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Karine Van der Straeten

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LABORATOIRE D'ECONOMETRIE

1rue Descartes F-75005 Paris

(33) 1 55558215

http://ceco.polytechnique.fr/

mailto:labecox@poly.polytechnique.fr

Product differentiation when consumers may choose not to buy: Hotelling's convergence result revisited

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Résumé: Nous étudions le modèle spatial de différentiation d'Hotelling, mais au lieu de supposer que le marché est couvert quelles que soient les localisations des deux firmes, nous supposons qu'il existe une distance maximale (éventuellement infinie) qu'un consommateur est prêt à parcourir pour acheter le bien. Cette généralisation apparemment anodine des hypothèses d'Hotelling modifie complètement les résultats : le principe de "différentiation minimale" d'Hotelling n'est plus vérifié en général. A l'équilibre, les firmes s'engagent dans un processus de différentiation maximale, intermédiaire ou minimale, selon la forme de la distribution des positions des consommateurs et cette distance critique qu'un consommateur est prêt à parcourir pour acquérir le bien.

Abstract: This paper studies Hotelling's spatial competition between two firms, but rather than assuming that consumers are ready to buy the good whatever the locations of the firms are, it is assumed here that there is an upper limit (possibly infinite) to the distance a consumer is ready to cover to buy the good. Under this slight generalization of Hotelling's assumptions, Hotelling's ``minimal differentiation principle" does not hold in general. At equilibrium, firms choose ``minimal", ``intermediate" or ``complete" differentiation, depending on this critical distance a consumer is ready to cover and on the shape of the distribution of consumers' locations.

Mots clés : modèle de compétition spatiale d'Hotelling, différentiation des produits

Key Words : Hotelling's spatial competition model, product differentiation

Classification JEL: C72, D43, L11, L13

[1] Laboratoire d'Econométrie, CNRS et Ecole polytechnique

1 Introduction

Product differentiation is a central issue in Industrial Organization. A large number of studies on this topic build on Hotelling's seminal model of location or spatial-differentiation (See Hotelling 1929).

In Hotelling's location model, consumers are located at different places on a line. Two firms – selling the same product at the same price – simultaneously choose a location on this line. Consumers are ready to buy one unit – and only one unit – of the good and they pay transportation costs when going and purchasing the good. Hotelling's "miminal differentiation principle" (1929) states that at Nash equilibrium, both firms choose the "median consumer" location (i.e. the location such that one half of the consumers lay at his left-hand side, and one half of the consumers lay at his right-hand side) – hence the name of "miminal differentiation principle" given to Hotelling's result.

D'Aspremont *et al.* (1979) challenge this convergence result, arguing that the assumption that both firms sell at the same price can not be derived as an equilibrium result, if sellers are not far enough one from the other. More specifically, if firms simultaneously choose their locations in a first stage, and then, once locations are observed, simulteously choose the price at which they sell the good, then no price equilibrium exists, when both sellers are locatted exactly at the same place – as is the case in Hotelling's result. They provide an example of a slightly modified version of Hotelling's framework (explicitly introducing prices together with transportation costs in consumers' preferences), in which this two-stage game between the firms always have an equilibrium. They show in that example than "maximum differentiation" can be observed between firms, firms choosing to locate at the two ends of the line[1]. This two-stage model where firms choose products first and price second offers the standard explanation in Industrial organization as to why "firms generally do not want to locate at the same place in the product space. The reason is simply the Bertrand paradoxe: Two firms producing perfect substitutes face unbridled price competition (at least in a static framework). In contrast, product differentiation establishes clienteles ("market niches", in the business terminology) and allows firms to enjoy some market power over these clienteles. Thus, firms usually wish to differentiate themselves from other firms" (Tirole 1997, p. 278).

In the present paper, Hotelling's convergence result will also be challenged, but on different grounds. Softening of price competition is not the only force which may drive firms to differentiate. Indeed, even with no price competition, it will be shown that firms may choose to differentiate. I will keep Hotelling's assumption that both firms sell the product at the same price, whatever their locations are. As d'Aspremont *et al.* (1979) have shown, this assumption can not be supported as an equilibrium result. Rather I simply take the view that prices are imposed upon the firms; for legal or technical reasons, price is not a free parameter in the competition.[2]. I rather alter another assumption in

[1] See also Osborne and Pitchik (1987) for further discussion on Hotelling's result in a two-stage game.

[2] As recalled by Tirole (1997), "There may exist legal or technical reasons why the scope

Hotelling's model: namely that the market is always covered. In Hotelling's model, consumers are ready to buy one unit of the good, whatever the location of the firms are. I assume instead that if both firms are too far away from his location, a consumer prefers not buying the good, rather than paying the transportation costs. All consumers equally value the good, and choose to buy the good only if the transportation costs do not exceed the valuation of the good. Of course I keep the assumption that when both firms are within "acceptable distance", the consumer selects the firms which is the closest. I only add the restriction that there is an upper limit to the distance a consumer is ready to cover to buy the good. This distance will characterize the width of the "attraction zone" of the firms.

This may seem an innocuous restriction, and yet, as will be shown below, it dramatically alters Hotelling's result. The convergence result may not hold anymore, and one may observe some "intermediate differentiation principle" or "maximal differentiation priciple". The intuition supporting this result is the following. All through this paper, a normal distribution of consumers's locations is assumed. Were a firm alone on the market, it would select the modal position, where the larger number of consumers are located, according to a "Be where the demand is" principle. Now, in the two-firm competition, the convergence to the modal position may not be observed.[3] Indeed, suppose that a firm has chosen the modal position. In that case, if its opponent also selects this central position, they will share exactly the same attraction zone and thus each will attract one half one the consumers who are located within acceptable distance of the central position. The latter firm may fare better in that case by avoiding complete competition, and prefer moving somewhat to the left or the right. In doing so, it will move from the modal position – thus move away from the situation where the concentration of consumers is the highest – but on the other hand, it will avoid splitting in half the number of potential consumers. We expect the incentives to move away to be greater when the distribution is flatter (less concentration in the modal position) and when the width of the attraction zone is larger. It will be shown to be the case indeed. Depending of the size of the width of the attraction zone compared to the standard deviation of the distribution of consumers' location, we can observe convergence in the central position, intermediate differentiation, or complete differentation (in the sense that no consumer is located at equilibrium within acceptable distance of both firms). In particular, some necessary and sufficient conditions on this ratio are provided for the convergence result to hold.

The paper is organized as follows. The model is presented in more details in section 2. Section 3 presents the firms' best responses. Section 4 describes the best response sets and section 5 characterizes all Nash equilibria in which

of price competition is limited. For instance, the prices of airline tickets in the United States (before deregulation) where determined exogenously, as the price of gas and books in France once were" (p. 287).

[3] Note that contrary to what happened in Hotelling's setting, we expect here a "modal consumer" theorem to hold rether than a "median consumer" theorem. Since the distribution is assumed to be symmetric, the two notions coincide in this paper.

firms play pure strategies. Section 6 briefly concludes The proofs are relegated in an appendix in section 7. [Note to the referee : In the present version of the paper, exhaustive proofs are provided. Since several parts of the proofs rely on similar arguments or calculus, some of them might be ommitted in a published version of the paper.]

2 The model

Two firms, labelled A and B, sell the same good at the same price, and choose location on the real line where consumers are distributed. Consumer locations are assumed to have a normal distribution, with mean zero and standard deviation σ. The consumers have unit demand; i.e. each consumer can either buy one unit of the good (from either firm), or abstain from buying. We assume quadratic transportation costs: when a consumer located at t goes to a firm located atn $\widehat{t}$, he pays a transportation cost $c(t,\widehat{t}) = -(t-\widehat{t})^2$. All consumers derive the same net intrinsic positive utility δ^2 from bying the good. If a consumer located at t goes to a firm located at $\widehat{t}$ and buy the good, he gets the total utility $\delta^2 - (t-\widehat{t})^2$. If he does not buy the good and incurs no transporation costs, he gets the utility zero.

Suppose firm A chooses location t^A, and firm B chooses location t^B. When facing the couple of location (t^A, t^B), a consumer located at t has three options: buy the good from firm A, buy the good from firm B, or abstain from buying. Given the assumption on preferences made above, four cases are to be dostinguished.

Case 1: $\left|t^A - t\right| \leq \delta$ and $\left|t^B - t\right| \leq \delta$, both firms are within acceptable distance[4], meaning that he prefers buying the good from either firm rather than not consuming the good. In that case, he buys the good from the closest firm, just as in Hotelling's model. If $\left|t^A - t\right| < \left|t^B - t\right|$, he buys the good from firm A; if $\left|t^A - t\right| > \left|t^B - t\right|$, he buys the good from firm B; if $\left|t^A - t\right| = \left|t^B - t\right|$ he is indifferent between buying from firm A and buying from firm B, and we assume that in that case he chooses both firms with equal probability 1/2.

Case 2: $\left|t^A - t\right| \leq \delta$ and $\left|t^B - t\right| > \delta$, only firm A is within acceptable distance. In that case he buys the good from firm A.

Case 3: $\left|t^A - t\right| > \delta$ and $\left|t^B - t\right| \leq \delta$, only firm B is within acceptable distance. In that case he buys the good from firm B.

Case 4: $\left|t^A - t\right| > \delta$ and $\left|t^B - t\right| > \delta$, both firms are beyond acceptable distance. In that case, the consumer rather abstains and does not buy the good.

Note that when $\delta \to +\infty$, we are in the standard Hotelling's situation: consumers consume one unit of the good whatever the location of the firms are, and buy from the closest firm.

[4] When a consumer is indifferent between buying and nor bying the good, we make the assumption that he does buy it. This assumption plays no role in solving the game for the set of consumers that might be in that case is of measure zero.

The firms meet any demand they face. The objective of the firms is to sell as many units of good as they can. Given the behavior of consumers described above, payoff functions for the firms are easily derived; they are explicitly presented in the next section.

3 Firms' payoff functions

Let us denote by $G^B\left(t;t^A\right)$ the payoff obtained by firm B when choosing location t, against location t^A by firm A, and $G^A\left(t;t^B\right)$ the payoff of firm A when choosing location t, against location t^B by firm B. We now derive a close form expression for teh payoff function $G^B\left(.;t^A\right)$. The expression of the payoff functions depends on whether the attraction zones of the two firms intersect or not.

When the two attraction zones do not intersect, that is, when $\left|t^A - t\right| > 2\delta$, firm B attracts all consumers that are located in its attraction zone. These are all consumers whose location belongs to the interval $[t-\delta, t+\delta]$. In that case, the payoff of firm B is $F(t+\delta) - F(t-\delta)$, where F is the cumulative function of the normal distribution with mean zero and standard deviation σ.

When the attraction zones do intersect, that is, when $\left|t^A - t\right| \leq 2\delta$, consumers who are located in the intersection will choose the firm which is closest to their own location. The explicit expression for the payoff function $G^B\left(t;t^A\right)$ depends on the relative position of t^A and t: as long as $t < t^A$, $G^B\left(t;t^A\right) = F((\frac{1}{2}\left(t+t^A\right))) - F(t-\delta)$; when $t > t^A$, $G^B\left(t;t^A\right) = F(t+\delta) - F((\frac{1}{2}\left(t+t^A\right)))$; when $t = t^A$, the two firms share the consumers located in the interval $\left[t^B - \delta, t^B + \delta\right]$ and in that case $G^B\left(t^A;t^A\right) = \frac{1}{2}\left[F(t^A+\delta)) - F(t^A - \delta)\right]$. Note that $G^B\left(t;t^A\right)$ as a function of t is continuous in $t^A - 2\delta$ and $t^A + 2\delta$, but that in general, it is not continuous in t^A. Lemma 1 and remark 1 below sum up the results so far.

Lemma 1

$$
\begin{array}{ll}
\text{If } t \leq t^A - 2\delta, & G^B\left(t;t^A\right) = F(t+\delta) - F(t-\delta), \\
\text{If } t^A - 2\delta \leq t < t^A, & G^B\left(t;t^A\right) = F((\frac{1}{2}\left(t+t^A\right))) - F(t-\delta), \\
\text{If } t = t^A, & G^B\left(t;t^A\right) = \frac{1}{2}\left[F(t^A+\delta)) - F(t^A-\delta)\right], \\
\text{If } t^A < t \leq t^A + 2\delta, & G^B\left(t;t^A\right) = F(t+\delta) - F((\frac{1}{2}\left(t+t^A\right))), \\
\text{If } t \geq t^A + 2\delta, & G^B\left(t;t^A\right) = F(t+\delta) - F(t-\delta).
\end{array}
$$

As noted above, $G^B\left(t;t^A\right)$ as a function of t is in general not continuous in t^A. Remark 1 tells more on this point.

Remark 1

$G^B\left(t;t^A\right)$ as a function of t is not continuous in t^A, except for $t^A = 0$: when $t^A < 0$, $\lim\limits_{\substack{t \to t^A \\ t < t^A}} G^B(t;t^A) < G^B(t^A;t^A) < \lim\limits_{\substack{t \to t^A \\ t > t^A}} G^B(t;t^A)$, when $t^A = 0$, $\lim\limits_{\substack{t \to t^A \\ t < t^A}} G^B(t; t^A) = G^B(t^A; t^A) = \lim\limits_{\substack{t \to t^A \\ t > t^A}} G^B(t; t^A) = 1/2\,[F(\delta) - F(-\delta)]$,

when $t^A > 0$, $\lim\limits_{\substack{t \to t^A \\ t < t^A}} G^B(t; t^A) > G^B(t^A; t^A) > \lim\limits_{\substack{t \to t^A \\ t > t^A}} G^B(t; t^A)$.

Indeed, when $t^A < 0$, $\lim\limits_{\substack{t \to t^A \\ t < t^A}} G^B(t; t^A) = F(t^A + \delta) - F(t^A)$ and $G^B(t^A; t^A) = \frac{1}{2}\left[F(t^A + \delta) - F(t^A - \delta)\right]$. Therefore $\lim\limits_{\substack{t \to t^A \\ t > t^A}} G^B(t; t^A) - G^B(t^A; t^A) = -1/2\{\left[F(t^A + \delta) - F(t^A)\right] - \left[F(t^A) - F(t^A - \delta)\right]\}$ which is strictly negative when $t^A < 0$. Similar arguments are used in the other cases.

4 Description of the best responses

Given the symmetry between firms, we only need to characterize the best responses for firm B. Let us denote by $BR^B(t)$ the set of best responses of firm B against location t by firm A.

Note first that it is sufficient to describe the best responses against $t \leq 0$, since best responses against $t \geq 0$ can then be very easily derived. Indeed, given the symmetry of the distribution of consumers' locations, for any $t \geq 0$, $x \in BR^B(t)$ if and only if $-x \in BR^B(-t)$.

Proposition 1 characterizes the set of best responses against any value of $t \leq 0$. Three cases are to be considered.

Proposition 1

Case 1. If $\delta^2/\sigma^2 \leq \frac{1}{4}\ln(2)$,

if $t \leq -2\delta$,	$BR^B(t) = \{0\}$
if $-2\delta \leq t < 0$,	$BR^B(t) = \{t + 2\delta\}$
if $t = 0$,	$BR^B(t) = \{2\delta, -2\delta\}$

Case 2. If $\frac{1}{4}\ln(2) < \delta^2/\sigma^2 < 2\ln(2)$,

if $t \leq -2\delta$,	$BR^B(t) = \{0\}$
if $-2\delta \leq t \leq -a$,	$BR^B(t) = \{t + 2\delta\}$
if $-a \leq t < 0$,	$BR^B(t) = \{t_2(t)\}$
if $t = 0$,	$BR^B(t) = \{t_2(0), -t_2(0)\}$

Case 3. If $\delta^2/\sigma^2 \geq 2\ln(2)$,

if $t \leq -2\delta$,	$BR^B(t) = \{0\}$
if $-2\delta \leq t \leq -a$,	$BR^B(t) = \{t + 2\delta\}$
if $-a \leq t < -b$,	$BR^B(t) = \{t_2(t)\}$
if $-b \leq t < 0$,	$BR^B(t) = \emptyset$[5]
if $t = 0$,	$BR^B(t) = \{0\}$

[5] Firm B would like to play $t + \varepsilon$, $\varepsilon > 0$ as small as possible

where

$$\begin{aligned} a &= \frac{4\delta^2-\sigma^2\ln(2)}{2\delta}, && (1)\\ b &= \frac{\delta^2-2\sigma^2\ln(2)}{2\delta}, && (2)\\ \text{and } t_2 &: R\to R \\ t &\mapsto t_2(t)=\frac{1}{3}\left(t-4\delta+2\sqrt{(t-\delta)^2+6\sigma^2\ln(2)}\right). && (3)\end{aligned}$$

The proof of proposition 1 is provided in the appendix in sub-section 7.1. Before commenting on the results stated in proposition 1, remark 2 describes some of the properties of the function t_2.

Remark 2. Properties of the function t_2

Elementary calculus shows that the function t_2 satisfies the properties summed up in the table below[6]:

t	$-\infty$		$\delta-\sqrt{2\sigma^2\ln(2)}$		$+\infty$
$t_2(t)$		$\searrow$	$-\left(\delta-\sqrt{2\sigma^2\ln(2)}\right)$	$\nearrow$	
$t_2'(t)$	$-2/3$	$\nearrow$	0	$\nearrow$	$+2/3$
$t_2''(t)$			$+$		

Besides, $t_2(-a)=-a+2\delta$, $t_2(-b)=-b$.

Therefore the monotonicity of t_2 on the interval $[-a,0]$ depends on the relative positions of $\delta-\sqrt{2\sigma^2\ln(2)}$, a, and 0, and thus on the ratio δ^2/σ^2.

Case 2a. When $1/4\ln(2)<\delta^2/\sigma^2\leq 1/2\ln(2)$, t_2 is increasing on $[-a,0]$.

Case 2b. When $1/2\ln(2)\leq\delta^2/\sigma^2<2\ln(2)$, t_2 is decreasing on $\left[-a,\delta-\sqrt{2\sigma^2\ln(2)}\right]$ and increasing on $\left[\delta-\sqrt{2\sigma^2\ln(2)},0\right]$.

Case 3. When $\delta^2/\sigma^2\geq 2\ln(2)$, t_2 is decreasing on $[-a,0]$.

Note also that in all cases $\forall t\in\,]-a,0]\,, t_2(t)<t+2\delta$ (indeed, $t_2(-a)=-a+2\delta$ and for $t\in[-a,0]\,, t_2'(t)<1$). Besides, in cases 2a and 2b, $\forall t\in[-a,0]\,,$ $t_2(t)>0$.

Comments on Proposition 1.

[6] For $t\in R$, $t_2'(t)=1/3\left(1+\frac{2(t-\delta)}{\sqrt{(t-\delta)^2+6\sigma^2\ln(2)}}\right)$ and $t_2''(t)=\frac{4\sigma^2\ln(2)}{\left((t^A-\delta)^2+6\sigma^2\ln(2)\right)^{3/2}}>0$. Note also that $\delta^2/\sigma^2\leq 1/2\ln(2)\iff\delta-\sqrt{2\sigma^2\ln(2)}\leq -a$; and $\delta^2/\sigma^2\geq 2\ln(2)\iff\delta-\sqrt{2\sigma^2\ln(2)}\geq 0$.

Case 1. These are situations where the width of the attraction zone (δ) is small compared to the standard deviation of the distribution of consumers location (σ).

In that case, firm B always avoids direct confrontation with firm $A:$ the two attraction zones do not intersect. We will use the term of "maximal differentiation" to describe such situations, where positions of teh firm are at distance 2δ one from the other.

If firm A is far away enough from the center ($t \leq -2\delta$), then firm's B best response is to locate at the modal central position, where it gets the maximal possible payoff.

If firm A chooses a location within 2δ of the central location, the best firm B can do is still to avoid direct confrontation and not to intersect its attraction zone ; in that case, firm B chooses $t + 2\delta$, which is, among all location such that the two attraction zones do not intersect, that which is closest to the central location.

Case 2. These are intermediate situations as to the width of the attraction zone compared to the standard deviation of the distribution of consumers location.

In that case, firm B avoids direct confrontation with firm A only when firm A is far enough from the central location ($t \leq a$).

When firm A is within distance a of the central location it becomes most profitable to seek direct confrontation, and have the two attraction zones partially intersect.[7]

In all these situations, the best response for firm B is always to choose a location such that the two firms are on opposite sides, relative to the central position.

Case 3. These are situations where the width of the attraction zone is large compared to the standard deviation of the distribution of consumers location.

As in case 2, firm B avoids direct confrontation with firm A only when firm A is far enough from the central location ($t \leq -a$).

When firm A chooses some intermediate location ($-a \leq t \leq -b$), it becomes most profitable to seek partial confrontation, and have the two attraction zones partially intersect. In that case, since the function t_2 is decreasing on the interval $[-a, -b]$ (see remark 2), firm B moves to the left hand side as firm A moves to the right towards the central position. Confrontation gets fiercer as firm A moves towards the central location.

If firm A is close enough to the center (has a location within b of the central location but still different from the center), there is no best response for firm B. Ideally, firm B would like to seek almost complete confrontation, that is to stick to firm A, but slightly on the right hand side. Ideally, firm B would like to choose position $t + \varepsilon$, with $\varepsilon > 0$ as small as possible.

[7] Indeed, in that case, firm B chooses $t_2(t) < t + 2\delta$ (see Remark 2) in response to firm A's choice of location t.

When firm A chooses the central location, the best firm B can do is to select this central location as well, facing in that case complete confrontation, in the sense that the attraction zones of the two firms exactly coincide.

Note that when δ goes to infinity, we find here the standard properties of the Hotelling model. Indeed, in that case both a and b go to infinity and the best response against $t = 0$ is zero . If $t < 0$, there is no best response: firm B would like to stick to firm A, but slightly on the right hand side. Ideally, firm B would like to choose a position $t + \varepsilon$, with $\varepsilon > 0$ as small as possible.

5 Characterization of Nash equilibria

Theorem 1 characterizes all Nash equilibria in which both firms play pure strategies.

Theorem 1

This games always has (at least) one equilibrium in pure strategies.

The shape of the set of Nash equilibria depends on the value of the ratio δ^2/σ^2. Let us denote by $\mathrm{N}\left(\delta^2/\sigma^2\right)$ the set of Nash equilibria, depending on the ratio δ^2/σ^2.

Case 1. If $\delta^2/\sigma^2 \leq \frac{1}{4}\ln(2)$,
(t^A, t^B) is a Nash equilibrium if and only if:
either $t^A \in [-2\delta, 0]$ and $t^B = t^A + 2\delta$,
or $t^A \in [0, +2\delta]$ and $t^B = t^A - 2\delta$.

Case 2a. If $\frac{1}{4}\ln(2) < \delta^2/\sigma^2 \leq \frac{1}{2}\ln(2)$,
(t^A, t^B) is a Nash equilibrium if and only if:
either $t^A \in [-2\delta + a, -a]$ and $t^B = t^A + 2\delta$,
or $t^A \in [+a, +2\delta - a]$ and $t^B = t^A - 2\delta$,
where

$$a = \frac{4\delta^2 - \sigma^2\ln(2)}{2\delta} \in \left]0, \delta\right].$$

Case 2b. If $\frac{1}{2}\ln(2) \leq \delta^2/\sigma^2 < 2\ln(2)$,
(t^A, t^B) is a Nash equilibrium if and only if:
either $t^A = -\left(\sqrt{2\sigma^2\ln(2)} - \delta\right)$ and $t^B = -t^A$,
or $t^A = \sqrt{2\sigma^2\ln(2)} - \delta$ and $t^B = -t^A$, where. $\sqrt{2\sigma^2\ln(2)} - \delta \in [0, \delta[$

Case 3. If $\delta^2/\sigma^2 \geq 2\ln(2)$,
$(t^A, t^B) = (0, 0)$ is the unique Nash equilibrium.

Comments on Theorem 1.

Case 1. There is a continuum of Nash equilibria. In all these equilibria, the firms are located on opposite sides from the central position, always at distance 2δ one from the other, meaning that the attraction zones do not intersect. These may be called situations of "maximal differentiation". The two firms generally get different payoffs, the firm closer to the central modal position attracting more consumers than its competitor. The only symmetric situations are the situations in which one firm chooses location $t = -\delta$ and the other firm location $t = +\delta$.

Note that in that case, the "minimal differentiation principle" does not hold. The situation in which the two firms propose the median location ($t^A = t^B = 0$) is not an equilibrium. When one firm proposes $t = 0$, the best that the other firm can do is to move either to the left or to the right of its competitor, far enough so that the two attraction zones do not intersect. The mass of consumers located at the modal central position is not sufficient to trigger direct confrontation.

Case 2a. Equilibria in that case are a subset of equilibria in case 1. In all equilibria, the two firms are still at distance 2δ one from the other, and on opposite sides from the central position. The difference with case 1 is that no firm can be within distance a of the central location.

As in case 1, payoffs of the two firms are generally different, the only symmetric situations are the situations in which one firm proposes $t = -\delta$ and the other firm offers $t = +\delta$.

The "minimal differentiation principle" still does not hold in that case.

Case 2b. There is a unique Nash equilibrium, in which the two firms engage in "intermediate differentiation". The differentiation is intermediate in the sense that firms do not converge to the same position, while at the same time having there attraction zones intersect: some consumers, in particular the median consumer, are located within acceptable distance of the two firms. In that situation, both firms get exactly the same payoff.

Case 3. There is a unique Nash equilibrium, at which both firms converge to the central modal position. In that case, we find the "minimal differentiation principle" to hold.

The proof of Theorem 1 is given in the appendix in sub-section 7.2.

6 Conclusion

In this paper, I study Hotelling's spatial competition between two firms, with a slight generalization of Hotelling's assumptions. Rather than assuming that consumers are ready to buy one unit of the good, whatever the location of the firms are, I assume that there is an upper limit (possibly infinite) to the distance a consumer is ready to cover to buy the good.

What may seem an innocuous change in the assumptions dramatically alters the results. It is shown that Hotelling's "minimal differentiation principle" does not hold in general in that case, firms engaging in intermediate or complete differentiation. Other papers in the literature have already demonstrated that firms' differentiation was to be expected – be it maximal or intermediate – see for example d'Aspremont *et al.* (1979) or Economides (1986). In these papers, product differentiation is the result of two opposite forces: a direct demand effect that drives the firms to locate near the center where the demand is and an indirect strategic effect that drives the firms to differentiate in order to soften price competition.

In the present paper, another motive for differentiation is put forward, which is also a direct demand effect, created by the assumption that a consumer may choose not to buy if the two firms are too far away from his location. Indeed, suppose that a firm has chosen the modal position. If its opponent also selects this central position, they will share exactly the same attraction zone and thus each will attract one half of the consumers who are located within acceptable distance of the central position. When the market is covered whatever the locations of the firms (Hotellin's assumption), each firm then attracts exactly one half of the consumers, and may only reduce its market share by moving either to the left or to the right. This is no more the case when consumers may choose not to buy the good. The latter firm may fare better in that case by avoiding complete competition, and prefer moving somewhat to the left or the right. In doing so, it will move away from the modal position – thus move away from the location where the concentration of consumers is the highest – but on the other hand, it will avoid splitting in two the number of potential consumers. This latter "split the pie in two" effect may be strong enough to counterbalance the force that drives the firm to the central location, where the demand is.

As noticed in the introduction, the intuition suggests that Hotelling's convergence result will hold, provided that the distance a consumer is ready to cover to buy the good is large enough (δ large) or that consumers are sufficiently numerous around the center (the standard deviation σ of the assumed centered normal distribution of consumers is small). Theorem 1 provides a precise quantification for these conditions: The situation where both firms choose the central location is an equilibrium if and only if $\delta^2/\sigma^2 \geq 2\ln(2)$. Note that $\sqrt{2\ln(2)}$ is approximately equal to 1.18. This shows that, for the minimal differentiation principle to hold, the total length of a firm's attraction zone (2δ) has to be approximately at least as large as 2.35 time the standard deviation of the distribution of ideal location. This figure is quite high. By instance, when $\delta/\sigma = \sqrt{2\ln(2)}$, the attraction zone of a firm located at the center covers over 75% of the population. Theorem 1 also provides conditions for a "maximal differentiation principle" to hold – that is for situations where the two firms do not compete for the same consumers. At equilibrium the attraction zones do not intersect if and only if $\delta^2/\sigma^2 \leq \frac{1}{2}\ln(2)$. When $\delta/\sigma = \sqrt{1/2\ln(2)}$[8], the attraction zone of a firm located at the center covers about 45% of the population. When a unique firm located at the central location covers between 45 and 75% of the market, at equilibrium firms engage in intermediate differentiation, the

[8] $\sqrt{1/2\ln(2)}$ is approximately equal to 0.59.

attraction zones of the two firms only partially intersecting.

7 Appendix

7.1 Proof of Proposition 1

Since the distribution of consumers is symmetric, we can restrict our attention to studying the best response against any $t \leq 0$. The proof of proposition 1 builds on the following three lemmas. Lemma 2 describes the best response of firm B against any t by firm A such that $t \leq -2\delta$. Lemma 3 describes the best response of firm B against any t by firm A such that $-2\delta < t < 0$ and lemma 4 describes the best response of firm B when firm A chooses location $t = 0$.

Lemma 1. Best responses against $t \leq -2\delta$
If $t \leq -2\delta$, $BR^B(t) = \{0\}$.

Lemma 2. Best responses against t, for $-2\delta < t < 0$.
Three cases are to be considered:

- If $\delta^2/\sigma^2 \leq \frac{1}{4}\ln(2)$, $BR^B(t) = \{t + 2\delta\}$,
- If $\frac{1}{4}\ln(2) < \delta^2/\sigma^2 < 2\ln(2)$,

If $t \leq -a$, $BR^B(t) = \{t + 2\delta\}$,

If $t \geq -a$, $BR^B(t) = \{t_2(t)\}$,

where a is defined in (1) and $t_2(.)$ is defined in (3).

- If $\delta^2/\sigma^2 \geq 2\ln(2)$,

If $t \leq -a$, $BR^B(t) = \{t + 2\delta\}$,

If $-a \leq t < -b$, $BR^B(t) = \{t_2(t)\}$, with $t_2(t) < t + 2\delta$ if $t > -a$,

If $t \geq -b$, $BR^B(t) = \emptyset$. Firm B would like to play $t + \varepsilon$, $\varepsilon > 0$ as small as possible,

where b is defined in (2).

Lemma 3. Best responses against $t = 0$

- If $\delta^2 \leq \frac{1}{4}\sigma^2\ln(2)$, $BR^A(0) = \{-2\delta, 2\delta\}$.
- If $\frac{1}{4}\sigma^2\ln(2) < \delta^2 < 2\sigma^2\ln(2)$, $BR^A(0) = \{-t_2(0), t_2(0)\}$, where $t_2(.)$ is defined in (3).
- If $\delta^2 \geq 2\sigma^2\ln(2)$, $BR^A(0) = \{0\}$.

Proof of lemma 1.

The proof of lemma 1 is straightforward. When firm A chooses a location t^A such that $t^A \leq -2\delta$, if firm B chooses the central modal position $t = 0$, the two attraction zones do not intersect and firm B obtains the maximum possible payoff: $F(\delta) - F(-\delta)$. It is obviously the unique best response.

Proof of lemma 2.

Note first that when $-2\delta < t^A < 0$, any best response for firm B against t^A lies in $\left[t^A, t^A + 2\delta\right]$.

Indeed, it is straightforward to note that $G^B(t; t^A)$ is strictly increasing in t on $\left]-\infty, t^A - 2\delta\right]$ and strictly decreasing in t on $\left[t^A + 2\delta, +\infty\right[$. Therefore any best response necessarily lies in the interval $\left[t^A - 2\delta, t^A + 2\delta\right]$.

Thus it remains to be shown that no best response lies in $\left[t^A - 2\delta, t^A\right[$. This is easily done by checking that if $t \in \left[t^A - 2\delta, t^A\right[$, then $G^B(2t^A - t; t^A) > G^B(t; t^A)$, where $2t^A - t$ is the symmetric of t relative to t^A and lies in $\left]t^A, t^A + 2\delta\right]$. Informally, when selecting t in $\left[t^A - 2\delta, t^A\right[$, firm B attracts all consumers located in the interval $\left[t - \delta, \frac{1}{2}\left(t + t^A\right)\right]$, whereas when selecting location $2t^A - t$, it attracts all consumers located in the interval $\left[\frac{3t^A - t}{2}, 2t^A - t + \delta\right]$. Both intervals have the same "length" ($\frac{1}{2}\left(t^A - t\right) + \delta$) but the latter is "closer" to the modal position[9], thus contains more consumers.

This preliminary remark allows us to restrict our attention to to the interval $\left[t^A, t^A + 2\delta\right]$. By remark 1, we also know that, since $t^A < 0$, $t = t^A$ is not a best response for firm B against location t^A by firm A. So if firm B has a best response against t^A, it necessarily lies in the interval $\left]t^A, t^A + 2\delta\right]$ and it is the value of t that maximizes G^B on this interval.
For $t \in \left]t^A, t^A + 2\delta\right]$, $G^B(t; t^A) = F(t + \delta) - F(\frac{1}{2}(t + t^A))$ and
for $t \in \left]t^A, t^A + 2\delta\right[$, $\partial G^B(t; t^A)/\partial t = f(t + \delta) - \frac{1}{2} f(\frac{1}{2}(t + t^A))$, where f is the density function: $f(t) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\frac{t^2}{\sigma^2}}$.

Some elementary algebra show that the derivative of G^B with respect to t is non negative if and only if $3t^2 + 2(4\delta - t^A)t + (4\delta^2 - (t^A)^2 - 8\sigma^2 \ln(2)) \leq 0 \Leftrightarrow t_1\left(t^A\right) \leq t \leq t_2\left(t^A\right)$,
where $t_1\left(t^A\right) = \frac{1}{3}\left(t^A - 4\delta - 2\sqrt{\left(t^A - \delta\right)^2 + 6\sigma^2 \ln(2)}\right)$

[9]Meaning that the distance between the middle of the interval and location zero is smaller. Indeed, the "middle" of the former interval is $\frac{1}{2}\left[(t - \delta) + \frac{1}{2}\left(t + t^A\right)\right] = \frac{1}{4}\left[t^A + 3t - 2\delta\right]$, whereas the "middle" of the latter interval is $\frac{1}{2}\left[\left(\frac{3t^A - t}{2}\right) + \left(2t^A - t + \delta\right)\right] = \frac{1}{4}\left[7t^A - 3t + 2\delta\right]$. We want to show that $\left|\frac{1}{4}\left[7t^A - 3t + 2\delta\right]\right| < \left|\frac{1}{4}\left[t^A + 3t - 2\delta\right]\right|$. Now, it is straightforward to see that $\frac{1}{4}\left[t^A + 3t - 2\delta\right] < 0$ and $\frac{1}{4}\left[t^A + 3t - 2\delta\right] < \frac{1}{4}\left[7t^A - 3t + 2\delta\right]$. So if $\frac{1}{4}\left[7t^A - 3t + 2\delta\right] < 0$, this inequality necessarily holds. If $\frac{1}{4}\left[7t^A - 3t + 2\delta\right] > 0$, one can check that $\frac{1}{4}\left[7t^A - 3t + 2\delta\right] < -\frac{1}{4}\left[t^A + 3t - 2\delta\right] \Leftrightarrow t^A < 0$.

and $t_2\left(t^A\right)=\frac{1}{3}\left(t^A-4\delta+2\sqrt{\left(t^A-\delta\right)^2+6\sigma^2\ln(2)}\right)$.

Remember we restrict our search for best responses to the interval $\left]t^A,t^A+2\delta\right]$. Let us first determine the relative position of $t_1\left(t^A\right)$, $t_2\left(t^A\right)$, relative to t^A and $t^A+2\delta$. One can easily check that $t_1\left(t^A\right)<t^A$. Besides, some elementary algebra show that $t_2\left(t^A\right)\leq t^A\Leftrightarrow t^A\geq -b$, and $t_2\left(t^A\right)\leq t^A+2\delta\Leftrightarrow t^A\geq -a$, where a and b are respectively defined in (1) and (2). Note that $b<a$.

To sum up the results so far:

- if $t^A\leq -a$, $t_1(t^A)<t^A<t^A+2\delta\leq t_2(t^A)$ and thus $G^B(t;t^A)$ is strictly increasing in t on $\left]t^A,t^A+2\delta\right]$. In that case, G^B reaches its maximum in t on the interval $\left]t^A,t^A+2\delta\right]$ in $t=t^A+2\delta$.
- if $-a\leq t^A<-b$, $t_1(t^A)<t^A\leq t_2(t^A)\leq t^A+2\delta$ and thus $G^B(t;t^A)$ is increasing in t on $\left]t^A,t_2(t^A)\right]$ and decreasing on $\left[t_2(t^A),t^A+2\delta\right[$. In that case, G^B reaches its maximum in t on the interval $\left]t^A,t^A+2\delta\right]$ in $t=t_2(t^A)$. (Note that $t_2(t^A)=t^A+2\delta$ for $t^A=-a$).
- if $t^A\geq -b$, $t_1(t^A)<t_2(t^A)\leq t^A<t^A+2\delta$ and thus $G^B(t;t^A)$ is decreasing in t on $\left]t^A,t^A+2\delta\right]$. In that case, G^B has no well defined maximum in t on the interval $\left]t^A,t^A+2\delta\right]$.

Now, remember we focus on the case $t^A\in\left]-2\delta,0\right[$. Therefore we now need to determine the positions of $-a$, $-b$, relative to -2δ and 0. One can easily check that both $-a$ and $-b$ are strictly greater than -2δ, with $-a<-b$. Besides, $b\geq 0\Leftrightarrow \delta^2/\sigma^2\geq 2\ln(2)$ and $a\geq 0\Leftrightarrow \delta^2/\sigma^2\geq \frac{1}{4}\ln(2)$.

Collecting all these results, one can conclude that:

- If $\delta^2/\sigma^2\leq\frac{1}{4}\ln(2)$, $-2\delta<0\leq -a<-b$. In that case, $t^A\leq -a$ and therefore, the best response against t^A is $t^A+2\delta$.
- If $\frac{1}{4}\ln(2)<\delta^2/\sigma^2<2\ln(2)$, $-2\delta<-a<0<-b$. In that case, if $t^A\leq -a$, the best response against t^A is $t^A+\delta$.; if $t^A\geq -a$, the best response against t^A is $t_2(t^A)$. Note that $t_2\left(t^A\right)<t^A+2\delta$ if $t^A>-a$.
- If $\delta^2/\sigma^2\geq 2\ln(2)$, $-2\delta<-a<-b\leq 0$. In that case, if $t^A\leq -a$, the best response against t^A is $t^A+\delta$. If $-a\leq t^A<-b$, the best response against t^A is $t_2(t^A)$. Note that $t_2\left(t^A\right)<t^A+2\delta$ if $-a<t^A<-b$. If $t^A\geq -b$, $G^B(t;t^A)$ is decreasing in t on $\left]t^A,t^A+2\delta\right]$. But by remark 1, when $t^A<0$, $\underset{\substack{t\to t^A\\ t>t^A}}{Lim}G^B(t;t^A)>G^B(t^A;t^A)$.Thus firm B would like to play $t^A+\varepsilon$, $\varepsilon>0$ as small as possible and $BR^B(t^A)$ is not well defined in that case.

This concludes the proof of lemma 2.

Proof of lemma 3.

Given the symmetry of the distribution of consumers' locations, when firm A chooses the location $t^A = 0$, firm B attracts exactly the same number of consumers by playing opposite policies t or $-t$: $\forall t \in \mathsf{R}$, $G^B(t;0) = G^B(-t;0)$. We can therefore restrict our study of $G^B(t;0)$ to the non-negative value of t and look for the best response (best responses) on R_+. Note first that the pay-off function $G^A(t;0)$ is decreasing in t on the interval $[2\delta, +\infty[$. Therefore any best response lies in the interval $[0, 2\delta]$.

A straightforward adaptation of the proof of lemma 3 for $t^A = 0$ allows to get the following results for $t^A = 0$:

- If $\delta^2/\sigma^2 \leq \frac{1}{4}\ln(2)$, $G^B(t;0)$ is increasing in t on $[0, 2\delta]$. Therefore, the best non-negative response against $t^A = 0$ is $+2\delta$.
- If $\frac{1}{4}\ln(2) < \delta^2/\sigma^2 < 2\ln(2)$, the best non negative response against $t^A = 0$ is $t_2(0)$ with $t_2(0) < 2\delta$.
- If $\delta^2/\sigma^2 \geq 2\ln(2)$, $G^B(t;0)$ is continuous and decreasing in t on $[0, 2\delta]$. Therefore the best non negative response against 0 is 0.

This completes the proof of lemma 3.

7.2 Proof of Theorem 1

Given the symmetry of the game, we can restrict our attention to studying the existence of Nash equilibria where $t^A \leq 0$.

We now consider in turn the three cases distinguished in Proposition 1.

Case 1. $\delta^2/\sigma^2 \leq \frac{1}{4}\ln(2)$

We explore in turn the existence and characterization of Nash equilibria (t^A, t^B) where **(a)** $t^A \leq -2\delta$, **(b)** $-2\delta < t^A < 0$, **(c)** $t^A = 0$.

(a) Suppose that (t^A, t^B) is a Nash equilibrium where $t^A \leq -2\delta$. Then it must be the case by proposition 1 (case 1) that $t^B = 0$. But now the best response of firm A against 0 is either -2δ or $+2\delta$. Therefore, $(t^A, t^B) = (-2\delta, 0)$ is a Nash equilibrium, and it is the unique Nash equilibrium where $t^A \leq -2\delta$.

(b) Suppose now that (t^A, t^B) is a Nash equilibrium where $-2\delta < t^A < 0$. The best response of firm B against t^A is $t^A + 2\delta$ (see proposition 1 case 1). Then it must be the case that $t^B = t^A + 2\delta$, which lies in the interval $]0, 2\delta[$.

Now the best response against any t^B in this interval is $t^B - 2\delta$. This shows that all pairs $(t^A, t^A + 2\delta)$ where $t^A \in]-2\delta, 0[$ are Nash equilibria, and that these are the only Nash equilibria where A selects a location in the interval $]-2\delta, 0[$.

(c) Suppose now that there is a Nash equilibrium with $t^A = 0$. The best response of firm B against $t^A = 0$ is is either 2δ or -2δ. Then it must be the case that $t^B \in \{-2\delta, 2\delta\}$. But the best response against $t^B \in \{-2\delta, 2\delta\}$ is 0. This shows that the pairs $(0, -2\delta)$ and $(0, +2\delta)$ are both Nash equilibria, and that these are the only Nash equilibria where A selects the central position $t^A = 0$.

Case 2: $\frac{1}{4}\ln(2) < \delta^2/\sigma^2 < 2\ln(2)$

We explore in turn the existence and characterization of Nash equilibria (t^A, t^B) wherer **(a)** $t^A \leq -2\delta$, **(b)** $-2\delta < t^A \leq -a$, **(c)** $-a < t^A < 0$, **(d)** $t^A = 0$.

(a) Suppose that (t^A, t^B) is a Nash equilibrium where $t^A \leq -2\delta$. Then it must be the case by proposition 1 that $t^B = 0$. But now the best response of firm A against 0 is either $-t_2(0)$ or $t_2(0)$. Since by remark 2, $0 < t_2(0) < 2\delta$, this is a contradiction with $t^A \leq -2\delta$. Thus, there is no Nash (t^A, t^B) equilibrium where $t^A \leq -2\delta$.

(b) Suppose now that (t^A, t^B) is a Nash equilibrium where $-2\delta < t^A \leq -a$. The best response of firm B against t^A is $t^A + 2\delta$. Then it must be the case that $t^B = t^A + 2\delta$, which lies in the interval $]0, -a + 2\delta]$.

If $0 < t^A + 2\delta < a$, by proposition 1, the best response of firm A against such a location by firm B is $-t_2(-(t^A + 2\delta))$. Since $0 < t^A + 2\delta < a$, $-a < -(t^A + 2\delta) < 0$ and $t_2(-(t^A + 2\delta)) < -t^A$ (see remark 2). Therefore $-t_2(-(t^A + 2\delta)) > t^A$, which contradicts the fact that (t^A, t^B) is a Nash equilibrium.

Now, if $a \leq t^A + 2\delta \leq -a + 2\delta$, the only best response of firm A against $t^A + 2\delta$ is t^A. Therefore (t^A, t^B) is a Nash equilibrium where $-2\delta < t^A \leq -a$ if and only if $t^B = t^A + 2\delta$ and $a - 2\delta \leq t^A \leq 0$. The intersection of the intervals $[-2\delta, -a]$ and $[a - 2\delta, 0]$ is empty when $a > \delta$ and is the interval $[a - 2\delta, -a]$ when $a \leq \delta$.

The condition $a \leq \delta$ also writes $2\delta^2 \leq \sigma^2 \ln 2$. Thus, if $2\delta^2 > \sigma^2 \ln 2$, there is no Nash equilibrium where $-2\delta < t^A \leq -a$, and if $2\delta^2 \leq \sigma^2 \ln 2$, any (t^A, t^B) where $t^B = t^A + 2\delta$ and $a - 2\delta \leq t^A \leq -a$ are Nash equilibria, and these are the only Nash equilibria where $-2\delta < t^A \leq -a$.

(c) Suppose now that (t^A, t^B) is a Nash equilibrium where $-a < t^A < 0$. The unique best response of firm B against t^A is $t_2(t^A) \in \left]0, t^A + 2\delta\right[\subset]0, 2\delta[$. The best response against $t_2(t^A)$ is $-t_2(-t_2(t^A))$ if $t_2(t^A) \leq a$ and $t_2(t^A) - 2\delta$ if $t_2(t^A) \geq a$. Therefore (t^A, t^B) is a Nash equilibrium where $-a < t^A < 0$ iff either $(t^A = -t_2(-t_2(t^A))$ and $t_2(t^A) \leq a)$ or $(t^A = t_2(t^A) - 2\delta$ and $t_2(t^A) \geq a)$.

Let us first show that there is no equilibrium of the latter kind. By remark 2, the function t_2 satisfies the property that for $t \in]-a, 0[$, $t_2(t) < t + 2\delta$. Therefore there is no t^A, $-a < t^A < 0$, such that $t^A = t_2(t^A) - 2\delta$.

Let us now turn to the existence of equilibria of the former kind. $(t^A, t^B) = (t^A, t_2(t^A))$ is an equilibrium of the former kind iff $-a < t^A < 0$, $t^A = -t_2(-t_2(t^A))$ and $t_2(t^A) \leq a$.

Let us consider the equation $t^A = -t_2(-t_2(t^A))$. Let us define the function $G : R \to R$, $G(t) = t_2(-t_2(t)) + t$. For all $t \in [-a, 0]$, $G'(t) = t_2'(t)t_2'(-t_2(t)) - 1$. By remark 2, for any $t \in R$, $|t_2'(t)| < 1$ and $|t_2'(-t_2(t))| < 1$. This shows that $G'(t) > 0$ and so G is strictly increasing on R and therefore there is at most one value for which G is equal to zero. One can easily check that $G\left(\delta - \sqrt{2\sigma^2 \ln(2)}\right) = 0$. Therefore, a necessary and sufficient condition for (t^A, t^B) to be an equilibrium with $-a < t^A < 0$ is that $t^A = \delta - \sqrt{2\sigma^2 \ln(2)}$, $t^B = t_2(t^A) = -\left(\delta - \sqrt{2\sigma^2 \ln(2)}\right)$, and $-a < \delta - \sqrt{2\sigma^2 \ln(2)} < 0$. One can easily check that $-a < \delta - \sqrt{2\sigma^2 \ln(2)} \Leftrightarrow \delta^2/\sigma^2 > 1/2 \ln(2)$ and it is always the case that $\delta - \sqrt{2\sigma^2 \ln(2)} < 0$. Therefore there is no Nash equilibrium where $-a < t^A < 0$ when $\delta^2/\sigma^2 \leq 1/2 \ln(2)$ and there is a unique Nash equilibrium where $-a < t^A < 0$ when $\delta^2/\sigma^2 > 1/2 \ln(2)$, that is $(t^A, t^B) = \left(\delta - \sqrt{2\sigma^2 \ln(2)}, -\delta + \sqrt{2\sigma^2 \ln(2)}\right)$.

(d) Suppose now that (t^A, t^B) is a Nash equilibrium where $t^A = 0$. The best response of firm B against t^A is either $t_2(0) \in [0, 2\delta]$ or $-t_2(0)$. Then it must be the case that $t^B \in \{-t_2(0), t_2(0)\}$. By symmetry, $(0, t_2(0))$ is a Nash equilibrium if and only if $(0, -t_2(0)$ is a Nash equilibrium. One can therefore study the former situation only. The best response against $t_2(0) \in]0, 2\delta[$ can not be the central location (see proposition 1), therefore there is no Nash equilibrium with $t^A = 0$ when $\delta^2/\sigma^2 < 2 \ln(2)$.

Case 3: $\delta^2/\sigma^2 \geq 2 \ln(2)$

Since 0 is a best response against location at 0 by the opponent firm, the situation where both firms choose the central location is obviously a Nash equilibrium.

It remains to be shown that this is the unique equilibrium in that case. Note first that since 0 is the unique best response against location at 0 by the opponent firm, $(t^A, t^B) = (0, 0)$ is the only Nash equilibrium where $t^A = 0$. We will prove in turn that **(a)** there is no nash equilibrium in which $t^A \leq -2\delta$, **(b)** there is no Nash equilibrium in which $-2\delta < t^A < -a$, **(c)** there is no Nash equilibrium in which $-a \leq t^A < -b$ and **(d)** there is no Nash equilibrium in which $-b \leq t^A < 0$.

(a) Suppose that (t^A, t^B) is a Nash equilibrium where $t^A \leq -2\delta$. Then it must be the case by proposition 1 that $t^B = 0$. But now the best response of firm A against 0 is 0, which is a contradiction with $t^A \leq -2\delta$.Thus there is no Nash (t^A, t^B) equilibrium where $t^A \leq -2\delta$.

(b) Suppose now that (t^A, t^B) is a Nash equilibrium where $-2\delta < t^A < -a$. The best response of firm B against t^A is $t^A + 2\delta$. Then it must be the case that $t^B = t^A + 2\delta$, where which lies in the interval $]0, 2\delta - a[\subset]0, a[$ Since there is no best response against any $t \in]0, b]$, a necessary condition for $(t^A, t^A + 2\delta)$ to be an equilibrium with $-2\delta < t^A < -a$ is that $t^A = -t_2(-t^A - 2\delta)$. But $0 < t^A + 2\delta < a$ implies by remark 2 that $t_2(-t^A - 2\delta) < -t^A$, which contradicts $t^A = -t_2(-t^A - 2\delta)$. This shows that there is no Nash equilibrium (t^A, t^B) where $-2\delta < t^A < -a$.

(c) Suppose now that (t^A, t^B) is a Nash equilibrium where $-a \leq t^A < -b$. The unique best response of firm B against t^A is $t_2(t^A)$. Then it must be the case that $t^B = t_2(t^A)$. By remark 2, t_2 is strictly decreasing on the interval $[-a, -b]$, therefore $t_2(t^A) \in]t_2(-b), t_2(-a)] =]-b, -a + 2\delta] \subset]-b, a]$. The best response against $t_2(t^A) \subset]-b, a]$ is 0 if $t_2(t^A) = 0$, $-t_2(-t_2(t^A))$ if $b < t_2(t^A) \leq a$ and is not defined otherwise. Therefore a necessary condition for (t^A, t^B) to be a Nash equilibrium where $-a \leq t^A < -b$ is that either $t^B = 0$ or $-t_2(-t_2(t^A)) = t^A$. Since the best response for firm A against $t^B = 0$ is 0, there is no equilibrium where $-a \leq t^A < -b \leq 0$ and $t^B = 0$. Let us now consider the equation $t^A = -t_2(-t_2(t^A))$. We have shown above that this equation has a unique solution on R: $t^A = \delta - \sqrt{2\sigma^2 \ln(2)}$. But when $\delta^2/\sigma^2 \geq 2\ln(2)$, $\delta - \sqrt{2\sigma^2 \ln(2)} \geq 0$. Therefore there is no equilibrium where $-a \leq t^A < -b \leq 0$ and $t^A = -t_2(-t_2(t^A))$. This shows that there is no Nash equilibrium where $-a \leq t^A < -b$.

(d) Since there is no well defined best response for firm B against any location in $[-b, 0[$ by firm A, there is no Nash equilibrium where $-b \leq t^A < 0$. This completes the proof of theorem 1.

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