mathcal{C}; z_t\big) = -\int_{\tau=t}^{+\infty} \exp\Big\{-\int_{s=t}^{\tau} \Theta[c(s), z(s)]ds\Big\}d\tau,$$
$$\text{for } z(s) = \sigma\int_t^s \exp[\sigma(x-s)]c(x)dx + z_t\exp[-\sigma(s-t)].$$

As $\partial z(s)/\partial z_t = \exp[-\sigma(s-t)]$, it thus derives that :

$$\begin{aligned}\beta(t) &= -\frac{\partial U}{\partial z_t}\\ &= \int_{\tau=t}^{+\infty} \exp\Big\{-\int_{s=t}^{\tau} \Theta[c(s), z(s)]ds\Big\}\\ &\qquad\qquad \times\Big\{-\int_{s=t}^{\tau} \Theta'_z[c(s), z(s)]\exp[-\sigma(s-t)]ds\Big\}d\tau.\end{aligned}$$

(ii) The total Volterra differential is introduced as

$$\delta U\big({}_t\mathcal{C}; z_t\big) := \lim_{\varepsilon\to 0}\frac{d}{d\varepsilon}U\big({}_t\mathcal{C}+\varepsilon\Phi; z_t\big),$$

for $\Phi := \big(\phi(s)\big)_{s\geq t}$ a function defined for $s\geq t$ into $\mathbb{R}$.
This restates as:

$$\begin{aligned}\frac{d}{d\varepsilon}U\big({}_t\mathcal{C}+\varepsilon\,\Phi; z_t\big)\Big|_{\varepsilon=0} &= \frac{d}{d\varepsilon}\Bigg[-\int_{u=t}^{+\infty}\exp\Big\{-\int_{s=t}^{u}\Theta\Big[c(s)+\varepsilon\phi(s),\\ &\quad \sigma\int_t^s \exp[\sigma(x-s)]\big(c(x)+\varepsilon\phi(x)\big)dx + z_t\exp[-\sigma(s-t)]\Big]ds\Big\}du\Bigg]\Bigg|_{\varepsilon=0}\\ &= -\int_{u=t}^{+\infty}\exp\Big\{-\int_{s=t}^{u}\Theta[c(s), z(s)]ds\Big\}\\ &\quad\times\Big\{-\int_{s=t}^{u}\Big(\Theta'_c[c(s), z(s)]\phi(s)\\ &\qquad + \sigma\int_{x=t}^{s}\exp[-\sigma(x-s)]\phi(x)\Theta'_z[c(s), z(s)]dx\Big)ds\Big\}du.\end{aligned}$$

The expression of the Volterra derivative at a given date $\tau\geq t$ can be derived by letting ϕ tend towards a dirac function at date τ. One then obtains :

$$\int_{s=t}^{u}\Theta'_c[c(s), z(s)]\phi(s)ds = \begin{cases} 0 & \text{for}\quad u<\tau,\\ \Theta'_c[c(\tau), z(\tau)] & \text{for}\quad u\geq\tau.\end{cases}$$

and

$$\int_{x=t}^{s}\exp[\sigma(x-s)]\phi(x)\Theta'_z[c(s), z(s)]dx = \begin{cases} 0 & \text{for}\quad s<\tau,\\ \exp[\sigma(\tau-s)]\Theta'_z[c(s), z(s)] & \text{for}\quad s\geq\tau.\end{cases}$$

Whence:

$$
\begin{aligned}
U'\big({}_t\mathcal{C}; z_t; \tau\big) = &\left(-\int_{u=\tau}^{+\infty} \exp\left\{-\int_{s=t}^{u} \Theta[c(s), z(s)]ds\right\}du\right)\left\{-\Theta'_c[c(\tau), z(\tau)]\right\} \\
&- \int_{u=\tau}^{+\infty} \exp\left\{-\int_{s=t}^{u} \Theta[c(s), z(s)]ds\right\} \\
&\qquad \times \left(-\sigma \int_{s=t}^{u} \exp[\sigma(\tau - s)]\Theta'_z[c(s), z(s)]ds\right\}\Bigg) du \\
= &- \exp\left\{-\int_{s=t}^{\tau} \Theta[c(s), z(s)]ds\right\} \\
&\times \left(-\int_{u=\tau}^{+\infty} \exp\left\{-\int_{s=\tau}^{u} \Theta[c(s), z(s)]ds\right\}du\left\{\Theta'_c[c(\tau), z(\tau)]\right\}\right. \\
&\quad - \sigma \int_{u=\tau}^{+\infty} \exp\left\{-\int_{s=\tau}^{u} \Theta[c(s), z(s)]ds\right\} \\
&\qquad \left. \times \left\{\int_{s=t}^{u} \Theta'_z[c(s), z(s)] \exp[\sigma(\tau - s)]ds\right\}du\right).
\end{aligned}
$$

(iii) Straightforward. △

VIII.3 – Proof of Lemma 3.

(i) A direct computation delivers :

$$
\begin{aligned}
\frac{\partial \varrho\big[{}_\tau\mathcal{C}, z(\tau)\big]}{\partial z(\tau)} = \Theta'_z - &\left\{-\Theta''_{cz}\sigma\xi + \Theta'_c\left(\Theta'_z\xi + \Theta\frac{\partial \xi}{\partial z}\right)\right. \\
&\left. + \sigma\left[\frac{\partial \beta}{\partial z}(\Theta + \sigma) + \beta\Theta'_z - \Theta''_{zz}\xi - \Theta'_z\frac{\partial \xi}{\partial z}\right]\right\} \Big/ \{\Theta'_c\xi + \sigma\beta\}.
\end{aligned}
$$

From Lemma 2, $d\xi/dz = -\beta$ and, for a stationary path, $\xi = -1/\Theta$ and $\beta = -\Theta'_z/\Theta(\sigma + \Theta)$. The component $\partial\beta/\partial z$ in the above expression however remains to be computed. For that purpose and recalling that

$$
\begin{aligned}
\beta(\tau) = -\int_{x=\tau}^{+\infty} &\exp\left\{-\int_{s=\tau}^{x} \Theta[c(s), z(s)]ds\right\} \\
&\times \left\{\int_{s=\tau}^{x} \Theta'_z[c(s), z(s)] \exp[-\sigma(s - \tau)]ds\right\}dx
\end{aligned}
$$

and relying to $\partial z(s)/\partial z(\tau) = \exp[-\sigma(s - \tau)]$, at the steady state, it is obtained that :

$$
\begin{aligned}
\frac{\partial \beta(\tau)}{\partial z(\tau)} = &-\int_{x=\tau}^{+\infty} \exp[-\Theta(x - \tau)]\left\{\left(-\int_{s=\tau}^{x} \Theta'_z \exp[-\sigma(s - \tau)]ds\right)\left(\int_{s=\tau}^{x} \Theta'_z \exp[-\sigma(s - \tau)]ds\right)\right. \\
&\left. + \int_{s=\tau}^{x} \Theta''_{zz} \exp[-2\sigma(s - \tau)]ds\right\}dx \\
= &\int_{x=\tau}^{+\infty} \exp[-\Theta(x - \tau)]\left\{\frac{(\Theta'_z)^2}{\sigma^2}\left\{1 - 2\exp[\sigma(\tau - x)] + \exp[2\sigma(\tau - x)]\right\}\right. \\
&\left. - \Theta''_{zz}\frac{1 - \exp[2\sigma(\tau - x)]}{2\sigma}\right\}dx.
\end{aligned}
$$

…A.3

Thus

$$\frac{\partial \beta(\tau)}{\partial z(\tau)} = \frac{2(\Theta'_z)^2}{\Theta(\Theta+\sigma)(\Theta+2\sigma)} - \frac{\Theta''_{zz}}{\Theta(\Theta+2\sigma)}.$$

Then replacing in the early expression of $\partial\varrho[{}_\tau\mathcal{C}, z(\tau)]/\partial z(\tau)$, it is finally obtained that :

$$\frac{\partial\varrho[z(\tau), {}_\tau\mathcal{C}]}{\partial z(\tau)} = \left\{ \frac{\Theta\Theta'_z}{\Theta+\sigma}\left(\Theta'_c + \frac{\sigma\Theta'_z}{\Theta+2\sigma}\right) + \frac{\sigma}{\Theta+2\sigma}\Big[(\Theta+2\sigma)\Theta''_{cz} + \sigma\Theta''_{zz}\Big]\right\} \Big/ \left(\Theta'_c + \frac{\sigma}{\Theta+\sigma}\Theta'_z\right).$$

(ii) Computing the Volterra derivation of $\partial\varrho[{}_\tau\mathcal{C}, z(\tau)]/\partial c(x)$ for $x \geq \tau$, it is obtained that :

$$\frac{\partial\varrho[{}_\tau\mathcal{C}, z(\tau)]}{\partial c(x)} = -\left\{\Theta\Theta'_c\frac{\partial\xi(\tau)}{\partial c(x)} + \sigma\frac{\partial\beta(\tau)}{\partial c(x)}(\Theta+\sigma) - \sigma\Theta'_z\frac{\partial\xi(\tau)}{\partial c(x)}\right\} \Big/ (\Theta'_c\xi + \sigma\beta),$$

where $\partial\xi(\tau)/\partial c(x)$ is available from Lemma 2 as :

$$\frac{\partial\xi(\tau)}{\partial c(x)} = -\exp\left\{-\int_{s=\tau}^{x}\Theta[c(s), z(s)]ds\right\} \times \left\{\Theta'_c[c(x), z(x)]\xi(x) + \sigma\beta(x)\right\}$$

whilst $\partial\beta(\tau)/\partial c(x)$ remains to be computed. It is however known that $\beta(\tau) = -\partial\xi(\tau)/\partial z(\tau)$; this allows to write

$$\begin{aligned}\frac{\partial\beta(\tau)}{\partial c(x)} &= -\frac{\partial}{\partial z(\tau)}\left[\frac{\partial\xi(\tau)}{\partial c(x)}\right] \\ &= \frac{\partial}{\partial z(\tau)}\left[\exp\left\{-\int_{s=\tau}^{x}\Theta[c(s), z(s)]ds\right\} \times \left\{\Theta'_c[c(x), z(x)]\xi(x) + \sigma\beta(x)\right\}\right],\end{aligned}$$

whence, omitting arguments :

$$\frac{\partial\beta(\tau)}{\partial c(x)} = \exp[-\Theta(x-\tau)]\left(\left\{-\int_{s=\tau}^{x}\Theta'_z\exp[-\sigma(s-\tau)]ds\right\} \times \left\{\Theta'_c\xi + \sigma\beta\right\} + \Theta''_{cz}\exp[-\sigma(x-\tau)]\xi - \Theta'_c\beta\exp[-\sigma(x-\tau)] + \sigma\frac{\partial\beta(x)}{\partial z(\tau)}\right),$$

where

$$\partial\beta(x)/\partial z(\tau) = \exp[-\sigma(x-\tau)]\partial\beta(x)/\partial z(x),$$

the expression of $\partial\beta(x)/\partial z(x)$ being available from the previous calculations. Finally replacing the various expressions within the coefficient $\partial\varrho[z(\tau), {}_\tau\mathcal{C}]/\partial c(x)$ for $x \geq \tau$, it derives

$$\frac{\partial\varrho[z(\tau), {}_\tau\mathcal{C}]}{\partial c(x)} = \exp[-\Theta(x-\tau)]\left[\Theta(\Theta'_c + \Theta'_z) - \exp[-\sigma(x-\tau)](\Theta+\sigma)\frac{\partial\varrho[z(\tau), {}_\tau\mathcal{C}]}{\partial z(\tau)}\right]$$

and the statement follows. △

VIII.4 – Proof of Proposition 1.

(i)-(ii) An immediate implication of Assumption 2 states as $\widetilde{c} > c^*$. Indeed,

$$\begin{aligned}
[\Theta'_c(\widetilde{c},\widetilde{c}) + \Theta'_z(\widetilde{c},\widetilde{c})] &= \Theta'_c(\widetilde{c},\widetilde{c}) + \Theta'_z(\widetilde{c},\widetilde{c})\frac{\sigma}{\Theta(\widetilde{c},\widetilde{c})+\sigma} + \frac{\Theta(\widetilde{c},\widetilde{c})\Theta'_z(\widetilde{c},\widetilde{c})}{\Theta(\widetilde{c},\widetilde{c})+\sigma} \\
&= \frac{\Theta(\widetilde{c},\widetilde{c})\Theta'_z(\widetilde{c},\widetilde{c})}{\Theta(\widetilde{c},\widetilde{c})+\sigma} \qquad (14)\\
&< 0.
\end{aligned}$$

The statement follows. △

VIII.5 – Proof of Lemma 4.

Letting

$$\begin{aligned}
\dot{c} &:= \Delta_{\dot{c}}(c,\beta,z,\xi,a),\\
\dot{\beta} &:= \Delta_{\dot{\beta}}(c,\beta,z,\xi,a),\\
\dot{z} &:= \Delta_{\dot{z}}(c,\beta,z,\xi,a),\\
\dot{\xi} &:= \Delta_{\dot{\xi}}(c,\beta,z,\xi,a),\\
\dot{a} &:= \Delta_{\dot{a}}(c,\beta,z,\xi,a)
\end{aligned}$$

feature the five components of the dynamical system and considering its linearised form in the neighbourhood of an unsatiated steady state, it is readily checked that the components of the Jacobian matrix list as:

$$\begin{bmatrix}
\Delta_{\dot{c}c} & \Delta_{\dot{c}\beta} & \Delta_{\dot{c}z} & \Delta_{\dot{c}\xi} & \Delta_{\dot{c}a}\\
\Delta_{\dot{\beta}c} & \Delta_{\dot{\beta}\beta} & \Delta_{\dot{\beta}z} & \Delta_{\dot{\beta}\xi} & \Delta_{\dot{\beta}a}\\
\Delta_{\dot{z}c} & \Delta_{\dot{z}\beta} & \Delta_{\dot{z}z} & \Delta_{\dot{z}\xi} & \Delta_{\dot{z}a}\\
\Delta_{\dot{\xi}c} & \Delta_{\dot{\xi}\beta} & \Delta_{\dot{\xi}z} & \Delta_{\dot{\xi}\xi} & \Delta_{\dot{\xi}a}\\
\Delta_{\dot{a}c} & \Delta_{\dot{a}\beta} & \Delta_{\dot{a}z} & \Delta_{\dot{a}\xi} & \Delta_{\dot{a}a}
\end{bmatrix}
= \begin{bmatrix}
0 & \dfrac{\sigma r(r+\sigma)}{\Theta''_{cc}} & \dfrac{\sigma(\Theta''_{cz}+\Theta''_{zz})}{\Theta''_{cc}} & \dfrac{r(r\Theta'_c-\sigma\Theta'_z)}{\Theta''_{cc}} & 0\\
\beta\Theta'_c+\Theta''_{cz}/r & r+\sigma & \beta\Theta'_z+\Theta''_{zz}/r & -\Theta'_z & 0\\
\sigma & 0 & -\sigma & 0 & 0\\
-\Theta'_c/r & 0 & -\Theta'_z/r & r & 0\\
-1 & 0 & 0 & 0 & r
\end{bmatrix}$$

Then computing the associated characteristic polynomial $\mathcal{P}(\lambda)$, it is obtained that :

$$\mathcal{P}(\lambda) = (r-\lambda)\mathcal{Q}(\lambda),$$

for

$$\begin{aligned}
\mathcal{Q}(\lambda) == \sigma \begin{vmatrix} \Delta_{\dot{c}\beta} & \Delta_{\dot{c}z} & \Delta_{\dot{c}\xi} \\ r+\sigma-\lambda & \Delta_{\dot{\beta}z} & -\Theta'_z \\ 0 & \Delta_{\dot{\xi}z} & r-\lambda \end{vmatrix} - (\sigma+\lambda) \begin{vmatrix} -\lambda & \Delta_{\dot{c}\beta} & \Delta_{\dot{c}\xi} \\ \Delta_{\dot{\beta}c} & r+\sigma-\lambda & -\Theta'_z \\ \Delta_{\dot{\xi}c} & 0 & r-\lambda \end{vmatrix} \\
= \lambda^4 - 2r\lambda^3 + \left[r^2 - r\sigma - \sigma^2 - \sigma\Delta_{\dot{c}z} - \Delta_{\dot{\xi}c}\Delta_{\dot{c}\xi} - \Delta_{\dot{\beta}c}\Delta_{\dot{c}\beta}\right]\lambda^2 \\
- \left[\sigma\Delta_{\dot{c}\beta}\Delta_{\dot{\beta}z} + \sigma\Delta_{\dot{c}\xi}\Delta_{\dot{\xi}z} - \sigma\Delta_{\dot{c}z}(2r+\sigma) - \sigma r(r+\sigma)\right. \\
\left. - \Delta_{\dot{c}\beta}\Theta'_z\Delta_{\dot{\xi}c} - \Delta_{\dot{\xi}c}\Delta_{\dot{c}\xi}(r+\sigma) + \Delta_{\dot{\xi}c}\Delta_{\dot{c}\xi}\sigma + (1-r\sigma)\Delta_{\dot{\beta}c}\Delta_{\dot{c}\beta}\right]\lambda \\
+ \sigma r\Delta_{\dot{c}\beta}\Delta_{\dot{\beta}z} + \sigma\Delta_{\dot{c}\xi}\Delta_{\dot{\xi}z}(r+\sigma) + \sigma\Delta_{\dot{c}\beta}\Delta_{\dot{\xi}z}\Theta'_z - \sigma\Delta_{\dot{c}z}r(r+\sigma) \\
+ \sigma\Delta_{\dot{\xi}c}\Delta_{\dot{c}\beta}\Theta'_z + \sigma\Delta_{\dot{\xi}c}\Delta_{\dot{c}\xi}(r+\sigma) + \sigma\Delta_{\dot{c}\beta}\Delta_{\dot{\beta}c}r.
\end{aligned}$$

In order to prove that the real and complex roots of the fourth-order polynomial equation $\mathcal{Q}(\lambda) = 0$ do actually correspond to the ones of the statement, it suffices to establish that if λ is a root of $\mathcal{Q}(\lambda) = 0$, then $r - \lambda$ is similarly a root of $\mathcal{Q}(\lambda) = 0$.

First assuming that $\mathcal{Q}(\lambda) = 0$, a direct computation gives $\mathcal{Q}(r-\lambda) - \mathcal{Q}(\lambda) = 0$, that in turn implies $\mathcal{Q}(r-\lambda) = 0$. The roots of $\mathcal{Q}(\lambda) = 0$ thus exhibit a paired root structure along $\lambda_i, r-\lambda_i, i = 1, 2$. Denoting $\mu_i := \lambda_i(r-\lambda_i)$, $\mathcal{Q}(\lambda)$ reformulates as :

$$\begin{aligned}
\mathcal{Q}(\lambda) &= (\lambda-\lambda_1)(\lambda-r+\lambda_1)(\lambda-\lambda_2)(\lambda-r+\lambda_2) \\
&= \lambda^4 - 2r\lambda^3 + (r^2+\mu_1+\mu_2)\lambda^2 - r(\mu_1+\mu_2)\lambda + \mu_1\mu_2.
\end{aligned}$$

Identifying with the computed expression of $\mathcal{Q}(\lambda)$, it derives that:

$$\begin{aligned}
\mu_1+\mu_2 &= -\frac{\sigma}{\Theta''_{cc}}\left[(r+\sigma)\Theta''_{cc} + (2\sigma+r)\Theta''_{cz} + \sigma\Theta''_{zz}\right] + \frac{r\,(\Theta'_c)^2}{\Theta''_{cc}}, \\
\mu_1\mu_2 &= -\frac{\sigma r}{\Theta''_{cc}}\left(\Theta'_c+\Theta'_z\right)\left[\sigma\Theta'_z + (r+\sigma)\Theta'_c\right].
\end{aligned}$$

The details of the statement follow. △

VIII.6 – Proof of Proposition 2.

The subsequent statement is first intended to provide a general picture of the structure of the set of the eigenvalues when they display the particular structure raised by Lemma 4.

Lemma A.6.1. *Let $r > 0$ and consider the four unknowns λ_1, $r-\lambda_1$, λ_2, $r-\lambda_2$ defined by $\mu_1 = \lambda_1(r-\lambda_1)$, $\mu_2 = \lambda_2(r-\lambda_2)$, where $\mu_1, \mu_2 \in \mathbb{C}$ and satisfy $\mu_1+\mu_2 = \mathcal{S}$ and $\mu_1\mu_2 = \mathcal{P}$, $(\mathcal{S},\mathcal{P}) \in \mathbb{R}^2$ given. Consider then the following $(\mathcal{S},\mathcal{P})$ plane boundaries graphed over Figure 2:*

$$\begin{aligned}
\mathcal{F}_1 &:= \left\{(\mathcal{S},\mathcal{P}) \in \mathbb{R}^2_+ \mid \mathcal{P} = \mathcal{S}(\mathcal{S}/2+r^2)/2\right\}, \\
\mathcal{F}_2 &:= \left\{(\mathcal{S},\mathcal{P}) \in \mathbb{R}\times\mathbb{R}_+ \mid \mathcal{P} = \mathcal{S}^2/4\right\}, \\
\mathcal{F}_3 &:= \left\{(\mathcal{S},\mathcal{P}) \in \mathbb{R}^2 \mid \mathcal{P} = r^2(\mathcal{S}-r^2/4)/4\right\},
\end{aligned}$$

where it is already noticed that $\mathcal{F}_2$ assumes a tangency with $\mathcal{F}_3$ at $\left(r^2/2, r^4/16\right)$. Consider then the areas:

$$
\begin{aligned}
\mathcal{R}_1 &:= \left\{(\mathcal{S},\mathcal{P}) \in \mathbb{R}\times\mathbb{R}_+ \mid \mathcal{P} > \mathcal{S}\left(\mathcal{S}/2 + r^2\right)/2 \text{ for } \mathcal{S} > 0, \quad \mathcal{P} > \mathcal{S}^2/4 \text{ for } \mathcal{S} < 0\right\},\\
\mathcal{R}_2 &:= \left\{(\mathcal{S},\mathcal{P}) \in \mathbb{R}_+^2 \mid \mathcal{S}^2/4 < \mathcal{P} < \mathcal{S}\left(\mathcal{S}/2 + r^2\right)/2\right\},\\
\mathcal{R}_2' &:= \left\{(\mathcal{S},\mathcal{P}) \in \mathbb{R}_+^2 \mid \mathcal{P} > r^2\left(\mathcal{S} - r^2/4\right)/4, \quad \mathcal{S} > r^2/2, \quad \mathcal{P} < \mathcal{S}^2/4\right\},\\
\mathcal{R}_3 &:= \left\{(\mathcal{S},\mathcal{P}) \in \mathbb{R}_+^2 \mid \mathcal{P} < \mathcal{S}^2/4, \quad \mathcal{P} > r^2\left(\mathcal{S} - r^2/4\right)/4, \mathcal{S} < r^2/2\right\},\\
\mathcal{R}_4 &:= \left\{(\mathcal{S},\mathcal{P}) \in \mathbb{R}_+^2 \mid \mathcal{S} > r^2/2, \quad \mathcal{P} < r^2\left(\mathcal{S} - r^2/4\right)/4\right\},\\
\mathcal{R}_5 &:= \left\{(\mathcal{S},\mathcal{P}) \in \mathbb{R}\times\mathbb{R}_- \mid \mathcal{P} < r^2\left(\mathcal{S} - r^2/4\right)/4, \quad \mathcal{P} < 0\right\},\\
\mathcal{R}_6 &:= \left\{(\mathcal{S},\mathcal{P}) \in \mathbb{R}\times\mathbb{R}_- \mid r^2\left(\mathcal{S} - r^2/4\right)/4 < \mathcal{P} < 0\right\},\\
\mathcal{R}_7 &:= \left\{(\mathcal{S},\mathcal{P}) \in \mathbb{R}_-\times\mathbb{R}_+ \mid \mathcal{P} < \mathcal{S}^2/4\right\}.
\end{aligned}
$$

The 4-uple of unknowns $\lambda_1, r-\lambda_1, \lambda_2, r-\lambda_2$ then satisfies:

- *within $\mathcal{R}_1$: four complex solutions, two with positive real parts, two with negative real parts ;*
- *within $\mathcal{R}_2$: four complex solutions with positive real parts ;*
- *within $\mathcal{F}_1$: boundary between $\mathcal{R}_1$ and $\mathcal{R}_2$, with four complex roots, two with positive real parts, two purely imaginary eigenvalues ;*
- *within $\mathcal{R}_2'$: four complex solutions with positive real parts ;*
- *within $\mathcal{R}_3$: four positive real roots ;*
- *within $\mathcal{R}_4$: two positive real roots, two complex roots with positive real parts ;*
- *within $\mathcal{R}_5$: two complex roots with positive real parts, one positive real root, one negative real root ;*
- *within $\mathcal{R}_6$: three positive real roots, one negative real root ;*
- *within $\mathcal{R}_7$: two positive real roots, two negative real roots.*

<u>Proof :</u> Consider the defining equation of μ_1 and μ_2, *i.e.*, $\mu^2 - \mathcal{S}\mu + \mathcal{P} = 0$, and remember that $\mathcal{F}_2$ features the separation between the area delimited by $\mathcal{P} > \mathcal{S}^2/4$ where μ_1 and μ_2 are complex from the one for which $\mathcal{P} < \mathcal{S}^2/4$ where μ_1 and μ_2 are real.

1– First consider the area $\mathcal{R}_7$: the holding of $\mathcal{P} < \mathcal{S}^2/4$ implies that μ_1 and μ_2 are real whilst the simultaneous holding of $\mathcal{P} > 0$ and $\mathcal{S} < 0$ implies that μ_1 and μ_2 are strictly negative. The defining equation $\mu_i = \lambda_i\left(r - \lambda_i\right)$ restating as $\left(\lambda_i\right)^2 - r\lambda_i + \mu_i = 0$ assumes two real roots, the first being positive whilst the second one, due to the holding of $\mu_i < 0$, is negative.

2– Consider the areas $\mathcal{R}_1$ and $\mathcal{R}_2$. Both of these zones being located above $\mathcal{F}_2$, μ_1 and μ_2 are complex conjugate and thus express according to $\mu_1 = \lambda_1\left(r - \lambda_1\right)$, $\mu_2 = \bar{\mu}_1 = \bar{\lambda}_1\left(r - \bar{\lambda}_1\right)$. For a given eigenvalue, *e.g.*, λ_1, the expressions of the other elements of the spectrum immediatly follow : $r - \lambda_1$, $\bar{\lambda}_1$ and $r - \bar{\lambda}_1$. Hence letting $\lambda_1 = \alpha + i\beta$, it derives that:

$$\mu_1 = \alpha(r - \alpha) + \beta^2 + i\beta(r - 2\alpha),$$
$$\mu_2 = \alpha(r - \alpha) + \beta^2 - i\beta(r - 2\alpha),$$
$$\mathcal{P} = \left[\alpha(r - \alpha) + \beta^2\right]^2 + \beta^2(r - 2\alpha)^2,$$
$$\mathcal{S} = 2\left[\alpha(r - \alpha) + \beta^2\right].$$

Consider then the expression that defines the boundary $\mathcal{F}_1$:

$$\mathcal{P} - \mathcal{S}(\mathcal{S}/2 + r^2)/2 = \alpha(\alpha - r)\left[4\beta^2 + r^2\right].$$

For $\mathcal{P} > \mathcal{S}(\mathcal{S}/2 + r^2)/2$, $\alpha < 0$ (and $r - \alpha > 0$) or $\alpha > r$ (and $r - \alpha < 0$). It derives that two amongst the four roots $\{\lambda_1, r - \lambda_1, \bar{\lambda}_1, r - \bar{\lambda}_1\}$ have positive real parts whilst the others have negative real parts.
For $\mathcal{P} < \mathcal{S}(\mathcal{S}/2 + r^2)/2$, $0 < \alpha < r$ and the four roots $\{\lambda_1, r - \lambda_1, \bar{\lambda}_1, r - \bar{\lambda}_1\}$ have positive real parts.
For $\mathcal{P} = \mathcal{S}(\mathcal{S}/2 + r^2)/2$, $\alpha = 0$ or $\alpha = r$ and two amongst the four roots $\{\lambda_1, r - \lambda_1, \bar{\lambda}_1, r - \bar{\lambda}_1\}$ have a nil real part.

3– Consider the areas underneath $\mathcal{F}_2$: μ_1 and μ_2 being real, the areas $\mathcal{R}_2'$, $\mathcal{R}_3$ and $\mathcal{R}_4$ are all defined for $\mathcal{S} > 0$ and $\mathcal{P} > 0$, that implies $\mu_1 > 0$ and $\mu_2 > 0$. The two equations to be solved are $\mu_1 = \lambda_1(r - \lambda_1)$ and $\mu_2 = \lambda_2(r - \lambda_2)$, the associated discriminants being expressed by $\Delta_1 = r^2 - 4\mu_1$ and $\Delta_2 = r^2 - 4\mu_2$. Consider then the expression:

$$X := \mathcal{P} - r^2(\mathcal{S} - r^2/4)/4,$$
$$= \mu_1\mu_2 - r^2(\mu_1 + \mu_2 - r^2/4)/4$$
$$= (\mu_1 - r^2/4)(\mu_2 - r^2/4).$$

3.(i)– Consider the area $\mathcal{R}_2'$: as $\mathcal{S} > r^2/2$ and $X > 0$ simultaneously hold, it derives that $\mu_1 > r^2/4$ and $\mu_2 > r^2/4$, that in turn implies a structure with four pairwise conjugate eigenvalues with $\bar{\lambda}_1 = r - \lambda_1$ and $\bar{\lambda}_2 = r - \lambda_2$. The real part α of λ_i being to satisfy $\alpha = r - \alpha$, it is obtained that $\alpha = r/2 > 0$. This finally gives rise to a structure with four complex eigenvalues with positive real parts.

3.(ii)– Consider the area $\mathcal{R}_3$: as $\mathcal{S} < r^2/2$ and $X > 0$ simultaneously hold, it derives that $\mu_1 < r^2/4$ and $\mu_2 < r^2/4$, hence a structure with four real roots with $\mu_1 > 0$ and $\mu_2 > 0$. The sum of λ_i and $r - \lambda_i$ being $r > 0$, the associated product being $\mu_i > 0$, it is finally obtained that the system assumes four positive real roots.

3.(iii)– Consider the area $\mathcal{R}_4$: as $X < 0$, one has $\mu_1 > r^2/4$ and $\mu_2 < r^2/4$ (or $\mu_1 < r^2/4$ and $\mu_2 > r^2/4$, μ_1 and μ_2 assuming a symmetrical role). Both λ_1 and $r - \lambda_1$ being complex conjugate eigenvalues, their real part is equal to $r/2$. Both λ_2 and $r - \lambda_2$ being real eigenvalues, their sum is given by $r > 0$ whilst their product expresses as $\mu_2 > 0$, that finally lead to two positive real eigenvalues. To sum up, over $\mathcal{R}_4$, the system exhibits two complex eigenvalues with positive real parts and two positive real eigenvalues.

4– Consider the zone located under the boundary $\mathcal{F}_2$, associated with real values for μ_1 and μ_2 and the areas $\mathcal{R}_5$ and $\mathcal{R}_6$. As $\mathcal{P} < 0$, it is obtained, that $\mu_1 > 0$ and $\mu_2 < 0$.

4.(i)– Consider then the area $\mathcal{R}_5$: as $X < 0$ $\left(\mathcal{R}_5\right.$ is located under $\left.\mathcal{F}_3\right)$ and $\mu_2 - r^2/4 < 0$ imply $\mu_1 < r^2/4$, both λ_1 and $r - \lambda_1$ are complex conjugate. Their real part is in its turn necessarily given by $r/2$. λ_2 and $r - \lambda_2$ being further real with a product given by $\mu_2 < 0$, one of the two is negative whilst the remaining one is positive. To sum up, the area $\mathcal{R}_5$ is associated with two complex eigenvalues with positive real parts and two real eigenvalues, one of the two is negative while the outstanding one is positive.

4.(ii)– Finally consider the area $\mathcal{R}_6$: as $X > 0$ $\left(\mathcal{R}_6\right.$ is located above $\left.\mathcal{F}_3\right)$ and $\mu_2 - r^2/4 < 0$ imply $\mu_1 < r^2/4$, $\Delta_1 > 0$ and $\Delta_2 > 0$. The equation $\mu_1 = \lambda_1(r - \lambda_1)$ gives rise to two positive real eigenvalues, their sum being given by $r > 0$ whilst their product is given by $\mu_1 > 0$. In parallel to this, the equation $\mu_2 = \lambda_2(r - \lambda_2)$ gives rise to two real eigenvalues, one of which is positive whilst the outstanding one is negative, their sum and their product being respectively given by r and $\mu_2 < 0$. △

Proof of Proposition 2.

(i) Under Assumption 2, the high level unsatiated steady state $(\bar{c}_2, \bar{\beta}_2, \bar{z}_2, \bar{\xi}_2, \bar{a}_2)$ is characterised by the holding of $\mu_1\mu_2 < 0$. But, and from Lemma 4, this is to correspond to areas $\mathcal{R}_5$ and $\mathcal{R}_6$ over Figure 2, whence the result.

(ii) Under Assumption 2, the high level unsatiated steady state $(\bar{c}_1, \bar{\beta}_1, \bar{z}_1, \bar{\xi}_1, \bar{a}_1)$ is characterised by the holding of $\mu_1\mu_2 > 0$ whilst both $\mu_1 + \mu_2 > 0$ and $\mu_1 + \mu_2 \leq 0$ remain admissible. Case a/ is then obtained for areas $\mathcal{R}_7$ and $\mathcal{R}_1$ over Figure 2 whilst case b/ is available for areas $\mathcal{R}_2$, $\mathcal{R}_2'$, $\mathcal{R}_3$ and $\mathcal{R}_4$ over Figure 2 and, finally, case c/ emerges for the boundary $\mathcal{F}_1$ over Figure 2. △

VIII.7 – Proof of Proposition 3.

(i) Reformulating the subsystem associated to satiated equilbrium dynamics according to:

$$\dot{c} = \Delta_{\dot{c}}(c, z, \xi),$$
$$\dot{z} = \Delta_{\dot{z}}(c, z, \xi),$$
$$\dot{\xi} = \Delta_{\dot{\xi}}(c, z, \xi),$$

considering its linearised form in the neighbourhood of a satiated steady state, it is readily checked that the components of the Jacobian matrix list as:

$$\begin{bmatrix} \Delta_{\dot{c}c} & \Delta_{\dot{c}z} & \Delta_{\dot{c}\xi} \\ \Delta_{\dot{z}c} & \Delta_{\dot{z}z} & \Delta_{\dot{z}\xi} \\ \Delta_{\dot{\xi}c} & \Delta_{\dot{\xi}z} & \Delta_{\dot{\xi}\xi} \end{bmatrix} = \begin{bmatrix} \Theta + \sigma & \dfrac{(\Theta + 2\sigma)\Theta''_{cz} + \sigma\Theta''_{zz}}{\Theta''_{cc}} & \dfrac{\Theta'_c\Theta^2}{\Theta''_{cc}} \\ \sigma & -\sigma & 0, \\ -\dfrac{\Theta'_c}{\Theta} & -\dfrac{\Theta'_z}{\Theta} & \Theta \end{bmatrix}$$

Computing the associated characteristic polynomial $\mathcal{P}(\lambda)$ delivers:

$$\begin{aligned} \mathcal{P}(\lambda) &= -\lambda^3 + \left[\Theta + \Delta_{\dot{c}c} - \sigma\right]\lambda^2 + \left[\sigma\Delta_{\dot{c}c} + \sigma\Theta - \Delta_{\dot{c}c}\Theta + \Delta_{\dot{c}\xi}\Delta_{\dot{\xi}c} + \sigma\Delta_{\dot{c}z}\right]\lambda \\ &\qquad - \sigma\Theta\Delta_{\dot{c}c} + \sigma\Delta_{\dot{\xi}z}\Delta_{\dot{c}\xi} + \sigma\Delta_{\dot{\xi}c}\Delta_{\dot{c}\xi} - \sigma\Delta_{\dot{c}z}\Theta \\ &= (\Theta - \lambda)\,\Upsilon(\lambda), \\ \Upsilon(\lambda) &= \lambda^2 - \Theta\lambda - \frac{1}{\Theta''_{cc}}\left[\Theta''_{cc}\sigma(\Theta + \sigma) + \sigma(\Theta + 2\sigma)\Theta''_{cz} + \sigma^2\Theta''_{zz} - \Theta\left(\Theta'_c\right)^2\right]. \end{aligned}$$

This establishes that, aside from the obvious eigenvalue Θ, the sum of the two outstanding ones is also given by Θ while their product is given by $-\sigma\left[\Theta''_{cc}(\Theta+\sigma)+(\Theta+2\sigma)\Theta''_{cz}+\sigma\Theta''_{zz} - \Theta\left(\Theta'_c\right)^2/\sigma\right]/\Theta''_{cc}$.

(ii) It is first remarked that the sign of $\nu_1\nu_2$ is given by the one of $\sigma\left(\Theta''_{cc}+\Theta''_{zz}+2\Theta''_{cz}\right)+\left(\Theta''_{cc}+\Theta''_{cz}\right)\Theta-\Theta\left(\Theta'_c\right)^2/\sigma$. Further noticing that, at the satiated steady state $\tilde{c}$, $\Theta'_c(\Theta'_c+\Theta'_z)=\Theta'_c\left[\Theta'_c-\Theta'_c(\Theta+\sigma)/\sigma\right]=-\Theta\left(\Theta'_c\right)^2/\sigma$, under Assumption 2, it is finally obtained that $\nu_1\nu_2=\Omega'(\tilde{c})<0$. Since $\nu_1+\nu_2=\Theta(\tilde{c},\tilde{c})>0$, ν_1 and ν_2 are two real eigenvalues of opposite signs. The statement follows. △

VIII.8 – Proof of Lemma 5.

Considering the dynamical system associated with the Jacobian Matrix at the unsatiated steady state $\left(\bar{c}_1,\bar{\beta}_1,\bar{z}_1,\bar{\xi}_1,\bar{a}_1\right)$, its solving allows for expressing z as :

$$z(t)=\bar{c}_1+a_1\exp\left(\lambda_1 t\right)+b_1\exp\left(\lambda_2 t\right),$$

for λ_1 and λ_2 the two stable eigenvalues (of negative signs in the real case or with negative real parts in the complex case) while a_1 and b_1 stand for two constant terms. Remark that for $\lambda_1,\lambda_2\in\mathbb{C}$, they are complex conjugate and one then gets $\bar{a}_1=b_1$, so that $z(t)$ is real.
A related expression similarly becomes available for a :

$$a(t)=\frac{\bar{c}_1-w}{r}+a_2\exp\left(\lambda_1 t\right)+b_2\exp\left(\lambda_2 t\right),$$

with a_2 and b_2 two constant terms.
Then looking for expressing $c(t)$ as a function of $z(t)$ and $a(t)$, the defining equation (1) for $z(t)$ first delivers :

$$c(t)=\bar{c}_1+a_1\left(1+\lambda_1/\sigma\right)\exp\left(\lambda_1 t\right)+b_1\left(1+\lambda_2/\sigma\right)\exp\left(\lambda_2 t\right).$$

Similarly, the law of motion of $a(t)$ available as (6) in turn gives :

$$c(t)=\bar{c}_1+a_2\left(r-\lambda_1\right)\exp\left(\lambda_1 t\right)+b_2\left(r-\lambda_2\right)\exp\left(\lambda_2 t\right).$$

Then stating $c(t)=xz(t)+ya(t)$, for x and y two constant terms, and finally identifying :

$$a_1\left(1+\lambda_1/\sigma\right)=xa_1+ya_2,$$
$$b_1\left(1+\lambda_2/\sigma\right)=xb_1+yb_2$$

with the relations

$$a_1\left(1+\lambda_1/\sigma\right)=a_2\left(r-\lambda_1\right),$$
$$b_1\left(1+\lambda_2/\sigma\right)=b_2\left(r-\lambda_2\right),$$

one obtains that :

$$1+\lambda_1/\sigma=x+y\frac{1+\lambda_1/\sigma}{r-\lambda_1},$$
$$1+\lambda_2/\sigma=x+y\frac{1+\lambda_2/\sigma}{r-\lambda_2},$$

that delivers :

$$x=\left(1+\frac{\lambda_1}{\sigma}\right)\left(1+\frac{\lambda_2}{\sigma}\right)\frac{\sigma}{r+\sigma}.$$

Whence $x>0\iff\left(1+\lambda_1/\sigma\right)\left(1+\lambda_2/\sigma\right)>0$ and the statement follows. △

VIII.9 – Proof of Proposition 4.

From Lemma 5, addiction is directly related to the properties of $(1 + \lambda_i/\sigma)$, λ_i, $i = 1, 2$, denoting the stable eigenvalues. In order to characterise these properties, let $x = 1 + \lambda/\sigma$ and replace λ by $\sigma(x - 1)$ in the expression of the characteristic polynomial. It is then obtained that x solves $Q(x) = 0$ for

$$Q(x) = \sigma^4 x^4 - 2\sigma^3(2\sigma + r)x^3 + \sigma^2\big(6\sigma^2 + 6r\sigma + r^2 + \mathcal{S}\big)x^2 \\ - \sigma(2\sigma + r)[2\sigma(r + \sigma) + \mathcal{S}]x + \mathcal{P} + (r + \sigma)\sigma\mathcal{S} + \sigma^2(r + \sigma)^2,$$

and $\mathcal{P} = \mu_1\mu_2$ and $\mathcal{S} = \mu_1 + \mu_2$.

It is first readily checked that the constant term in the above polynomial formulates as :

$$\mathcal{P} + (r + \sigma)\sigma\mathcal{S} + \sigma^2(r + \sigma)^2 = \frac{\partial\varrho}{\partial z}\left\{\left(\Theta'_c + \frac{\sigma}{r + \sigma}\Theta'_z\right)(r + 2\sigma)(r + \sigma)\sigma\right\} \Big/ \left(-\Theta''_{cc}\right)$$

whose sign is given by $\partial\varrho/\partial z$.

When $\partial\varrho/\partial z = 0$, the defining equation of the straightline $\mathcal{F}_4$ becomes available as $\mathcal{P} + (r + \sigma)\sigma\mathcal{S} + \sigma^2(r + \sigma)^2 = 0$. It is also noticed that the four solutions $x_i = 1 + \lambda_i/\sigma$ of the equation $\mathcal{Q}(x) = 0$ assume a product whose sign is the one of $\partial\varrho/\partial z$.

Besides, the frontier $\mathcal{F}_4$ being tangent from below to $\mathcal{F}_2$ at the point $\mathcal{S} = -2\sigma(r + \sigma)$, it is hence located within the area $\mathcal{R}_7$ where the eigenvalues λ_i are real, two of them being negative while the other ones are positive.

When $\lambda_i > 0$ holds, one also obtains that $x_i > 0$. In opposition to this and when $\lambda_i < 0$, both $x_i < 0$ and $x_i > 0$ are conceivable.

On the L.H.S. of $\mathcal{F}_4$, since $\partial\varrho/\partial z < 0$ holds therein and since $\mathcal{Q}(x) = 0$ assumes at least two positive solutions, it is derived that $\mathcal{Q}(\cdot)$ assumes three positive eigenvalues and one negative eigenvalue.

On the R.H.S. of $\mathcal{F}_4$, a root of $\mathcal{Q}(\cdot)$ is to undergo a change in sign. In order to more precisely characterise this case, one focuses on the variation with respect to $\partial\varrho/\partial z$ of the root x_i that is nil over $\mathcal{F}_4$ and subsequently changes sign. It is obtained that :

$$\left.\frac{dx_i}{d(\partial\varrho/\partial z)}\right|_{x_i = 0} = \left\{\left(\Theta'_c + \frac{\sigma}{r + \sigma}\Theta'_z\right)(r + \sigma)\sigma\right\} \Big/ \left(-\Theta''_{cc}\right)\left[\mathcal{S} + 2\sigma(r + \sigma)\right]$$

whose sign is given by the one of $\mathcal{S} + 2\sigma(r + \sigma)$.

It is then derived that the R.H.S. of the boundary $\mathcal{F}_4$ features the inception of the addiction zona associated with four positive values of x_i for $\mathcal{S} > -2\sigma(r + \sigma)$ but two negative and two positive ones for $\mathcal{S} < -2\sigma(r + \sigma)$.

When the boundary $\mathcal{F}_2$ is in its turn attained, the two stable eigenvalues λ_i, $i = 1, 2$, are complex conjugate and their product $(1 + \lambda_1/\sigma)(1 + \lambda_2/\sigma)$ is of positive sign. △

VIII.10 – Proof of Lemma 6.

(i) Obvious.
(ii) a/ Obvious. b/ Letting $x = \sqrt{-\Theta''_{zz}}$ and analysing the function

$$g(x) = (r+\sigma)\Theta''_{cc} - \sigma x^2 + (2\sigma + r)\sqrt{-\Theta''_{cc}}x - r\big(\Theta'_c\big)^2/\sigma,$$

it derives that the latter assumes a global maximum for $g'_x(x) = 0$, that is $-2\sigma x + (2\sigma + r)\sqrt{-\Theta''_{cc}} = 0$ and

$$x = (1 + r/2\sigma)\sqrt{-\Theta''_{cc}}.$$

There then exists a value of $\Theta''_{zz} < 0$ satisfying $\mathcal{X}\big(\Theta''_{zz};\Theta''_{cc},\Theta'_c\big) > 0$ as soon as $g\big[(1 + r/2\sigma)\sqrt{-\Theta''_{cc}}\big] > 0$, that reformulates along $-\Theta''_{cc}r > 4\big(\Theta'_c\big)^2$. △

VIII.11 – Proof of Proposition 5.

(i) This directly follows from Lemma 6.
(ii) This is obtained by taking advantage of the set of equations that relate $\mathcal{P}$ and $\mathcal{S}$ to $(\partial\varrho/\partial c)c/\varrho$, $(\partial\varrho/\partial z)z/\varrho$ and Σ, namely

$$\mathcal{P} + \sigma(r+\sigma)\mathcal{S} + \sigma^2(r+\sigma)^2 = \left(\frac{\partial\varrho}{\partial z}\frac{z}{\varrho}\right)\Sigma\sigma(r+\sigma)(r+2\sigma)r,$$
$$\mathcal{P} = \left[\frac{\partial\varrho}{\partial c}\frac{c}{\varrho} + (r+\sigma)\frac{\partial\varrho}{\partial z}\frac{z}{\varrho}\right](r+\sigma)\sigma r\Sigma.$$

Strong addiction requires the satisfaction of $\mathcal{S} > 0$ for $\mathcal{P}$ sufficiently close from 0, that is obtained if :

$$\frac{\partial\varrho}{\partial z}\frac{z}{\varrho} > \frac{\sigma(r+\sigma)}{(r+2\sigma)r}\frac{1}{\Sigma},$$
$$\frac{\partial\varrho}{\partial c}\frac{c}{\varrho} + (r+\sigma)\frac{\partial\varrho}{\partial z}\frac{z}{\varrho} \longrightarrow 0.$$

The statement follows. △

VIII.12 – Derivation of the Characteristic Polynomial in a Production Economy.

The Jacobian Matrix in a neighbourhood of a unsatiated steady state $(\bar{c},\bar{\beta},\bar{z},\bar{\xi},\bar{x})$ is available as :

$$\begin{bmatrix} 0 & \dfrac{\sigma\Theta(\Theta+\sigma)}{\Theta''_{cc}} & \dfrac{\sigma\big(\Theta''_{cz}+\Theta''_{zz}\big)}{\Theta''_{cc}} & \dfrac{\Theta\big(\Theta\Theta'_c - \sigma\Theta'_z\big)}{\Theta''_{cc}} & -\dfrac{f''_{xx}}{\Theta''_{cc}}\left[\Theta'_c + \sigma\Theta'_z/(\Theta+\sigma)\right] \\ \beta\Theta'_c + \dfrac{\Theta''_{cz}}{\Theta} & \Theta+\sigma & \beta\Theta'_z + \frac{\Theta''_{zz}}{\Theta} & -\Theta'_z & 0 \\ \sigma & 0 & -\sigma & 0 & 0 \\ -\frac{\Theta'_c}{\Theta} & 0 & -\frac{\Theta'_z}{\Theta} & \Theta & 0 \\ -1 & 0 & 0 & 0 & f'_x \end{bmatrix}$$

where it is further recalled that an unsatiated steady stade is characterised by the holding of $f'_x = \Theta$.

It derives that the expression of the characteristic polynomial $\mathcal{Z}(\lambda)$ is nothing but the one of the partial equilibrium analysis computed through Appendix 5, namely $\mathcal{P}(\lambda)$, once it is augmented by an extra component that makes explicit account of the concavity of the production technology :

$$\mathcal{Z}(\lambda) = \mathcal{P}(\lambda) - \frac{f''_{xx}}{\Theta''_{cc}}\left[\Theta'_c + \sigma\Theta'_z/(\Theta+\sigma)\right](\Theta+\sigma-\lambda)(-\sigma-\lambda)(\Theta-\lambda).$$

It hence remains possible to complete a factorisation through $\Theta - \lambda$:

$$\mathcal{Z}(\lambda) = (\Theta-\lambda)\left\{\mathcal{Q}(\lambda) - \frac{f''_{xx}}{\Theta''_{cc}}\left[\Theta'_c + \sigma\Theta'_z/(\Theta+\sigma)\right](\Theta+\sigma-\lambda)(-\sigma-\lambda)\right\},$$

the modified expressions of $\mathcal{S}$ and $\mathcal{P}$ being then immediate. △

VIII.13 – Proof of Proposition 6.

(i) The method of proof will essentially mimic the one followed for the Proof of Proposition 5. First consider the expression of $\mathcal{S}$:

$$\mathcal{S} = -\frac{\sigma}{\Theta''_{cc}(\bar{c},\bar{c})}\left\{\left[\Theta(\bar{c},\bar{c})+\sigma\right]\Theta''_{cc}(\bar{c},\bar{c}) + \left[\Theta(\bar{c},\bar{c})+2\sigma\right]\Theta''_{cz}(\bar{c},\bar{c}) + \sigma\Theta''_{zz}(\bar{c},\bar{c})\right\}$$
$$+\frac{\Theta(\bar{c},\bar{c})\left[\Theta'_c(\bar{c},\bar{c})\right]^2}{\Theta''_{cc}(\bar{c},\bar{c})} - \frac{\Theta'_c(\bar{c},\bar{c}) + \sigma\Theta'_z(\bar{c},\bar{c})/\left[\sigma+\Theta(\bar{c},\bar{c})\right]}{\Theta''_{cc}(\bar{c},\bar{c})} f''_{xx}(\bar{x}),$$

From Appendix 10, the first component of $\mathcal{S}$ reaches its maximum for $\Theta''_{cz} = \sqrt{\Theta''_{cc}\Theta''_{zz}}$ for $\sqrt{-\Theta''_{zz}}$ selected so as to satisfy

$$\sqrt{-\Theta''_{zz}} = (1+\Theta/2\sigma)\sqrt{-\Theta''_{cc}}.$$

Under such a configuration for Θ''_{cz} and Θ''_{zz}, it is then obtained, omitting arguments, that :

$$\mathcal{S} = \frac{\Theta^2}{4} + \frac{\Theta\left(\Theta'_c\right)^2}{\Theta''_{cc}} - \left(\Theta'_c + \frac{\sigma\Theta'_z}{\sigma+\Theta}\right)\frac{f''_{xx}}{\Theta''_{cc}}.$$

Finally, and in order to obtain $\mathcal{P} \leq \mathcal{S}(\mathcal{S}/2+\Theta^2)/2$, one is to let $\mathcal{P}$ arbitrarily low by letting Θ'_z tend towards its minimal admissible value, namely $-\Theta'_c + f''_{xx}/f'_x$. For this value of Θ'_z, $\mathcal{S}$ is to be positive, or :

$$\frac{\Theta^2}{4} + \frac{\Theta\left(\Theta'_c\right)^2}{\Theta''_{cc}} - \left[\Theta'_c + \frac{\sigma}{\sigma+\Theta}\left(-\Theta'_c + \frac{f''_{xx}}{f'_x}\right)\right]\frac{f''_{xx}}{\Theta''_{cc}} > 0.$$

Simplifying, this eventually gives :

$$-\frac{\Theta''_{cc}\Theta}{4} - \left(\Theta'_c\right)^2 + \left(\Theta\Theta'_c + \sigma\frac{f''_{xx}}{f'_x}\right)\frac{f''_{xx}}{f'_x}\frac{1}{(\Theta+\sigma)} > 0.$$

(ii)a/ Equation (17) being let unmodified by the consideration of a production economy, the property identified through the earlier partial equilibirum argument continues to prevail.

(ii)b/ The result follows from the reexpression of $\mathcal{P}$ along :

$$\mathcal{P} = \sigma(\Theta+\sigma)\Theta\Sigma\left\{\frac{\partial\varrho}{\partial c}\frac{c}{\varrho} + (\Theta+\sigma)\frac{\partial\varrho}{\partial z}\frac{z}{\varrho} - \frac{f''_{xx}}{f'_x}\right\}.$$

Local unstablity then proceeds by letting $\mathcal{P}$ tend towards 0 that in turn delivers the condition of the main text. △

VIII.14 – A Constructive Example

This section will consider an example of a function $\Theta(\cdot,\cdot)$ that exhibits the range of properties that were listed in the course of the main text. As this will soon appear, this is a straightforward task as far as sections II and III are concerned. In opposition to this, the look for an appropriate parametric form reveals as being more involving when one is further to integrate the class of properties detailed in Section V.

The approach shall essentially proceed as follows. A simple parametric form that fits with the assumptions of sections II and III is first going to be introduced. It will later on be amended in order to make an explicit account of the extra properties considered through Section V. Both addiction and strong addiction phenomena happen to be conceivable.

a / A Benchmark Example

Let the function $\Theta(\cdot,\cdot)$ be defined as :

$$\Theta(c,z) = C_0 + Ac^{\alpha} - B(z+b)^{\beta},$$

where C_0, A, B and b denote positive constants, $0<\alpha<1$ and $\beta>1$, b being further such that $C_0 - Bb^{\beta} > 0$.

The function $\Theta(\cdot,\cdot)$ is strictly concave, of class C^2, increases as a function of c but decreases as a function of z, whence the satisfaction of Assumption 1. That $\Theta(c,c)$ is a unimodal function is similarly checked while noticing that it reaches its maximum for c^* that solves $A\alpha\left(c^*\right)^{\alpha-1} = B\beta\left(c^*+b\right)^{\beta-1}$. It thus satisfies Assumption 2(i). As for Assumption 2(ii), it is satisfied if the following inequality holds :

$$C_0 < r < \Theta\left(c^*, c^*\right).$$

The first inequality can always be satisfied through sufficiently small values for C_0 (and thus for a sufficiently small value for b so that $Bz^{\beta} < C_0$).[5]

Finally facing with Assumption 2(iii), the existence and the uniqueness of $\tilde{c}$ can be established. Reformulating the definition of $\tilde{c}$ along

$$\Theta\left(\tilde{c},\tilde{c}\right) + \sigma = -\sigma\frac{\Theta'_z\left(\tilde{c},\tilde{c}\right)}{\Theta'_c\left(\tilde{c},\tilde{c}\right)},$$

[5] It is worthwhile noticing that the constant b does not assume a direct role at that stage and could admittedly be put to 0. Its importance will however emerge when the examination will be concerned with the topics of Section V.

or

$$C_0 + A(\tilde{c})^{\alpha} - B(\tilde{c} + b)^{\beta} + \sigma = \sigma \frac{B\beta(\tilde{c} + b)^{\beta-1}}{A\alpha(\tilde{c})^{\alpha-1}}.$$

Then recalling that, for c^*, the equality $\Theta'_c(c^*, c^*) = -\Theta'_z(c^*, c^*)$ holds, at that point, the R.H.S. of the above equation hence summarises to $\sigma < \sigma + \Theta(c^*, c^*)$. The R.H.S. further is a function that strictly increases with $\tilde{c}$ while the L.H.S. is a function that strictly decreases with this value over the interval $[c^*, c_{\text{Max}}]$, with $\Theta(c_{\text{Max}}, c_{\text{Max}}) = 0$. The existence and the uniqueness of $\tilde{c}$ are thus established.

To sum up and under the constraints that have just been found, the function $\Theta(\cdot,\cdot)$ satisfies the Assumptions 1 and 2 that underlie the exposition of sections II and III. One will then be allowed to apply to such a $\Theta(\cdot,\cdot)$ the results that emerged from section IV. Firstly noticing that $\Theta''_{cz} = 0$, $\forall c, z$, according to Proposition 2, the stationary point $\bar{c}_1$ is to exhibit a saddlepoint property since the sum $\mathcal{S} = \mu_1 + \mu_2$ considered at that point is of negative sign. From Section V, no addiction phenomenon can be obtained as long as $\mathcal{S} < 0$ prevails. In order to achieve strong addiction in the neighbourhood of the steady state $\bar{c}_1$, the benchmark example will be modified to allow for a positive sign for Θ''_{cz}.

b / A Modified Example with Strong Addiction.

This eventual subsection aims at building an explicit example for which the stationary state $\bar{c}_1$ displays the properties gathered in Proposition 5 and thus provides an explicit parametric illustration of strong addiction.

Let the function $\Theta(\cdot,\cdot)$ be defined as :

$$\Theta(c, z) = C_0 + Ac^{\alpha} - B(z + b)^{\beta} + C \int_{\bar{c}_1-\epsilon}^{c} \int_{\bar{c}_1-\epsilon}^{z} \text{Max}\left[0, 1 - \frac{(x - \bar{c}_1)^2}{\varepsilon^2} - \frac{(y - \bar{c}_1)^2}{\varepsilon^2}\right] dxdy,$$

with $\varepsilon > 0$, C being a positive constant.

The function $\text{Max}[0, 1 - (x - \bar{c}_1)^2/\varepsilon^2 - (y - \bar{c}_1)^2/\varepsilon^2]$ is everywhere nil but for $(x, y) \in \mathcal{B}((\bar{c}_1, \bar{c}_1), \epsilon)$, for $\mathcal{B}((\bar{c}_1, \bar{c}_1), \epsilon)$ that denotes the open ball with a center of $(\bar{c}_1, \bar{c}_1)$ and a radius of ε over the plane (c, z). As this is illustrated by Figure 7, for $\mathcal{B}((\bar{c}_1, \bar{c}_1), \epsilon)$, this function depicts a part of an elliptical paraboloïd whose summit is $(\bar{c}_1, \bar{c}_1)$. By construction, $\Theta''_{cz}(\bar{c}_1, \bar{c}_1) = C$ and $\Theta''_{cz}(c, z) = 0$ for every $(c, z) \notin \mathcal{B}((\bar{c}_1, \bar{c}_1), \epsilon)$.

— Please insert Figure 7 —

This new formulation for $\Theta(\cdot,\cdot)$ is hence to be understood as a modified version of the benchmark one when the latter is subject to a perturbation whose size is related to the value of ε and tends towards zero with ε. This perturbation coefficient is indeed non-negative and upper-bounded by $C \times 4\varepsilon^2$. The first-order derivatives of $\Theta(\cdot,\cdot)$ are :

$$\Theta'_c(c, z) = A\alpha c^{\alpha-1} + C \int_{\bar{c}_1-\epsilon}^{z} \text{Max}\left[0, 1 - \frac{(c - \bar{c}_1)^2}{\varepsilon^2} - \frac{(y - \bar{c}_1)^2}{\varepsilon^2}\right] dy,$$

$$\Theta'_z(c, z) = -B\beta(z + b)^{\beta-1} + C \int_{\bar{c}_1-\epsilon}^{c} \text{Max}\left[0, 1 - \frac{(x - \bar{c}_1)^2}{\varepsilon^2} - \frac{(z - \bar{c}_1)^2}{\varepsilon^2}\right] dx.$$

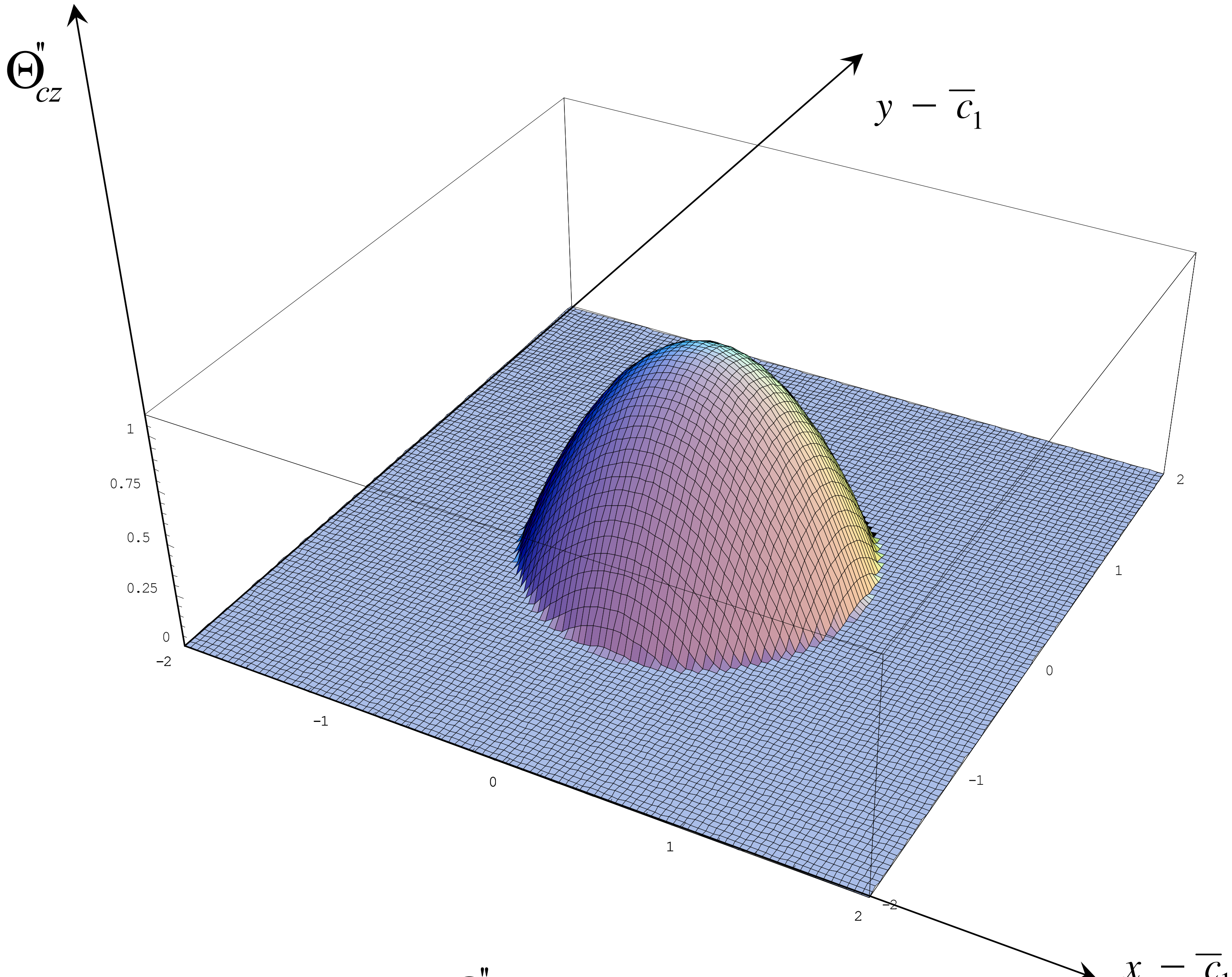

Figure 7: the shape of Θ''_{cz} for $\varepsilon = 1$

The extra components in these two expressions are both non negative and upper-bounded by $C \times 2\varepsilon$. While it is immediate that $\Theta'_c > 0$, a sufficient condition for the obtention of $\Theta'_z < 0$ writes down as :

$$-B\beta b^{\beta-1} + C \times 2\varepsilon < 0,$$

an inequality that will hold for small enough values of ε.
Facing then with the second-order derivatives of $\Theta(\cdot,\cdot)$, it is obtained that :

$$\Theta''_{cc} = -A\alpha(1-\alpha)c^{\alpha-2},$$
$$\Theta''_{zz} = -B\beta(\beta-1)(z+b)^{\beta-2},$$
$$\Theta''_{cz} = \begin{cases} 0 & \text{for} \quad (c,z) \notin \mathcal{B}\big((\bar{c}_1,\bar{c}_1),\epsilon\big) \\ C\left[0, 1 - \dfrac{(c-\bar{c}_1)^2}{\varepsilon^2} - \dfrac{(z-\bar{c}_1)^2}{\varepsilon^2}\right] & \text{for} \quad (c,z) \in \mathcal{B}\big((\bar{c}_1,\bar{c}_1),\epsilon\big), \end{cases}$$

where $\Theta''_{cz}(\bar{c}_1,\bar{c}_1) = C$. The strict concavity of $\Theta(\cdot,\cdot)$ is ensured for $\Theta''_{cc}\Theta''_{zz} > (\Theta''_{cz})^2$, that currently states as :

$$A\alpha(1-\alpha)B\beta(\beta-1)c^{\alpha-2}(z+b)^{\beta-2} > (\Theta''_{cz})^2.$$

Two cases are to be considered. For $(c,z) \notin \mathcal{B}\big((\bar{c}_1,\bar{c}_1),\epsilon\big)$, $\Theta''_{cz} = 0$. Recalling then that $0 < \alpha < 1$ and $\beta > 1$, it is immediate that this concavity condition is satisfied. For $(c,z) \in \mathcal{B}\big((\bar{c}_1,\bar{c}_1),\epsilon\big)$, a sufficient condition for concavity is obtained as the joint satisfaction of :

$$(i) \quad A\alpha(1-\alpha)B\beta(\beta-1)(\bar{c}_1)^{\alpha-2}(\bar{c}_1+b)^{\beta-2} > C^2 \qquad \text{and} \quad \varepsilon \quad \text{sufficiently small.}$$

Condition (i) imposes an upper bound on C. It is then immediate to check that the whole range of conditions listed through Assumptions 1 and 2 keep on being satisfied for the perturbated expression of $\Theta(\cdot,\cdot)$ as long as ε is sufficiently small.
Proposition 5 and Lemma 6 formulate conditions for the obtention of strong addiction. These conditions will here translate as restrictions on the parameters of $\Theta(\cdot,\cdot)$.
The condition $-\Theta''_{cc}r > 4(\Theta'_c)^2$ at $(\bar{c}_1,\bar{c}_1)$ will be satisfied if

$$rA\alpha(1-\alpha)(\bar{c}_1)^{\alpha-2} > 4\big[A\alpha(\bar{c}_1)^{\alpha-1}\big]^2 \qquad \text{and} \quad \varepsilon \quad \text{sufficiently small,}$$

that gives

$$(ii) \quad (1-\alpha)r > 4A\alpha(\bar{c}_1)^{\alpha} \qquad \text{and} \quad \varepsilon \quad \text{sufficiently small.}$$

This condition being ensured, Lemma 6 then establishes that Θ''_{zz} is to be selected so that $\Theta''_{zz} = (1 + r/2\sigma)^2\Theta''_{cc}$, that in turn gives :

$$(iii) \quad \beta(\beta-1)(\bar{c}_1+b)^{\beta-2} = \left(1+\frac{r}{2\sigma}\right)^2 A\alpha(1-\alpha)(\bar{c}_1)^{\alpha-2}.$$

...A.16...

Similarly, Θ'_z is to be sufficiently close from $-\Theta'_c$ $\left(\text{with } \Theta'_c > -\Theta'_z\right)$. From the expressions of Θ'_c and Θ'_z, it derives that for

$$(iv) \quad A\alpha\left(\bar{c}_1\right)^{\alpha-1} = B\beta\left(\bar{c}_1 + b\right)^{\beta-1},$$

$\Theta'_c\left(\bar{c}_1, \bar{c}_1\right)$ tends towards $-\Theta'_z\left(\bar{c}_1, \bar{c}_1\right)$, with $\Theta'_c\left(\bar{c}_1, \bar{c}_1\right) + \Theta'_z\left(\bar{c}_1, \bar{c}_1\right) > 0$, when ε tends towards 0.
Finally, from the definition of $\bar{c}_1$:

$$(v) \quad \Theta\left(\bar{c}_1, \bar{c}_1\right) = r,$$

and

$$(vi) \quad \lim_{\varepsilon \to 0} \Theta\left(\bar{c}_1, \bar{c}_1\right) = C_0 + A\left(\bar{c}_1\right)^{\alpha} - B\left(\bar{c}_1 + b\right)^{\beta}.$$

It remains to prove that there does exist parameters values such that all of the conditions (i), (ii),(iii), (iv) and (v) are simultaneously satisfied.
Firstly combining (iii) and (iv), it is obtained that :

$$(vii) \quad (\beta - 1)\bar{c}_1 = (1 - \alpha)\left(\bar{c}_1 + b\right)\left(1 + \frac{r}{2\alpha}\right)^2,$$

where it is noticed that the system $\left((iii), (iv)\right)$ is equivalent to the system $\left((iv), (vii)\right)$. Then muliplying (iv) and (vii) component by component, one gets :

$$(viii) \, A\alpha\left(\bar{c}_1\right)^{\alpha}(\beta - 1) = (1 - \alpha)B\beta\left(\bar{c}_1 + b\right)^{\beta}\left(1 + \frac{r}{2\sigma}\right)^2.$$

Making now use of (v) and taking advantage of (vi), condition (ii) will be satisfied as soon as

$$(ix) \quad (1 - \alpha)\left[C_0 + A\left(\bar{c}_1\right)^{\alpha} - B\left(\bar{c}_1 + b\right)^{\beta}\right] > 4A\alpha\left(\bar{c}_1\right)^{\alpha} \qquad \text{and} \quad \varepsilon \quad \text{sufficiently small,}$$

The formal condition in (ix) can be reformulated to :

$$(1 - \alpha)C_0 + A\left(\bar{c}_1\right)^{\alpha}(1 - 5\alpha) > (1 - \alpha)B\left(\bar{c}_1 + b\right)^{\beta}.$$

A stronger condition is obtained by letting $C_0 = 0$:

$$A\left(\bar{c}_1\right)^{\alpha}(1 - 5\alpha) > (1 - \alpha)B\left(\bar{c}_1 + b\right)^{\beta}$$

Finally, taking advantage of $(viii)$:

$$(x) \quad (1 - 5\alpha)\left(1 + \frac{r}{2\sigma}\right)^2 \beta > \alpha(\beta - 1).$$

To sum up, it has just been established that, for low enough values of ε, if (i), (x), (vii), (iv) and (v) are satisfied, then (i), (ii), (iii), (iv) and (v) are satisfied.

It remains to prove that there exists parameters values that satisfy all these conditions, that will fortunately reveal as a rather simple task.
Firstly selecting α and β so that they satisfy $\alpha < 1/5$ and $\beta > 2$ $\big($and thus $\beta > 2 - \alpha\big)$ and retaining any value of $\bar{c}_1$ that satisfies $\bar{c}_1 > 0$, (vii) formulates as

$$\frac{\beta - 1}{1 - \alpha} \frac{\bar{c}_1}{\bar{c}_1 + b} = \left(1 + \frac{r}{2\sigma}\right)^2.$$

Fixing b at a sufficiently small value, so that the L.H.S. is greater than 1, (vii) defines a unique value for the ratio r/α, that in particular increases as a function of α. Letting $(r/\sigma)_{\text{lim}}$ denote the limit value of this ratio when $\alpha \to 0$, and considering (x), it clearly appears that, for a small value of α,

$$(1 - 5\alpha)\left[1 + \frac{1}{2}\left(\frac{r}{\sigma}\right)_{\text{lim}}\right]^2 \beta > \alpha(\beta - 1).$$

Thus, for a sufficiently small value of α, condition (x) is satisfied, whereas (vii) defines a unique value of r/σ.
The parameters α and (r/σ) being henceforth fixed so that (vii) and (x) are jointly satisfied, one selects an arbitrary value for A. Equation (iv) will then determine a unique value for B. Equation (i) will fix the maximum admissible value for C. Finally, (v) fixes the value of r that is compatible with the value of $\bar{c}_1$ as a candidate for the steady state. The value of r being then determined and r/σ being fixed from (vii), the value of σ happens to be similarly fixed.
It has thus been shown that all of the conditions of Proposition 5 can simultaneously be satisfied, that guarantees the occurrence of a strong addiction phenomenon. It may further be noticed that, in the course of the construction of this example, two parameters, namely $\bar{c}_1$ and A, have been selected in a purely arbitrary way, *i.e.*, without any constraint. For this particular example, the occurrence of a strong addiction phenomenon in the neighbourhood of the stationary state $\bar{c}_1$ was easily obtained.