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Jean-François Laslier

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LABORATOIRE D'ECONOMETRIE
1rue Descartes F-75005 Paris
(33) 1 55558215
http://ceco.polytechnique.fr/
mailto:labecox@poly.polytechnique.fr

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Résumé: Cet article se place dans le cadre d'un modèle de pure politique redistributive entre trois groupes d'électeurs. Il compare deux variantes de la compétition électorale entre deux partis, l'objectif d'un parti étant soit la probabilité de victoire ("jeu du tournoi majoritaire"), soit le nombre de voix obtenues ("jeu de la pluralité"). On exhibe des équilibres en stratégies mixtes pour ces deux variantes. En moyenne tous les individus sont traités de la même manière dans le jeu de la pluralité, alors que le jeu du tournoi majoritaire favorise les individus appartenant aux petits groupes.

Abstract: In the "divide a dollar" framework of distributive politics among three pivotal groups of unequal size, the paper compares two variants of two-party competition, the objective of a party being the probability of winning (``Majority Tournament" game) or the expected number of votes ("Plurality" game). At a mixed equilibrium, all individuals are, on expectation, treated alike in the Plurality Game while the Tournament Game favors individuals in small groups.

Mots clés : compétition électorale, stratégies mixtes, Blotto, objectif des partis

Key Words : Electoral Competition, Divide a Dollar, Mixed Strategies, Blotto, Party Objective

Classification JEL: C72, D72

[1] Laboratoire d'Econométrie, CNRS et Ecole polytechnique

Party objectives in the 'divide a dollar' electoral competition

Jean-François Laslier
Laboratoire d'Econométrie,
Ecole Polytechnique
1 rue Descartes
75005 Paris

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Abstract

In the "divide a dollar" framework of distributive politics among three pivotal groups of unequal size, the paper compares two variants of two-party competition, the objective of a party being the probability of winning ("Majority Tournament" game) or the expected number of votes ("Plurality" game). At a mixed equilibrium, all individuals are, on expectation, treated alike in the Plurality Game while the Tournament Game favors individuals in small groups.

1 Introduction

In basic models of two-party Downsian political competition, maximizing the number of vote (the "plurality") or trying to win the election by whatever margin (the "majority tournament") are equivalent objectives for the parties in the sense that they lead to the same equilibria. But the equivalence result is often lost in richer models that take into account abstention, uncertainty, or a non-trivial decision structure for parties.

The hypothesis that parties maximise the expected number of votes rather than the probability of winning usually (but not always) gives rise to more mathematically tractable models; for that reason, as noticed by Coughlin, 1992 [11], "In most of the Public Choice literature, it is assumed that each candidate wants to maximize his expected plurality". A notable exception is Roemer (see 2001, [29]) for whom the "probability of winning" hypothesis is important. It may be the case that specific models can be interpreted in one way or another, like for instance, the model of Aragones and Palfrey, 2002 [1].

The equivalence result is very sensitive to various kind of uncertainty. Uncertainty in electoral competition may arise from exogenous reasons such that parties having imperfect information on the voters' preferences or technical impossibility to communicate with the voters without noise. Uncertainty may moreover come out for endogenous strategic reasons.

Models of imperfect information in Politics are surveyed by Banks, 1991 [5], where instances can be found in which payoffs are uncertain, be it for exogenous or endogenous reasons. According to the Condorcetian philosophy, gathering information in an uncertain world is one reason of being for democratic institutions, as expressed in the Condorcet Jury theorem. The "Jury" framework is used by Austen-Smith and Banks, 1996 [2] to discuss in a modern fashion information aggregation by direct voting. With the same framework, Laslier and Van der Straeten, 2003 [22] discuss information aggregation by electoral competition and the equivalence problem.

The equivalence problem is studied with great details by Patty, 1999, and Duggan, 2000, [27, 12] in "probabilistic voting" models. Here the uncertainty about the number of votes to be received by a party derives from a random noise added to each voter's utility function and the probability distribution of this noise is exogenous.

The question of endogenous strategic uncertainty is of particular importance because electoral competition games usually have no pure strategy equilibria (the book Austen-Smith and Banks, 1999 [3], contains a modern survey of the "chaos" theorems of McKelvey, Schofield, and others). The question is thus raised of the mixed strategy equilibria of these games.

Laffond, Laslier and Le Breton, 1994 [15] give a simple example with a finite set of alternatives in which the Majority Tournament game and the Plurality game each have unique equilibria and the two equilibria are completely different. Dutta and Laslier, 1999 [13] provide a more detailed comparison of the social choice correspondences based on the Majority Tournament and the Plurality. Laslier, 2000 [18] proposes an unusual interpretation of mixed strategies in these specific games. There are two standard interpretations for the mixed strategy probability $p(x)$ associated to a proposed alternative x. (i) Random choice: $p(x)$ is the probability that the party chooses x at random. (ii) Belief: $p(x)$ is the belief of the other party as to the choice of x. In electoral competition games, one can furthermore interpret $p(x)$ as the fraction of the electorate who judges the party according to proposition x, or probability that a voter understands that the party will implement x. The equilibrium of the electoral competition under this third interpretation can curiously be the equilibrium of the payoff matrix of the Plurality game even if the parties maximise the probability of winning[1] This is a theoretical point in favor of the "Plurality" model and against the "Majority Tournament" one.

Another point in favor of the Plurality model is that it has a natural extension to the case of more than two parties, under the "Borda electoral competition", even in the presence of a mixed equilibrium (Laffond, Laslier and LeBreton, 2000 [16]. Laslier, 2003 [20] studies the robustness of mixed strategy equilibria in electoral competition with respect to the voters disliking ambiguity and the parties prudent behavior.

On economic domains, mixed strategy equilibria are usually hard to compute. Technical difficulties abound, no general existence theorem is available, and the inclusion of the support of optimal strategies in the uncovered set, which is easy to prove in the finite setting, is no longer a trivial matter (see McKelvey, 1986, Banks, Duggan and Le Breton, 2002 [23, 6]).

One important economic model, which serves as a benchmark in Social Choice Theory and Formal Political Science, is the "Divide a dollar" model. When voting is not mediated by parties, Banks and Gasmi, 1987, and Penn, 2002 [7, 28] have studied the three-voter divide a dollar game, following the ideas developed in the "sophisticated voting" literature (see the survey Miller, 1995 [24]) which led to the definition of the "Banks set" (Banks, 1985 [4]).

When voting is mediated by parties, the "Divide a dollar" model is equivalent to a problem of strategic resource allocation that has been studied by game theorists (Borel, 1938, Gross and Wagner, 1950, Owen, 1982 [8, 14, 26]). It is used in Political Science as a model of pure redistributive politics in order to study the question of the treatment of minorities by democratic rules (Myerson, 1993, Laslier and Picard, 2002, Laslier, 2002 [25, 21, 19]).

The present paper considers the following "Divide a dollar" framework: A fixed amount of money has to be divided among three groups of individuals of unequal size. I exhibit a mixed strategy equilibrium for the associated Tournament game and compare it with the mixed strategy equilibrium for the associated Plurality game that is studied in Laslier, 2002 [19]. The optimal strategy for the Plurality game always treats individuals in the same way, whatever group they belong to; but the optimal strategy for the Tournament game treats better the individuals belonging to small groups. To highlight the difference, I further consider the situation with two large

[1] The key point is here: do the various voters judge a party fuzzy proposal independently the ones from the others?

groups of equal size and one small “pivotal” group. By computing expected values for two indices of inequality (variance and Gini index) I show that the optimal strategy of the Tournament game generates more inequality than the one of the Plurality game.

The paper is organized as follows: after this introduction, section 2 presents the model. Section 3 describes optimal strategies and contains the statement (proposition 8) about the shares received by individuals in different groups. Section 4 is devoted to the analysis of inequality in the case of one small pivotal group. Section 5 is a short conclusion.

2 The model

The population N is formed of n individuals and is partitioned in 3 groups N_i, $i = 1, 2, 3$. There are n_i individuals in group N_i :

$$N = \bigcup_{i=1}^{3} N_i, \ \sum_{i=1}^{3} n_i = n.$$

The partition (N_1, N_2, N_3) *contains a majority* if there exists an i such that $n_i \geq n/2$; the majority is *strict* if the previous inequality is strict.

A quantity $Q \in I\!R_{++}$ of money must be divided among the N individuals but individuals belonging to the same group must receive the same amount. Let x_i be the amount received by an individual in group i, the set of possible outcomes is:

$$\Delta^{(Q)} = \{x \in I\!R^3_+ : \sum_{i=1}^{3} n_i x_i = Q\}.$$

For two outcomes $x, y \in \Delta^{(Q)}$ the *net plurality* for x against y can be written:

$$g(x, y) = \sum_{i=1}^{3} n_i sgn(x_i - y_i),$$

where, for any number $t \in I\!R$, $sgn(t)$ equals -1 if $t < 0$, 0 if $t = 0$ and $+1$ if $t > 0$. Call *Plurality Game* the two-player, symmetric, zero-sum game defined by g. The net plurality $g(x, y)$ is the gain for the strategy x against strategy y. A mixed strategy is a regular probability distribution on $\Delta^{(Q)}$; if p and q are two mixed strategies the payoff to p against q is:

$$g(p, q) = \int_{x \in X} \int_{y \in X} g(x, y) dp(x) dq(y),$$

That is the expected margin of votes.

For two outcomes $x, y \in \Delta^{(Q)}$ the *net majority* for x against y is by definition $+1$ if a majority of individuals prefer x to y, -1 in the opposite case, and 0 in case of a tie. It can be written:

$$m(x,y) = sgn[g(x,y)] = sgn\left[\sum_{i=1}^{3} n_i sgn(x_i - y_i)\right].$$

The two-player, symmetric, zero-sum game defined by m is called the Majority Tournament Game or simply the *Tournament Game.* The net majority $m(x,y) \in \{-1, 0, +1\}$ is the payoff for the strategy x against the strategy y. If p and q are two mixed strategies the payoff to p against q is:

$$\begin{aligned} m(p,q) &= \int_{x\in X}\int_{y\in X} m(x,y)\; dp(x)\; dq(y) \\ &= \Pr_{p\otimes q}\left[g(x,y) > 0\right] - \Pr_{p\otimes q}\left[g(x,y) < 0\right], \end{aligned}$$

If ties can be neglected, that is if $p \otimes q\,[g(x,y) = 0] = 0$, then $m(p,q)$ is, up to constants, equal to the probability of winning.

3 Optimal strategies

The two games considered in this paper are zero-sum, therefore the solution concept is *min-max* equilibrium. Recall that, in a zero-sum game with payoff function g, one defines the *value for player* 1, or "gain-floor" as:

$$v_1 = \max_p \min_q g(p,q).$$

The symmetric definition for player 2 is $v_2 = \max_q \min_p (-g(p,q))$ and it is easy to see that:

$$v_2 = -\min_q \max_p g(p,q).$$

The values for the two players are well-defined numbers, provided for instance that the payoff function is bounded, and are such that $v_1 + v_2 \leq 0$.

If $v_1 + v_2 = 0$, which also writes: $\max_p \min_q g(p,q) = \min_q \max_p g(p,q)$, one simply says that v_1 *is the value of the game.* According to von Neumann's min-max theorem, finite games in mixed strategies have a value, but infinite games may have no value, even in mixed strategies. A strategy p^* for player 1 is *optimal* if

$$\min_q g(p^*,q) = \max_p \min_q g(p,q) = \min_q \max_p g(p,q);$$

that is: p^* is optimal if the game has a value and p^* guarantees this value to player 1. The definition is symmetric for the other player. A min-max equilibrium is a pair of optimal strategies[2]. When the game is symmetric, that is if both players have the same strategy set and $g(x, y) = -g(y, x)$, optimal strategies, when they exist, are obviously the same for both players and the value of the game can only be 0. (All games in the present paper are symmetric.)

3.1 Cases with pure strategy equilibrium

It is a standard result that, in pure strategies, the two objectives for parties (Plurality or Majority) are equivalent.

Proposition 1 *The Plurality and Tournament games have the same optimal pure strategies.*

But a pure equilibrium appears only when the partition contains a majority; more precisely, Laslier, 2002 [19], notices:

Proposition 2 *(i) If the partition of the population contains a strict majority then each party has a unique optimal strategy. This strategy is pure, it consists in giving all the money to the largest group.*

(ii) The same result holds if one group contains exactly half of the population and there are at least two other groups.

(iii) If the partition is (N_1, N_2) with $n_1 = n_2 = n/2$ then any strategy is optimal.

Proposition 3 *If the partition of the population contains no majority then no pure strategy is optimal, and the Plurality Game as well as the Tournament Game have no pure strategy equilibrium.*

3.2 Cases with no pure strategy equilibrium

We now turn to the cases where no pure strategy equilibrium exists. None of the three groups is a majority:

$$\begin{aligned} n &= n_1 + n_2 + n_3 \\ 0 &< n_1 \leq n_2 \leq n_3 < n/2. \end{aligned}$$

[2]See Owen, 1982 [26] for an introduction to the theory of zero-sum games. For these games, a strategy profile is a Nash equilibrium if and only if each player's strategy is optimal.

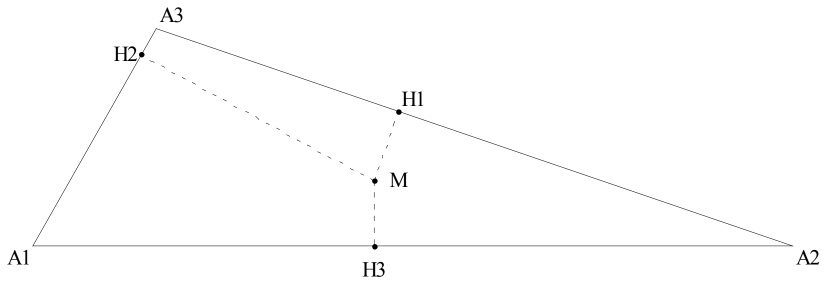

Figure 1: Representing divisions by heights in a triangle

The set $\Delta^{(Q)}$ of possible divisions can then be represented as a triangle (A_1, A_2, A_3) with sides $A_1A_2 = n_3$, $A_2A_3 = n_1$, $A_3A_1 = n_2$. For a point M inside the triangle, let H_i, $i = 1, 2, 3$, be the projections of M on the sides, as in Figure 1. Then for any M, $\sum_{i=1}^{3} n_i MH_i = 2S$, where S is the surface of (A_1, A_2, A_3). Changing the unit of money so that $Q = 2S$, we can represent any division of Q by a point M. The height MH_i is the amount of money that each individual in group i gets.

We look for optimal strategies in the Tournament Game and in the Plurality Game. We now define a strategy that will prove to be optimal for the Tournament Game:

Definition 4 (An optimal strategy for the Tournament Game) *Let φ and θ be two independent random variables such that φ has density $(1/2)\sin\varphi$ on $[0, \pi]$ and θ is uniform on $[0, 2\pi]$. Let for $i = 1, 2, 3$:*

$$a_i = (1/3)\left(1 - \sin\varphi\cos(\theta - 2i\pi/3)\right).$$

Let p^*_{ma} be the probability distribution of the variable (x_1, x_2, x_3) with:

$$x_i = \frac{a_i Q}{n_i}.$$

Proposition 5 *With three groups, p^*_{ma} is an optimal strategy for the Tournament Game.*

Proof. With probability one, $x \in \Delta^{(Q)}$. Consider a pure strategy $x' \in \Delta^{(Q)}$,

$$m(p^*_{\text{ma}}, x') = E\left[\sum_{i=1}^{3} sgn(x_i - x'_i)\right].$$

The events such that $\{x_i = x'_i\}$ have probability 0, thus the only case to be distinguished are of the form "two groups against one". Therefore the payoff of p^*_{ma} against x' in the Tournament game is[3]:

$$\begin{aligned} m(p^*_{\mathrm{ma}}, x') &= \sum_{i,j,k} \Pr\left[x_i > x'_i \text{ and } x_j > x'_j \text{ and } x_k < x'_k\right] \\ &\quad - \sum_{i,j,k} \Pr\left[x_i < x'_i \text{ and } x_j < x'_j \text{ and } x_k > x'_k\right]. \end{aligned}$$

where the sum $\sum_{i,j,k}$ goes over suitable distinct indices $i, j, k = 1, 2, 3$.

The random variable $a = (a_1, a_2, a_3)$ has the property that $\sum_{i=1}^{3} a_i = 1$ and, for $i = 1, 2, 3$, a_i is uniform on $[0, 2/3]$. From this, one can easily infer that:

$$\forall a' \in \Delta_3, E_a \left[\sum_{i=1}^{3} sgn(a_i - a'_i)\right] \geq 0$$

where E_a denotes expectation, here with respect to the probability distribution of a. (See Laslier and Picard, 2002 [21]).Write $a'_i = \frac{x'_i n_i}{Q}$, and notice that $x_i > x'_i$ is equivalent to $a_i > a'_i$, it follows that

$$\begin{aligned} m(p^*_{\mathrm{ma}}, x') &= \sum_{i,j,k} \Pr\left[a_i > a'_i \text{ and } a_j > a'_j \text{ and } a_k < a'_k\right] \\ &\quad - \sum_{i,j,k} \Pr\left[a_i < a'_i \text{ and } a_j < a'_j \text{ and } a_k > a'_k\right] \\ &= E\left[\sum_{i=1}^{3} sgn(a_i - a'_i)\right] \geq 0 \end{aligned}$$

Thus we proved that the mixed strategy p^*_{ma} has non-negative payoff against any pure strategy. In a symmetric and zero-sum game, this is sufficient to prove that p^*_{ma} is optimal. ■

To illustrate, Pictures 2 shows 1000 draws according to the probability distribution p^*_{ma}. The support is an ellipse, with more weight on the border than in the center region. In the case of the Plurality Game, Laslier, 2002 [19], exhibits an optimal strategy that can be compared with p^*_{ma}, and that can be described geometrically.

Definition 6 (An optimal stategy for the Plurality Game) *In the triangle (A_1, A_2, A_3), consider the inscribed circle and a hemisphere erected*

[3]This is the point where the proof given here is only valid for three groups.

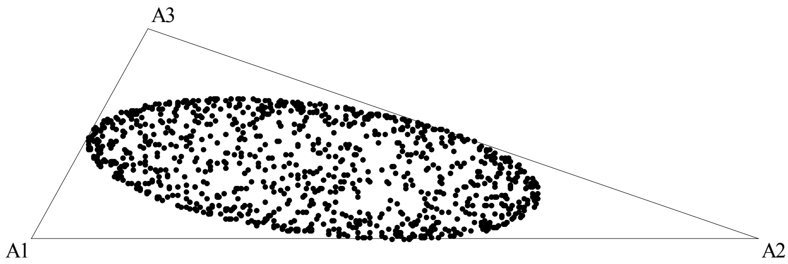

Figure 2: Draws from an optimal strategy for the tournament game

over that circle (see picture 3). Choose a point R at random uniformly on the hemisphere and denote by M the projection of R on the plane that contains A_1, A_2, A_3 (see picture 4). Let p^*_{pl} be the probability distribution of the variable (y_1, y_2, y_3) with

$$y_i = MH_i.$$

Proposition 7 *With three groups, p^*_{pl} is an optimal strategy for the Plurality Game.*

From the properties of the strategies p^*_{ma} and p^*_{pl}, one immediately deduces the following proposition.

Proposition 8 *With three groups:*

- *(Tournament Game) According to the optimal strategy p^*_{ma}, the individual share in group i is uniform on $[0, \frac{2Q}{3n_i}]$. On average, all groups receive the same amount and individual belonging to smaller groups receive more.*
- *(Plurality Game) According to the optimal strategy p^*_{pl}, the individual share in group i is uniform on $[0, \frac{2Q}{n}]$. On average, all individuals receive the same amount.*

Picture 4, to be compared with Picture 2, shows 1000 draws according to the probability distribution p^*_{pl}. The support is now a circle, again with more weight on the border than in the center region.

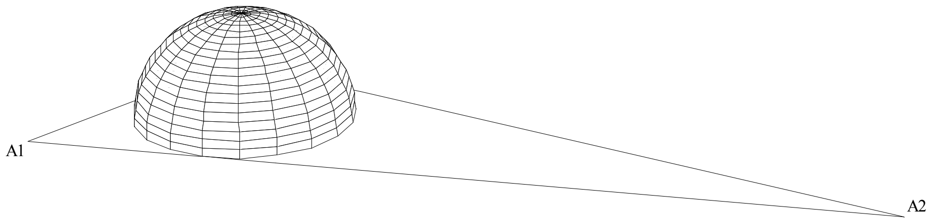

Figure 3: Hemisphere erected over the inscribed cercle

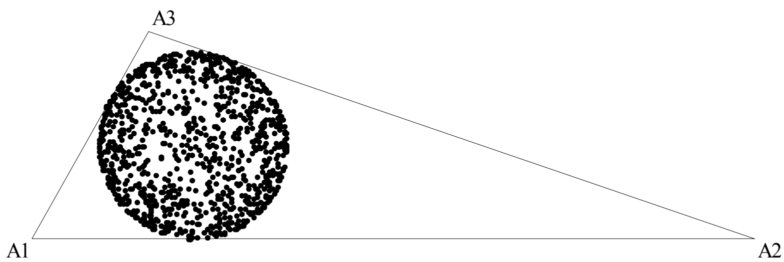

Figure 4: Draws from an optimal strategy for the plurality game

4 A small pivotal group

In order to underline the difference of between the two games, consider the following example: there are two groups of large, identical, size and one small "pivotal" group. For an overall population of size 1 and for a pivotal group of size $e \in]0, 1/3]$,

$$\begin{aligned} n_1 &= n_2 = (1-e)/2 \\ n_3 &= e. \end{aligned}$$

Looking at Pictures 2 and 4, one can hint that, as e gets close to zero, the support for the Tournament case looks like a long and flat ellipse while the support for the plurality case looks like a small circle. The two cases are thus obviously very different for the small group. It seems that, because the individuals in the small group occasionally get a large share, the Tournament case generates more inequality than the plurality case. This section shows that the intuition is right, despite the fact that the number of favored individuals vanishes as e gets small. To do so, two indices of inequality are considered: the variance and the Gini index.

4.1 Inequality according to the variance

In order to be able to make comparisons for different values of c, we consider that the amount of money to be divided is 1, therefore the amount that an individual gets in group i is $x_i = h_i/(2S)$, where h_i is the height and S is the surface of the triangle:

$$S = (1/4)\, e \sqrt{1-2e}.$$

The variance at point M is:

$$var(M) = \sum_{i=1}^{3} n_i x_i^2 - 1.$$

Proposition 9 *Expected value of the variance:*

- *(Tournament Game) According to the optimal strategy* p^*_{ma} *:*

$$E^*_{ma}\,[var] = \frac{4}{27e} + \frac{16}{27\,(1-e)} - 1.$$

It thus tends to $+\infty$ *when the size* e *of the pivotal group tends to* 0*.*

- *(Plurality Game) According to the optimal strategy* p_{pl}^* *:*

$$E_{pl}^*[var] = 1/3.$$

Proof. According to proposition 8, for p_{ma}^*, the individual share is uniform between 0 and $\frac{2Q}{3n_i} = \frac{2}{3n_i}$, therefore:

$$\begin{aligned} E_{\mathrm{ma}}^*[var] &= \sum_{i=1}^{3} n_i \frac{1}{3}\left(\frac{2}{3n_i}\right)^2 - 1 = \sum_{i=1}^{3} \frac{4}{27n_i} - 1 \\ &= \frac{4}{27(1-e)/2} + \frac{4}{27(1-e)/2} + \frac{4}{27e} - 1 \\ &= \frac{16}{27(1-e)} + \frac{4}{27e} - 1. \end{aligned}$$

Likewise, for p_{pl}^*, the individual share is uniform between 0 and $\frac{2Q}{n} = 2$, therefore:

$$E_{\mathrm{pl}}^*[var] = \sum_{i=1}^{3} n_i \frac{4}{3} - 1 = \frac{1}{3}.$$

■

4.2 Inequality according to the Gini index

When the shares have been ordered $x_{(1)} \leq x_{(2)} \leq x_{(3)}$, the Gini index is given by the following formula:

$$gini(M) = 1 - 2\left[\begin{array}{c} n_{(1)}\left(\frac{n_{(1)}x_{(1)}}{2}\right) + n_{(2)}\left(\frac{2n_{(1)}x_{(1)}+n_{(2)}x_{(2)}}{2}\right) \\ +n_{(3)}\left(\frac{2n_{(1)}x_{(1)}+2n_{(2)}x_{(2)}+n_{(3)}x_{(3)}}{2}\right) \end{array}\right]$$

which can be written in a more convenient way:

$$\begin{aligned} gini(M) &= 1 - \left(n_{(1)}^2 + 2n_{(1)}n_{(2)} + 2n_{(1)}n_{(3)}\right)x_{(1)} \\ &\quad - \left(n_{(2)}^2 + 2n_{(2)}n_{(3)}\right)x_{(2)} - n_{(3)}^2 x_{(3)} \end{aligned}$$

Proposition 10 *Expected value of the Gini index when e is small:*

- *(Tournament Game) Let* $E_{ma}^*[gini]$ *denote the expected value, according to* p_{ma}^**, of the Gini index,*

$$\lim_{e \to 0} E_{ma}^*[gini] = \frac{4+\sqrt{3}}{12} \simeq .48$$

- *(Plurality Game) Let $E^*_{pl}[gini]$ denote the expected value, according to p^*_{pl}, of the Gini index,*

$$\lim_{e\to 0} E^*_{pl}[gini] = .25$$

Proof. (i) For the Tournament game.

I will prove the stronger result that, when e is small, the expected value, according to p^*_{ma}, of the Gini index is approximately:

$$E^*_{\text{ma}}[gini] \simeq \frac{4+\sqrt{3}-(12+\sqrt{3})e}{12}.$$

Following the definition (4) of p^*_{ma}, one writes:

$$\begin{aligned} x_1 &= \frac{2}{3(1-e)}(1-\sin\varphi\cos(\theta-2\pi/3)) \\ x_2 &= \frac{2}{3(1-e)}(1-\sin\varphi\cos(\theta+2\pi/3)) \qquad (1) \\ x_3 &= \frac{1}{3e}(1-\sin\varphi\cos\theta) \end{aligned}$$

The Gini index is a piecewise affine function of x_1, x_2, x_3, depending on how they are ordered. First notice that, because of the symmetry between groups 1 and 2, one can compute expected values conditionally on $x_1 \leq x_2$.

$$E^*_{\text{ma}}[gini] = 2\int_{\{x_1\leq x_2\}} gini(M)\; dp^*_{\text{ma}}.$$

From (1), $x_1 \leq x_2$ if and only if $\cos(\theta-2\pi/3) \geq \cos(\theta+2\pi/3)$, which is equivalent to $0 \leq \theta \leq \pi$.

A Claim

As a second point, I claim that, according to p^*_{ma}, the probability of the event $x_3 > x_1, x_2$ tends to one when e tends to zero. To check that point, one studies the inequality $x_3 > x_2$, which writes equivalently:

$$\begin{aligned} \frac{1}{3e}(1-\sin\varphi\cos\theta) &> \frac{2}{3(1-e)}(1-\sin\varphi\cos(\theta+2\pi/3)) \\ 1-3e &> ((1-e)\cos\theta - 2e\cos(\theta+2\pi/3))\sin\varphi \\ 1-3e &> \left((1-e)\cos\theta - e\left[-\cos\theta-\sqrt{3}\sin\theta\right]\right)\sin\varphi \\ 1-3e &> \left(\cos\theta + e\sqrt{3}\sin\theta\right)\sin\varphi. \end{aligned}$$

Since $0 \leq \varphi \leq \pi$, one can see that the inequality holds as soon as

$$1 - 3e > \cos\theta + e\sqrt{3}\sin\theta. \tag{2}$$

Studying the function $\cos\theta + e\sqrt{3}\sin\theta$ for $\theta \in [0, \pi]$, one finds that (2) is equivalent to:

$$\theta > \overline{\theta}(e) = \arccos \frac{1 - 3e - 3e\sqrt{2e\,(1-e)}}{1 - 3e^2}.$$

Recall that the random variable θ is uniform on $[0, \pi]$; since $\overline{\theta}(e)$ tends to 0 with e, the probability of the event $\theta > \overline{\theta}(e)$ tends to 1 when e tends to 0. The claim follows.

An affine approximation

As a consequence, the expected value of the bounded function *gini* on the event $x_1 \leq x_2$ is approximately equal to the expected value of the function that coincides with *gini* on the event $x_1 \leq x_2 \leq x_3$. Let

$$\widetilde{gini}(M) = 1 - \left(n_1^2 + 2n_1n_2 + 2n_1n_3\right)x_1 - \left(n_2^2 + 2n_2n_3\right)x_2 - n_3^2x_3,$$

then

$$\lim_{e \to 0} \int_{\{x_1 \leq x_2\}} \left[gini(M) - \widetilde{gini}(M)\right]\, dp^*_{\mathrm{ma}} = 0.$$

One can write:

$$2\int_{\{x_1 \leq x_2\}} \widetilde{gini}(M)\, dp^*_{\mathrm{ma}} = 2\int_{\theta=0}^{\pi}\int_{\varphi=0}^{\pi} \widetilde{gini}(M)\, f(\varphi, \theta)\, d\varphi\, d\theta$$

where f is the density:

$$f(\varphi, \theta) = \frac{1}{4\pi}\sin\varphi.$$

The expression $\widetilde{gini}$ can be simplified given the expressions of n_i :

$$\widetilde{gini}(M) = 1 - \frac{(1-e)\,(3+e)}{4}x_1 - \frac{(1-e)\,(1+3e)}{4}x_2 - e^2x_3$$

Final computations

The integral to be computed is thus:

$$
\begin{aligned}
& 2\int_{\{x_1\le x_2\}} \widetilde{gini}(M)\ dp^*_{\mathrm{ma}} \\
= \ & 2\int_{\theta=0}^{\pi}\int_{\varphi 0}^{\pi} \widetilde{gini}(M)\ \frac{\sin\varphi}{4\pi}\ d\varphi\ d\theta \\
= \ & \int_0^{\pi}\int_0^{\pi} 1\ \frac{\sin\varphi}{2\pi}\ d\varphi\ d\theta \\
& -\frac{(1-e)\,(3+e)}{4}\int_0^{\pi}\int_0^{\pi} x_1\ \frac{\sin\varphi}{2\pi}\ d\varphi\ d\theta \\
& -\frac{(1-e)\,(1+3e)}{4}\int_0^{\pi}\int_0^{\pi} x_2\ \frac{\sin\varphi}{2\pi}\ d\varphi\ d\theta \\
& -e^2\int_0^{\pi}\int_0^{\pi} x_3\ \frac{\sin\varphi}{2\pi}\ d\varphi\ d\theta.
\end{aligned}
$$

Substituting x:

$$
\begin{aligned}
= \ & \int_0^{\pi}\int_0^{\pi} 1\ \frac{\sin\varphi}{2\pi}\ d\varphi\ d\theta \\
& -\frac{(6+2e)}{12}\int_0^{\pi}\int_0^{\pi} (1-\sin\varphi\cos(\theta-2\pi/3))\ \frac{\sin\varphi}{2\pi}\ d\varphi\ d\theta \\
& -\frac{(2+6e)}{12}\int_0^{\pi}\int_0^{\pi} (1-\sin\varphi\cos(\theta+2\pi/3))\ \frac{\sin\varphi}{2\pi}\ d\varphi\ d\theta \\
& -\frac{e}{3}\int_0^{\pi}\int_0^{\pi} (1-\sin\varphi\cos\theta)\ \frac{\sin\varphi}{2\pi}\ d\varphi\ d\theta.
\end{aligned}
$$

Integrating with respect to θ:

$$
\begin{aligned}
= \ & \pi\int_{\varphi=0}^{\pi} \frac{\sin\varphi}{2\pi}\ d\varphi \\
& -\frac{(6+2e)}{12}\int_0^{\pi} \left(\pi-\sqrt{3}\sin\varphi\right)\ \frac{\sin\varphi}{2\pi}\ d\varphi \\
& -\frac{(2+6e)}{12}\int_0^{\pi} \left(\pi+\sqrt{3}\sin\varphi\right)\ \frac{\sin\varphi}{2\pi}\ d\varphi \\
& -\frac{e}{3}\int_0^{\pi} \pi\frac{\sin\varphi}{2\pi}\ d\varphi.
\end{aligned}
$$

Integrating with respect to φ :

$$\begin{aligned} &= 1-\frac{e}{3} \\ &\quad -\frac{(3+e)}{12}\left(2-\frac{\sqrt{3}}{2}\right) \\ &\quad -\frac{(1+3e)}{12}\left(2+\frac{\sqrt{3}}{2}\right), \end{aligned}$$

and the result follows.

(ii) For the Plurality Game

The disk solution (definition 6) p^*_{pl} in that case can be defined by the equations:

$$\begin{aligned} x_1 &= r\ (1+\sin\varphi\cos(\theta+\alpha)) \\ x_2 &= r\ (1+\sin\varphi\cos(\theta-\alpha)) \\ x_3 &= r\ (1-\sin\varphi\cos\theta)\,, \end{aligned} \tag{3}$$

where r is the radius of the inscribed circle, α is the angle at A_1 (or A_2),

$$\begin{aligned} r &= \frac{1}{2}e\sqrt{1-2e}, \\ \cos\alpha &= \frac{e}{1-e}, \end{aligned}$$

and φ and θ are two independent random variables such that φ has density $(1/2)\sin\varphi$ on $[0,\pi]$ and θ is uniform on $[0,2\pi]$. This is easily seen graphically (picture **??**): C the center of the inscribed circle, $H=(A_1+A_2)/2$ is the foot of A_3, θ is the angle between the vectors $\overrightarrow{CH}$ and $\overrightarrow{CM}$, and $r\sin\varphi$ is the distance CM. Then, if H_1 is the projection of M on the side A_2A_3, with $x_1=MH_1$, one gets by considering the projection P of C on MH_1:

$$\begin{aligned} \widehat{HCP} &= \frac{\pi}{2}-\alpha,\ \widehat{HCM}=\theta, \\ \widehat{CMH_1} &= \theta+\alpha, \\ MH_1 &= PH_1+CM\ \cos(\theta+\alpha) \\ &= r+r\sin\varphi\cos(\theta+\alpha). \end{aligned}$$

There is no simpler way to write down equations (3), but the regions of integration, for computing the expected Gini index are very simple:

$$\begin{array}{llcll} \text{for} & 0 & <\theta< & \pi/2-\alpha & ,\quad x_3<x_1<x_2 \\ \text{for} & \pi/2-\alpha & <\theta< & \pi/2+\alpha & ,\quad x_1<x_3<x_2 \\ \text{for} & \pi/2+\alpha & <\theta< & \pi & ,\quad x_1<x_2<x_3 \end{array}$$

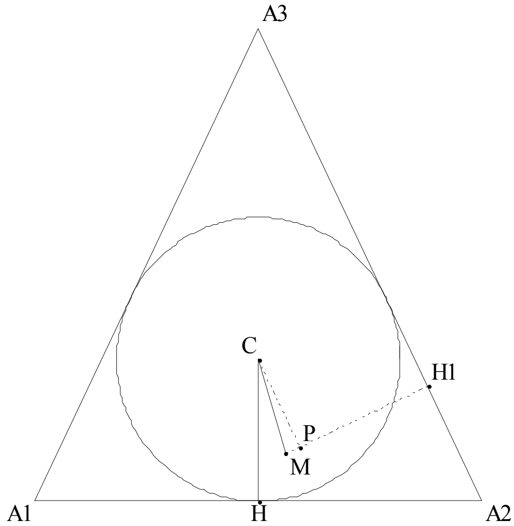

Figure 5: Construction for the proof of propopsition 10

and so on...

On each of these regions, integrating the Gini index with respect to the density $\frac{1}{4\pi}\sin\varphi$ is a long but straightforward computation, suitable for a symbolic computation software. Using Mathematica, one obtains that the expected value, according to p^*_{pl}, of the Gini index is :

$$E^*_{\mathrm{pl}}\left[var\right] = \sqrt{1-2e}\left[\frac{1-2e^2}{1-e}\right]/4 + \frac{e}{\sqrt{2-2e}}\left[\frac{1-2e+2e^2}{1-e}\right]/4$$

therefore:

$$\lim_{e\to 0} E^*_{\mathrm{pl}}\left[var\right] = .25$$

■

5 Conclusion

In a redistributive problem among three groups of unequal size, this paper showed that at a mixed equilibrium of the two-party electoral competition, all voters are, on expectation, treated symmetrically in the Plurality Game while the Tournament Game favors individuals in small groups. In this model, the Downsian competition is predicted to generate more or less inequality depending on an assumption on parties' objective, more inequality arising if parties do not care about their margin of victory.

One may wonder about the possibility that the "party objective" question be settled by observation. To that end, one would need differentiated testable predictions based on the kind of models proposed in the present paper. In the 70's, several scholars, following Brams and Davis, 1974 [9], tried to relate the amount spent during their campaign by presidential candidates in the different States in the USA, to the size of the States. Brams and Davis, used a "Plurality" objective and derive the conclusion that resources are allocated proportionally to the power 3/2 of the electoral vote. Lake, 1979 [17] used a "probability of winning" objective and found that candidates should spend a disproportionately large amount of their funds in *large* states. These analyses use a "local equilibrium" approach which is not totally satisfying, and the conformity of observations with prediction has been disputed (see Colantoni, Levesque and Ordeshook, 1975 [10], and the exchanges that followed; see also Young, 1978 [30]).

My opinion is that there is little hope, when trying to figure out what is the true objective of a party, to find a simple answer of the form "parties maximise the expected plurality and not the probability of winning", and this for two reasons. One obvious reason is that a party is not a single decision-maker but a complex organization whose members have different goals: winning the next election, but also winning other elections, promoting ideas, representing some part of the electorate, etc. Another reason is the following: When we simplify things to a point where parties are "Downsian" and the party objective is one of the two objectives under scrutiny, we are often confronted with mixed strategy equilibria, whose predictions are difficult to test. I therefore submit that the main thing we need is to better understand the nature of political competition in the absence of a Condorcet winner policy, which includes understanding how can a party behave in front of the voters when the party has no pure optimal strategy. I suspect that a key point here is to deepen our theoretical knowledge of the communication process that takes place between a candidate and a voter. Thus the "party objective problem" is probably a case in point where we are in need of theory before observations.

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