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CHARACTERIZATION OF EQUILIBRIUM PATHS IN A TWO-SECTOR ECONOMY WITH CES PRODUCTION FUNCTIONS AND SECTOR-SPECIFIC EXTERNALITY

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Characterization of Equilibrium Paths in a Two-Sector Economy with CES Production Functions and Sector-Specific Externality*

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Abstract

In this paper, we study a two-sector CES economy with sector-specific externality as described by Nishimura and Venditti [5]. We characterize the equilibrium paths in the case that allows negative externality as that equlibrium paths were not explicitly discussed by Nishimura and Venditti. We show how the degree of externality affects the local behavior of the equilibrium path around the steady state.

1 Introduction

This paper characterizes the local behavior of the equilibrium paths around the steady state in a two-sector model with CES production functions and sector-

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specific externality. It is well known that externalities may cause indeterminate equilibrium paths in an infinite horizon model. Benhabib and Farmer [1] have shown that indeterminacy could occur in a one-sector growth model with both externality and increasing returns. In their model, the production function has constant returns to scale from the private perspective, while it has increasing returns to scale from the social perspective. Since then, there have been many papers about the existence of indeterminate equilibria in dynamic general equilibrium models. However, until Benhabib and Nishimura [2, 3], most of the literature dealt with models in which the production function has increasing returns to scale from the social perspective. They proved indeed that indeterminacy may arise in a continuous time economy in which the production functions from the social perspective have constant returns to scale. Benhabib, Nishimura and Venditti [4] then studied a discrete-time two-sector model with sector-specific external effects. They assumed that each sector has a Cobb-Douglas production function with positive sector-specific externalities, that are a special case of CES functions, and there is an infinitely-lived representative agent with a linear utility function. Under these assumptions, they provided conditions in which indeterminacy may occur even if the production functions have decreasing returns to scale from the social perspective. Nishimura and Venditti [5] consider a CES economy with sector-specific external effects and partial depreciation of capital. They study the interplay between the elasticity of capital-labor substitution and the rate of depreciation of capital, and its influence on the local behavior of the equilibrium paths in the neighborhood of the steady state.

In this paper, we study the Nishimura and Venditti [5] model, focussing on the external effect of the capital-labor ratio in the pure capital good sector and characterize the equilibrium paths in the case that allows negative externality. That particular scenario was not explicitly discussed by Nishimura and Venditti. We assume indeed that the externality is given by the capital stock per capita. Such a formulation then implies that for a given level of capital, the externality from labor is negative. We demonstrate that the degree of externality affects the local behavior of the equilibrium path around the steady state.

In Section 2 we describe the model. We discuss the existence of a steady state and give the local characterization of the equilibrium paths around the steady state in Section 3. Section 4 contains some concluding comments.

2 The Model

We consider a two-sector model with an infinitely-lived representative agent. We assume that its single period linear utility function is given by

$$u\left(c_t\right) = c_t.$$

We assume that the consumption good, c, and the capital good, y, are produced with a CES technology.

$$c_t = \left[\alpha_1 K_{ct}^{-\rho_c} + \alpha_2 L_{ct}^{-\rho_c}\right]^{-\frac{1}{\rho_c}}, \tag{1}$$

$$y_t = \left[\beta_1 K_{yt}^{-\rho_y} + \beta_2 L_{yt}^{-\rho_y} + e_t\right]^{-\frac{1}{\rho_y}}, \tag{2}$$

where $\rho_c,\ \rho_y > -1$ and e_t represents the time-dependent externality in the investment good sector. Let the elasticity of capital-labor substitution in each sector be $\sigma_c = \frac{1}{1+\rho_c} \geq 0$ and $\sigma_y = \frac{1}{1+\rho_y} \geq 0$. We assume that the externalities are as follows:

$$e_t = b\bar{K}_{yt}^{-\rho_y} - b\bar{L}_{yt}^{-\rho_y}, \tag{3}$$

where $b > 0$, and $\bar{K}_y$ and $\bar{L}_y$ represents the economy-wide average values. The representative agent takes these economy-wide average values as given.

Definition 1 *We call* $y = \left[\beta_1 K_y^{-\rho_y} + \beta_2 L_y^{-\rho_y} + e\right]^{-\frac{1}{\rho_y}}$ *the production function from the private perspective, and* $y = \left[(\beta_1 + b)\, K_y^{-\rho_y} + (\beta_2 - b)\, L_y^{-\rho_y}\right]^{-\frac{1}{\rho_y}}$ *the production function from the social perspective.*

In the rest of the paper we will assume that $\alpha_1 + \alpha_2 = \beta_1 + \beta_2 = 1$. Thus, if $\hat{\beta}_1 = \beta_1 + b$ and $\hat{\beta}_2 = \beta_2 - b$, then $\hat{\beta}_1 + \hat{\beta}_2 = 1$.

Remark 1 Notice that the externality (3) may be expressed as follows

$$e = b\bar{L}_y^{-\rho_y}\left[\left(\frac{\bar{K}_y}{\bar{L}_y}\right)^{-\rho_y} - 1\right]. \tag{4}$$

Now consider the production function from the social perspective as given in Definition 1. Dividing both sides by L_y, we get denoting $k_y = K_y/L_y$ and $\tilde{y} = y/L_y$

$$\tilde{y} = \left[(\beta_1 + b)\, k_y^{-\rho_y} + (\beta_2 - b)\right]^{-\frac{1}{\rho_y}}. \tag{5}$$

From equations (4) and (5) we derive that the externality is given in terms of the capital-labor ratio in the investment good sector. This implies that for a given level of capital $\bar{K}_y$, the externality from labor $\bar{L}_y$ is negative.

The aggregate capital is divided between sectors,

$$k_t = K_{ct} + K_{yt},$$

and the labor endowment is normalized to one and divided between sectors,

$$L_{ct} + L_{yt} = 1.$$

The capital accumulation equation is

$$k_{t+1} = y_t,$$

Thus capital depreciates completely in one period. To simplify the formulation, we assume that both technologies are characterized by the same properties of substitution, i.e. $\rho_c = \rho_y = \rho$.

The consumer optimization problem is given by

$$\begin{aligned} \max \quad & \sum_{t=0}^{\infty} \delta^t \left[\alpha_1 K_{ct}^{-\rho} + \alpha_2 L_{ct}^{-\rho}\right]^{-\frac{1}{\rho}} \\ s.t. \quad & y_t = \left[\beta_1 K_{yt}^{-\rho} + \beta_2 L_{yt}^{-\rho} + e_t\right]^{-\frac{1}{\rho}} \\ & 1 = L_{ct} + L_{yt} \\ & k_t = K_{ct} + K_{yt} \\ & y_t = k_{t+1} \\ & k_0, \ \{e_t\}_{t=0}^{\infty} \quad given \end{aligned} \tag{6}$$

where $\delta \in (0,\ 1)$ is the discount factor. p_t, r_t, and w_t respectively denote the price of capital goods, the rental rate of the capital goods and the wage rate of labor at time $t \geq 0$.[1] For any sequences $\{e_t\}_{t=0}^{\infty}$ of external effects that the representative agent considers as given, the Lagrangian at time $t \geq 0$ is defined as follows:

$$\begin{aligned} \mathcal{L}_t \quad = \quad & \left[\alpha_1 K_{ct}^{-\rho} + \alpha_2 L_{ct}^{-\rho}\right]^{-\frac{1}{\rho}} + p_t \left[\left[\beta_1 K_{yt}^{-\rho} + \beta_2 L_{yt}^{-\rho} + e_t\right]^{-\frac{1}{\rho}} - k_{t+1}\right] \\ + \quad & r_t \left(k_t - K_{ct} - K_{yt}\right) + w_t \left(1 - L_{ct} - L_{yt}\right). \end{aligned} \tag{7}$$

Then the first order conditions derived from the Lagrangian are as follows:

$$\frac{\partial \mathcal{L}_t}{\partial K_{ct}} = \alpha_1 \left(\frac{c_t}{K_{ct}}\right) - r_t = 0, \tag{8}$$

$$\frac{\partial \mathcal{L}_t}{\partial L_{ct}} = \alpha_2 \left(\frac{c_t}{L_{ct}}\right) - w_t = 0, \tag{9}$$

$$\frac{\partial \mathcal{L}_t}{\partial K_{yt}} = p\beta_1 \left(\frac{y_t}{K_{yt}}\right) - r_t = 0, \tag{10}$$

$$\frac{\partial \mathcal{L}_t}{\partial L_{yt}} = p\beta_2 \left(\frac{y_t}{L_{yt}}\right) - w_t = 0. \tag{11}$$

From the above first order conditions, we derive the following equation,

$$\left(\frac{\alpha_1 / \alpha_2}{\beta_1 / \beta_2}\right) = \left(\frac{K_{ct} / L_{ct}}{K_{yt} / L_{yt}}\right)^{1+\rho}. \tag{12}$$

[1] We normalize the price of consumption goods to one.

If $\alpha_1/\alpha_2 > (<)\, \beta_1/\beta_2$, then the consumption (capital) good sector is capital intensive from the private perspective.

For any value of (k_t, y_t), solving the first order conditions with respect to K_{ct}, K_{yt}, L_{ct}, L_{yt} gives inputs as functions of capital stock at time t and $t+1$, and external effect, namely:

$$\begin{aligned} K_{ct} &= K_c(k_t, y_t, e_t), \quad L_{ct} = L_c(k_t, y_t, e_t), \\ K_{yt} &= K_y(k_t, y_t, e_t), \; L_{yt} = L_y(k_t, y_t, e_t). \end{aligned}$$

For any given sequence $\{e_t\}_{t=0}^{\infty}$, we define the efficient production frontier as follows:

$$T(k_t, k_{t+1}, e_t) = \left[\alpha_1 K_c(k_t, y_t, e_t)^{-\rho} + \alpha_2 L_c(k_t, y_t, e_t)^{-\rho}\right]^{-\frac{1}{\rho}}.$$

Using the envelope theorem we derive the equilibrium prices[2].

$$T_2(k_t, k_{t+1}, e_t) = -p_t, \tag{13}$$

$$T_1(k_t, k_{t+1}, e_t) = r_t. \tag{14}$$

Next we solve the intertemporal problem (6). In this model, the lifetime utility function becomes

$$U = \sum_{t=0}^{\infty} \delta^t T(k_t, k_{t+1}, e_t).$$

From the first order conditions with respect to k_{t+1}, we obtain the Euler equation

$$T_2(k_t, k_{t+1}, e_t) + \delta T_1(k_{t+1}, k_{t+2}, e_{t+1}) = 0. \tag{15}$$

The solution of equation (15) also has to satisfy the following transversality condition

$$\lim_{t\to+\infty} \delta^t k_t T_1(k_t, k_{t+1}, e_t) = 0. \tag{16}$$

We denote the solution of this problem $\{k_t\}_{t=0}^{\infty}$. This path depends on the choice of the sequence $\{e_t\}_{t=0}^{\infty}$. If the sequence $\{e_t\}_{t=0}^{\infty}$ satisfies

$$e_t = bK_y(k_t, y_t, e_t)^{-\rho} - bL_y(k_t, y_t, e_t)^{-\rho}, \tag{17}$$

then $\{\hat{k}_t\}_{t=0}^{\infty}$ is called an equilibrium path.

Definition 2 $\{k_t\}_{t=0}^{\infty}$ *is an equilibrium path if* $\{k_t\}_{t=0}^{\infty}$ *satisfies (15), (16) and (17).*

[2] See Takayama [6] for the envelope theorem, pp160-165. Using the envelope theorem, we get $\frac{\partial \mathcal{L}_t}{\partial k_t} = \frac{\partial T}{\partial k_t}$ and $\frac{\partial \mathcal{L}_t}{\partial k_{t+1}} = \frac{\partial T}{\partial k_{t+1}}$. This is equivalent to (13) and (14).

Solving equation (17) for e_t, we derive e_t as a function of $(k_t,\ k_{t+1})$, namely $e_t = \hat{e}\,(k_t, k_{t+1})$. Let us substitute $\hat{e}\,(k_t, k_{t+1})$ into equations (13) and (14) and define the equilibrium prices as

$$p_t = p_t\,(k_t, k_{t+1})\,,$$

$$r_t = r_t\,(k_t, k_{t+1})\,.$$

Then the Euler equation (15) evaluated at $\{k_t\}_{t=0}^{\infty}$ is

$$-p\,(k_t, k_{t+1}) + \delta r\,(k_{t+1}, k_{t+2}) = 0. \tag{18}$$

3 Steady State and Local Stability Properties

Definition 3 *A steady state is defined by $k_t = k_{t+1} = y_t = k^*$ and is given by the solution of $T_2\,(k^*, k^*, e^*) + \delta T_1\,(k^*, k^*, e^*) = 0$ with $e^* = \hat{e}\,(k^*, k^*)\,.$*

In the rest of the paper we assume the following restriction on parameters' values that guarantees all the steady state values are positive.

Assumption 1 *The parameters δ, β_1, b and ρ satisfy*

$$(\delta\beta_1)^{\frac{-\rho}{1+\rho}} < \beta_1 + b.$$

From the proof given in Nishimura and Venditti [5], we can obtain the steady state value.

Proposition 1 *In this model, there exists a unique stationary capital stock k^* such that:*

$$k^* = \left\{1 + \left(\tfrac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)^{\frac{-1}{1+\rho}} (\delta\beta_1)^{\frac{-1}{1+\rho}} \left[1 - (\delta\beta_1)^{\frac{1}{1+\rho}}\right]\right\}^{-1} \left[\tfrac{1-\hat{\beta}_1(\delta\beta_1)^{\frac{-\rho}{1+\rho}}}{\hat{\beta}_2}\right]^{\frac{1}{\rho}}. \tag{19}$$

To study the local behavior of the equilibrium path around the steady state k^*, we linearize the Euler equation (15) at the steady state k^* and obtain the following characteristic equation

$$\delta T_{12}\lambda^2 + [\delta T_{11} + T_{22}]\,\lambda + T_{21} = 0,$$

or

$$\delta\lambda^2 + \left[\delta\frac{T_{11}}{T_{12}} + \frac{T_{22}}{T_{12}}\right]\lambda + \frac{T_{21}}{T_{12}} = 0. \tag{20}$$

As shown in Nishimura and Venditti [5], the expressions of the characteristic roots are as follows:

Proposition 2 *The characteristic roots of equation (20) are*

$$\lambda_1 = \frac{1}{(\delta\beta_2)^{\frac{1}{1+\rho}}\left[\left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{1+\rho}} - \left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{1}{1+\rho}}\right]}, \tag{21}$$

$$\lambda_2(b) = \frac{(\delta\beta_2)^{\frac{1}{1+\rho}}\left[\frac{\beta_1+b}{\beta_1}\left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{1+\rho}} - \frac{\beta_2-b}{\beta_2}\left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{1}{1+\rho}}\right]}{\delta}.$$

The roots of the characteristic equation determine the local behavior of the equilibrium paths. The sign of λ_1 is determined by factor intensity differences from the private perspective.[3]

Remark 2 The capital/labor ratio externality, i.e. the negative labor externality, implies that the characteristic root $\lambda_2(b)$ in Proposition 2 is larger than the corresponding characteristic root from Nishimura and Venditti [5].

We now characterize the equilibrium paths in this model. In particular we can show that the local behavior of the equilibrium path around the steady state changes according to the degree of external effect in the capital good sector.

Definition 4 *A steady state k^* is called locally indeterminate if there exists $\varepsilon > 0$ such that for any $k_0 \in (k^* - \varepsilon, k^* + \varepsilon)$, there are infinitely many equilibrium paths converging to the steady state.*

As there is one pre-determined variable, the capital stock, local indeterminacy occurs if the stable manifold has two dimensions, i.e. if the two characteristic roots are within the unit circle. We will also present conditions for local determinacy (saddle-point stability) in which there exists a unique equilibrium path. Such a configuration occurs if the stable manifold has one dimension, i.e. if one root is outside the unit circle while the other is inside.

When the investment good is capital intensive, local indeterminacy cannot occur. However, depending on the size of the externality, saddle-point stability or total instability may occur.

Proposition 3 *Suppose that the capital good sector is capital intensive from the private perspective, i.e. $\alpha_2\beta_1 > \alpha_1\beta_2$. The following cases hold:*

(i) If $\rho \in (-1, 0)$, there is $b(\delta) > \beta_2$ such that the steady state is a saddle point for $b \in [0, b(\delta))$ and is totally unstable for $b > b(\delta)$.

(ii) If $\rho = 0$, the steady state is a saddle point for $b \in [0, \beta_2)$ and is totally unstable for $b > \beta_2$.

(iii) If $\rho > 0$, there exists $0 < \delta_2 < 1$ and $0 < b(\delta) < \beta_2$ such that when $\delta \in (\delta_2, 1)$, the steady state is a saddle point for $b < b(\delta)$ and is totally unstable for $b > b(\delta)$. When $\delta \in (0, \delta_2)$, the steady state is totally unstable.

[3] If $\alpha_2\beta_1 - \alpha_1\beta_2 > 0$, the capital good sector is capital intensive from the private perspective.

Proof. Notice that $\lambda_1 > 0$ and define $\delta_1 \equiv \beta_2^{-1}\left[(\beta_1/\beta_2)^{\frac{1}{1+\rho}} - (\alpha_1/\alpha_2)^{\frac{1}{1+\rho}}\right]^{-1-\rho} > 1$.[4] From this we obtain $\lambda_1 = (\delta_1/\delta)^{\frac{1}{1+\rho}} > 1$ for any $\delta \in (0,1)$. Since $(\beta_1 + b)/\beta_1 > 1$ and $(\beta_2 - b)/\beta_2 < 1$, $\lambda_2(b)$ is always positive. Thus $\lambda_2(b)$ is an increasing function of b. For $b = 0$, we get $\lambda_2(0) = \delta^{-\frac{\rho}{1+\rho}}\delta_1^{-\frac{1}{1+\rho}}$ and for $b = \beta_2$, we have $\lambda_2(\beta_2) = (\delta\beta_1)^{\frac{-\rho}{1+\rho}}$.

(i) If $-1 < \rho < 0$, $0 < \lambda_2(0) < 1$ and $\lambda_2(\beta_2) < 1$. Therefore there is $b(\delta) > \beta_2$ such that $0 < \lambda_2(b) < 1$ for $b \in (0, b(\delta))$, and $\lambda_2(b) > 1$ for $b > b(\delta)$.

(ii) If $\rho = 0$, $0 < \lambda_2(0) = \delta_1^{-1} < 1$ and $\lambda_2(\beta_2) = 1$. Therefore $\lambda_2(b) < 1$ for $b < \beta_2$ and $\lambda_2(b) > 1$ for $b > \beta_2$.

(iii) If $\rho > 0$, $0 < \lambda_2(0) < 1$ for $\delta_2 < \delta < 1$ and $\lambda_2(0) > 1$ for $\delta \in (0, \delta_2)$ with $\delta_2 = \delta_1^{-\frac{1}{\rho}}$. Moreover $\lambda_2(\beta_2) > 1$. Therefore if δ lies in $(\delta_2, 1)$, there is $b(\delta) \in (0, \beta_2)$ such that $\lambda_2(b) < 1$ for $b \in (0, b(\delta))$ and $\lambda_2(b) > 1$ for $b > b(\delta)$. If δ lies in $(0, \delta_2)$, then $\lambda_2(0) > 0$ and $\lambda_2(\beta_2) > 1$, and therefore $\lambda_2(b) > 1$ for all $b > 0$. ■

Remark 3 Consider the production function from the social perspective as given in Definition 1 and recall from equation (5) that we can express it as follows

$$\tilde{y} = \left[(\beta_1 + b)\, k_y^{-\rho_y} + (\beta_2 - b)\right]^{-\frac{1}{\rho_y}}. \tag{22}$$

According to $b \gtreqless \beta_2$, the following inequality holds: for any $\eta > 1$,

$$\begin{aligned}\left[(\beta_1 + b)(\eta k_y)^{-\rho} + (\beta_2 - b)\right]^{-\frac{1}{\rho}} &\gtreqless \left[(\beta_1 + b)(\eta k_y)^{-\rho} + \eta^{-\rho}(\beta_2 - b)\right]^{-\frac{1}{\rho}} \\ &= \eta\left[(\beta_1 + b)\, k_y^{-\rho} + (\beta_2 - b)\right]^{-\frac{1}{\rho}}\end{aligned}$$

If b is larger than β_2, the function $\tilde{y}$ exhibits increasing returns. If b is smaller than β_2 the function $\tilde{y}$ exhibits decreasing returns. Note that larger values of b contribute to have total instability.

Next we present our results assuming that the capital good is labor intensive from the private perspective, i.e. $\alpha_2\beta_1 - \alpha_1\beta_2 < 0$. This case results in local indeterminacy. By rewriting equation (21), the characteristic roots are,

$$\begin{aligned}\lambda_1 &= -\frac{1}{(\delta\beta_2)^{\frac{1}{1+\rho}}\left[\left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{1}{1+\rho}} - \left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{1+\rho}}\right]}, \\ \lambda_2(b) &= -\frac{(\delta\beta_2)^{\frac{1}{1+\rho}}\left[\frac{\beta_2 - b}{\beta_2}\left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{1}{1+\rho}} - \frac{\beta_1 + b}{\beta_1}\left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{1+\rho}}\right]}{\delta}.\end{aligned} \tag{23}$$

[4] Note that $\delta_1 = \alpha_2\left[(\alpha_2\beta_1)^{\frac{1}{1+\rho}} - (\alpha_1\beta_2)^{\frac{1}{1+\rho}}\right]^{-1-\rho} > \frac{\alpha_2}{(\alpha_2\beta_1)} = \frac{1}{\beta_1} > 1$.

To get $\lambda_1 \in (-1, 0)$, we need however to suppose a slightly stronger condition than simply ensuring the capital good sector to be labor intensive from the private perspective. The capital intensity difference $\alpha_1\beta_2 - \alpha_2\beta_1$ needs to be large enough and the discount factor has to be close enough to 1.

Proposition 4 *Assume that* $(\alpha_1\beta_2)^{\frac{1}{1+\rho}} - (\alpha_2\beta_1)^{\frac{1}{1+\rho}} > \alpha_2^{\frac{1}{1+\rho}}$ *and* $\delta \in (\delta_3, 1)$ *with* $\delta_3 = \beta_2^{-1}\left[(\beta_1/\beta_2)^{\frac{1}{1+\rho}} - (\alpha_1/\alpha_2)^{\frac{1}{1+\rho}}\right]^{-1-\rho} < 1$. *Then the following cases hold:*

(i) If $\rho \in (-1, 0)$, *there exist* $\underline{b}(\delta) \in (0, \beta_2)$ *and* $\overline{b}(\delta) > \beta_2$ *such that the steady state is a saddle point for* $b \in (0, \underline{b}(\delta)) \cup (\overline{b}(\delta), +\infty)$ *and is locally indeterminate for* $b \in \left(\underline{b}(\delta), \overline{b}(\delta)\right)$.

(ii) If $\rho = 0$, *the steady state is a saddle point for* $b \in (0, \beta_2 - 2\alpha_2) \cup (\beta_2, +\infty)$ *and is locally indeterminate for* $b \in (\beta_2 - 2\alpha_2, \beta_2)$.

(iii) If $\rho > 0$, *there are* $\underline{b}(\delta)$ *and* $\overline{b}(\delta)$ *in* $(0, \beta_2)$ *such that the steady state is a saddle point for* $b \in (0, \underline{b}(\delta)) \cup (\overline{b}(\delta), +\infty)$ *and is locally indeterminate for* $b \in \left(\underline{b}(\delta), \overline{b}(\delta)\right)$.

Proof. If $(\alpha_1\beta_2)^{\frac{1}{1+\rho}} - (\alpha_2\beta_1)^{\frac{1}{1+\rho}} > \alpha_2^{\frac{1}{1+\rho}}$ and $\delta \in (\delta_3, 1)$, then $-1 < \lambda_1 < 0$. The size of $\lambda_2(b)$ is determined in the following way. Recall that $\lambda_2(b)$ is increasing in b. For $b = 0$, $\lambda_2(0) = 1/\delta\lambda_1 < -1$ by the above hypothesis and for $b = \beta_2$, $\lambda_2(\beta_2) = (\delta\beta_1)^{\frac{-\rho}{1+\rho}}$.

(i) If $-1 < \rho < 0$, $\lambda_2(\beta_2) < 1$. Therefore there exist $\underline{b}(\delta) \in (0, \beta_2)$ and $\overline{b}(\delta) > \beta_2$ such that $\lambda_2 < -1$ for $b \in (0, \underline{b}(\delta))$, $-1 < \lambda_2 < 1$ for any $b \in \left(\underline{b}(\delta), \overline{b}(\delta)\right)$ and $\lambda_2 > 1$ for any $b > \overline{b}(\delta)$.

(ii) If $\rho = 0$, $\lambda_2(\beta_2) = 1$. Therefore $\lambda_2(b) < -1$ for $b \in (0, \beta_2 - 2\alpha_2)$, $-1 < \lambda_2(b) < 1$ for $b \in (\beta_2 - 2\alpha_2, \beta_2)$ and $\lambda_2(b) > 1$ for $b > \beta_2$.

(iii) If $\rho > 0$, $\lambda_2(\beta_2) > 1$. Therefore there exist $\underline{b}(\delta)$ and $\overline{b}(\delta)$ in $(0, \beta_2)$ such that $\lambda_2(b) < -1$ for $b \in (0, \underline{b}(\delta))$, $-1 < \lambda_2(b) < 1$ for $b \in \left(\underline{b}(\delta), \overline{b}(\delta)\right)$, and $\lambda_2(b) > 1$ for $b > \overline{b}(\delta)$. ■

Notice that intermediary values for b are necessary to get local indeterminacy. Indeed, values of b that are too small or too large imply saddle-point stability.

Remark 4 In Proposition 4 we consider restrictions implying that the first characteristic root λ_1 is negative and larger than -1. These restrictions also imply that the second characteristic root is such that $\lambda_2(0) < -1$. As the characteristic root $\lambda_2(b)$ is an increasing function of b, a large enough (but not too large) amount of externalities ensures the existence of local indeterminacy. Building on Remark 2, we also derive that local indeterminacy is more likely in our framework than in that of Nishimura and Venditti [5] in the sense that their framework requires a larger amount of externalities.

Next we still assume that capital goods are labor intensive from the private perspective with $\alpha_2\beta_1 - \alpha_1\beta_2 < 0$, but we make λ_1 an unstable root, i.e. $\lambda_1 < -1$.

Two cases need to be considered: $(\alpha_1\beta_2)^{\frac{1}{1+\rho}} - (\alpha_2\beta_1)^{\frac{1}{1+\rho}} > \alpha_2^{\frac{1}{1+\rho}}$ and $\delta \in (0, \delta_3)$, as well as $(\alpha_1\beta_2)^{\frac{1}{1+\rho}} - (\alpha_2\beta_1)^{\frac{1}{1+\rho}} < \alpha_2^{\frac{1}{1+\rho}}$.

Proposition 5 *Suppose that the capital good sector is labor intensive from the private perspective and let* $\delta_4 = \beta_2^{\frac{1}{\rho}} \left[(\beta_1/\beta_2)^{\frac{1}{1+\rho}} - (\alpha_1/\alpha_2)^{\frac{1}{1+\rho}} \right]^{\frac{1+\rho}{\rho}}$.

1 - If $(\alpha_1\beta_2)^{\frac{1}{1+\rho}} - (\alpha_2\beta_1)^{\frac{1}{1+\rho}} > \alpha_2^{\frac{1}{1+\rho}}$ *and* $\delta \in (0, \delta_3)$, *the following results hold:*

(i) *Let* $\rho \in (-1, 0)$. *If* $\delta \in (0, \delta_4)$, *then there exist* $\underline{b}(\delta) \in (0, \beta_2)$ *and* $\overline{b}(\delta) > \beta_2$ *such that the steady state is a saddle point for* $b \in \left(\underline{b}(\delta), \overline{b}(\delta)\right)$ *and is totally unstable for* $b \in (0, \underline{b}(\delta)) \cup (\overline{b}(\delta), +\infty)$. *If* $\delta \in (\delta_4, \delta_3)$, *then there exists* $\overline{b}(\delta) > \beta_2$ *such that the steady state is a saddle point for* $b \in \left(0, \overline{b}(\delta)\right)$ *and is totally unstable for* $b > \overline{b}(\delta)$.

(ii) *Let* $\rho = 0$. *Then the steady state is a saddle point for* $b \in (\beta_2 - 2\alpha_2, \beta_2)$ *and is totally unstable for* $b \in (0, \beta_2 - 2\alpha_2) \cup (\beta_2, +\infty)$.

(iii) *Let* $\rho > 0$. *Then there exist* $\underline{b}(\delta), \overline{b}(\delta) \in (0, \beta_2)$ *such that the steady state is a saddle point for* $b \in \left(\underline{b}(\delta), \overline{b}(\delta)\right)$ *and is totally unstable for* $b \in (0, \underline{b}(\delta)) \cup (\overline{b}(\delta), +\infty)$.

2 - If $(\alpha_1\beta_2)^{\frac{1}{1+\rho}} - (\alpha_2\beta_1)^{\frac{1}{1+\rho}} < \alpha_2^{\frac{1}{1+\rho}}$, *the following results hold:*

(i) *Let* $\rho \in (-1, 0)$. *Then there exists* $\overline{b}(\delta) > \beta_2$ *such that the steady state is saddle point for* $b \in \left(0, \overline{b}(\delta)\right)$ *and totally unstable for* $b > \overline{b}(\delta)$.

(ii) *Let* $\rho = 0$. *Then the steady state is saddle point for* $b \in (0, \beta_2)$ *and totally unstable for* $b > \beta_2$.

(iii) *Let* $\rho > 0$. *If* $\delta \in (0, \delta_4)$, *then there exist* $\underline{b}(\delta) \in (0, \beta_2)$ *and* $\overline{b}(\delta) > \beta_2$ *such that the steady state is saddle point for* $b \in \left(\underline{b}(\delta), \overline{b}(\delta)\right)$ *and totally unstable for* $b \in (0, \underline{b}(\delta)) \cup (\overline{b}(\delta), +\infty)$. *If* $\delta \in (\delta_4, 1)$, *then there exists* $\overline{b}(\delta) > \beta_2$ *such that the steady state is saddle point for* $b \in \left(0, \overline{b}(\delta)\right)$ *and totally unstable for* $b > \overline{b}(\delta)$.

Proof. We assume here that $\lambda_1 < -1$. Recall that $\lambda_2(b)$ increases with b. Moreover, for $b = 0$, we get $\lambda_2(0) = -\delta^{-\frac{\rho}{1+\rho}} \delta_3^{-\frac{1}{1+\rho}}$, with δ_3 as defined in Proposition 4, and for $b = \beta_2$, we get $\lambda_2(\beta_2) = (\delta\beta_1)^{\frac{-\rho}{1+\rho}}$. We easily get $\lambda_2(0) \in (-1, 0)$ if and only if $\delta^\rho > \delta_3^{-1}$. As shown above, two cases have to be considered:
1 - Let $(\alpha_1\beta_2)^{\frac{1}{1+\rho}} - (\alpha_2\beta_1)^{\frac{1}{1+\rho}} > \alpha_2^{\frac{1}{1+\rho}}$ and $\delta \in (0, \delta_3)$ with $\delta_3 < 1$.

(i) When $-1 < \rho < 0$, we get $\lambda_2(\beta_2) < 1$, and $\lambda_2(0) < -1$ if and only if $\delta < \delta_3^{-\frac{1}{\rho}} \equiv \delta_4$. It follows then that $\delta_4 < \delta_3$ as $\delta_3^{-\frac{1+\rho}{\rho}} < 1$. We conclude that $\lambda_2(0) < -1$ if $0 < \delta < \delta_4$, while $\lambda_2(0) > -1$ if $\delta \in (\delta_4, \delta_3)$. Therefore if δ lies in $(0, \delta_4)$, there exist $\underline{b}(\delta) \in (0, \beta_2)$ and $\overline{b}(\delta) > \beta_2$ such that $\lambda_2(b) < -1$ for $b \in (0, \underline{b}(\delta))$, $-1 < \lambda_2(b) < 1$ for $b \in \left(\underline{b}(\delta), \overline{b}(\delta)\right)$, and $\lambda_2(b) > 1$ for $b > \overline{b}(\delta)$. If δ lies in (δ_4, δ_3), there exists $b(\delta) > \beta_2$ such that $-1 < \lambda_2(b) < 1$ for $b \in (0, b(\delta))$ and $\lambda_2(b) > 1$ for $b > b(\delta)$.

(ii) If $\rho = 0$, $\lambda_2(0) < -1$ and $\lambda_2(\beta_2) = 1$. Therefore $-1 < \lambda_2(b) < 1$ for $b \in (\beta_2 - 2\alpha_2, \beta_2)$ and $\lambda_2(b) > 1$ for $b \in (0, \beta_2 - 2\alpha_2)$ or $b > \beta_2$.

(iii) If $\rho > 0$, then $\delta_4 > \delta_3$ and thus $\lambda_2(0) < -1$ for any $\delta \in (0, \delta_3)$. Moreover $\lambda_2(\beta_2) > 1$. Therefore, there exist $\underline{b}(\delta)$ and $\bar{b}(\delta)$ in $(0, \beta_2)$ such that $\lambda_2(0) < -1$ for $b \in (0, \underline{b}(\delta))$, $-1 < \lambda_2(b) < 1$ for $b \in \left(\underline{b}(\delta), \bar{b}(\delta)\right)$ and $\lambda_2(0) > 1$ for $b > \bar{b}(\delta)$.

2 - Let $(\alpha_1\beta_2)^{\frac{1}{1+\rho}} - (\alpha_2\beta_1)^{\frac{1}{1+\rho}} < \alpha_2^{\frac{1}{1+\rho}}$. In this case we have $\delta_3 > 1$.

(i) When $-1 < \rho < 0$, $\lambda_2(\beta_2) < 1$. Moreover, we get $\delta^\rho > 1 > \delta_3^{-1}$ so that $\lambda_2(0) > -1$. Then there exists $\bar{b}(\delta) > \beta_2$ such that $-1 < \lambda_2(b) < 1$ for $b \in \left(0, \bar{b}(\delta)\right)$, and $\lambda_2(b) > 1$ for $b > \bar{b}(\delta)$.

(ii) If $\rho = 0$, $-1 < \lambda_2(0) < 0$ and $\lambda_2(\beta_2) = 1$. Therefore $-1 < \lambda_2(b) < 1$ for $b \in (0, \beta_2)$ and $\lambda_2(b) > 1$ for $b > \beta_2$.

(iii) If $\rho > 0$, then $\lambda_2(0) > -1$ if and only if $\delta > \delta_3^{-\frac{1}{\rho}} \equiv \delta_4$ with $\delta_4 < 1$ and thus $\lambda_2(0) < -1$ for any $\delta \in (0, \delta_4)$. Moreover $\lambda_2(\beta_2) > 1$. Therefore if δ lies in $(0, \delta_4)$, then there exist $\underline{b}(\delta) \in (0, \beta_2)$ and $\bar{b}(\delta) > \beta_2$ such that $\lambda_2(b) < -1$ for $b \in (0, \underline{b}(\delta))$, $-1 < \lambda_2(b) < 1$ for $b \in \left(\underline{b}(\delta), \bar{b}(\delta)\right)$, and $\lambda_2(b) > 1$ for $b > \bar{b}(\delta)$. If δ lies in $(\delta_4, 1)$, there exists $b(\delta) > \beta_2$ such that $-1 < \lambda_2(b) < 1$ for $b \in (0, b(\delta))$ and $\lambda_2(b) > 1$ for $b > b(\delta)$. ■

Notice that the impact of b on the local stability properties of the steady state depends on the value of the discount factor δ. If δ is not too small, larger values of b contributes to total instability as in Proposition 3 (see cases 1-i) with $\delta \in (\delta_4, \delta_3)$ and 2-iii) with $\delta \in (\delta_4, 1)$ in Proposition 5). On the contrary, if δ is close enough to zero, local instability is obtained for low enough and large enough values of b.

4 Concluding Remarks

In this paper we have characterized the local dynamics of the equilibrium paths depending on the size of the external effects b. We have shown that when the investment good is capital intensive, large values of b contributes to total instability. When the consumption good is capital intensive, the effect of b on the local dynamics of the equilibrium path is more complex. If the capital intensity difference is large enough, local indeterminacy occurs for intermediary values of b while saddle-point stability is obtained when b is low enough or large enough. If the capital intensity difference is small, local indeterminacy cannot occur and the role of b depends on the size of the discount factor. When the discount factor is not too low, larger values of b again contributes to total instability. However, when the discount factor is close enough to zero, total instability is also obtained for small values of b. In this case indeed, saddle-point stability requires intermediary values of b.

In this paper, we have assumed that both sectors have the same elasticity of capital-labor substitution. It would be interesting to study the characterization of the equilibrium paths by introducing heterogeneity.

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