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Résumé: Ce travail utilise une version évolutionniste du modèle de Kiyotaki et Wright (1989) sur l'émergence de la monnaie. Le principal objectif de ce travail est d'étudier les conséquences de l'endogénéité du processus d'appariement des agents. Dans ce cas, les conditions de stabilité sont trouvées pour les deux types d'équilibres (équilibre fondamental ou spéculatif). Le second objectif est d'analyser la dynamique hors équilibre quand la distribution des stocks n'est pas supposée avoir sa valeur d'équilibre temporaire. On montre que, pour certain valeurs des paramètres, l'équilibre fondamental et l'équilibre spéculatif sont instables.

Abstract: This paper uses an evolutionary version of the commodity money model in Kiyotaki and Wright (1989). The main objective of this paper is to study the implications of endogenizing the matching process. Under the endogenous set up we find stability conditions for each kind of equilibrium (fundamental or speculative). The second objective is to analyse the disequilibrium dynamics, when the inventory distribution is not assumed to be continuously at its temporary equilibrium value. We prove that under this setting, for some values of the parameters the fundamental and speculative states are unstable.

Mots clés : monnaie, évolution, appariement, dynamique hors équilibre

Key Words : money, evolution, matching, disequilibrium dynamics

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1. Introduction

The neoclassical general equilibrium models get away with the difficulty of co-ordination of trade in a many person economy by a fictional coordinating and price setting central authority. The weakness of modelling trade among agents by assuming a central authority results in a failure to model the demand for money. The emergence of money as a medium of exchange is not explained and the demand for money is rather rationalised on the grounds that it is an asset of low risk and high liquidity. Search theoretic models overcome this failure by an explicit modelling of resource allocation process and provide a framework where the use of money depends on the degree of acceptance of money as a medium of exchange. The process of search and recruitment introduces trade frictions such as bilateral exchange, lack of commitment and memory. Such frictions generate in Kiyotaki and Wright (1989) an essential role for money with no particular constraint that it must be used in exchange[1].

This paper uses an evolutionary version of the commodity money model in Kiyotaki and Wright (1989) which explores the structure of barter trades in an econ-

[1] Kiyotaki and Wright (1989) has been the source of a fruitfull literature. This literature is by now large; a few examples include Kiyotaki and Wright (1991,1993), Aiyagari and Wallace (1991,1992), Kehoe, Kiyotaki and Wright (1993), Matsuyama, Kiyotaki, and Matsui (1993), Trejos and Wright (1995), Shi (1995).

omy with the following characteristics: There is a continuum of rational agents living a finite but uncertain number of periods. The agents are specialised in consumption and production and meet pairwise and engage in bilateral exchange. The goods are indivisible and durable but costly to store. Since all goods are indivisible, there is one to one swap of inventories in case of mutually agreed upon trade. In this setting, certain goods emerge as media of exchange depending both on intrinsic properties and extrinsic beliefs. These two situations are referred to as *fundamental equilibrium* and *speculative equilibrium* by Kiyotaki and Wright (1989). *Fundamental equilibrium* is the situation where agents accept a non consumption good to facilitate further trade if it has a lower storage cost than the one currently held in inventory. On the other hand, *speculative equilibrium* requires agents to accept a good with a higher storage cost.

These results are obtained based on the standard assumption of rationality. Kiyotaki and Wright (1989) model and most of its extensions assume that the rational and optimizing agents are collectively able to locate an equilibrium of the model. However, the evolutionary approach suggests that the initial population consists of a variety of heterogeneous types reflecting all permissible behaviours with their related material rewards. The population evolves in such a manner that the population share of more highly rewarded behaviours grows relative to that of poorly rewarded behaviours. Then the asymptotically stable rest points of this dynamic selection process are identified.

The point of departure of this paper is Sethi (1999) which analyses Kiyotaki and Wright (1989) model of money within an evolutionary framework. In this version of the model with exogenous random matching, it is shown that *fundamental*, *speculative* as well as 'polymorphic' states can all be stable and there may exist a multiplicity of stable states. In order to show the evolutionary stability of these states, Sethi (1999) uses the following procedure. Given a behavioural population composition, the dynamics of inventory holdings are defined and the equilibrium values of inventories are expressed. Then, evolutionary selection dynamics are applied to various population states in order to establish their stability with respect to the evolutionary dynamics. The assumption behind this analysis is that the inventory distribution is expected to be continuously at its temporary equilibrium value even when the behavioural composition evolves.

The present paper deals with two issues. In Section 2, the relation of the random matching assumption with the emergence of media of exchange is explored. The standard search theoretic models assume that agents meet exogenously and at random. This assumption is rather unrealistic since agents do not choose their actions on the basis of random encounters. In order to make the matching process endogenous, matching probabilities are added to the model of Sethi (1999). The conditions of stability of fundamental, speculative and polymorphic states are defined as a function of the matching probabilities. As a second issue, in Section 3, the dynamics of population shares are analysed when the inventory distribution is not assumed to be continuously at its temporary equilibrium value. The assumption that the inventory distribution is continuously at its temporary equilibrium value implies that the effect of changes in the inventory distribution in response to disequilibrium is neglected. In order to take into account this effect, we propose a model with random matching in which the population is classified according to the inventory distribution and the behavioural distribution. This classification allows us to analyse the evolution of population shares through trade and evolutionary selection affecting the process at different rates. The convergence to fundamental and speculative states is not observed for some values of the parameters. Instead the population stays polymorphic.

2. Endogenous matching

In this section the results in Sethi (1999) are reviewed with a different matching setup. The notation of the original article is adopted. First, the environment is described. Then, given behavioural population distribution, temporary equilibrium values of inventory holdings are calculated. The existence and the stability of those values are discussed. Given the equilibrium values of inventories, the utilities corresponding to different behavioural situations are given and the evolutionary stability of these is analysed.

2.1. The model

Goods: There exist three indivisible goods indexed by i. They are durable. Storing the good i entails the cost c_i (in terms of instantaneous disutility). A one for one swap of inventories occurs in case of mutually agreed upon trade.

Time: Time is discrete and indexed by $t \in N$.

Economic agents: There exists three types of agents again indexed by i. Agents specialize in consumption and production. Agent i derives utility only from consuming good i and produces only good $i+1$ modulo 3[2]. Agents can store only one good at a time since goods are indivisible and stored at a cost. If an agent of type i gets through trade his consumption good i he consumes it immediately, gets one unit of utility and produces a new unit

[2] Agent 3 consumes good 3 and produces good 1.

of good $i+1$. Thus agents always have in stock one unit of one good other than their consumption good.

In period 0, n agents of each type, each endowed with one unit of their production good enter the market. The number of agents stays fixed thereafter.

For each type of agent there exist two strategies: Agents of type i can accept only their consumption good i when they meet another agent. These agents are denoted by αi. On the other hand agents denoted by βi can accept both goods i and $i+2$ if they trade. An agent of type αi has always in stock his production good $i+1$. In return agents of type βi can have either the good $i+1$ or $i+2$.

Matching: Each period one agent is selected randomly (with probability $\frac{1}{3n}$). Then this agent is supposed to contact an agent to trade from the other two populations. Agents of type i are assumed to contact agents of type $i+1$ with probability π_i and with agents of type $i+2$ with probability $1-\pi_i$. If we denote by I_i the set of agents of type i and by (x,y) the pair trading at period t then the following probabilities apply:

- $\Pr(x \in I_i) = \frac{1}{3}$
- $\Pr(y \in I_i | x \in I_i) = 0$

– $\Pr(y \in I_{i+1} | x \in I_i) = \pi_i$

– $\Pr(y \in I_{i+2} | x \in I_i) = 1 - \pi_i$

Therefore, the probability that agent i and agent $i+1$ are matched is $\frac{\pi_i + 1 - \pi_{i+1}}{3}$. In order to simplify the notation, we will denote

$$a_i = \frac{(1 - \pi_i + \pi_{i+2})}{3}.$$

The pair exchanges their inventories if it is mutually agreeable. The pair dissolves in case of both mutually agreed trade or rejection.

Information: The agents know the types of people they meet but have no information on the inventory holdings and the strategies of these agents.

In order to analyse this model from an evolutionary point of view we need to allow all behaviours from the part of agents and study the ones that are robust to a dynamic selection mechanism. Thus, the total population of agents can be characterised according to two criteria: the share of agents adopting a certain behaviour and the share of agents holding a certain good.

We denote the proportion of agents of type β among agents of type i by s_i. Therefore the share of α agents is $1 - s_i$. The proportion of agents of type i having in stock their production good $i+1$ is denoted by p_i. Hence the share of agents of type βi holding $i+2$ is $1-p_i$ and the share of agents of type βi holding $i+1$ is p_i+s_i-1. The player types, their inventory holdings and population shares are represented in Table 1.

Table 1 Agent types, inventory holdings and population shares

Type	Inventory holding	Population shares among i types
αi	$i+1$	$1-s_i$
βi	$i+1$	s_i+p_i-1
	$i+2$	$1-p_i$

2.2. Temporary equilibrium

2.2.1. Inventory dynamics

Suppose the composition of behaviours $s \in [0,1]^3$ is given. We denote the set containing the values of inventory holdings by $\Delta(s)$ which is defined as follows:

$$\Delta(s) = [1-s_1, 1] \times [1-s_2, 1] \times [1-s_3, 1]$$

Agents of type αi have always in stock their production good $i+1$ as they accept only their consumption good. Therefore, agents of type αi do not have an effect on the inventory distribution when they trade.

Agents of type βi holding their production good $i+1$ can decrease the share

p_i in their population if they exchange $i+1$ for $i+2$. This happens if an agent of type βi holding $i+1$ is selected and he in return selects an agent $\beta(i+1)$ holding $i+2$ (with probability $\frac{1}{3}(s_i+p_i-1)\pi_i p_{i+1}$) or an agent of type $\beta(i+1)$ holding $i+2$ is selected and he in return selects an agent βi holding $i+1$ (with probability $\frac{1}{3}p_{i+1}(1-\pi_{i+1})(s_i+p_i-1)$).

On the other hand, agents of type βi holding the good $i+2$ can increase the share p_i in their population if they exchange $i+2$ for $i+1$. This happens if an agent of type βi holding $i+2$ is selected and he in return selects an agent $\beta(i+2)$ holding i (with probability $\frac{1}{3}(1-p_i)(1-\pi_i)p_{i+2}$) or an agent of type $\beta(i+2)$ holding i is selected and he in return selects an agent βi holding $i+2$ (with probability $\frac{1}{3}p_{i+2}\pi_{i+2}(1-p_i)$).

The resulting inventory dynamics is given by the following equation:

$$\dot{p}_i = (1-p_i)p_{i+2}a_i - (s_i+p_i-1)p_{i+1}a_{i+1} \tag{2.1}$$

The inventory dynamics do not cross the boundary of $\Delta(s)$. In other words if $p(0)\in\Delta(s)$, then $p(t)\in\Delta(s)$ for all t. To see this, notice the following limits:

$$\lim_{p_i\uparrow 1}\dot{p}_i = -s_i p_{i+1}a_{i+1} \leqslant 0$$
$$\lim_{p_i\downarrow 1-s_i}\dot{p}_i = s_i p_{i+2}a_i \geqslant 0$$

2.2.2. Temporary equilibria and their stability

For any given vector of population shares $s \in [0,1]^3$, the rest points of equation (2.1) are defined as:

$$\Phi(s) = \{p \in \Delta(s) \mid \dot{p} = 0\}$$

A state s is *monomorphic* if individuals belonging to the same population are of the same behavioural type. States which are not monomorphic are *polymorphic*. The rest points of equation (2.1) are calculated for monomorphic populations and the values are provided at Table 2.

Table 2 Equilibrium inventories for monomorphic populations

s	$\rho(s)$
$(0,0,0)$	$(1,1,1)$
$(0,1,0)$	$(1, \frac{a_2}{a_3+a_2}, 1)$
$(1,1,0)$	$(\rho_1^3, \rho_2^3, 1)$
$(1,1,1)$	$(\rho_1^4, \rho_2^4, \rho_3^4)$

Notice that for $s = (1,1,0)$ the equilibrium inventories is given by $(\rho_1^3, \rho_2^3, 1)$ where

$$\rho_1^3 = \frac{a_1}{2a_2(a_1+a_2)} \left(a_2 - a_3 + \sqrt{\frac{4a_2^2 a_3}{a_1} + (a_2+a_3)^2} \right) \tag{2.2}$$

and

$$\rho_2^3 = \frac{a_1(1-\rho_1^3)}{a_2\rho_1^3} \tag{2.3}$$

For $s = (1,1,1)$ the equilibrium inventories is $(\rho_1^4, \rho_2^4, \rho_3^4)$ where

$$\rho_1^4 = \frac{a_3\left[(\rho_3^4 - 1)a_2 + (\rho_3^4)^2 a_1\right]}{\rho_3^4 a_1 a_2} \tag{2.4}$$

and

$$\rho_2^4 = 1 - \frac{(\rho_3^4)^2 a_1}{(1-\rho_3^4)a_2} \tag{2.5}$$

and ρ_3^4 is given by the following fourth degree equation:

$$\begin{aligned} 0 \;=\; & \rho_3^4(a_1a_2a_3 - 2a_2^2a_3) + (\rho_3^4)^2(-4a_1a_2a_3 + a_1^2a_2 + a_2^2a_3) \\ & +(\rho_3^4)^3(3a_1a_2a_3 - a_1^2a_2 - a_1^2a_3) + 2(\rho_3^4)^4a_1^2a_3 + a_2^2a_3 \end{aligned} \tag{2.6}$$

The set $\Phi(s)$ of rest points calculated given a behavioural population composition has at least one element for each admissible composition. The analogous of the following propositions have been proved in Sethi (1999) for the case of random matching. The proof of these are provided in the appendix.

Proposition 2.1. *The set of rest points* $\Phi(s)$ *is non-empty for all values of* s_i *for all* $i = 1, 2, 3$.

Proposition 2.2. *Suppose* $s \neq (1, 1, 1)$, *the set of rest points* $\Phi(s)$ *contains a single element.*

Proposition 2.3. *Suppose* $s = (1, 1, 1)$, $\pi_i \neq 0$ *and* $\pi_i \neq 1$. *The set of rest points* $\Phi(s)$ *contains exactly two elements, exactly one of which is stable with respect to the dynamics of equation (2.1).*

Finally, we will provide a technical proposition that will be useful in the sequel.

Proposition 2.4. *Suppose* $\pi_i \neq 0$ *and* $\pi_i \neq 1$. *The function* $\rho(s)$ *is continuous at all monomorphic states and at all polymorphic states of the type* $s = (x, 1, 0)$ *where* $x \in (0, 1)$.

2.3. Evolutionary stability

Given that inventories are continuously at their equilibrium values, we proceed with the evolution of the behavioural composition s. The evolutionary approach analyses the population distribution of behaviours (decision rules, strategies) subject to specific selection dynamics. Consequently, we will allow for all permissible behaviours on the part of agents and analyse the behaviours which will survive the dynamic selection process.

2.3.1. Expected payoffs

We denote the utility of agent i by u_i. According to the strategies the agents adopt, the utility of an agent of type αi is denoted by $u_{\alpha i}$ and the utility of an agent of type βi is denoted by $u_{\beta i}$. The expected payoffs to each type of player are functions of the population composition s and the corresponding equilibrium inventories $\rho(s)$. The expected payoff of an agent of type αi is:

$$u_{\alpha i}(s) = (1 - \rho_{i+1}(s))a_{i+1} + (s_{i+2} + \rho_{i+2}(s) - 1)a_i - c_{i+1} \tag{2.7}$$

where the first two terms indicate the expected payoff from consumption and the last term is the cost of storing good $i + 1$. Agents of type αi will have in stock the good $i + 1$ whether they trade or not.

Agents of type βi have in stock either their production good $i+1$ or the good $i + 2$. Since the good held by these agents changes over time through trade, we need to define the probability of holding the good $i+1$ and the good $i+2$ in order to compute the expected payoff of agent of type βi. The probability of having $i+1$ in inventory is denoted by $\tau_i(s)$. The probability of having $i+2$ in inventory is consequently equal to $1-\tau_i(s)$. The probability of having $i+1$ in stock, $\tau_i(s)$ is given by:

$$\tau_i(s) = \frac{\rho_{i+2}(s)a_i}{\rho_{i+2}(s)a_i + \rho_{i+1}(s)a_{i+1}}$$

The expected payoff of an agent of type βi holding $i+1$ is:

$$u_{\beta i}^{i+1}(s) = (1-\rho_{i+1}(s)(1+c_{i+2}-c_{i+1}))a_{i+1} + (s_{i+2}+\rho_{i+2}(s)-1)a_i - c_{i+1} \quad (2.8)$$

The expected payoff of an agent of type βi holding $i+2$ is:

$$u_{\beta i}^{i+2}(s) = \rho_{i+2}(s)a_i(1-c_{i+1}) - (1-\rho_{i+2}(s)a_i)c_{i+2} \quad (2.9)$$

Then the expected payoff of an agent of type βi will be:

$$u_{\beta i}(s) = \tau_i(s)u_{\beta i}^{i+1}(s) + (1-\tau_i(s))u_{\beta i}^{i+2}(s) \quad (2.10)$$

Given the payoffs to each strategy in each population, define the mean payoff in population i as:

$$\bar{u}_i(s) = (1 - s_i)u_{\alpha i}(s) + s_i u_{\beta i}(s) \tag{2.11}$$

2.3.2. Definition of the replicator dynamics

In the evolutionary setting, there are interactions among boundedly rational agents from each of the three populations. These agents have little or no information about the environment. In each population, we allow for all types of behaviours on the part of agents. Evolutionary pressures select better performing behaviours in the long run. The selection dynamics governing change are in continuous time and are regular selection dynamics. Given the payoffs to each of the two behavioral types in each of the three sub-populations, the evolution of the behavioural composition of the population is given by the following system of continuous-time differential equations: $\dot{s}_i = \xi_i(s)$. The function ξ is said to yield a monotonic selection dynamic if the following conditions are satisfied:

i. ξ is Lipschitz continuous

ii. $s_i = 0 \Rightarrow \xi_i(s) \geqslant 0$ and $s_i = 1 \Rightarrow \xi_i(s) \leqslant 0$

iii. $\lim\limits_{s_i \to 0} \frac{\xi_i(s)}{s_i}$ exists and is finite.

iv. $u_{\beta i}(s) > (=)\ u_{i\alpha}(s) \Rightarrow \frac{\xi_i(s)}{s_i} > (=)\ 0$

These conditions ensure that s_i remains in $[0, 1]$, its growth rates are defined and continuous at all points $s \in [0,1]^3$ and the growth of the share of β types in population i is proportional to its relative payoff. Taylor and Jonker (1978) defined a special case of the class of monotonic selection dynamics as the replicator dynamics.

$$\frac{\dot{s}_i}{s_i} = u_{\beta i}(s) - \bar{u}_i(s) \tag{2.12}$$

2.3.3. Asymptotic stability for the replicator dynamics

Note that all monomorphic population states are rest points of monotonic dynamics. Asymptotically stable monomorphic rest points will now be described. Notice that results analogous to the following two propositions have been proved in Sethi (1999) for the case of random matching. The proof of these propositions are provided in the appendix.

Proposition 2.5. *Suppose $c_1 < c_2 < c_3$.*

1. *(Fundamental equilibrium) If $(c_3 - c_2) > a_1 - a_2 + \frac{a_2^2}{a_3+a_2}$ there is an asymptotically stable rest point at $s = (0,1,0)$.*

2. *(Speculative equilibrium) If $(c_3 - c_2) < a_1 - a_2 + \rho_2^3 a_2$ there is an asymptotically stable rest point at $s = (1,1,0)$.*

The first part of the proposition describes the case where there is a fundamental equilibrium. On the other hand, the second part of the proposition gives the stability condition of the speculative equilibrium. In Sethi (1999), the simulations are done based on the cost vector

$$c = (0.01, 0.04, 0.09)$$

and the replicator dynamics. At this cost vector, the population of agents converge to a speculative equilibrium. Through holding c_1 and c_2 constant and raising c_3, Sethi (1999) shows that the speculative equilibrium loses stability at $c_3 = 0.18$ and the population converges to a polymorphic state where some agents of type 1 use fundamental strategies while others speculate. At $c_3 = 0.21$, there is a stable fundamental equilibrium.

In this paper, the conditions of stability are expressed in terms of matching probabilities. In order to simplify the analysis, we will suppose that agents of type 1 and 2 will randomly choose their trading partners. Then the conditions of stability for the first part of the proposition becomes:

$$3(c_3 - c_2) > \pi_3 - 0.5 + \frac{1}{2.5 - \pi_3}$$

A graphical analysis will show that based on the same cost vector, up to $\pi_3 = 0.21$, the fundamental equilibrium is stable (the intersection of the dashed curve and the solid horizontal line representing $3(c_3 - c_2)$ in Figure (2.1)). On the other hand, the conditions of stability for the second part of the proposition becomes the following complicated equation:

$$3(c_3 - c_2) < \frac{-3(1-\pi_3)^2 + \sqrt{9\pi_3 - 2.5\pi_3^2 - 4\pi_3^3 + \pi_3^4 + 4.5625} - 1.25}{3 - 2\pi_3}$$

In order to visualise this function, we provide the graph of the right hand side by the solid curve in Figure (2.1). It shows that above $\pi_3 = 0.38$, the speculative equilibrium is stable based on the same cost vector (the intersection of the solid curve and the solid horizontal line representing $3(c_3 - c_2)$).

Proposition 2.6. *Suppose* $c_1 < c_2 < c_3$.

If $a_1 - a_2 + \frac{-a_1(1-a_1)+\sqrt{a_1^2(1-a_1)^2+4a_1a_2^2(1-a_2-a_1)}}{2(1-a_1-a_2)} < (c_3 - c_2) < a_1 - a_2 + \frac{a_2^2}{1-a_1}$ there is an asymptotically stable rest point at $s = (x, 1, 0)$ where $x \in (0, 1)$.

When we suppose that the agents of type 1 and 2 will randomly choose their trading partners, the previous condition of stability becomes:

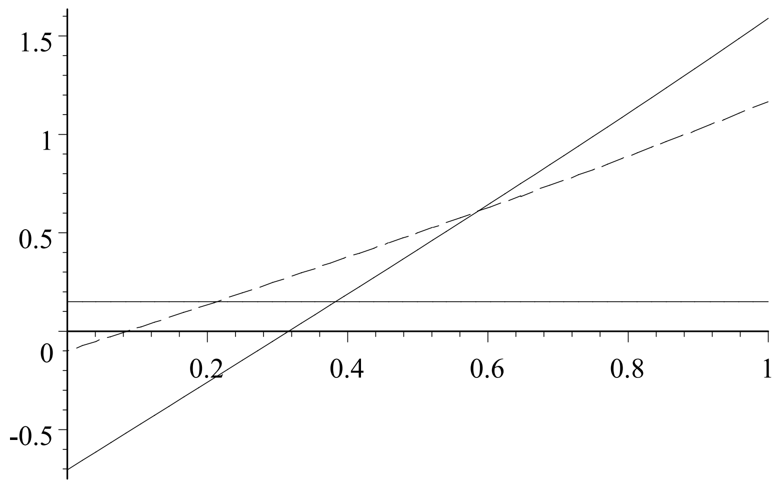

Figure 2.1: The first and second condition of stability in terms of π_3

$$
\begin{aligned}
&\frac{-3(1-\pi_3)^2+\sqrt{9\pi_3-2.5\pi_3^2-4\pi_3^3+\pi_3^4+4.5625}-1.25}{3-2\pi_3} \\
< \;\; & 3(c_3-c_2) < \pi_3 + \frac{1}{2.5-\pi_3} - 0.5
\end{aligned}
$$

In order to visualise this function, we can use the same graph we used for the stability of the fundamental and speculative equilibrium (2.1) since the right and left hand sides turn out to be the same equations as the previous equations for the stability of the fundamental and speculative equilibrium. We can also conclude that when π_3 varies between 0.21 and 0.38, the polymorphic equilibrium is stable.

3. Disequilibrium dynamics

In the previous section, given various initial distributions of strategies, the rest points of inventory dynamics are determined. Then the robustness of these distributions of strategies is checked. The assumption that the inventory distribution is continuously at its temporary equilibrium value implies that the effect of changes in the inventory distribution in response to disequilibrium is neglected. In this section, the population shares change according to trade and evolutionary selection affecting the process at different rates. Thus the inventories are allowed to be in disequilibrium. The dynamics are studied with $\pi_i = 0.5$ for all $i = 1, 2, 3$.

3.1. The model

The description of the previous section will be used. In order to analyse this model from an evolutionary point of view, we consider that the agents may change their strategies. The total population of agents in a group can be characterised according to two criteria: the strategy they adopt (either α or β) and the good they have in stock (either $i+1$ or $i+2$). Consequently, each group is composed of four categories. We denote the proportion of agents of type αi having in stock their production good $i+1$ by e_i and the proportion of agents of type αi having in stock the good $i+2$ by f_i. The share of agents of type βi holding $i+1$ is denoted by g_i and the share of agents of type βi holding $i+2$ by h_i. The player types, their inventory holdings and population shares are represented in Table 3.

Table 3 Agent types, inventory holdings and population shares

Type	Inventory holding	Population shares among i types
αi	$i+1$	e_i
αi	$i+2$	f_i
βi	$i+1$	g_i
βi	$i+2$	h_i

Given the previous definition of population shares, the population of agents can be represented by the matrix $r = (\sigma_1, \sigma_2, \sigma_3)$ where σ_1, σ_2 and σ_3 are the row vectors $\sigma_1 = (e_1, f_1, g_1)$, $\sigma_2 = (e_2, f_2, g_2)$ and $\sigma_3 = (e_3, f_3, g_3)$. We define the utility of agent i according to the group they belong (for example the utility of an agent of type αi holding in stock the good $i+1$ will be denoted by $u_{e_i}(r)$). Consequently there will be four utility functions for each group.

The expected payoff to an agent of type αi holding the good $i+1$ is:

$$u_{e_i}(r) = \frac{1}{3}(f_{i+1} + h_{i+1}) + \frac{1}{3}g_{i+2} - c_{i+1} \tag{3.1}$$

The expected payoff to an agent of type αi holding the good $i+2$ is:

$$u_{f_i}(r) = \frac{1}{3}(e_{i+2} + g_{i+2})(1 - c_{i+1}) - (1 - \frac{1}{3}(e_{i+2} + g_{i+2}))c_{i+2} \tag{3.2}$$

The expected payoff to an agent of type βi holding the good $i+1$ is:

$$u_{g_i}(r) = \frac{1}{3}(f_{i+1}+h_{i+1})+\frac{1}{3}(e_{i+2}+g_{i+2})-\frac{1}{3}(e_{i+1}+g_{i+1})c_{i+2}-(1-\frac{1}{3}(e_{i+1}+g_{i+1}))c_{i+1} \tag{3.3}$$

The expected payoff of an agent of type βi holding the good $i+2$ is:

$$u_{h_i}(r) = \frac{1}{3}(e_{i+2} + g_{i+2})(1 - c_{i+1}) - (1 - \frac{1}{3}(e_{i+2} + g_{i+2}))c_{i+2} \tag{3.4}$$

3.2. Evolution and trade

The change in population composition results from the change in stocks due to trade and from the change in strategies which is due to the evolutionary mechanism. We denote the change due to trade by Δ^{trade} and the change due to the evolutionary mechanism by Δ^{imit}. These changes do not occur at the same rate. In this paper, it is supposed that agents have a chance to trade more often than they have a chance to revise their strategies. Accordingly, the rate at which the agents trade, v_1 is greater than the rate at which the agents revise their strategies, v_2 ($v_1 > v_2$). Without loss of generality, let $v_1 = 1$ and $v_2 = v < 1$.

The change in e_i is given by the following equation:

$$\dot{e}_i = \Delta_{e_i}^{trade} + v\Delta_{e_i}^{imit} \tag{3.5}$$

Agents of type αi having in stock their production good $i+1$ do not affect the inventory distribution when they trade as they accept only their consumption good whenever they trade. On the other hand, agents of type αi having in stock the good $i+2$ matched with agents of type $\alpha(i+2)$ and $\beta(i+2)$ having in stock the good i will increase the share e_i and consequently decrease the share f_i. Notice that $\Delta_{e_i}^{trade} = -\Delta_{f_i}^{trade}$.

$$3\Delta_{e_i}^{trade} = f_i(e_{i+2} + g_{i+2}) \tag{3.6}$$

The evolutionary dynamics modelled in this section is replication by imitation. In the previous section the replicator dynamics allowed replication by way of biological reproduction where each agent reproduces according to his relative fitness measured in terms of the payoff for his strategy and each offspring inherits his single parent's strategy. In case of replication by imitation, agents live forever but review their pure strategies. Each reviewing agent samples another agent at random from his player population.

In this model, the distribution of agents in each population does not exactly represent the distribution of strategies since the differences of stocks are taken into account. Each reviewing agent is supposed to meet at random an agent from his population and imitate the strategy of the agent he meets if it is better performing. The contribution of our model rises from the fact that the performance of the strategies depends also on the inventory holdings of the agents. If an agent of type αi having in stock $i+1$ meets an agent of type βi having in stock $i+2$ and finds out that the strategy β performs better, he will imitate this agent and adopt the strategy β but in return he will not become part of agents βi having in stock $i+2$. The imitation will not affect the share h_i.

As a result we have to define four regions and write the dynamics due to imitation in these regions.

Regions	
A	$u_{f_i}(r) > u_{g_i}(r)$ and $u_{h_i}(r) > u_{e_i}(r)$
B	$u_{f_i}(r) > u_{g_i}(r)$ and $u_{h_i}(r) < u_{e_i}(r)$
C	$u_{f_i}(r) < u_{g_i}(r)$ and $u_{h_i}(r) > u_{e_i}(r)$
D	$u_{f_i}(r) < u_{g_i}(r)$ and $u_{h_i}(r) < u_{e_i}(r)$

Note two following points.

1. $u_{h_i}(r) > u_{e_i}(r)$ whenever $u_{f_i}(r) > u_{g_i}(r)$.

$u_{h_i}(r) > u_{e_i}(r) \Longrightarrow \frac{\frac{1}{3}(f_{i+1}+h_{i+1}-e_{i+2})}{1-\frac{1}{3}(e_{i+2}+g_{i+2})} < -(c_{i+2} - c_{i+1})$

$u_{f_i}(r) > u_{g_i}(r) \Longrightarrow -(c_{i+2} - c_{i+1}) > \frac{\frac{1}{3}(f_{i+1}+h_{i+1})}{1-\frac{1}{3}(e_{i+2}+g_{i+2}+e_{i+1}+g_{i+1})}$

$u_{h_i}(r) > u_{e_i}(r)$ whenever $u_{f_i}(r) > u_{g_i}(r)$ since we have

$\frac{\frac{1}{3}(f_{i+1}+h_{i+1}-e_{i+2})}{1-\frac{1}{3}(e_{i+2}+g_{i+2})} < \frac{\frac{1}{3}(f_{i+1}+h_{i+1})}{1-\frac{1}{3}(e_{i+2}+g_{i+2}+e_{i+1}+g_{i+1})}$

Consequently it is impossible to be in the B region.

2. $u_{f_i}(r) < u_{g_i}(r)$ and $u_{h_i}(r) > u_{e_i}(r)$.

$u_{f_i}(r) < u_{g_i}(r) \Longleftrightarrow -(c_{i+2} - c_{i+1}) < \frac{\frac{1}{3}(f_{i+1}+h_{i+1})}{1-\frac{1}{3}(e_{i+2}+g_{i+2}+e_{i+1}+g_{i+1})}$

$$u_{h_i}(r) > u_{e_i}(r) \iff \frac{\frac{1}{3}(f_{i+1}+h_{i+1}-e_{i+2})}{1-\frac{1}{3}(e_{i+2}+g_{i+2})} < -(c_{i+2}-c_{i+1})$$

Denote by $k^* \frac{\frac{1}{3}(f_{i+1}+h_{i+1})}{1-\frac{1}{3}(e_{i+2}+g_{i+2}+e_{i+1}+g_{i+1})}$ and by $k_* \frac{\frac{1}{3}(f_{i+1}+h_{i+1}-e_{i+2})}{1-\frac{1}{3}(e_{i+2}+g_{i+2})}$. We can rewrite the conditions as functions of the costs.

Regions	
A	$-(c_{i+2}-c_{i+1}) > k^*$
C	$k_* < -(c_{i+2}-c_{i+1}) < k^*$
D	$-(c_{i+2}-c_{i+1}) < k_*$

The dynamics of replication by imitation for e_i is given by the following equation:

$$3\Delta_{e_i}^{imit} = 2e_i g_i(u_{e_i}(r)-u_{g_i}(r)) + \begin{cases} f_i g_i(u_{f_i}(r)-u_{g_i}(r)) - e_i h_i(u_{h_i}(r)-u_{e_i}(r)) & A \\ -e_i h_i(u_{h_i}(r)-u_{e_i}(r)) & C \\ 0 & D \end{cases} \tag{3.7}$$

The change in f_i is given by the following equation:

$$\dot{f_i} = \Delta_{f_i}^{trade} + v\Delta_{f_i}^{imit} \tag{3.8}$$

Notice again that $\Delta_{e_i}^{trade} = -\Delta_{f_i}^{trade}$.

$$3\Delta_{f_i}^{trade} = -f_i(e_{i+2} + g_{i+2}) \tag{3.9}$$

The dynamics of replication by imitation for f_i is given by the following equation:

$$3\Delta_{f_i}^{imit} = 2f_ih_i(u_{f_i}(r) - u_{h_i}(r)) + \begin{cases} 0 & A \\ -f_ig_i(u_{g_i}(r) - u_{f_i}(r)) & C \\ e_ih_i(u_{e_i}(r) - u_{h_i}(r)) - f_ig_i(u_{g_i}(r) - u_{f_i}(r)) & D \end{cases} \tag{3.10}$$

The change in g_i is given by the following equation:

$$\dot{g}_i = \Delta_{g_i}^{trade} + v\Delta_{g_i}^{imit} \tag{3.11}$$

Agents of type βi holding their production good $i+1$ can decrease the share g_i, consequently increase the share h_i in their population if they exchange $i+1$ for $i+2$. This happens if an agent of type βi holding $i+1$ and an agent $\beta(i+1)$ holding $i+2$ are selected (with probability $\frac{1}{3}g_i(e_{i+1} + g_{i+1})$). If agents of type βi holding their production good $i+1$ exchange $i+1$ for $i+2$, they decrease the share g_i and consequently increase the share h_i.

On the other hand, agents of type βi holding the good $i+2$ can increase the share g_i and consequently decrease the share h_i in their population if they exchange $i+2$ for $i+1$. This happens if an agent of type βi holding $i+2$ and an agent $\beta(i+2)$ holding i are selected (with probability $\frac{1}{3}h_i(e_{i+2}+g_{i+2})$). Notice that $\Delta_{g_i}^{trade} = -\Delta_{h_i}^{trade}$.

The resulting dynamics is given by the following equation:

$$3\Delta_{g_i}^{trade} = h_i(e_{i+2}+g_{i+2}) - g_i(e_{i+1}+g_{i+1}) \tag{3.12}$$

The dynamics of replication by imitation for g_i is given by the following equation:

$$3\Delta_{g_i}^{imit} = 2e_i g_i(u_{g_i}(r) - u_{e_i}(r)) + \begin{cases} f_i g_i(u_{g_i}(r) - u_{f_i}(r)) + e_i h_i(u_{h_i}(r) - u_{e_i}(r)) & A \\ e_i h_i(u_{h_i}(r) - u_{e_i}(r)) & C \\ 0 & D \end{cases} \tag{3.13}$$

The change in h_i is given by the following equation:

$$\dot{h}_i = \Delta_{h_i}^{trade} + v\Delta_{h_i}^{imit} \tag{3.14}$$

Notice again that $\Delta_{g_i}^{trade} = -\Delta_{h_i}^{trade}$.

$$3\Delta_{h_i}^{trade} = -h_i(e_{i+2} + g_{i+2}) + g_i(e_{i+1} + g_{i+1}) \tag{3.15}$$

The dynamics of replication by imitation for h_i is given by the following equation:

$$3\Delta_{h_i}^{imit} = 2f_ih_i(u_{h_i}(r) - u_{f_i}(r)) + \begin{cases} 0 & A \\ f_ig_i(u_{g_i}(r) - u_{f_i}(r)) & C \\ e_ih_i(u_{h_i}(r) - u_{e_i}(r)) + f_ig_i(u_{g_i}(r) - u_{f_i}(r)) & D \end{cases} \tag{3.16}$$

3.3. Sufficient conditions of instability

The fundamental and speculative equilibria are rest points of the dynamics given by equations 3.5-3.16. At this point, we will study the stability of these equilibria. Again, to simplify the analysis c_1 will be 0.01, and c_2 will be 0.04. There are as a result two parameters with respect to which we can analyse their stability. In order to determine the conditions under which they are stable, the characteristic polynomial of the Jacobian has to be computed. This polynomial turns out to be a ninth degree polynomial and we can not define the roots of the characteristic polynomial as explicit functions of the parameters. Remark that in order to conclude to instability, we need to prove that there exists at least one positive eigenvalue. Consequently, the product of the eigenvalues is checked. Since there is an odd number of roots and the coefficients of the polynomial are real, if the product is positive, there exists at least one positive root. This allows us to determine in terms of the parameters, regions where the fundamental and speculative equilibria fail to be stable.

First, we will analyse the conditions under which the fundamental equilibrium is stable. The fundamental equilibrium is represented by the population states $\sigma_1 = (1, 0, 0)$ for the agents of type 1, $\sigma_2 = (0, 0, 0.5)$ for the agents of type 2 and $\sigma_3 = (1, 0, 0)$ for the agents of type 3. The conditions for the evolutionary dynamics are computed for these population states.

A	$-(c_3 - c_2) > \frac{1}{3}$	$-(c_1 - c_3) > 0$	$-(c_2 - c_1) > 0$
C	$-\frac{1}{4} < -(c_3 - c_2) < \frac{1}{3}$	$-\frac{1}{2} < -(c_1 - c_3) < 0$	$0 < -(c_2 - c_1) < 0$
D	$-(c_3 - c_2) < -\frac{1}{4}$	$-(c_1 - c_3) < -\frac{1}{2}$	$-(c_2 - c_1) < 0$

Note that only A is satisfied for the agents of type 2, and only D is satisfied for the agents of type 3. For the agents of type 1, C and D are satisfied.

As a result, there are two cases to analyse:

1. $-\frac{1}{4} < -(c_3 - c_2)$

2. $-(c_3 - c_2) < -\frac{1}{4}$

For both cases, the analysis is done based on the characteristic polynomial. The product of the eigenvalues are computed based on the characteristic polynomial. The case where the product of the eigenvalues is zero is depicted by a curve in order to visualise the combinations of the parameters for which the product of the eigenvalues is positive. The first case is depicted by the Figure (3.1). The product of the eigenvalues is negative at the left of the graph which covers the parameters range in consideration. As a result, the instability of the fundamental equilibrium can not be concluded by the product of the eigenvalues for the first case. The second case is depicted by Figure (3.2). The product of the eigenvalues is positive at the right of the curve. As a result, the fundamental equilibrium is not stable for this combination of parameters.

Second, we will analyse the conditions under which the speculative equilibrium is stable. The speculative equilibrium is represented by the population states $\sigma_1 = (0, 0, \sqrt{2}/2)$ for the agents of type 1, $\sigma_2 = (0, 0, \sqrt{2} - 1)$ for the agents

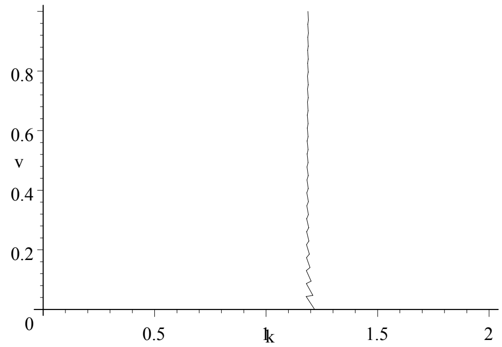

Figure 3.1: Case 1 Fundamental equilibrium

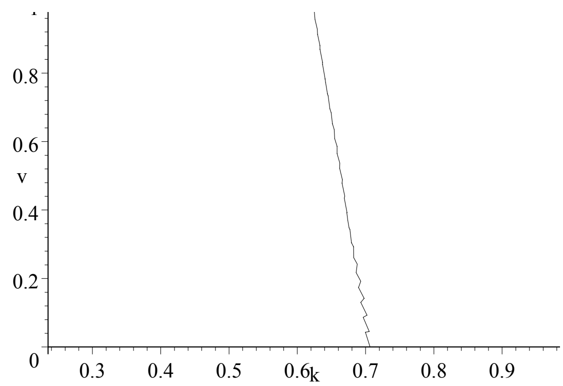

Figure 3.2: Case 2 Fundamental equilibrium

of type 2 and $\sigma_3 = (1, 0, 0)$ for the agents of type 3. The conditions for the evolutionary dynamics are computed for these population states.

A	$-(c_3 - c_2) > 0.37$	$-(c_1 - c_3) > 0$	$-(c_2 - c_1) > 0.16$
C	$-0.21 < -(c_3 - c_2) < 0.37$	$0 < -(c_1 - c_3) < 0$	$0.11 < -(c_2 - c_1) < 0.16$
D	$-(c_3 - c_2) < -0.21$	$-(c_1 - c_3) < 0$	$-(c_2 - c_1) < 0.11$

Note that only A is satisfied for the agents of type 2, and only D is satisfied for the agents of type 3. For the agents of type 1, C and D are satisfied.

As a result, there are two cases to analyse:

1. $-0.21 < -(c_3 - c_2)$

2. $-(c_3 - c_2) < -0.21$

For both cases, the product of the eigenvalues are computed based on the characteristic polynomial. The case where the product of the eigenvalues is zero is depicted by a curve in order to visualise the combinations of the parameters for which the product of the eigenvalues is positive. The first case is depicted by the Figure (3.3). The product of the eigenvalues is negative at the left of the graph which covers the parameters range in consideration. As a result, the instability of

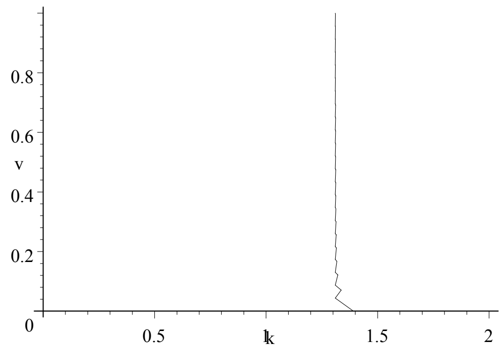

Figure 3.3: Case 1 Speculative equilibrium

the speculative equilibrium can not be concluded by the product of the eigenvalues for the first case. The second case is depicted by Figure (3.4). The product of the eigenvalues is positive at the left of the curve. As a result, the speculative equilibrium is not stable for this combination of parameters.

4. Conclusion

The standard search theoretic models of the emergence of money assumes random bilateral encounters between a large number of agents. The evolutionary extensions of these models retain the random matching assumption. The first objective of this paper was to explore a version of the commodity money model in Kiyotaki and Wright (1989) with endogenous meetings and to study the implications of

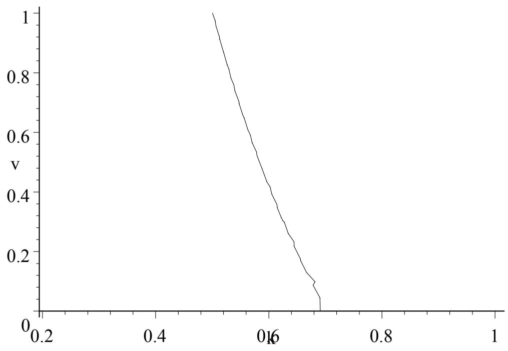

Figure 3.4: Case 2 Speculative equilibrium

endogenizing the matching process. The *fundamental equilibrium* in Kiyotaki and Wright (1989) in which every exchange involves either agents trading for their consumption goods or trading a higher storage cost good for a lower storage cost good, *the speculative equilibrium* requiring some individuals to trade their production good for one with a higher storage cost can be stable under the endogenous set up.

The second objective was to analyse the disequilibrium dynamics, when the inventory distribution is not expected to be continuously at its temporary equilibrium value. We found sufficient conditions for the instability of the fundamental and speculative equilibria. The findings are not in accordance with the earlier results in the literature. For instance, for the same range of parameter values as in Sethi (1999), the disequilibrium dynamics defined in this paper, result in instability.

5. Appendix

For the convenience, the propositions are restated before their proofs.

(Proposition 2.1) *The set of rest points* $\Phi(s)$ *is non-empty for all values of* s_i *for all* $i = 1, 2, 3$.

Proof. Given $s \in [0,1]^3$, define a continuous function $F(p)$ as follows:

$F_i(p) = p_i + (1 - p_i)p_{i+2}a_i - (s_i + p_i - 1)p_{i+1}a_{i+1}$

Since $p_{i+2} \geqslant 1 - s_{i+2}$ and $p_{i+1} \leqslant 1$ we have the following inequality:

$F_i(p) \geqslant p_i + (1 - p_i)(1 - s_{i+2})a_i - (s_i + p_i - 1)a_{i+1}$

$= p_i(1 - a_{i+1}) + (1 - p_i)(1 - s_{i+2})a_i + (1 - s_i)a_{i+1}$

To check whether $F_i(p) \geqslant 1 - s_i$:

$p_i(1 - a_i) + (1 - p_i)(1 - s_{i+2})a_i + (1 - s_i)a_{i+1} \overset{?}{\geqslant} 1 - s_i$

$p_i(1 - a_i) + (1 - p_i)(1 - s_{i+2})a_i \overset{?}{\geqslant} (1 - a_{i+1})(1 - s_i)$

Dividing both sides by $(1 - a_{i+1})$ we get the following inequality.

$p_i + (1 - p_i)(1 - s_{i+2})\frac{a_i}{1-a_{i+1}} \overset{?}{\geqslant} (1 - s_i)$

As $p_i \geqslant (1 - s_i)$ and $(1 - p_i)(1 - s_{i+2})\frac{a_i}{1-a_{i+1}} \geqslant 0$ from the fact that p_i and s_{i+2} lie both in the unit interval and $\pi_i \leqslant 1$ for all $i = 1, 2, 3$ we conclude that

$F_i(p) \geqslant 1 - s_i$.

Since $p_{i+1} \geqslant 1 - s_{i+1}$ and $p_{i+2} \leqslant 1$ we have the following inequality:

$F_i(p) \leqslant p_i + (1 - p_i)a_i - (s_i + p_i - 1)(1 - s_{i+1})a_{i+1}$

$= p_i(1 - a_i) + a_i - (s_i + p_i - 1)(1 - s_{i+1})a_{i+1}$

To check whether $F_i(p) \leqslant 1$:

$p_i(1 - a_i) + a_i - (s_i + p_i - 1)(1 - s_{i+1})a_{i+1} \overset{?}{\leqslant} 1$

$p_i(1 - a_i) - (s_i + p_i - 1)(1 - s_{i+1})a_{i+1} \overset{?}{\leqslant} 1 - a_i$

Diving both sides by $(1 - a_i)$ we get the following inequality:

$p_i - (s_i + p_i - 1)(1 - s_{i+1})\frac{a_{i+1}}{1-a_i} \overset{?}{\leqslant} 1$

As $p_i \leqslant 1$ and $(s_i + p_i - 1)(1 - s_{i+1})\frac{a_{i+1}}{1-a_i} \geqslant 0$ from the fact that $s_i + p_i - 1$ and s_{i+1} lie both in the unit interval and $\pi_i \leqslant 1$ for all $i = 1, 2, 3$ we conclude that $F_i(p) \leqslant 1$.

Hence $F : \Delta^s \to \Delta^s$. Since Δ^s is a compact set and F is continuous, we have by Brouwer's Fixed Point Theorem the existence of a p^* satisfying $F(p^*) = p^*$. At any such point we have $(1 - p_i)p_{i+2}a_i - (s_i + p_i - 1)p_{i+1}a_{i+1} = 0$ by the definition of F so p^* is a fixed point of the inventory dynamics defined by equation (2.1). ■

(Proposition 2.2) *Suppose $s \neq (1, 1, 1)$, the set of rest points $\Phi(s)$ contains a single element.*

The proof of three preliminary results will lead to the proof of the proposition.

Lemma 1 *Suppose* $s \neq (1,1,1)$*,* $\pi_i \neq 0$ *and* $\pi_i \neq 1$ *for all* $i = 1,2,3$*. If* $p \in \Phi(s)$ *then* $p_i > 0$ *for all* $i = 1,2,3$*.*

Proof. If $p \in \Phi(s)$ then from equation (2) we have

$(1 - p_i)p_{i+2}a_i - (s_i + p_i - 1)p_{i+1}a_{i+1} = 0$

Suppose $p_1 = 0$. This requires that $s_1 = 1$. Then the above equation results in $p_3 = 0$ implying $s_3 = 1$. If $p_3 = 0$ from the above equation we have $p_2 = 0$ implying $s_2 = 1$. This contradicts the assumption $s \neq (1,1,1)$. ■

Lemma 2 *Suppose* $s \neq (1,1,1)$*,* $\pi_i \neq 0$ *and* $\pi_i \neq 1$*. If* $p \in \Phi(s)$ *and* $p' \in \Phi(s)$ *then either* $p = p'$ *or* $p_i \neq p'_i$ *for all* $i = 1,2,3$*.*

Proof. As p and $p' \in \Phi(s)$ they satisfy equation (2.1). Rearranging the terms we have the following equation that is well defined under the assumptions and $p_i > 0$ from Lemma 1.

$$s_i = (1 - p_i)\left[1 + \frac{a_i p_{i+2}}{a_{i+1}p_{i+1}}\right] \tag{5.1}$$

Suppose that p and p' have at least one common element. Without loss of generality, suppose $p_1 = p'_1$. Then either (i) $p_2 = p'_2$ or (ii) $p_2 < p'_2$ or (iii) $p_2 > p'_2$. In case (i) it must also be true that $p_3 = p'_3$ otherwise equation (5.1) could not be satisfied for both p and p'. In case (ii), equation (5.1) implies that $p_3 < p'_3$. But

$p_2 < p'_2$ and $p_3 < p'_3$ are inconsistent if $p = p'$, so case (ii) is impossible. In case (iii), equation (5.1) implies that $p_3 > p'_3$. But $p_2 > p'_2$ and $p_3 > p'_3$ are inconsistent with equation (5.1) if $p = p'$, so case (iii) is impossible. Hence if p and p' have a common element, they are identical. ■

Lemma 3 *Suppose $s \neq (1,1,1)$, $\pi_i \neq 0$ and $\pi_i \neq 1$. If $p \in \Phi(s)$ and $p' \in \Phi(s)$ then either $p > p'$ or $p < p'$ or $p = p'$.*

Proof. Suppose $s \neq (1,1,1)$ and $p \in \Phi(s)$ and $p' \in \Phi(s)$. Suppose $p \neq p'$. Then from the previous lemma, we know that all their elements differ. Consequently, at least two elements of one vector must be strictly greater than the corresponding two elements of the other. Suppose, without loss of generality, that $p_1 > p'_1$ and $p_2 > p'_2$. This implies that $p_3 > p'_3$ and $p > p'$. Reversing the inequalities yields the remainder of the lemma. Suppose $p \in \Phi(s)$ and $p' \in \Phi(s)$, with $p \neq p'$. Following the previous lemma, we have either $p > p'$ or $p < p'$. Without loss of generality, assume that $p < p'$ with $p_1 < p'_1$. This implies $\frac{p_3}{p_2} < \frac{p'_3}{p'_2}$. By equation (5.1) we have $\frac{p_3}{p_2} > \frac{p'_3}{p'_2}$. If $p \in \Phi(s)$ and $p' \in \Phi(s)$, $p \neq p'$ is impossible. ■

(Proposition 2.3) *Suppose $s = (1,1,1)$, $\pi_i \neq 0$ and $\pi_i \neq 1$. The set of rest points $\Phi(s)$ contains exactly two elements, exactly one of which is stable with respect to the dynamics of equation (2.1).*

Proof. If $s = (1,1,1)$ then $\dot{p}_i = (1-p_i)p_{i+2}a_i - p_i p_{i+1} a_{i+1}$. Setting $\dot{p}_i = 0$ we get two rest points: $p = (0,0,0)$ and $(\rho_1^4, \rho_2^4, \rho_3^4)$. In order to check if these rest points are asymptotically stable we calculate the eigenvalues of the following Jacobian:

$$\frac{\partial F}{\partial p} = \begin{bmatrix} -a_1p_3 - a_2p_2 & -a_2p_1 & a_1(1-p_1) \\ a_2(1-p_2) & -a_2p_1 - a_3p_3 & -a_3p_2 \\ -a_1p_3 & a_3(1-p_3) & -a_3p_2 - a_1p_1 \end{bmatrix}$$

At $p = (0,0,0)$ the eigenvalues of the Jacobian are w, $w(-1/2 \pm i\sqrt{3}/2)$ where $w = (a_1a_2a_3)^{1/3} \geqslant 0$ as π_i lie in the unit interval for all $i = 1,2,3$. p is unstable at $p = (0,0,0)$. At $p = (\rho_1^4, \rho_2^4, \rho_3^4)$ the eigenvalues of the Jacobian are $w' - \frac{w''}{3}$, $-\frac{1}{2}w' - \frac{w''}{3} \pm i\frac{\sqrt{3}}{2}w'$ where $w'' = (a_1+a_2)\rho_1^4 + (a_2+a_3)\rho_2^4 + (a_1+a_3)\rho_3^4 \geqslant 0$ and as π_i and p_i lie in the unit interval for all $i = 1,2,3$. So p is stable at $p = (\rho_1^4, \rho_2^4, \rho_3^4)$. $-\frac{2w''}{3} < w' < \frac{w''}{3}$ ■

(Proposition 2.4) *Suppose $\pi_i \neq 0$ and $\pi_i \neq 1$. The function $\rho(s)$ is continuous at all monomorphic states and at all polymorphic states of the type $s = (x,1,0)$ where $x \in (0,1)$.*

Proof. Define the function $F(s,p) : R^3 \times R^3 \to R^3$ as follows:

$F_i(s,p) = (1-p_i)p_{i+2}a_i - (s_i + p_i - 1)p_{i+1}a_{i+1}$ for all $i = 1,2,3$.

Since $F(s,p)$ is continuously differentiable at any (s,p), we have the following matrix.

$$\frac{\partial F}{\partial p} = \begin{bmatrix} -a_1p_3 - a_2p_2 & -a_2(s_1 + p_1 - 1) & a_1(1 - p_1) \\ a_2(1 - p_2) & -a_2p_1 - a_3p_3 & -a_3(s_2 + p_2 - 1) \\ -a_1(s_3 + p_3 - 1) & a_3(1 - p_3) & -a_3p_2 - a_1p_1 \end{bmatrix}$$

If this matrix is invertible at (s, p), then by the implicit function theorem, there are open sets $U \in R^3$ and $V \in R^3$ with $(s, p) \in U \times V$ and a unique continuously differentiable function, $\rho : U \to V$ such that $F(s', \rho(s')) = 0$ for all $s' \in U$. $\rho(s)$ is continuous at all monomorphic population states if $\frac{\partial F}{\partial p}$ is invertible at all such states. By symmetry, it is sufficient to check this for states $s = (0, 0, 0)$, $s = (0, 1, 0)$, $s = (1, 1, 0)$, $s = (1, 1, 1)$. At the values of inventories corresponding to these states (Table 2), the determinant of $\frac{\partial F}{\partial p}$ is equal to $-(a_1 + a_2)(a_1 + a_3)(a_2 + a_3)$, $(1 - a_2)^{-2}(a_2(1 - a_2) + a_3^2)(a_1 + a_3)(a_2^2 - a_1a_3 - a_2)$, $-(a_1\rho_3^3 + a_2\rho_2^3)(a_1a_2 + a_1a_3\rho_3^3 + a_2a_3\rho_2^3 + a_3^2\rho_2^3)$, $a_1a_2a_3 - a_1a_2a_3\rho_1^4 - a_1a_2a_3\rho_2^4 - a_1a_2a_3\rho_3^4 - a_1a_2^2(\rho_1^4)^2 - a_2a_3^2(\rho_2^4)^2 + a_1a_2a_3\rho_1^4\rho_2^4 + a_1a_2a_3\rho_1^4\rho_3^4 + a_1a_2a_3\rho_2^4\rho_3^4 - a_2^2a_3\rho_1^4\rho_2^4 - a_1a_3^2\rho_2^4\rho_3^4 - a_1^2a_2\rho_1^4 \rho_3^4 - a_1^2a_3(\rho_3^4)^2 - 4a_1a_2a_3\rho_1^4\rho_2^4\rho_3^4$ respectively. In each case the determinant is non zero, so $\frac{\partial F}{\partial p}$ is invertible.

We now prove continuity at all points $s = (z, 1, 0)$. If $s = (z, 1, 0)$, then $p = (\frac{a_3w}{a_2(1-w)}, w, 1)$ where w is the root of the following second degree equation:

$w^2(a_2^2(z-1) - a_2a_3) + w(-a_1(a_2+a_3) - a_2^2(z-1)) + a_1a_2$. At these values, the determinant of the Jacobian is:

$$-wa_2a_3(1-w)^{-2}\left(a_2(1-w)+a_1\right)\times$$

$$\left(a_2^2\left(1-z\right)-a_1a_3+2a_2w\left(a_2\left(z-1\right)-a_3\right)+a_2w^2\left(a_2\left(1-z\right)+a_3\right)\right)$$

. Since the determinant can not be zero for all values of $0 < w < 1$, the Jacobian is invertible. ■

(Proposition 2.5) *Suppose* $c_1 < c_2 < c_3$.

1. *(Fundamental equilibrium) If* $(c_3 - c_2) > a_1 - a_2 + \frac{a_2a_3}{a_3+a_2}$ *there is an asymptotically stable rest point at* $s = (0, 1, 0)$.

2. *(Speculative equilibrium) If* $(c_3 - c_2) < a_1 - a_2 + \rho_2^3 a_2$ *there is an asymptotically stable rest point at* $s = (1, 1, 0)$.

To prove the proposition, we need to prove the following lemma.

Lemma 4 *Given a monomorphic population state* s *and a monotonic selection dynamic* ξ_i, s *is asymptotically stable if* $(2s_i - 1)(u_{\beta i}(s) - u_{\alpha i}(s)) > 0$ *for all* i.

Proof. Suppose $J = \{i : s_i^* = 1\}$ and $K = \{i : s_i^* = 0\}$. If for all i, $(2s_i - 1)(u_{i\beta}(s) - u_{i\alpha}(s)) > 0$, then $u_{i\beta}(s^*) > u_{i\alpha}(s^*)$ when $i \in J$ and $u_{i\beta}(s) < u_{i\alpha}(s)$ when

$i \in K$. By continuity of payoffs in population shares, there exists a neighborhood N of s^* such that, for all $s \in N - s^*$, $u_{i\beta}(s^*) > u_{i\alpha}(s^*)$ when $i \in J$ and $u_{i\beta}(s) < u_{i\alpha}(s)$ when $i \in K$. If the dynamics $f_i(s)$ are monotonic, then $f_i(s) > 0$ when $i \in J$ and $f_i(s) < 0$ when $i \in K$. ■

The Lemma states simply that a monomorphic population state s is asymptotically stable if $u_{\beta i}(s^*) > u_{i\alpha}(s^*)$ in those sub-populations i which consist exclusively of β-types, and $u_{\beta i}(s^*) < u_{i\alpha}(s^*)$ in those sub-populations i which consist exclusively of α-types. The proposition now may be proved.

Proof. For $s = (0, 1, 0)$ to be asymptotically stable we need the following conditions: $u_{1\alpha}(s) > u_{1\beta}(s)$, $u_{2\alpha}(s) < u_{2\beta}(s)$, $u_{3\alpha}(s) > u_{3\beta}(s)$. For $s = (0, 1, 0)$, the equilibrium inventory holdings are $\rho(s) = (1, \rho_2^2, 1)$ where ρ_2^2 is equal to $\frac{a_2}{a_3+a_2}$.

$u_{1\alpha}(s) - u_{1\beta}(s) = \frac{\left(a_2-a_1-c_2+c_3-a_2\rho_2^2\right)a_2\rho_2^2}{a_1+a_2\rho_2^2}$

$u_{2\alpha}(s) - u_{2\beta}(s) = -\frac{(a_2+c_3-c_1)a_3}{a_3+a_2}$

$u_{3\alpha}(s) - u_{3\beta}(s) = \frac{(c_2-c_1)a_1}{a_1+\rho_2^2 a_3}$

For $s = (1, 1, 0)$ to be asymptotically stable we need the following conditions: $u_{1\alpha}(s) < u_{1\beta}(s)$, $u_{2\alpha}(s) < u_{2\beta}(s)$, $u_{3\alpha}(s) > u_{3\beta}(s)$. For $s = (1, 1, 0)$, the equilibrium inventory holdings are $\rho(s) = (\rho_1^3, \rho_2^3, 1)$ where ρ_1^3 is given by the equation (2.2) and ρ_2^3 is by the equation (2.3).

$u_{1\alpha}(s) - u_{1\beta}(s) = \frac{\left(a_2-a_1-c_2+c_3-a_2\rho_2^3\right)a_2\rho_2^3}{a_1+a_2\rho_2^3}$

$u_{2\alpha}(s) - u_{2\beta}(s) = -\frac{(c_3-c_1)a_3}{a_3+\rho_1^3 a_2}$

$u_{3\alpha}(s) - u_{3\beta}(s) = \frac{(a_1-c_1+c_2-a_1\rho_1)a_1\rho_1}{a_1\rho_1^3+\rho_2^3 a_3}$ ■

(Proposition 2.6) *Suppose* $c_1 < c_2 < c_3$.

If $a_1 - a_2 + \frac{-a_1(1-a_1)+\sqrt{a_1^2(1-a_1)^2+4a_1a_2^2(1-a_2-a_1)}}{2(1-a_1-a_2)} < (c_3 - c_2) < a_1 - a_2 + \frac{a_2^2}{1-a_1}$ *there is an asymptotically stable rest point at* $s = (x, 1, 0)$ *where* $x \in (0, 1)$.

Proof. The resulting equilibrium inventories for $s = (x, 1, 0)$ is $p = (\rho_1^5, \rho_2^5, 1)$ where

$$\rho_1^5 = \frac{\rho_2^5 a_3}{(1-\rho_2^5)a_2}$$

and ρ_2^5 is given by the following equation:

$$a_2(a_3 + a_2(1-x))(\rho_2^5)^2 + (a_1(a_2+a_3) - a_2^2(1-x))\rho_2^5 - a_1a_2 = 0$$

>From this equation we get:

$$\rho_2^5 = \frac{-a_1\frac{a_2+a_3}{a_2} + a_2(1-x) + \sqrt{(a_1\frac{a_2+a_3}{a_2} - a_2(1-x))^2 + 4a_1(a_3+a_2(1-x))}}{2(a_3+a_2(1-x))}$$

Rest points of the evolutionary dynamics require all surviving strategies to have equal payoffs. Then for an interior point such as $s = (x, 1, 0)$ where $x \in (0, 1)$, we must have $u_{1\alpha}(s) = u_{1\beta}(s)$. At $s = (x, 1, 0)$ and $p = (\rho_1^5, \rho_2^5, 1)$:

$u_{1\alpha}(s) = (1-\rho_2^5)a_2 - c_2$

$u_{1\beta}(s) = \frac{-a_1c_2+a_1a_2-a_2c_3\rho_2^5}{a_1+a_2\rho_2^5}$

Solving $u_{1\alpha}(s) - u_{1\beta}(s) = 0$ we get:

$$x = \frac{(c_3 - c_2 + a_2)\left((1-a_1)(c_3-c_2+a_2-a_1) - a_2^2\right)}{a_2\left(c_3-c_2-a_1\right)\left(c_3-c_2+a_2-a_1\right)}$$

We must ensure that $x > 0$ and $x < 1$. As $x \in (0,1)$, we have $\rho_1^5 > \rho_1^3$. ■

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