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Abstract

This article explores individual incentives to produce information on communication networks. In our setting, efforts are strategic complements along communication paths with convex decay. We analyze Nash equilibria on a set of networks which are unambiguous in terms of centrality. We first characterize both dominant and dominated equilibria. Second, we examine the issue of social coordination in order to reduce the social dilemma.

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1 Introduction

On communication networks, agents obtain both direct and indirect informational spillovers. Typically, agents may benefit from the information of their neighbors, the neighbors of their neighbors, and so on. The related theoretical literature is mainly concerned with the strategic formation of links. In this paper, we study the individual incentives to produce information, when agents produce efforts which are strategic complements[1] along communication paths. Let us present two series of relevant examples.

The first one concerns protective measures, like computer network security, airline baggage screening, fire safety in apartment buildings, infectious disease vaccination, protection against bankruptcy and theft (see Heal and Kunreuther [2003] and Kearns [2005]). To give a flavour, consider the resources devoted by air carriers to luggage screening for explosives; the network structure arises from baggage transfers between air carriers. Because risks are correlated, more protective measures set by my neighbor may improve the return of own increased protection. Further, the level of protection of my neighbor's neighbor has an incidence upon my own security level through the combination of her protection levels and that of my neighbor (the probability that a bomb passes from my neighbor's neighbor to me without being detected is a decreasing function of the protection level of her and of my neighbor).

A second class of applications concerns knowledge production, the cost of which may be lowered by social coordination. In the context of joint investments in research, some empirical literature is consistent with the existence of synergies along communication paths. Breschi and Lissoni (2006) and Singh (2005) suggest that knowledge spillovers may transit through the social network of inventors (insisting that the probability of knowledge flow decreases with social distance); Hanaki *et al.* (2007) recently show that the probability of alliance formation between firms in IT industry is increasing in the number of alliances situated at short distance from the partners (typically until distance 4). Education at school is another case. For example, Hoxby (2000) finds evidence of synergies in the performance of students in a classroom, and Calvo *et al.*

[1] Traditionally, economists conceive two categories of effort spillovers: efforts are either strategic complements or strategic substitutes. In the former (resp. latter) case, more spillovers increase (resp. decrease) incentives to produce effort.

(2006) present evidence of networked effects.

The existence of strategic complementarities in efforts naturally exhibits a social dilemma, in the sense that there exists Pareto-ranked equilibria, and particularly a Pareto-dominant equilibrium. Given that externalities are *networked*, we analyze the impact of the position of players on the issue of social dilemma. In particular, we ask the following questions: firstly, does there exist a simple characterization of the Pareto-dominant equilibrium in terms of players' positions on the network? Secondly, starting from a Pareto-dominated equilibrium, can we provide some appropriate procedure that would take account of the network topology, and which would lead to an equilibrium reducing the social dilemma? Our general answer is that players' centrality may help solving both questions.

To address these issues, we set up a model in which agents produce costly synergic efforts on a fixed communication network. The originality of our approach is that synergies exist between efforts of both directly and indirectly connected agents. More precisely, players aggregate the choice of others through an aggregator (which we name 'inflow of spillovers') which satisfies simple decay assumptions, and payoffs are strategic complements in the players' own actions and the aggregation of the other players' actions. The inflow of spillover is a function of the efforts of other agents along communication paths. We capture the communication aspect by assuming convex decay along communication paths. Our restrictions are mild; in particular, they encompass standard geometric decay (with possible upper bound on communication path length).

Some well-know results can be derived from our model. Notably, a dominant equilibrium exists (at the dominant equilibrium, every agent produces more effort than what she produces at any other equilibrium), and it Pareto-dominates all other equilibria. For the sake of clarity, we begin our analysis with the line network[2]. We first detect an equilibrium such that the ranking of efforts is not aligned with agents' centrality. Can the dominant equilibrium be such? A first proposition states that under very general conditions on utility functions, both dominant and dominated equilibria satisfy that more central agents produce larger effort levels (property P thereafter). This result,

[2] In our analysis, efforts are defined over a compact set of $\mathbb{R}$. Additional results in the context of binary effort choice on the line network are available upon request to the authors.

which may sound as rather intuitive at first glance, is technically hard to show.
First, fixing one equilibrium which both is symmetric (*i.e.*, agents with same structural position have same effort level) and does not satisfy property P, we build the smallest - resp. greatest - configuration among those that both dominate - resp. are dominated by - the equilibrium and satisfy property P; we name this latter configuration the *cover* - resp. *covered* - configuration of the former one. The first lemma establishes that if we initiate a simultaneous best-response algorithm (SBRA thereafter) from that cover - resp. covered - configuration, all individual efforts increase - resp. decrease - at each step of the algorithm. In short, the lemma presents a simple procedure, which uses the centrality of agents, and which enables to escape from an equilibrium that does not satisfy property P. A second lemma addresses the characterization of equilibria reached through a SBRA. It shows that a SBRA starting from a symmetric configuration converges to some equilibrium satisfying property P if the equilibrium dominates the cover configuration of the initial one. Finally, a proposition states that, when the initial configuration is the cover (resp. covered) configuration of a symmetric equilibrium, the 'if' part of the second lemma holds. The first proposition follows.
The first proposition characterizes the most socially desirable and undesirable equilibria in terms of the centralities of the agents. One immediate consequence is that, if one detects some equilibrium such that the respective rankings of efforts and centralities are not aligned, then for sure a social dilemma exists, although we are not in the worst case. The second proposition provides another message: if a social dilemma exists, and specifically if the equilibrium is both symmetric and does not satisfy property P, then some initial coordinated impulse to certain agents' efforts, and a dynamic procedure of myopic revisions, is sufficient to reach a configuration which is beneficial to all; furthermore this latter configuration satisfies property P so there is no chance to improve upon by replicating the procedure; last, the initial coordination of impulses is a task which is not demanding for the agents: it can be done by asking agents to adjust once their level of effort to that of one of their neighbor.

Keeping up the degree of generality of utility functions, we extend our results to a class of networks the individual centralities of which are unambiguous. We name it *hierarchical communities*. Roughly speaking, this architecture is a mixture of trees and symmetric networks, in such a way that neighbors have themselves a close number of neighbors[3]. The proofs exhibit no specific difficulty, except that we have to tackle with the following point: inflows are now a function of the values of all paths between agents (typically, one can think about functions like max, min, sum, average, $\cdots$). Then, selecting either max or sum functions, we generalize the proofs established on the line network to hierarchical communities networks.

Related literature: This model is related to coordination failures inherent to synergies and spillovers (Cooper and John [1988]). More specifically, this work inserts in two literatures.

First, our model is related to communication networks (the spillover aspect). In contrast with our article, the literature, issued from the pioneering works of Jackson and Wolinski (1996) and Bala and Goyal (2000), mainly focuses on strategic network formation, and does not assume endogenous efforts. Few models of communication network formation have an explicit treatment of decay[4]. With respect to this literature, our paper provides results under very mild restrictions on spillovers, that is we only need to assume that decay is convex.

Second, a literature addresses the issue of good production on networks (the endogenous efforts aspect). In Bramoullé and Kranton (2007) and Ballester *et al.* (2006), agents produce a costly effort and benefit from the effort of their neighbors. From these works, our model extends spillovers' channels to a communication network setting. Our model also encompasses the formulation of utility functions described in Ballester *et al.* (2006) in the case all efforts are strategic complements. One major interest of their paper is to show that, if the level of interdependencies is sufficiently low, there is a unique equilibrium in which efforts are proportional to a Bonacich-centrality measure; this measure, as well as uniqueness, is clearly tied to the choice of the utility function. In contrast, we provide results for a wide range of utility functions, and we allow multiple equilibria and related coordination failure issues. This generalization precipitates the restriction of our results to architectures which are unambiguous in terms of centrality measures.

[3]Interestingly, networks with high level of assortativity in degree are empirically documented in many social network analyses, like the internet network, research collaboration networks, etc.

[4]See Bloch and Dutta (2007), Fery (2007), Hojman and Szeidl (2006), Matsubayashi and Yamakawa (2006), Rogers (2005).

Finally, we mention that some papers address the issue of both network formation and endogenous effort, like Cho (2006), Goyal and Moraga (2001), Goyal and Vega-Redondo (2005), Cabrales *et al.* (2007), and Galeotti and Goyal (2007). This last article incorporates indirect spillovers with possible decay (in a context where efforts are strategic substitutes).

The article is organized as follows. The next section presents the model, section 3 is devoted to the characterization of equilibria on the line network, while section 4 extends the results to more general networks. Section 5 concludes. All proofs are included in the appendix.

2 The model

Let $N = \{1, ..., n\}$ be a finite set of agents, with $n \geq 3$. The effort of agent i is denoted δ_i. The definition set Δ is a compact in $\mathbb{R}$ and it is common to all agents. We denote by δ^l (resp. δ^h) the lower (resp. upper) bound of Δ. The effort level may represent the amount of time researchers spend inventing a new product. Throughout the article, superscripts refer to effort levels, subscripts to agents. A strategy profile $\vec{\delta} = (\delta_1, \cdots, \delta_n)$ may be denoted (δ_i, δ_{-i}) for convenience. Selecting effort level $\delta_i \in \Delta$, agent i incurs a fixed cost $c(\delta_i)$, with $c(.)$ strictly increasing and $c(\delta^l) = 0$.

Agents are placed on specific networks. Nodes represents agents, edges between nodes represent communication links. For the sake of clarity, we will devote next section to the finite line network (section 4 generalizes results to other architectures which we formally define thereafter). Let index i quote for the position of agent i on the line. Links are undirected. A link between agents i and j is written $i : j$. The set of links of the line network is $\{1 : 2, 2 : 3, \cdots, n-1 : n\}$. The path $p_{i,j}$ from agent i to agent j, with $j > i$ (resp. $j < i$) without loss, is the sequence of distinct nodes $\{i+1, i+2, \cdots, j\}$ (resp. $\{i-1, i-2, \cdots, j\}$). We denote by $\vec{\delta}(p)$ the vector of effort levels of the agents placed on path p. A configuration of effort $\vec{\delta}$ is *symmetric* if any pair of agents with same structural position have same effort level; that is, $\vec{\delta}_i = \vec{\delta}_{n-(i-1)}$.

Let $\vec{\delta}_k$ denote a vector of effort levels with k elements. Denote by $\mathcal{R}$ the space of all possible vectors $\vec{\delta}_k$, for $k \in \{1, 2, \cdots, n\}$.

We define the value of a path p as a function v related to effort levels $\vec{\delta}(p)$ on the path:

$$\mathcal{R} \to R^+$$

$$\vec{\delta} \mapsto v(\vec{\delta})$$

The value $v(\vec{\delta}(p_{i,j}))$ may be interpreted as the amount of externality that agent i captures from joining agent j through path $p_{i,j}$. We may abuse the language by evoking the value of a path rather than the value of effort profile associated with a path.

Throughout the paper, we provide function v with three assumptions which grasp the communication aspect of the model. Firstly, we assume that function v is increasing in its arguments. Assumption 1 states this formally:

Assumption 1 *For all $\vec{\delta}_k, \vec{\delta}'_k$ in $\mathcal{R}^2$ such that $\vec{\delta}_k \leq \vec{\delta}'_k$, we have $v(\vec{\delta}_k) \leq v(\vec{\delta}'_k)$.*

Put differently, the value of a path increases if agents on the path produce more effort. Secondly, information travels with possible decay:

Assumption 2 *For any sequence of effort levels $(\delta^a, \delta^b, \cdots, \delta^r) \in \mathcal{R}$, for every $\delta^q \in \Delta$,*

$$v(\delta^q, \delta^a, \delta^b, \cdots, \delta^r) \leq v(\delta^a, \delta^b, \cdots, \delta^r)$$

Assumption 2 states that the value of a path decreases if we add a new intermediary at the beginning of the path. Note that the assumption expresses weak inequality. Notably, this formulation encompasses bounded communication.

Assumption 3 *If $v(\delta^{a_1}, \delta^{b_1}, \cdots, \delta^{r_1}) \leq v(\delta^{a_2}, \delta^{b_2}, \cdots, \delta^{r_2})$, then for all $\delta^q \in \Delta$,*

$$v(\delta^{a_1}, \delta^{b_1}, \cdots, \delta^{r_1}) - v(\delta^q, \delta^{a_1}, \delta^{b_1}, \cdots, \delta^{r_1}) \leq v(\delta^{a_2}, \delta^{b_2}, \cdots, \delta^{r_2}) - v(\delta^q, \delta^{a_2}, \delta^{b_2}, \cdots, \delta^{r_2})$$

Assumption 3 states that the loss which results from extending the path with some given effort at the beginning, is increasing in the value of the path. In a word, decay increases with the value of the path. This condition is quite general; in particular, it is satisfied in the case of geometric decay.

Let $I(\delta_{-i})$ denote the global externality received by agent i (indifferently labeled the inflow of agent i). For simplicity, we assume an additive formulation of the global externalities that agents capture from the network:

$$\Delta^{n-1} \to R^+$$

$$(\delta_1, \cdots, \delta_{i-1}, \delta_{i+1}, \cdots, \delta_n) \mapsto I(\delta_{-i}) = \sum_{j \neq i} v(\vec{\delta}(p_{i,j}))$$

Individual payoffs then are computed as follows:

$$\Delta^n \to R^+$$

$$\vec{\delta} \mapsto \Pi_i(\delta_i, \delta_{-i}) = \pi(\delta_i, I(\delta_{-i})) - c(\delta_i)$$

To be consistent with our communication context, we assume that function $\pi(.,.)$ is increasing in both arguments. Furthermore, the profit function satisfies a standard definition of synergic efforts:

Definition *Function π is increasing in differences if: for all $i \in \{1, 2, \cdots, n\}$ and every pair $(a, b) \in \Delta^2$ with $a < b$, if $I(\delta_{-i}) \leq I(\delta'_{-i})$, then*

$$\pi(b, I(\delta_{-i})) - \pi(a, I(\delta_{-i})) \leq \pi(b, I(\delta'_{-i})) - \pi(a, I(\delta'_{-i}))$$

We shall say that efforts are strategic complements along communication paths if function π satisfies the increasing difference property.

This definition expresses that when an agent increases his effort level, the increase of his benefit is strictly larger, the higher the value of the inflow that he receives[5].

Example 1 *Geometric decay with possible upper bound $B \in \mathcal{N}$ on the length of communication paths:*

$$v(\delta_{i_1}, \delta_{i_2}, \cdots, \delta_{i_q}) = \Pi_{k=1}^{\min(B,q)} \delta_{i_k}$$

The profit function is written:

$$\pi(\delta_i, \delta_{-i}) = \delta_i \times I(\delta_{-i}) - c(\delta_i)$$

with $\delta^m \leq 1$. This formulation exhibits strategic complementarity in individual efforts. Note that the value function satisfies assumptions 1 and 2. As producing effort, agents may for instance access some valuable knowledge, with all pieces of knowledge being complementary: the return of the effort of one agent is increasing in others' amount of knowledge. For instance, researchers may be more productive when the knowledge

[5]The 'strict' refinement means that no Nash equilibrium contains two agents with same inflow and different effort levels.

they receive from the community is increased. If efforts are exogenous ($m = 1$), we obtain the payoffs of connections' model (net of link formation costs) (Jackson and Wolinsky [1996]).

Example 2 *Spillovers from direct neighbors (Ballester et al. [2006]):*

$$v(\delta_{i_1}, \delta_{i_2}, \cdots, \delta_{i_q}) = \delta_{i_1} \text{ if } q = 1, \ 0 \text{ otherwise.}$$

The profit function is written:

$$\pi(\delta_i, \delta_{-i}) = \delta_i - \frac{\sigma}{2}\delta_i^2 + \gamma\delta_i \times I(\delta_{-i})$$

We analyze Nash equilibria in pure strategies: a strategy profile is Nash if for every agent, her current strategy is a best-response to the current strategies of all other agents. Formally, a profile of individual strategies $\vec{\delta}^* = (\delta_1^*, \cdots, \delta_n^*)$ is a Nash equilibrium of the game on the network g if and only if, for every agent $i \in N$, if $\delta_i \neq \delta_i^*$, $\pi_i(\delta_i^*, \delta_{-i}^*; g) \geq \pi_i(\delta_i, \delta_{-i}^*; g)$. Note that individual participation constraints are always satisfied in the game since the smallest effort level is costless. A dominant equilibrium, say $\vec{\delta}^d$, is such that for all existing equilibrium $\vec{\delta}^*$, $\vec{\delta}_i^* \leq \vec{\delta}_i^d$ for all i. Symmetrically, a dominated equilibrium, say $\vec{\delta}^{dd}$, is such that for all existing equilibrium $\vec{\delta}^*$, $\vec{\delta}_i^* \geq \vec{\delta}_i^{dd}$ for all i.

Finally, we briefly introduce some notations related to the SBRA. Define an infinite sequence of rounds $r = 0, 1, 2, \cdots$ Fix some initial configuration $\vec{\delta}^0$. Then, at each round $r \geq 1$, let all agents revise simultaneously their strategy in response to the configuration $\vec{\delta}^{r-1}$, and denote $\vec{\delta}^r$ this new configuration. The SBRA may eventually converge to some equilibrium configuration which we denote $\vec{\delta}^\infty$.

3 Results on the line

Games with strategic complementarity contain often social dilemmas. Notably, a dominant equilibrium, socially attractive, often exists.

3.1 Preliminary

We begin the analysis with recalling a well-known result about existence of Nash equilibria and dominant equilibria in our game.

Preliminary result 1 *There always exists a Nash equilibrium in pure strategies. Furthermore, a dominant Nash equilibrium exists, as well as a dominated one. In both, symmetric agents produce the same effort level. The dominant (resp. dominated) equilibrium is easily accessed through a SBRA with initial efforts set at maximal (resp. minimal) level.*

(we omit the formal proof, see Topkis [1979]) To give a flavor, existence is related to the strategic complementarity of the game. Starting from the configuration where all agents exert a maximal effort level, we apply the SBRA. Firstly, this algorithm converges since effort levels are bounded below and we have the increasing differences property: at the end of each iteration, the effort level of every agent does not exceed the one he had at the beginning of the iteration. Secondly, at each step of iteration, symmetric agents produce the same effort level since they simultaneously revise their strategy. Thirdly, the SBRA converges to the dominant equilibrium: at each stage of the algorithm, no agent selects some effort level below the one he exerts at any equilibrium, due to increasing return property; in a word, no agent can 'bore' a configuration which is an equilibrium. This result is more general than our networked context, and only assumption 1, in combination with the increasing difference property, are required.

3.2 A counter-intuitive example

Since agents are generally not symmetrically positioned on the line[6], equilibrium configurations of efforts may not be homogenous. One basic observation is that for any homogenous configuration of efforts, more central agents receive more externality. Then, we may expect that equilibria satisfy the following property:

Property P *More central agents produce higher effort levels.*

We name P-equilibrium an equilibrium satisfying property P. Actually, equilibria, and even the dominant and the dominated one, need not necessarily be P-equilibria.

[6]Every agent $i \in N$ has a unique symmetric agent, who is agent $n - i + 1$.

For instance, consider the case of the dominant equilibrium, which is accessible when running the SBRA with all initial efforts set at the upper bound. At the end of first round, more central agents clearly produce more effort. However, inflows are not necessarily increasing toward the center of the line. *The larger the agent's effort, the greater the inflow of her neighbors; therefore property P needs not being satisfied at each stage of the algorithm.* And actually, a simple example show that there exist equilibria which do not satisfy property P. This example obtains under geometric decay (which satisfies the three assumptions). Consider $n = 8$, $\Delta = \{\delta^1, \delta^2, \delta^3\}$ (hence $\delta^1 = \delta^l$ and $\delta^3 = \delta^h$), with $\delta^1 = 0.02$, $\delta^2 = 0.32$, $\delta^3 = 0.45$. Consider the symmetric profile of efforts $(\delta^2, \delta^3, \delta^2, \delta^1, \delta^1, \delta^2, \delta^3, \delta^2)$. Direct computation indicates that the inflows of agents 1 to 4 are $I_1 \simeq 0.59, I_2 \simeq 0.65, I_3 \simeq 0.61, I_4 \simeq 0.54$ (the inflow of other agents is deduced by symmetry). Thus, there is the same ordinal ranking between efforts and inflows. We conclude that there exists a cost profile under which this configuration is stable; for instance a Nash equilibrium obtains if $c_1 = 0, c_2 = 0.179, c_3 = 0.260$.

3.3 Main results

Among all equilibria, the dominant equilibrium is socially desirable. Indeed, since individual payoffs are increasing in the inflows, the dominant equilibrium Pareto-dominates all other Nash equilibria. Next proposition makes a link between the position of agents on the line and their level of effort at both dominant and dominated equilibria:

Proposition 1 *Both dominant and dominated equilibria are symmetric P-equilibria.*

To establish the proposition, we prove two successive lemmata and a second proposition.

Before presenting the lemmata, we define the notions of *cover* and *covered* configurations associated with any symmetric configuration. Consider a symmetric configuration $\vec{\delta}$. The cover configuration $\vec{\delta}^H$ is the unique *smallest element of the set of configurations which both dominate $\vec{\delta}$ and satisfy property P*; the covered configuration $\vec{\delta}^L$ that we set up is the *largest element of the set of configurations that both satisfy property P and are dominated by $\vec{\delta}$.* Figure 1 illustrates how we build the required configurations. We will see later on that the cover configuration of an equilibrium is crucial to our analysis.

Technically, we apply the following algorithm: denote by i_0 the index of the central agent: $i_0 = \frac{n+1}{2}$ if n is odd, $i_0 = \frac{n}{2} + 1$ otherwise. First, we set $\delta_1^H = \delta_1$. Then, stage 1 proceeds as follows: consider the efforts of agents 1 and 2. If $\delta_2 \leq \delta_1$, we set $\delta_2^H = \delta_1$, otherwise we set $\delta_2^H = \delta_2$. Stage 2 replicates stage 1 with agents 2 and 3; that is, If $\delta_3 \leq \delta_2$, we set $\delta_3^H = \delta_2$, otherwise we set $\delta_3^H = \delta_3$. Then we iterate the process until agent i_0. Since the configuration δ is symmetric, we do the same at the right side of agent i_0, that is we set $\delta_{n+1-i}^H = \delta_i^H$ for all $i < i_0$ (if n is even, we also set $\delta_{i_0+1}^H = \delta_{i_0}^H$). We proceed similarly with the covered configuration, except that we start from the center to the extremities of the line (decreasing the effort of agents such that their more central neighbor is lower). Note that both the cover and covered configurations are symmetric. Further, this algorithm enables to obtain those configurations after a unique revision of individual strategies.

We are now able to state our first lemma (which contains the main technical difficulties):

Lemma 1 *Consider a symmetric equilibrium $\vec{\delta}$. Then, if a SBRA starts from its cover (resp. covered) configuration, all agents produce a higher (resp. lower) effort level after round 1. Formally, if $\vec{\delta}^0 = \vec{\delta}^H$ (resp. $\vec{\delta}^0 = \vec{\delta}^L$), then $\vec{\delta}^1 \geq \vec{\delta}^0$ (resp. $\vec{\delta}^1 \leq \vec{\delta}^0$).*

(Proof in the appendix) Remark that, by increasing difference property, it is actually true that $\vec{\delta}^r \geq \vec{\delta}^{r-1}$ (resp. $\vec{\delta}^r \leq \vec{\delta}^{r-1}$) for all $r = 1, 2, \cdots, \infty$. The lemma may not hold if the initial configuration is not an equilibrium, even if it satisfies property P. Again, we would like to insist that, although intuitive at first glance, the result is not trivial. More precisely, the shape of the cover configuration (like an 'Inca' pyramid) enables the comparison of inflows received by agents with *same* level of effort, which is critical to obtain our result. In opposite, if we start from a configuration that dominates the equilibrium, we are not sure that it converges to an equilibrium which *strictly* dominates the initial equilibrium.

Lemma 1 provides sufficient conditions for escaping from some equilibrium; indeed, if the equilibrium that we start from does not satisfy property P, its cover configuration is distinct from it, and for sure we escape from the initial equilibrium. We now turn to the characterization of equilibria toward which a SBRA converges. Next lemma gives some condition under which the SBRA starting from a symmetric configuration (not necessarily an equilibrium) converges to a P-equilibrium.

Lemma 2 *Initiating the SBRA from a symmetric configuration, if the converging configuration dominates its cover configuration, then the converging configuration is a P-equilibrium.*

(Proof in the appendix) This lemma is rather general, but conjectural. The next proposition provides sufficient conditions for the 'if' part of lemma 2 to be valid:

Proposition 2 *Consider a symmetric equilibrium $\vec{\delta}$ that does not satisfy property P. Starting from its cover (resp. covered) configuration, the SBRA converges to a symmetric P-equilibrium.*

(proof omitted) Proposition 2 is a direct application of lemma 1 and lemma 2. Proposition 1 stems immediately from lemma 1 and proposition 2 (proof omitted).

The message behind proposition 1 is that if we detect an equilibrium which does not satisfy property P, then for sure the equilibrium is not dominant. Then, Proposition 2 provides our second main message. In a word, if we detect a symmetric equilibrium $\vec{\delta}$ which does not satisfy property P, providing an appropriate collection of impulses in efforts, and thus letting agents myopically revise their strategies without any central coordination, the procedure reaches an equilibrium which is beneficial to all. This result is interesting from the point of view of the social dilemma. Furthermore, the converging configuration is a P-equilibrium, which means that we cannot do better (since the new equilibrium confounds with its cover configuration). Finally, starting from an equilibrium, accessing its cover configuration is not very demanding. Indeed, following the algorithm of the construction of the cover configuration, it is sufficient to ask agents a unique adjustment of their effort level to their more peripheral neighbor.

4 Hierarchical communities

In this section, we develop our results established on the line to some other networks. We only present the extension of lemma 1. The other lemma and the two propositions established in the preceding section follow directly.

Example. We build the following class of trees. There is a sequence of q integers $p_1, p_2, \cdots, p_q$ with $p_1 \geq p_2 \geq \cdots \geq p_q$. This class of tree is built recursively as follows.

Start with one agent say i_0, denote by class-1 the set $\{i_0\}$, and build p_1 links involving agent i_0; label agent i_0's neighbors class-2 agents. Then, considering every class-2 agent, build p_2 links involving her. The new agents are by construction order-2 neighbors from agent i_0; denote them as class-3 agents. Continue until reaching class-q. This class of trees has two main features: first, any pair of agents in a given class-k, $k = 2, 3, \cdots, q$ are symmetric; second, if agent i belongs to class-k and agent j to class-k' with $k < k'$, then agent i's degree (*i.e.* the number of nodes agent i is involved in) is not smaller than that of agent j. For instance, the line network is such that $p_1 = p_2 = \cdots = p_q = 1$ and $n = 2q - 1$. We name this class a *hierarchy* (see figure 2 - Left).

A definition of centrality for this class. *Agents with lower index k have higher centrality.*
(for an explanation, see the last remark of the section)

Result 1 *For any hierarchy such that $p_1 > p_2$, lemma* 1 *holds.*

We omit the detail of the proof. We explain briefly why the lemma still holds (we only mention the part related to the cover configuration). First, we build the cover configuration similarly to the case of the line: recalling that symmetric agents produce the same effort level at any equilibrium, we start with class-1 agent; consider in the rest of the society the set of agents with maximum effort, and among them select one with lowest index. Then put the effort of agents with lower index at this effort level. Replicate the process accordingly until the end of the network. Having built the cover configuration, the proof extends straightforwardly. Consider some agent i belonging to some class-k. Let us suppose that agent i's effort is smaller than one agent say j of class-$(k + r)$ at the equilibrium. Then they produce same effort on the cover configuration. Now it is easy to see that the inflow of agent i is greater than that of agent j on the cover configuration. The networks depicted in figure 3 illustrate the point in the case $r = 1$ and $k > 1$. The agents providing externalities to agent i and j can be divided in two separate regions. One region consists in two subtrees of similar pattern: the first contains agent j as top agent and contains the (direct and indirect) descendants of agent j; the second is a replication of this subtree with agent i as top agent. If $p_k > p_{k+1}$ (figure 3-Left), it is sufficient to consider a subtree containing exclusively descendants of agent i; if $p_k = p_{k+1}$ (figure 3-Right), this subtree contains one branch with the class-$(k-1)$ neighbor of agent i. Note that if $k=1$, the latter case does not arise since we impose $p_1 > p_2$. Then by construction, with any path toward a class-$(k+z)$ descendant of agent j we can associate a path toward a class-$(k+z-1)$ neighbor of agent i, with all efforts of intermediaries greater on the latter path. Using assumptions 1 and 3 is then sufficient to find that agent i receives more inflow than agent j in this region. The complementary region poses no difficulty: decay (assumption 2) basically induces that agent i receives more inflow from this part of the network than agent j. Summing all externalities, we find that agent i receives more inflow than agent j on the cover configuration. Then, starting from the cover configuration, the SBRA would converge to a configuration that dominates the cover configuration.

We remark that the condition $p_1 > p_2$ is necessary. Indeed, if $p_1 = p_2$, the top agent has p_1 direct neighbors while each of her neighbors has $p_1 + 1$ neighbors (see figure 4). If decay is strong (a limiting case is when only direct neighbors provide externalities), the top agent may receive less inflow than her neighbors.

Definition. We build the following class of networks. We consider a collection of x identical hierarchies of length say q. Then, we add (i) any symmetric and connected structure between all class-1 agents (a circle, a complete network, $\cdots$), with average degree $d_1 \geq 1$ in this subnetwork; (ii) any symmetric and connected structure between all class-k agents of a same hierarchy, with same average degree d_k for all hierarchies. We impose that $d_k \geq d_{k+1}$ for $k = 1, 2, \cdots, q-1$. We name this class a *hierarchical community* (see figure 5). Of course, this class contains hierarchies as defined in the preceding example. The definition of centrality for this class is identical: Agents with lower index k are more central.

On general networks, there may exist more than one path between any pair of agents. Thus, we need to define how we derive some inflow from the collection of paths between two agents. Of course, which function is selected (like max or sum) matters. We restrict attention to functions *max* (*i.e.* the externality that agent i receives from agent j is the greatest value over all paths linking them) and *sum* (*i.e.* the externality that agent i receives from agent j is the sum of path values over all paths linking them):

Result 2 *Suppose that agents receive inflow through either function max or function* *sum. Any hierarchical community with* $d_1 \geq p_2$ *is such that lemma* 1 *holds.*

Figure 6 illustrates the point with function *max*. To see why the result holds, build the cover configuration of a symmetric equilibrium as usual. Suppose that agent i is a class-1 agent (which is the case in figure 6). Then $d_1 \geq p_2$ guarantees that the equivalent tree to that of descendants of agent j (the red one in figure) can be obtained from agent i (red-doted nodes) with all agents producing more (on the cover configuration) node by node. Considering all paths between pairs of agents, function max ensures that agent i obtains more inflow than agent j on the cover configuration. Now suppose that agent i is not central. Then if $p_i > p_j$, only descendants of agent i can be used to find the good tree, and we are done. If $p_i = p_j$, then one uses one direct 'parent' of agent i as a source of one branch. Finally, the subnetwork delineated by the doted line and excluding agent i provides either equal inflow to agents i and j, or is favorable to agent i by decay. We do not present the case of function *sum*; basically it may only increase the difference of inflows in favor of agent i.

5 Conclusion

This article has studied individual incentives to produce synergic efforts, the returns of which depend on spillovers spreading on a network. In that situation, a social dilemma exists, *i.e.* some equilibria are Pareto-ranked. First, we provide a characterization of the equilibrium which Pareto-dominates all others, under the form of a simple property (property P). This characterization holds under general conditions on the propagation of spillovers. Second, this characterization is useful for addressing the issue of social coordination. Considering any equilibrium which is not the most desirable and which does not possess property P, we provide a simple procedure, which exploits property P, to reduce the social dilemma.

Due to the level of generality of our assumptions regarding the transmission of information, our analysis is restricted to specific network architectures, for which centrality ranking is unambiguous. Future research may explore further the relationship between centrality indexes and efforts on more general network architectures, perhaps under more specific decay assumptions. Moreover, the strategic formation of both efforts and links is a challenging issue.

APPENDIX: PROOFS

Proof of lemma 1. Suppose that a symmetric equilibrium $\vec{\delta}$ does not satisfy property P. if a SBRA starts from its cover (resp. covered) configuration $\vec{\delta}^H$, all agents produce a higher (resp. lower) effort level after round 1.
Denote i_0 the index of the central agent: $i_0 = \frac{n+1}{2}$ if n is odd, $i_0 = \frac{n}{2} + 1$ otherwise. Considering one symmetric equilibrium δ and its cover configuration δ^H, we define the set $\mathcal{Z}(\delta) = \{1, n\} \cup \{i \in \{2, 3, \cdots, i_0\} / \delta_i^H = \delta_i \text{ and } \delta_i > \delta_{i-1}\} \cup \{i \in \{i_0+1, i_0+2, \cdots, n-1\} / \delta_i^H = \delta_i \text{ and } \delta_i > \delta_{i+1}\}$. In words, the set $\mathcal{Z}$ contains agents such that their effort level on the cover configuration is the same as their effort in the equilibrium configuration δ, and such that their less central neighbor produces a strictly lower effort on the cover configuration.

Part **(A)** shows that, in the cover configuration $\vec{\delta}^H$, if we have two agents, say i and j, with same effort level and such that agent $i \in \mathcal{Z}$ while $j \notin \mathcal{Z}$ (then agent j is more central than agent i by construction of $\mathcal{Z}$), then agent i receives a lower inflow. Part **(B)** shows that the SBRA converges to a configuration which dominates $\vec{\delta}$.

(A): Recall that $\vec{\delta}^H$ is symmetric. Fix, in $\vec{\delta}^H$, any agent i_0+r, $r \geq 0$. Consider the unique agent $i_0+z(r) \in \mathcal{Z}$, such that $\delta^H_{i_0+r} = \delta^H_{i_0+z(r)}$ (superscript H quotes here for the effort in profile $\vec{\delta}^H$). Note that $r \leq z(r)$. Having fixed r, Let $k(r) = 2i_0 + r - (n-z)$; agent $k(r)$, placed at the left side of agent i_0+r, is such that $(i_0+r) - k(r) = n - (i_0 + z(r))$. Note that $k(r) > 1$ if $r > 0$. Writing the difference $I^H_{i_0+r} - I^H_{i_0+z(r)}$ and rearranging, we find:

$$I^H_{i_0+r} - I^H_{i_0+z(r)} = \underbrace{\sum_{k=1}^{k=k(r)-1} [v(p_{i_0+r,k}) - v(p_{i_0+z(r),k})]}_{(E1)}$$

$$+ \underbrace{\sum_{k=k(r)}^{k=i_0+r-1} [v(p_{i_0+r,k}) - v(p_{i_0+z(r),k})] - \sum_{k=i_0+z(r)+1}^{k=n} [v(p_{i_0+z(r),k}) - v(p_{i_0+r,k})]}_{(E2)}$$

$$+ \underbrace{\sum_{k=i_0+r+1}^{k=i_0+z(r)} v(p_{i_0+r,k}) - \sum_{k=i_0+r}^{k=i_0+z(r)-1} v(p_{i_0+z(r),k})}_{(E3)}$$

. Expression $(E1)$: for all $k = 1, 2, \cdots, k(r) - 1$, $v(p_{i_0+r,k}) \geq v(p_{i_0+z(r),k})$ by the assumption 2 on decay. Indeed, for all such k, agent $i_0 + r$ is intermediary between agent k and agent $i_0 + z(r)$ on the line. Then we find $(E1) \geq 0$.

. Expression $(E2)$: for each $q = 1$ to $n - i_0 - z(r)$, $v(p_{i_0+z(r),i_0+z(r)+q}) \leq v(p_{i_0+r,i_0+r-q})$ by assumption 1. Assumption 3 therefore predicts that for all such indexes q,

$$v(p_{i_0+z(r),i_0+z(r)+q}) - v(p_{i_0+r,i_0+z(r)+q}) \leq v(p_{i_0+r,i_0+r-q}) - v(p_{i_0+z(r),i_0+r-q})$$

Summing all inequalities we obtain $(E2) \geq 0$.

. Expression $(E3)$: By construction of $\vec{\delta}^H$, $\delta_{i_0+r} = \delta_{i_0+r+1} = \cdots = \delta_{i_0+z(r)}$, so clearly $(E3) = 0$.
Thus, $I^H_{i_0+r} \geq I^H_{i_0+z(r)}$.

(B): Let us apply the SBRA with profile $\vec{\delta}^H$ as initial condition ($\vec{\delta}^H = \vec{\delta}^0$), and let $\vec{\delta}^t$ denote the value of efforts at the end of round $t = 1, 2, \cdots$. Recall that:

. every agent in $\mathcal{Z}$ produces the same effort level in both configurations δ and δ^H,

. $\vec{\delta}$ is an equilibrium profile,

. $\vec{\delta}^H$ dominates $\vec{\delta}$,

. payoff functions satisfy increasing differences.

Then, at the end of the first round of the SBRA, we derive that $I^H_{i_0+z} \geq I_{i_0+z}$ for all $i_0 + z \in \mathcal{Z}$. Then, by increasing differences property, we have $\delta^1_{i_0+z} \geq \delta^H_{i_0+z}$ (where superscript '1' quotes here for first round). Now, consider some agent $i_0 + r$. By **(A)**, $I^H_{i_0+r} \geq I^H_{i_0+z(r)}$, where $i_0 + z(r) \in \mathcal{Z}$ is such that $\delta^H_{i_0+r} = \delta^H_{i_0+z(r)}$. Then, the increasing differences property induces that $\delta^1_{i_0+r} \geq \delta^1_{i_0+z(r)}$. Since $\delta^H_{i_0+r} = \delta^H_{i_0+z(r)}$, we have $\delta^1_{i_0+r} \geq \delta^H_{i_0+r}$. Hence, all agents increase their efforts at the end of round 1. $\diamond$

Proof of lemma 2. Consider a symmetric configuration $\vec{\delta}^0$. Suppose that the configuration $\vec{\delta}^\infty$ it converges to through SBRA does not satisfy property P. Since $\vec{\delta}^{0H} \leq \vec{\delta}^\infty$, we also have $\vec{\delta}^{0H} \leq \vec{\delta}^{\infty L}$, and thus $\vec{\delta}^0 \leq \vec{\delta}^{\infty L}$. Now, applying the SBRA with $\vec{\delta}^0$ and $\vec{\delta}^{\infty L}$ as initial configurations, it is true that for all rounds $r \geq 1$, $\vec{\delta}^r \leq \vec{\delta}^{\infty L_r}$ (by increasing difference property). Therefore, $\vec{\delta}^\infty \leq (\vec{\delta}^{\infty L})^\infty$ (where $(\vec{\delta}^{\infty L})^\infty$ is the configuration the SBRA converges to if we put $\vec{\delta}^{\infty L}$ as initial condition). Remarking that $\vec{\delta}^\infty$ is a symmetric equilibrium, we can apply lemma 1 which says that from the covered configuration of a symmetric equilibrium, SBRA converges toward a configuration dominated by the covered configuration. We find that $(\vec{\delta}^{\infty L})^{\infty} \leq \vec{\delta}^{\infty L}$, inducing finally $\vec{\delta}^{\infty} \leq (\vec{\delta}^{\infty L})^{\infty} \leq \vec{\delta}^{\infty L} < \vec{\delta}^{\infty}$, a contradiction. □

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FIGURES

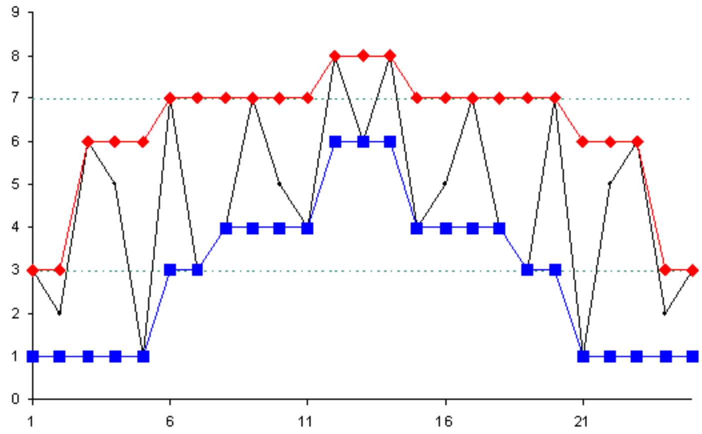

Figure 1: Black: a configuration $\vec{\delta}$; Red-diamond: $\vec{\delta}^H$; Blue-square: $\vec{\delta}^L$

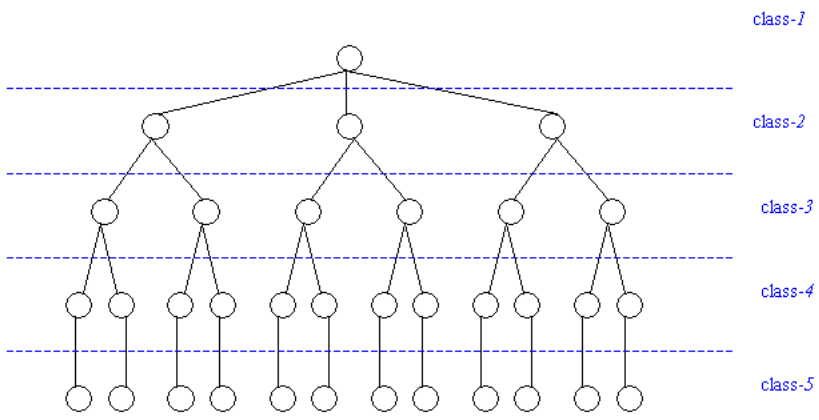

Figure 2: A hierarchy ($q = 5$)

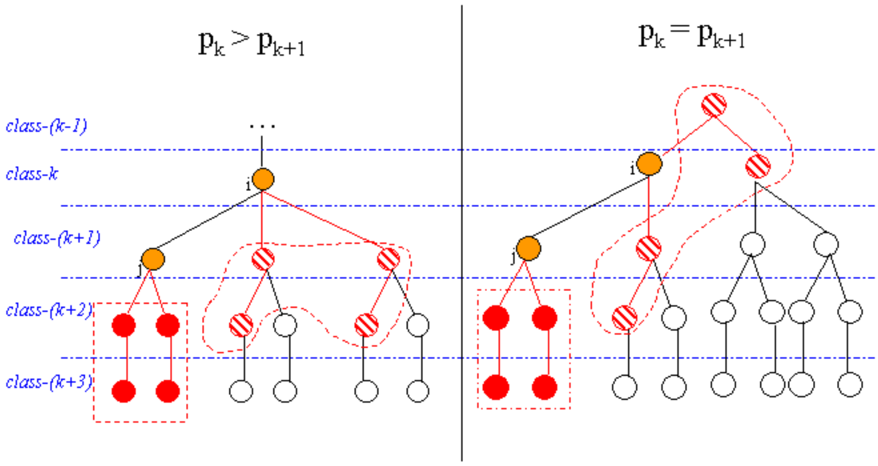

Figure 3: $k > 1$; Left: $p_k > p_{k+1}$ - Right: $p_k = p_{k+1}$

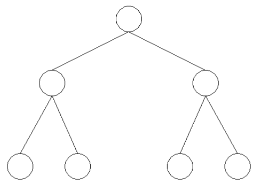

Figure 4: A hierarchy such that $p_1 = p_2$

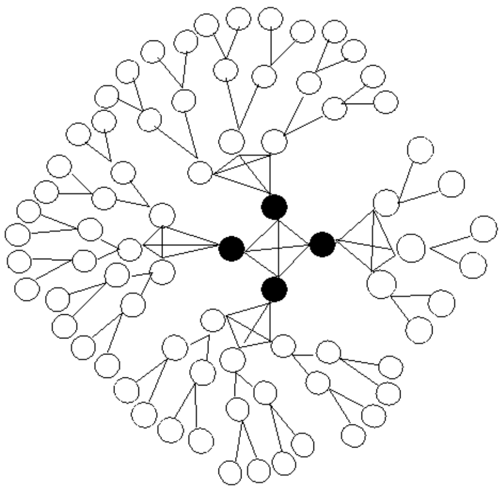

Figure 5: A hierarchic community ($x = 4$, $q = 3$, $p_1 = 3$, $p_2 = p_3 = 2$, $d_1 = 3$, $d_2 = 2$, $d_3 = d_4 = 0$) - black nodes are class-1 agents in hierarchies

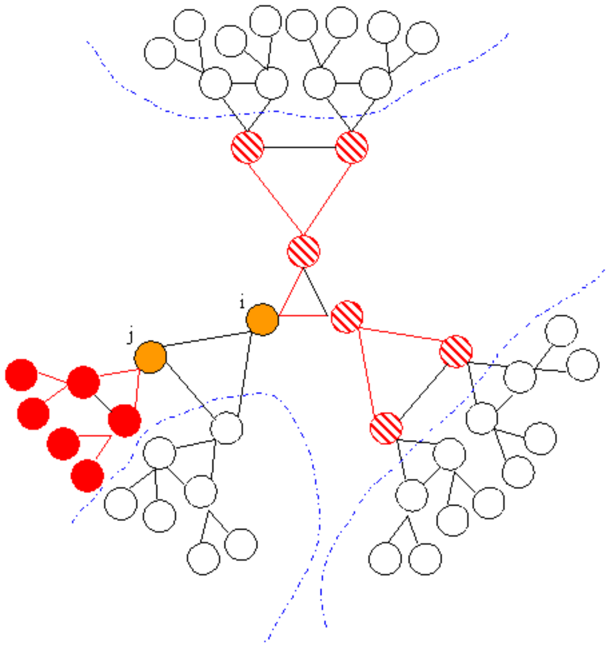

Figure 6: A hierarchic community ($x = 3$, $q = 4$, $p_1 = p_2 = p_3 = 2$, $d_1 = 2$, $d_2 = d_3 = 1$, $d_4 = 0$)