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Résumé: Cet article analyse l'équilibre d'un marché d'assurances où les individus qui souscrivent une police d'assurance ont une obligation de bonne foi lorsqu'ils révèlent une information privée sur leur risque. Les assureurs peuvent, à un certain coût, vérifier le type des assurés qui présentent une demande d'indemnité et ils sont autorisés à annuler rétroactivement le contrat d'assurance s'il est établi que l'assuré avait présenté son risque de manière incorrecte lorsqu'il avait souscrit la police d'assurance. Toutefois les assureurs ne peuvent s'engager sur leur stratégie de vérification du risque. L'article analyse la relation entre l'optimalité de Pareto de second rang et l'équilibre concurrentiel du marché de l'assurance dans un cadre de théorie des jeux. Il caractérise les contrats offerts à l'équilibre, les choix de contrat par les individus ainsi que les conditions d'existence de l'équilibre.

Abstract: This paper analyzes the equilibrium of an insurance market where applicants for insurance have a duty of good faith when they reveal private information about their risk type. Insurers can, at some cost, verify the type of insureds who file a claim and they are allowed to retroactively void the insurance contract if it is established that the policyholder has misrepresented his risk when the contract was taken out. However, insurers cannot precommit to their risk verification strategy. The paper analyzes the relationship between second-best Pareto-optimality and the insurance market equilibrium in a game theoretic framework. It characterizes the contracts offered at equilibrium, the individuals' contract choice as well as the conditions under which an equilibrium exists.

Mots clés : Assurance, Asymétrie d'information, Bonne foi, Vérification du risque, Crédibilité.

Key Words : Insurance, Asymmetric information, Good faith, Risk verification, Credibility.

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1 Introduction

Under the law of contracts, an insurer is bound by the provisions of an insurance policy insofar as the policyholder has not deliberately concealed relevant information about his risks when the insurance was taken out. The contract is automatically rescinded in case of risk misrepresentation or non-disclosure of material facts affecting risk unless the policyholder's good faith is established[1].

Dixit (2000) studies the consequences of the duty of good faith in the setting of a competitive insurance market à la Rothschild-Stiglitz (1976). He shows that the good faith principle achieves a Pareto improvement by allowing the insurers to better separate low risk individuals from high risks ones. If verifying the accident probability is not too costly, then a random *ex post* investigation should be carried out when an alleged low risk individual files a claim, no indemnity being paid to a policyholder caught lying. Dixit also shows that a larger insurance indemnity should be paid to a (truthful) low risk individual in case of verification than when the claim is not verified. Furthermore, the good faith principle extends the range of high risk and low risk proportions for which a competitive equilibrium exists. Dixit and Picard (2003) extend Dixit's results to a setting where individuals may have only partial information about their risk level : they only perceive a signal of their risk. Bad faith and good faith then respectively correspond to intentional or unintentional misrepresentation of risk and insurers can verify risk type, perceived signal or both.

In these papers, insurers commit to a random investigation policy and all policyholders reveal their information truthfully at equilibrium : in other words, they are all in good faith. Consequently, when verification is costly, insurers may be tempted not to verify the policyholders' types or perceived signals with the preannounced frequency, which implies that the insurers' verification strategy is weakened by credibility problems. The present paper will focus attention on this issue of the credibility of the insurers' verification strategy when insurance applicants have a duty of good faith.

It is hardly likely that full commitment on a verification strategy can be recovered thanks to repeated relationships. Firstly, the duration of an insurer-policyholder relationship is finite and random. In particular, an increase in the customers' turnover (i.e. a larger probability to quit at each period of time) is equivalent to an increase in the discount rate : a large turnover rate will prevent the insurer to reach a full commitment. Secondly, for a given policyholder the frequency of an accident is usually too low for commitment to be sustainable in a long run relationship. This is all the more likely because the optimal verification strategy is probabilistic which makes the detection of deviations even more difficult. Thirdly, a policyholder usually has imperfect informa-

[1]See Clarke (1997) on the duty of good faith in the law of insurance contracts. Colquitt and Hoyt (1997) show that most of the reasons provided by US life insurers for resisting claims are linked to the bad faith of insured, mainly material risk misrepresentation, hidden preexisting condition, misstatement of age or of medical history.

tion about the verification frequency of other policyholders which reduces the insurers' ability to build a reputation for frequent auditing.

In this paper, to make the problem more easily tractable, but also to look at the good faith principle from the *a priori* less favorable point of view, we will consider a one-shot insurer-policyholder relationship in which any commitment through repeated relationship is dismissed. The setting is similar to the model of Rothschild and Stiglitz (1976). There is a large number of risk averse individuals who have private information on their accident probability : there are low risk individuals and high risk individuals. These individuals seek for insurance on a competitive market. Each insurer offers a menu of two contracts, one of them being reserved for low risk individuals. Individuals choose the contract they prefer. A high risk individuals may lie: he may announce that he is a low risk in order to benefit from cheaper insurance. Insurers may carry out a costly verification of the risk type of alleged low risk individuals who file a claim. If investigation reveals that the individual was not truthful, then the good faith principle allows the insurer to canceal the contract and to deny any indemnity. The remaining player is nature : it chooses the risk type of each individual and whether he has an accident or not. Insurers, individuals and nature play a multistage game whose (perfect Bayesian) equilibrium is the insurance market equilibrium. In particular, at equilibrium the insurer's verification probability is the best response to the policyholder's contract choice strategy.

Our first stage will be to characterize the second-best Pareto optimal allocations, i.e. the allocations that are efficient under the additional constraints imposed by the asymmetry of information between insureds and insurers and by the unability of insurers to commit on their auditing strategy. This will be an important step toward the characterization of the market equilibrium because, in the present model, the equilibrium allocation (i.e. the players' decisions on the equilibrium path of the insurance market game) are second-best Pareto optimal[2]. It will be shown that second-best Pareto optimal allocations, and in particular equilibrium allocations, are symmetric, in the sense that generically all insurers offer the same menu of contract. Furthermore, at equilibrium there is no cross-subsidization between contracts. These results will require refinements of the Perfect Bayesian Equilibrium concept (trembling-hand perfection and a Markov-type restriction on the low risk individual strategy), but from the point of view of realism we really think that these are innocuous restrictions and that they do not really affect the relevance of the results. Nevertheless, for the sake of completeness, we will show through examples that the correspondance between second-best Pareto-optimal allocations and equilibrium allocations vanishes when the refinements are not postulated.

[2]Readers who are familiar with the Rothschild-Stiglitz (1976) model may be surprised at such a statement because, in the basic version of the Rothschild-Stiglitz model, the equilibrium allocation is not necessarily second-best Pareto-optimal. Indeed in this version, each insurer only offers one contract and at equilibrium insurers specialize in high risk or low risk customers. When each insurer can offer a menu, as in the present model, then the Rothschild-Stiglitz equilibrium is second-best Pareto optimal.

From a methodological standpoint, our model shows that the characterization of second-best Pareto optimal allocations is a fruitful roundabout means to analyze the competitive market equilibrium. Hopefully this is a result of general interest for other markets with adverse selection where players (e.g. lenders and borrowers, employers and employees,...) interact after the initial contract offer by the uninformed parties. Concerning the analysis of the insurance market itself (under the refined equilibrium concept), our main results are the following. Firstly, the good faith principle is still Pareto-improving in this no-commitment setting. Secondly, an equilibrium exists for a larger set of parameters than in the standard Rothschild-Stiglitz model. Thirdly, the equilibrium may be separating or semi-separating. At a separating equilibrium, different types purchase different contracts : there is full coverage for high risks and partial coverage for low risks and no auditing is implemented at equilibrium. A separating equilibrium in fact coincides with the Rothschild-Stiglitz equilibrium: high-risk individuals are indifferent between buying full insurance at fair premium and choosing the insurance contract which is intended for low-risks individuals. By contrast, at a semi-separating equilibrium, high-risks randomize between both contracts (we may say that they are in bad faith with positive probability) and the risk type is verified with positive probability for alleged low risk individuals who have filed a claim. Furthermore, a separating equilibrium involves partial coverage for low risks individuals (as in the Rothschild-Stiglitz model) but they are overinsured at a semi-separating equilibrium. Fourthly, we shall define the conditions of validity of the different regimes that may prevail (separating equilibrium, semi-separating equilibrium or no equilibrium) according to the values of two parameters : the fraction of high risk individuals in the population and the cost of risk type verification.

The paper is organized as follows. Section 2 sets out the model. Section 3 introduces the definition and the basic properties of an insurance market equilibrium with risk verification. Section 4 characterizes second-best Pareto-optimal allocations. Section 5 provides our results about the existence and the features of a market equilibrium. Section 6 concludes. The proofs are gathered in the appendix.

2 The model

We consider a large population represented by a continuum of individuals facing idiosyncratic risks of accident. All individuals are risk averse : they maximize the expected utility of wealth $u(W)$, where W denotes wealth and the (twice continuously differentiable) utility function u is such that $u' > 0$ and $u'' < 0$. If no insurance policy is taken out, we have $W = W_N$ in the no-accident state and $W = W_A$ in the accident state; $A = W_N - W_A$ is the loss from an accident. Individuals differ according to their probability of accident π and they have private information on their own accident probability. We have $\pi = \pi_\ell$ for a low-risk individual (or ℓ-type) and $\pi = \pi_h$ for a high-risk individual (or h-type) with $0 < \pi_\ell < \pi_h < 1$. The fraction of high-risk individuals is λ with $0 < \lambda < 1$ and $\overline{\pi} = \lambda\pi_h + (1-\lambda)\pi_\ell$ is the average probability of loss.

Insurance contracts are offered by n insurers ($n \geq 2$) indexed by $i = 1, ..., n$ and we assume that each individual can buy only one contract. An insurance contract is written as (k, x) where k is the insurance premium and x is the net payout in case of an accident. Hence $x + k$ is the indemnity. The expected utility of a policyholder is then written as

$$Eu = (1-\pi)u(W_N - k) + \pi u(W_A + x), \tag{1}$$

where $\pi \in \{\pi_\ell, \pi_h\}$. $C_\ell^* = (k_\ell^*, x_\ell^*) = (\pi_\ell A, A - \pi_\ell A)$ and $C_h^* = (k_h^*, x_h^*) = (\pi_h A, A - \pi_h A)$ are the actuarially fair full insurance contracts, respectively for low risk and high risk.

Let us begin with a brief presentation of the Rothschild-Stiglitz (1976) model. An equilibrium in the sense of Rothschild and Stiglitz consists of a set of contracts such that, when individuals choose contracts to maximize expected utility, (i): Each contract in the equilibrium set makes non-negative expected profit, and (ii): There is no contract outside the equilibrium set that, if offered in addition to those in the equilibrium set, would make strictly positive expected profits. This concept of equilibrium may be understood as a pure strategy subgame perfect equilibrium of a game where insurers simultaneously offer contracts and individuals respond by choosing one of the contracts (or refusing them all). At equilibrium, each contract makes zero profit and there is no profitable deviation at the contract offering stage, given the subsequent reaction of the insurance purchasers.

Rothschild and Stiglitz show that there cannot be a pooling equilibrium where both groups would buy the same contract. Only a separating equilibrium can exist : different types then choose different contracts. Rothschild and Stiglitz establish that the only candidate separating equilibrium is such that high risk individuals purchase full insurance at fair price, i.e. they choose C_h^*, and low risk individuals purchase a contract C_ℓ^{**} with partial coverage. C_ℓ^{**} is the contract that low risk individuals most prefer in the set of (fairly priced) contracts that do not attract high risk individuals: $C_\ell^{**} = (k_\ell^{**}, x_\ell^{**}) = (\pi_\ell A', A' - \pi_\ell A')$ with $A' \in (0, A)$ given by

$$u(W_N - \pi_h A) = (1-\pi_h)u(W_N - \pi_\ell A') + \pi_h u(W_A + (1-\pi_\ell)A'). \tag{2}$$

The Rothschild-Stiglitz equilibrium is illustrated in Figure 1, with state-dependent wealth on each axis[3]. $W^1 = W_N - k$ and $W^2 = W_A + x$ respectively denote final wealth in the no-accident state and in the accident state. The no-insurance situation corresponds to point E with coordinates $W^1 = W_N$ and $W^2 = W_A$. The high risk and low risk fair-odds line go through E, with slopes (in absolute value) respectively equal to

[3]Because no ambiguity may occur, we use the same notation for insurance contracts (k, x) and their images in the (W^1, W^2) plane.

$\pi_h/1-\pi_h$ and $\pi_\ell/1-\pi_\ell$. At C_h^* the h-type indifference curve is tangent to the high risk fair-odds line EH. Similarly, C_ℓ^* is at a tangency point between a ℓ-type indifference curve and the low risk fair-odds line EL. C_ℓ^{**} is at the intersection between EL and the high-risk indifference curve that goes through C_h^*. EA in Figure 1 corresponds to the average fair-odds line with slope $\overline{\pi}/1-\overline{\pi}$.

Figure 1

Rothschild and Stiglitz also show that the candidate equilibrium C_h^*, C_ℓ^{**} is actually an equilibrium (in the sense of the above definition) if and only if λ is larger than a threshold $\widehat{\lambda}$, with $\widehat{\lambda} \in (0,1)$. When $\lambda = \widehat{\lambda}$, the low-risk indifference curve that goes through C_ℓ^{**} is just tangent to EA. When $\lambda < \widehat{\lambda}$, there exist contracts that, if offered in addition to C_h^*, C_ℓ^{**}, would attract high and low-risk individuals and that would make a positive expected profit. Hence, an equilibrium in the sense of Rothschild and Stiglitz only exists if $\lambda \geq \widehat{\lambda}$ as represented in Figure 1.

The above given definition of an equilibrium assumes that each insurer can only offer one contract. At equilibrium some insurers offer C_h^* and others offer C_ℓ^{**}. When insurers are allowed to offer a menu of contract, which is certainly a more realistic assumption, then the definition of an equilibrium in the sense of Rothschild and Stiglitz consists of a set of menus that break even on average, such that there is no menu of contracts outside the equilibrium set that, if offered in addition, would make strictly positive expected profits. At an equilibrium, the menu C_h^*,C_ℓ^{**} is offered by all insurers: h-types choose C_h^* and ℓ-types choose C_ℓ^{**}. Hence the set of equilibrium contracts is unchanged. However, the possibility of offering a menu increases the critical proportion of high risk individuals above which an equilibrium exist: there exists λ^* in $(0,1)$, with $\lambda^* > \widehat{\lambda}$ such that an equilibrium exists if and only if $\lambda \geq \lambda^*$[4].

In what follows, we will modify the Rothschild-Stiglitz model by considering the consequences of the good faith principle. Applicants for insurance have a duty of good faith, which stipulates that they should reveal their risk type truthfully and provides that if an investigation reveals that a high risk individual passed himself off as a low risk, then the insurance contract may be rescinded. It will be assumed that no third party can verify whether a risk type investigation has actually been carried out, except when risk misrepresentation has been established. In other words, only the proof of risk misrepresentation is verifiable information. Under this assumption, a supposedly low-risk policyholder receives the same insurance indemnity when the truthfulness of his assertion has been verified by the insurer and when no investigation has been carried out[5]. In such a framework, an insurance contract is still written as (k, x) where k is the insurance premium and x is the net payout in case of an accident, and the expected

[4]The fact that λ^* is larger than $\widehat{\lambda}$ was pointed out by Rothschild and Stiglitz (1976) themselves.

[5]This restriction on the set of admissible contracts is in line with the common practice of insurers.

utility of a truthful policyholder is given by (1).

At the equilibrium of the insurance market, a contract $C_\ell^i = (k_\ell^i, x_\ell^i)$ is offered by insurer i to low-risk individuals only, while another contract $C_h^i = (k_h^i, x_h^i)$ offered by insurer i may be taken out by any individual whatever his type. In case of an accident, a policyholder who has taken out the C_ℓ^i contract will be investigated with probability $p^i \in [0,1]$. Through investigation, the insurer gets information about the type of the policyholder. This information is verifiable if the policyholder has misrepresented his type in which case the insurer voids the contract, which means that no indemnity is paid and the premium is refunded to the policyholder, and the latter pays a fine $F > 0$ to the Government[6]. Verifying the insureds' type costs c to the insurer. Individuals may also opt out of purchasing insurance. For notational simplicity, this no-insurance choice corresponds to an additional (fictitious) insurer $i = 0$, with $C_h^0 = C_\ell^0 \equiv (0,0)$ and $p^0 \equiv 0$.

σ_{hh}^i and $\sigma_{h\ell}^i$ respectively denote the probability for a h-type individual to choose C_h^i or C_ℓ^i and $\sigma_{\ell h}^i$ and $\sigma_{\ell\ell}^i$ respectively denote the probability for a ℓ-type individual to choose C_h^i or C_ℓ^i, with $\sum_{i=0}^n(\sigma_{hh}^i + \sigma_{h\ell}^i) = 1$ and $\sum_{i=0}^n(\sigma_{\ell h}^i + \sigma_{\ell\ell}^i) = 1$. Let $C = (C_h^1, C_\ell^1, ..., C_h^n, C_\ell^n), p = (p^1, ..., p^n), \sigma_h = (\sigma_{hh}^0, \sigma_{h\ell}^0, ..., \sigma_{hh}^n, \sigma_{h\ell}^n)$ and $\sigma_\ell = (\sigma_{\ell h}^0, \sigma_{\ell\ell}^0, .., \sigma_{\ell h}^n, \sigma_{\ell\ell}^n)$. In what follows, $\{C, p, \sigma_h, \sigma_\ell\}$ is called an *allocation*: it is a complete description of the decisions of firms and individuals in the insurance market.

3 Definition and basic properties of a market equilibrium

An equilibrium of the insurance market is a perfect Bayesian equilibrium of a five stage game, called the market game. At stage 1, nature chooses the type of each individual : he is a h-type with probability λ or a ℓ-type with probability $1-\lambda$. At the second stage, each insurer $i > 0$ decides whether to offer a menu of contracts and, if so, she chooses the specification C_h^i, C_ℓ^i of each contract in the menu. At the third stage, each individual decides whether to accept a contract, and if so, which contract in the proposed menus. At the fourth stage, for each individual, nature decides whether an accident occurs or not with probability $(\pi_h, 1-\pi_h)$ or $(\pi_\ell, 1-\pi_\ell)$ according to the individual's type. The insured policyholders who have suffered an accident file a claim. At the fifth stage, each insurer chooses whether or not to verify the type of the alleged low-risk individuals who have filed a claim (insurer i verifies with probability p^i) and, depending on the results of the investigation, she pays the indemnity or returns the premium.

For any insurance contract (k, x), be it a C_h or a C_ℓ contract, the expected utility of ℓ-type individuals is

$$U_\ell(k,x) \equiv (1-\pi_\ell)u(W_N - k) + \pi_\ell u(W_A + x).$$

[6]Assuming $F > 0$ simplifies matters, but F may be arbitrarily small.

The expected utility of h-type individuals is

$$U_h(C_h^i) \equiv (1-\pi_h)u(W_N - k_h^i) + \pi_h u(W_A + x_h^i) \quad \text{for } C_h^i,$$
$$U_{h\ell}(C_\ell^i, p^i) \equiv (1-\pi_h)u(W_N - k_\ell^i) + \pi_h[(1-p^i)u(W_A + x_h^i) + p^i u(W_A - F)] \quad \text{for } C_\ell^i.$$

Let $\overline{U}_h \equiv U_h(0,0), \overline{U}_\ell \equiv U_\ell(0,0), U_h^* \equiv U_h(C_h^*) = u(W_N - \pi_h A)$ and $U_\ell^* \equiv U_\ell(C_\ell^*) = u(W_N - \pi_\ell A)$. For C_h^i, insurer i's expected profit is

$$\Pi_h(C_h^i) = (1-\pi_h)k_h^i - \pi_h x_h^i \quad \text{for a } h\text{-type policyholder},$$
$$\Pi_\ell(C_h^i) = (1-\pi_\ell)k_h^i - \pi_\ell x_h^i \quad \text{for a } \ell\text{-type policyholder}.$$

For a C_ℓ^i, the expected profit is

$$\Pi_{\ell h}(C_\ell^i, p^i) = (1-\pi_h)k_\ell^i - \pi_h[(1-p^i)x_\ell^i + p^i c] \quad \text{for a } h\text{-type policyholder},$$
$$\Pi_{\ell\ell}(C_\ell^i, p^i) = (1-\pi_\ell)k_\ell^i - \pi_\ell(x_\ell^i + p^i c) \quad \text{for a } \ell\text{-type policyholder}.$$

Let us now define the players' strategies and beliefs in the market game. The strategy of insurer $i > 0$ is defined by $C^i \in \mathbb{R}_+^4$ and $p^i(.) : \mathbb{R}_+^{4n} \to [0,1]$ where $C^i = (C_h^i, C_\ell^i)$ and $p^i(C)$ is the audit probability for C_ℓ^i at stage 5 when $C = (C^1, ..., C^n)$ is offered in the market at stage 2. Let $p(.) = (p^1(.), ..., p^n(.))$ be the profile of auditing strategy and $p^0(.) \equiv 0$.

The strategy of h-type individuals is $\sigma_h(.) : \mathbb{R}_+^{4n} \to S^{2n+1}$ where $\sigma_h(C) = (\sigma_{hh}^0(C), \sigma_{h\ell}^0(C), ..., \sigma_{hh}^n(C), \sigma_{h\ell}^n(C))$ describes the h-types' contract choices when C is offered and $S^{2n+1} = \{t = (t_1, t_2, ...t_{2n+2}) \in \mathbb{R}_+^{2n+2}, \sum_{j=1}^{2n+2} t_j = 1\}$ is the $2n+1$ dimensional simplex. In words, a h-type individual chooses C_h^i with probability $\sigma_{hh}^i(C)$ and he chooses C_ℓ^i with probability $\sigma_{h\ell}^i(C)$ when C is offered. Likewise, the strategy of ℓ-types is $\sigma_\ell(.) : \mathbb{R}_+^{4n} \to S^{2n+1}$ where $\sigma_\ell(C) = (\sigma_{\ell h}^0(C), \sigma_{\ell\ell}^0(C), ..., \sigma_{\ell h}^n(C), \sigma_{\ell\ell}^n(C))$ specifies the contract choices by ℓ-type individuals. Hence the profile of strategy is denoted by $\{C, p(.), \sigma_h(.), \sigma_\ell(.)\}$.

Auditing decisions depend on the insurers'beliefs about the policyholders' type. Beliefs depend on the set of contracts that are offered in the market. At stage 5, the beliefs of insurer $i > 0$ are defined by by $\mu^i(.) : \mathbb{R}_+^{4n} \to [0,1]$ where $\mu^i(C)$ is the probability that a C_ℓ^i-claimant is a h-type when C is offered at stage 2. The system of beliefs is denoted by $\mu(.) = (\mu^1(.), ..., \mu^n(.))$. We have assumed that insurers refund premium but do not pay anything else if investigation reveals that an alleged ℓ-type individual was in fact a h-type. Thus, given the beliefs $\mu^i(C)$ and the audit probability p^i, the expected cost of a claim filed by a C_ℓ^i-policyholder is $x_\ell^i + p^i[c - \mu^i(C)x_\ell^i]$.

Definition 1. *A profile of strategies* $\widetilde{C}, \widetilde{p}(.), \widetilde{\sigma}_h(.), \widetilde{\sigma}_\ell(.)$ *and system of beliefs* $\widetilde{\mu}(.)$ *is a*

Perfect Bayesian Equilibrium of the market game if it has the following properties:

$$\sum_{i=0}^{n}[\widetilde{\sigma}^i_{hh}(C)U_h(C^i_h)+\widetilde{\sigma}^i_{h\ell}(C)U_{h\ell}(C^i_\ell,\widetilde{p}^i(C))] \geq \sum_{i=0}^{n}[\sigma^i_{hh}U_h(C^i_h)+\sigma^i_{h\ell}U_{h\ell}(C^i_\ell,\widetilde{p}^i(C))]$$
$$\textit{for all } \sigma_h \in S^{2n+1} \textit{ and all } C \in \mathbb{R}^{4n}_+, \tag{3}$$

$$\sum_{i=0}^{n}[\widetilde{\sigma}^i_{\ell h}(C)U_\ell(C^i_h)+\widetilde{\sigma}^i_{\ell\ell}(C)U_\ell(C^i_\ell)] \geq \sum_{i=0}^{n}[\sigma^i_{\ell h}U_\ell(C^i_h)+\sigma^i_{\ell\ell}U_\ell(C^i_\ell)]$$
$$\textit{for all } \sigma_\ell \in S^{2n+1} \textit{ and all } C \in \mathbb{R}^{4n}_+, \tag{4}$$

$$\widetilde{p}^i(C)[\widetilde{\mu}^i(C)x^i_\ell - c] \geq p^i[\widetilde{\mu}^i(C)x^i_\ell - c]$$
$$\textit{for all } p^i \in [0,1], \textit{ all } C \in \mathbb{R}^{4n}_+ \textit{ and all } i=1,...,n, \tag{5}$$

$$\begin{aligned}
&\lambda[\widetilde{\sigma}^i_{hh}(\widetilde{C})\Pi_h(\widetilde{C}^i_h)+\widetilde{\sigma}^i_{h\ell}(\widetilde{C})\Pi_{\ell h}(\widetilde{C}^i_\ell,\widetilde{p}^i(\widetilde{C}))]\\
&+(1-\lambda)[\widetilde{\sigma}^i_{\ell h}(\widetilde{C})\Pi_\ell(\widetilde{C}^i_h)+\widetilde{\sigma}^i_{\ell\ell}(\widetilde{C})\Pi_{\ell\ell}(\widetilde{C}^i_\ell,\widetilde{p}^i(\widetilde{C}))]\\
&\geq \lambda[\widetilde{\sigma}^i_{hh}(C^i,\widetilde{C}^{-i})\Pi_h(C^i_h)+\widetilde{\sigma}^i_{h\ell}(C^i,\widetilde{C}^{-i})\Pi_{\ell h}(C^i_\ell,\widetilde{p}^i(C^i,\widetilde{C}^{-i}))]\\
&+(1-\lambda)[\widetilde{\sigma}^i_{\ell h}(C^i,\widetilde{C}^{-i})\Pi_\ell(C^i_h)+\widetilde{\sigma}^i_{\ell\ell}(C^i,\widetilde{C}^{-i})\Pi_{\ell\ell}(C^i_\ell,\widetilde{p}^i(C^i,\widetilde{C}^{-i}))\\
&\textit{for all } C^i=(C^i_h,C^i_\ell)\in\mathbb{R}^4_+ \textit{ and all } i=1,...,n,
\end{aligned} \tag{6}$$

$$\widetilde{\mu}^i(C) = \frac{\lambda\pi_h\widetilde{\sigma}^i_{h\ell}(C)}{\lambda\pi_h\widetilde{\sigma}^i_{h\ell}(C)+(1-\lambda)\pi_\ell\widetilde{\sigma}^i_{\ell\ell}(C)}$$
$$\textit{for all } C \textit{ such that } \widetilde{\sigma}^i_{h\ell}(C)+\widetilde{\sigma}^i_{\ell\ell}(C)>0 \textit{ and all } i=1,...,n. \tag{7}$$

(3) means that $\widetilde{\sigma}_h(.)$ is an optimal contract choice strategy for h-types, given the profile of insurers' auditing strategy. (4) says that $\widetilde{\sigma}_\ell(.)$ is an optimal contract choice strategy for ℓ-types. From (5), $\widetilde{p}^i(.)$ is an optimal auditing strategy given insurer i's beliefs. Together (3),(4) and (5) mean that for any contract offer C made at stage 2, then $\{\widetilde{\sigma}_h(C),\widetilde{\sigma}_\ell(C),\widetilde{p}(C)\}$ is a Nash equilibrium of the corresponding continuation subgame, given beliefs $\widetilde{\mu}(C)$. (6) means that $\widetilde{C}^i$ is an optimal offer by insurer i when $\widetilde{C}^{-i}=(\widetilde{C}^1,...,\widetilde{C}^{i-1},\widetilde{C}^{i+1},...,\widetilde{C}^n)$ is offered by the other insurers, given the continuation equilibrium strategy. In other words, (3)-(6) says that the strategy profile $\widetilde{C},\widetilde{p}(.),\widetilde{\sigma}_h(.),\widetilde{\sigma}_\ell(.)$ is sequentially rational given the belief system $\widetilde{\mu}(.)$. Finally (7) states that $\widetilde{\mu}(.)$ is derived from strategy profile $\widetilde{\sigma}_h(.),\widetilde{\sigma}_\ell(.)$ through Bayes law whenever possible. For brevity, in what follows a Perfect Bayesian Equilibrium of the market game is simply called an equilibrium (or, more explicitly, an insurance market equilibrium). Lemmas 1 and 2

establish basic properties of any equilibrium.

Lemma 1. *At an equilibrium* $\mathcal{E} = \{\widetilde{C}, \widetilde{p}(.), \widetilde{\sigma}_h(.), \widetilde{\sigma}_\ell(.), \widetilde{\mu}(.)\}$, *for all* $C \in \mathbb{R}_+^{4n}$ *and all* i *such that* $\widetilde{\sigma}^i_{h\ell}(C) + \widetilde{\sigma}^i_{\ell\ell}(C) > 0$, *we have* $\widetilde{p}^i(C) < 1$ *and*

$$\widetilde{p}^i(C)[\widetilde{\mu}^i(C)x^i_\ell - c] = 0. \tag{8}$$

When $\widetilde{\sigma}^i_{h\ell}(C) = 0$ *and* $\widetilde{\sigma}^i_{\ell\ell}(C) > 0$, *we have* $\widetilde{p}^i(C) = 0$.

Lemma 1 first says that auditing is stochastic. Indeed, if $\widetilde{p}^i(C) = 1$, then h-types do not choose C^i_ℓ and insurer i has no incentive to audit, hence a contradiction. Equation (8) states that insurer i may audit types with positive probability only if (according to her beliefs) the proportion of h-types among C^i_ℓ-purchasers is equal to a threshold $\widetilde{\mu}^i(C) = c/x^i_\ell$. For such a threshold, the expected benefit of auditing $\widetilde{\mu}^i(C)x^i_\ell$ is equal to the audit cost c. In particular, when C^i_ℓ is chosen by ℓ-types only, then the claimants are not audited.

Lemma 2. *At an equilibrium* $\mathcal{E} = \{\widetilde{C}, \widetilde{p}(.), \widetilde{\sigma}_h(.), \widetilde{\sigma}_\ell(.), \widetilde{\mu}(.)\}$, *we have*

$$\begin{aligned} & \lambda\widetilde{\sigma}^i_{h\ell}(C)\Pi_{\ell h}(C^i_\ell, \widetilde{p}^i(C)) + (1-\lambda)\widetilde{\sigma}^i_{\ell\ell}(C)\Pi_{\ell\ell}(C^i_\ell, \widetilde{p}^i(C)) \\ = \; & \lambda\widetilde{\sigma}^i_{h\ell}(C)\Pi_h(C^i_\ell) + (1-\lambda)\widetilde{\sigma}^i_{\ell\ell}(C)\Pi_\ell(C^i_\ell) \quad \textit{for all } C \in \mathbb{R}_+^{4n} \textit{ and all } i = 1, ..., n. \end{aligned} \tag{9}$$

Lemma 2 shows that the expected profit made on any contract C^i_ℓ can be written as a function of the individuals' strategy $\widetilde{\sigma}^i_{h\ell}(C)$ and $\widetilde{\sigma}^i_{\ell\ell}(C)$ only: the equilibrium auditing strategy $\widetilde{p}^i(C)$ does not appear explicitly in the right-hand-side of (9).

4 Second-best Pareto-optimal allocations

The next step toward the characterization of an insurance market equilibrium consists in listing the properties of the equilibrium allocation (i.e. of the strategies played on the equilibrium path of the game). For an allocation $\{C, p, \sigma_h, \sigma_\ell\}$, they are written as follows:

$$\lambda\pi_h\sigma^i_{h\ell}x^i_\ell = c[\lambda\pi_h\sigma^i_{h\ell} + (1-\lambda)\pi_\ell\sigma^i_{\ell\ell}] \quad \text{if} \quad p^i > 0, i > 0 \tag{10}$$

$$\lambda\pi_h\sigma^i_{h\ell}x^i_\ell \leq c[\lambda\pi_h\sigma^i_{h\ell} + (1-\lambda)\pi_\ell\sigma^i_{\ell\ell}] \quad \text{if} \quad p^i = 0, i > 0 \tag{11}$$

$$\lambda[\sigma^i_{hh}\Pi_h(C^i_h) + \sigma^i_{h\ell}\Pi_h(C^i_\ell)] + (1-\lambda)[\sigma^i_{\ell h}\Pi_\ell(C^i_h) + \sigma^i_{\ell\ell}\Pi_\ell(C^i_\ell)] \geq \Pi^i_0 \text{ if } i > 0, \tag{12}$$

$$\sum_{i=0}^{n}[\sigma^i_{hh}U_h(C^i_h) + \sigma^i_{h\ell}U_{h\ell}(C^i_\ell, p^i)] = \max\{U_h(C^i_h), U_{h\ell}(C^i_\ell, p^i), i = 0, ..., n\}, \tag{13}$$

$$\sum_{i=0}^{n}[\sigma^i_{\ell h}U_\ell(C^i_h) + \sigma^i_{\ell\ell}U_\ell(C^i_\ell)] = \max\{U_\ell(C^i_h), U_\ell(C^i_\ell), i = 0, ..., n\}, \tag{14}$$

$$C \in \mathbb{R}_+^{4n}, p \in [0,1]^n, \sigma_h \in S^{2n+1}, \sigma_\ell \in S^{2n+1}. \tag{15}$$

Any equilibrium $\{\widetilde{C}, \widetilde{p}(.), \widetilde{\sigma}_h(.), \widetilde{\sigma}_\ell(.), \widetilde{\mu}(.)\}$ leads to an equilibrium allocation $\{\widetilde{C}, \overline{p}, \overline{\sigma}_h, \overline{\sigma}_\ell\}$ defined by $\overline{p} = \widetilde{p}(\widetilde{C}), \overline{\sigma}_h = \widetilde{\sigma}_h(\widetilde{C}), \overline{\sigma}_\ell = \widetilde{\sigma}_\ell(\widetilde{C})$ which satisfies the conditions (10) to (15) with $\Pi_0^i = 0$ for all $i > 0$. (10) and (11) follow from (5), (7) and Lemma 1: in words, the expected benefit from auditing is equal to the audit cost when an audit is performed with positive probability and it is lower than the audit cost otherwise. Lemma 2 shows that the left-handside in (12) is equal to the equilibrium expected profit of insurer i. Thus when $\Pi_0^i = 0$, (12) means that each insurer makes non-negative profit: (6) shows that this will be actually the case for the equilibrium allocation since deviating to a zero-contract offer is a possible choice for the insurers. From (13) and (14), all individuals only choose the best contracts with positive probability.

Definition 2. *An allocation $\{C, p, \sigma_h, \sigma_\ell\}$ is feasible if it satisfies the equations (10) to (15) with $\Pi_0^i = 0$ for all $i > 0$. A feasible allocation $\{C, p, \sigma_h, \sigma_\ell\}$ is second-best Pareto-optimal if there is no other feasible allocation $\{C', p', \sigma_h', \sigma_\ell'\}$ such that*

$$\sum_{i=0}^{n}[\sigma_{hh}^{i\prime}U_h(C_h^{i\prime}) + \sigma_{h\ell}^{i\prime}U_{h\ell}(C_\ell^{i\prime}, p^{i\prime})] \geq \sum_{i=0}^{n}[\sigma_{hh}^{i}U_h(C_h^{i}) + \sigma_{h\ell}^{i}U_{h\ell}(C_\ell^{i}, p^{i})], \qquad (16)$$

$$\sum_{i=0}^{n}[\sigma_{\ell h}^{i\prime}U_\ell(C_h^{i\prime}) + \sigma_{\ell\ell}^{i\prime}U_\ell(C_\ell^{i\prime})] \geq \sum_{i=0}^{n}[\sigma_{\ell h}^{i}U_\ell(C_h^{i}) + \sigma_{\ell\ell}^{i}U_\ell(C_\ell^{i})], \qquad (17)$$

at least one of these inequalities being slack.

In Definition 2, Pareto-optimality is defined in the second-best sense because of informational asymmetries and no-commitment constraints: insurers do not observe the risk types (hence the self-selection constraints (13) and (14)) and they cannot precommit to their auditing decisions (hence the incentives constraints (10)-(11))[7]. In order to characterize second-best Pareto optimal allocations, let us consider the maximization problem, denoted $\mathbb{P}_1(\lambda, c, u_0, \Pi_0^1, ..., \Pi_0^n)$, in which the ℓ-types expected utility is maximized over the set of allocations that satify the conditions (10) to (15) and that provide at least expected utility u_0 to h-types, with $u_0 \geq \overline{U}_h$. This is written as:

$$\mathbb{P}_1(\lambda, c, u_0, \Pi_0^1, ..., \Pi_0^n): \text{ Maximize } \sum_{i=0}^{n}[\sigma_{\ell h}^{i}U_\ell(C_h^{i}) + \sigma_{\ell\ell}^{i}U_\ell(C_\ell^{i})],$$

with respect to $C, p, \sigma_h, \sigma_\ell$ subject to conditions (10) to (15) and

$$\sum_{i=0}^{n}[\sigma_{hh}^{i}U_h(C_h^{i}) + \sigma_{h\ell}^{i}U_{h\ell}(C_\ell^{i}, p^{i})] \geq u_0. \qquad (18)$$

Any second-best Pareto-optimal allocation $\{C, p, \sigma_h, \sigma_\ell\}$ is an optimal solution to $\mathbb{P}_1(\lambda, c, u_0, 0, ..., 0)$ with $u_0 = \sum_{i=0}^{n}[\sigma_{hh}^{i}U_h(C_h^{i}) + \sigma_{h\ell}^{i}U_{h\ell}(C_\ell^{i}, p^{i})]$. We shall see that the h-types'

[7] See Crocker and Snow (1985) and Henriet and Rochet (1990) on second-best Pareto optimality in the Rothschild-Stiglitz model.

expected utility is uniquely defined at the optimum of $\mathbb{P}_1(\lambda, c, u_0, 0, ..., 0)$. Hence symmetrically, any optimal solution to $\mathbb{P}_1(\lambda, c, u_0, 0, ..., 0)$ is second-best Pareto optimal. Obviously, any equilibrium allocation $\{\widetilde{C}, \overline{p}, \overline{\sigma}_h, \overline{\sigma}_\ell\}$ is a feasible solution to $\mathbb{P}_1(\lambda, c, U_h^e, \overline{\Pi}^1, ..., \overline{\Pi}^n)$ where U_h^e is the equilibrium expected utility of h-type individuals, i.e. $U_h^e = \sum_{i=0}^n [\overline{\sigma}_{hh}^i U_h(\widetilde{C}_h^i) + \overline{\sigma}_{h\ell}^i U_{h\ell}(\widetilde{C}_\ell^i, \overline{p}^i)]$, and $\overline{\Pi}^i$ is the equilibrium expected profit of insurer i, i.e

$$\overline{\Pi}^i = \lambda[\overline{\sigma}_{hh}^i \Pi_h(\widetilde{C}_h^i) + \overline{\sigma}_{h\ell}^i \Pi_h(\widetilde{C}_\ell^i)] + (1-\lambda)[\overline{\sigma}_{\ell h}^i \Pi_\ell(\widetilde{C}_h^i) + \overline{\sigma}_{\ell\ell}^i \Pi_\ell(\widetilde{C}_\ell^i)].$$

Let us consider problem $\mathbb{P}_2(\lambda, c, u_0, \Pi_0)$ obtained from $\mathbb{P}_1(\lambda, c, u_0, \Pi_0, ..., \Pi_0)$ by assuming that all individuals actually purchase insurance and by adding a symmetry assumption: all insurers offer the same contracts and individuals are evenly shared among the insurers:

$$\mathbb{P}_2(\lambda, c, u_0, \Pi_0) : \text{Maximize} \quad \widehat{\sigma}_{\ell h} U_\ell(C_h) + \widehat{\sigma}_{\ell\ell} U_\ell(C_\ell),$$

with respect to $C_h = (k_h, x_h), C_\ell = (k_\ell, x_\ell), \widehat{p}, , \widehat{\sigma}_{hh}, \widehat{\sigma}_{h\ell}, \widehat{\sigma}_{\ell h}, \widehat{\sigma}_{\ell\ell}$ subject to

$$\lambda[\widehat{\sigma}_{hh}\Pi_h(C_h) + \widehat{\sigma}_{h\ell}\Pi_h(C_\ell)] + (1-\lambda)[\widehat{\sigma}_{\ell h}\Pi_\ell(C_h) + \widehat{\sigma}_{\ell\ell}\Pi_\ell(C_\ell)] \geq \Pi_0, \tag{19}$$

$$\lambda\pi_h\widehat{\sigma}_{h\ell}x_\ell = c[\lambda\pi_h\widehat{\sigma}_{h\ell} + (1-\lambda)\pi_\ell\widehat{\sigma}_{\ell\ell}] \quad \text{if} \quad \widehat{p} > 0, \tag{20}$$

$$\lambda\pi_h\widehat{\sigma}_{h\ell}x_\ell \leq c[\lambda\pi_h\widehat{\sigma}_{h\ell} + (1-\lambda)\pi_\ell\widehat{\sigma}_{\ell\ell}] \quad \text{if} \quad \widehat{p} = 0, \tag{21}$$

$$\widehat{\sigma}_{hh}U_h(C_h) + \widehat{\sigma}_{h\ell}U_{h\ell}(C_\ell, \widehat{p}) = \max\{U_h(C_h), U_{h\ell}(C_\ell, \widehat{p})\}, \tag{22}$$

$$\widehat{\sigma}_{\ell h}U_\ell(C_h) + \widehat{\sigma}_{\ell\ell}U_\ell(C_\ell) = \max\{U_\ell(C_h), U_\ell(C_\ell)\}, \tag{23}$$

$$\widehat{\sigma}_{hh}U_h(C_h) + \widehat{\sigma}_{h\ell}U_{h\ell}(C_\ell, \widehat{p}) \geq u_0, \tag{24}$$

$$(C_h, C_\ell) \in \mathbb{R}_+^4, \widehat{p} \in [0,1], (\widehat{\sigma}_{hh}, \widehat{\sigma}_{h\ell}) \in S^1, \ (\widehat{\sigma}_{\ell h}, \widehat{\sigma}_{\ell\ell}) \in S^1. \tag{25}$$

In $\mathbb{P}_2(\lambda, c, u_0, \Pi_0)$, there are only two contracts : C_h and C_ℓ. $\widehat{p}$ is the audit probability for C_ℓ, $\widehat{\sigma}_{hh}$ is the probability for h-types to choose C_h, $\widehat{\sigma}_{h\ell}$ is the probability for h-types to choose C_ℓ, etc... Π_0 is the required profit per insured. Let $\Phi_2(\lambda, c, U_0, \Pi_0)$ be the value function of $\mathbb{P}_2(\lambda, c, u_0, \Pi_0)$[8].

Let us first consider $\mathbb{P}_2(\lambda, c, u_0, \Pi_0)$ under the additional constraint $\widehat{p} = 0$ or $\widehat{\sigma}_{\ell\ell} = \widehat{\sigma}_{h\ell} = 0$ which means that either there is no auditing or nobody chooses C_ℓ (so that $\widehat{p}$ is irrelevant). The corresponding maximization problem and value function are respectively denoted $\mathbb{P}_3(\lambda, c, u_0, \Pi_0)$ and $\Phi_3(\lambda, c, u_0, \Pi_0)$. They are characterized in Proposition 1 and Corollary 1. We respectively denote $\overline{C}_0 \equiv (\overline{\pi}A + \Pi_0, (1-\overline{\pi})A - \Pi_0)$ and $\overline{u}_0 \equiv u(W_N - \overline{\pi}A - \Pi_0)$ the full insurance pooling contract that provides expected profit Π_0 and the corresponding utility. Let also $U_h^0 = u(W_N - \pi_h A - \Pi_0)$. U_h^0 is the highest expected utility that h-types may reach through a (full coverage) insurance contract that provides profit at least equal to Π_0, with $U_h^0 = U_h^*$ if $\Pi_0 = 0$.

[8]We will later show that problem $\mathbb{P}_2(\lambda, c, u_0, \Pi_0)$ has an optimal solution when the set of its feasible solutions is non empty (i.e. when u_0 is not too large). For the time being, $\Phi_2(\lambda, c, u_0, \Pi_0)$ is defined as the smallest upper bound of ℓ-types expected utility among the allocations that are feasible in $\mathbb{P}_2(\lambda, c, u_0, \Pi_0)$ with $\Phi_2(\lambda, c, u_0, \Pi_0) = -\infty$ if there is no feasible solution. The same for Φ_3 and Φ_4 below.

Proposition 1. *(i) There is a unique optimal solution to* $\mathbb{P}_3(\lambda, c, u_0, \Pi_0)$ *and it is such that* $\widehat{p} = 0, \widehat{\sigma}_{hh} = \widehat{\sigma}_{\ell\ell} = 1, \widehat{\sigma}_{h\ell} = \widehat{\sigma}_{\ell h} = 0$ *and* $(C_h, C_\ell) \in \mathbb{R}^4_+$ *maximize* $U_\ell(C_\ell)$ *subject to*[9]

$$\lambda \Pi_h(C_h) + (1-\lambda)\Pi_\ell(C_\ell) \geq \Pi_0, \tag{26}$$

$$U_\ell(C_\ell) \geq U_\ell(C_h), \tag{27}$$

$$U_h(C_h) \geq U_h(C_\ell), \tag{28}$$

$$U_h(C_h) \geq u_0. \tag{29}$$

(ii) At this optimal solution $C_h = C_\ell = \overline{C}_0$ *if* $u_0 = \overline{u}_0$ *and* $C_h \neq C_\ell$ *if* $u_0 \neq \overline{u}_0$ *with:*

$$\begin{aligned} \text{If } u_0 &< \overline{u}_0 : x_h + k_h = A, k_\ell + x_\ell < A, U_h(C_\ell) = U_h(C_h) < \overline{u}_0, U_\ell(C_\ell) > \overline{u}_0 > U_\ell(C_h), \\ \text{If } u_0 &> \overline{u}_0 : x_h + k_h > A, k_\ell + x_\ell = A, U_h(C_\ell) > U_h(C_h) = u_0, U_\ell(C_\ell) = U_\ell(C_h) < \overline{u}_0. \end{aligned}$$

(iii) For all $\eta > 0$*, there exist* $\varepsilon, \varepsilon' > 0$ *and* $(C_h, C_\ell) \in \mathbb{R}^4_+, \widehat{p} = 0, \widehat{\sigma}_{hh} = \widehat{\sigma}_{\ell\ell} = 1, \widehat{\sigma}_{h\ell} = \widehat{\sigma}_{\ell h} = 0$ *feasible in* $\mathbb{P}_3(\lambda, c, u_0 + \varepsilon, \Pi_0 + \varepsilon')$ *such that* $U_\ell(C_\ell) > U_\ell(C_h), U_h(C_h) > U_h(C_\ell), U_h(C_h) \geq u_0 + \varepsilon$ *and* $U_\ell(C_\ell) \geq \Phi_3(\lambda, c, u_0, \Pi_0) - \eta$*.*

Proposition 1-*i* states that individuals do not randomize at a second-best Pareto-optimal allocation without auditing: C_h is chosen by h-types and C_ℓ by ℓ-types. Proposition 1-*ii* coincides with the results by Crocker and Snow (1985)[10]. When $u_0 < \overline{u}_0$, h-types are fully insured and ℓ-types have partial insurance, while the h-types self-selection constraint (28) is binding and the ℓ-types constraint (27) is slack. The results are reversed when $u_0 > \overline{u}_0$: then there is overinsurance for h-types and full insurance for ℓ-types, while (27) is binding and (28) is slack. A pooling allocation is optimal only when $u_0 = \overline{u}_0$. Proposition 1-*iii* states that if the ℓ-type expected utility is lower than $\Phi_3(\lambda, c, u_0, \Pi_0)$ and h-types reach expected utility u_0, then it is possible to improve the welfare of everybody while increasing profit through a pair of strictly incentive compatible contracts.

Corollary 1. *There exists* λ^* *in* $(0,1)$ *such that* $\Phi_3(\lambda, c, U_h^*, 0) > U_\ell(C_\ell^{**})$ *if* $0 \leq \lambda < \lambda^*$ *and* $\Phi_3(\lambda, c, U_h^*, 0) = U_\ell(C_\ell^{**})$ *if* $\lambda^* \leq \lambda \leq 1$*.*

From Corollary 1, when $\lambda < \lambda^*$ there exists a menu of contracts with cross-subsidization that Pareto-dominates th Rothschild-Stiglitz alloction. As already mentioned, in such a case there is no equilibrium in the Rothschild-Stiglitz model.

[9] Of course, by symmetry, there also exists an optimal solution to $\mathbb{P}_3(\lambda, u_0, \Pi_0)$ where ℓ-types choose C_h and h-types choose C_ℓ. The fact that ℓ-types chooses C_ℓ and h-types choose C_h is purely a notational convention since there is no auditing. It is the characterization of the contract chosen by each type that matters.

[10] The characterization of Crocker and Snow (1985) can be directly recovered from the Proposition since $\overline{u}_0 = u(W - \overline{\pi}A)$ when $\Pi_0 = 0$. They implicitly postulate that individuals do not randomize between contracts at a second-best Pareto-optimal allocation: Proposition 1-*i* establishes that this is actually the case.

Now let us consider problem $\mathbb{P}_2(\lambda, c, u_0, \Pi_0)$ under the additional constraints $\widehat{p} > 0$ and $\{\widehat{\sigma}_{\ell\ell} > 0$ or $\widehat{\sigma}_{h\ell} > 0\}$. This will be called problem $\mathbb{P}_4(\lambda, c, u_0, \Pi_0)$, with value function $\Phi_4(\lambda, c, u_0, \Pi_0)$. Note that (20) implies $\widehat{\sigma}_{h\ell} > 0$ when $\widehat{\sigma}_{\ell\ell} > 0$. Hence, any allocation feasible in $\mathbb{P}_4(\lambda, c, u_0, \Pi_0)$ is such that $\widehat{\sigma}_{h\ell} > 0$, which implies $U_{h\ell}(C_\ell, \widehat{p}) \geq u_0$ for such an allocation. $\widehat{\sigma}_{h\ell} > 0$ and $\widehat{p} > 0$ then give $C_h \neq C_\ell$. We also have $U_h(C_\ell) = U_{h\ell}(C_\ell, 0) > U_{h\ell}(C_\ell, \widehat{p}) \geq U_h(C_h)$: hence h-types strongly prefer C_ℓ to C_h.

The domain of $\mathbb{P}_4(\lambda, c, u_0, \Pi_0)$ is not a closed set and consequently this problem may not have any optimal solution. However, we are in fact interested in $\mathbb{P}_4(\lambda, c, u_0, \Pi_0)$ only when $\Phi_4(\lambda, c, u_0, \Pi_0) \geq \Phi_3(\lambda, c, u_0, \Pi_0)$ and, among other results, Proposition 2 shows that, under this restriction, $\mathbb{P}_4(\lambda, c, u_0, \Pi_0)$ has actually an optimal solution.

Proposition 2. *If* $\Phi_4(\lambda, c, u_0, \Pi_0) \geq \Phi_3(\lambda, c, u_0, \Pi_0)$ *then:*

(i) $u_0 < \overline{u}_0$,

(ii) There is an optimal solution to $\mathbb{P}_4(\lambda, c, u_0, \Pi_0)$. *It is such that:* $\widehat{\sigma}_{\ell\ell} = 1$, $0 < \widehat{\sigma}_{h\ell} < 1$ *and* $C_h, C_\ell, \widehat{\sigma}_{h\ell}$ *maximize* $U_\ell(C_\ell)$ *subject to :*

$$\lambda \Pi_h(C_h)(1 - \widehat{\sigma}_{h\ell}) + \lambda \widehat{\sigma}_{h\ell} \Pi_h(C_\ell) + (1 - \lambda)\Pi_\ell(C_\ell) \geq \Pi_0, \tag{30}$$

$$\widehat{\sigma}_{h\ell} = K(x_\ell, \lambda, c) \equiv \frac{(1 - \lambda)c\pi_\ell}{\lambda \pi_h(x_\ell - c)} \leq 1, \tag{31}$$

$$U_\ell(C_\ell) \geq U_\ell(C_h), \tag{32}$$

$$U_h(C_\ell) > U_h(C_h), \tag{33}$$

$$U_h(C_h) \geq u_0. \tag{34}$$

(iii) For this optimal solution, we have $U_h(C_h) = U_{h\ell}(C_\ell, \widehat{p}) = u_0 < \overline{u}_0 < U_\ell(C_\ell)$ *and* $k_h + x_h = A$. *Furthermore when* $u_0 = U_h^*$ *and* $\Pi_0 = 0$, *we have* $C_h = C_h^*$ *and* $k_\ell + x_\ell > A$.

(iv) For all $\eta > 0$, *there exist* $\varepsilon, \varepsilon' > 0$ *and* $(C_h, C_\ell) \in \mathbb{R}_+^4$ $\widehat{p} > 0, \widehat{\sigma}_{\ell\ell} = 1, \widehat{\sigma}_{h\ell} \in (0, 1]$ *feasible in* $\mathbb{P}_4(\lambda, c, u_0 + \varepsilon, \Pi_0 + \varepsilon')$ *such that* $U_\ell(C_\ell) > U_\ell(C_h), U_h(C_\ell) > U_h(C_h) \geq u_0 + \varepsilon$ *and* $U_\ell(C_\ell) \geq \Phi_4(\lambda, c, u_0, \Pi_0) - \eta$.

Proposition 2 characterizes an optimal allocation of $\mathbb{P}_4(\lambda, c, u_0, \Pi_0)$ when auditing matters, that is when $\Phi_4(\lambda, c, u_0, \Pi_0) \geq \Phi_3(\lambda, c, u_0, \Pi_0)$. The proposition first states that we necessarily have $u_0 < \overline{u}_0$ in such a case. This is an intuitive result because auditing is a way to deter h-types to choose the contract which is intended for ℓ-types. If the h-types minimal expected utility is larger than $\overline{u}_0$, then the problem we must face is in fact to deter ℓ-types to choose the contract intended for h-types and auditing is useless in such a case.

Proposition 2-*ii* states that $\widehat{\sigma}_{\ell\ell} = 1$ and $0 < \widehat{\sigma}_{h\ell} < 1$ at an optimal solution to $\mathbb{P}_4(\lambda, c, u_0, \Pi_0)$. Indeed if $\widehat{\sigma}_{\ell\ell} < 1$ then ℓ-types would weakly prefer C_h to C_ℓ, while h-types would (strongly) prefer C_ℓ to C_h. The proof shows that allocating contracts without auditing would be more efficient in such a case: in other words, we would have

$\Phi_4(\lambda, c, u_0, \Pi_0) < \Phi_3(\lambda, c, u_0, \Pi_0)$, hence a contradiction. $\widehat{\sigma}_{h\ell} = K(x_\ell, \lambda, c) > 0$ is just a consequence of (20) and $\widehat{\sigma}_{\ell\ell} = 1$: audit incentives require that h-types choose C_ℓ with probability $K(x_\ell, \lambda, c)$ when ℓ-types choose C_ℓ with probability 1. Furthermore we have $\widehat{\sigma}_{h\ell} < 1$ for otherwise we would get a pooling allocation where all individuals choose C_ℓ. Auditing would be useless in such a case and we know from Proposition 1 that this pooling allocation is dominated in $\mathbb{P}_3(\lambda, c, u_0, \Pi_0)$ when $u_0 \neq \overline{u}_0$. Proposition 2-*ii* then states that optimal contracts maximize the ℓ-type expected utility with profit larger or equal to Π_0, under self-selection constraints (ℓ-types weakly prefer C_ℓ to C_h and h-types strictly prefer C_ℓ to C_h) and C_h should provide at least utility u_0 to h-types. Equations (22) to (25) are then satisfied with $\widehat{\sigma}_{h\ell} \in (0,1), \widehat{\sigma}_{\ell\ell} = 1$ and $\widehat{p} \in (0,1)$ such that $U_h(C_h) = U_{h\ell}(C_\ell, \widehat{p})$. Hence h-types randomize between C_h and C_ℓ because the audit probability makes them indifferent between the two contracts, and their mixed strategy makes insurers indifferent between auditing and not auditing.

Proposition 2-*iii* states that C_h is a full insurance contract with utility u_0. Furthermore C_ℓ provides overinsurance (i.e. $k_\ell + x_\ell > A$) if $u_0 = U_h^*$ and $\Pi_0 = 0$. This contrasts sharply with the Rothschild-Stiglitz separating pair of contracts, where low risk individuals are underinsured. The optimal contracts are in fact obtained by deleting the ℓ-types self selection constraint (32) and by checking ex post that it is satisfied by the optimal solution of the relaxed problem. In this relaxed problem, it is optimal to choose $k_h + x_h = A$ and $U_h(C_h) = u_0$, which gives $U_\ell(C_h) = U_h(C_h) < \overline{u}_0 < U_\ell(C_\ell)$. Hence ℓ-types and h-types strictly prefer C_ℓ to C_h.

When $\Pi_0 = 0$ and $u_0 = U_h^*$, $C_h = C_h^*$ just breaks even. Using $\widehat{\sigma}_{h\ell} = K(x_\ell, \lambda, c)$, a straightforward calculation shows that C_ℓ also breaks even when $k_\ell = \phi(x_\ell)$, where

$$k_\ell = \phi(x_\ell) \equiv \frac{\pi_\ell \pi_h x_\ell^2}{(1-\pi_\ell)\pi_h x_\ell - c(\pi_h - \pi_\ell)}, \tag{35}$$

or equivalently, when $k_\ell = \pi_\ell[1 + \sigma(x_\ell, c)](k_\ell + x_\ell)$ where $\sigma(x_\ell)$ is given by

$$\sigma(x_\ell) \equiv \frac{\lambda c(\pi_h - \pi_\ell)}{\pi_h x_\ell - c(\pi_h - \pi_\ell)}. \tag{36}$$

Since $k_\ell + x_\ell$ is the indemnity for C_ℓ, $\sigma(x_\ell)$ can be interpreted as a loading factor in the insurance terminology. (36) shows that an increase in x_ℓ entails a decrease in the loading factor. In other words, the marginal price of insurance is decreasing with respect to coverage. This is just a consequence of the fact that $\widehat{\sigma}_{h\ell} = K(x_\ell, \lambda, c)$ is decreasing in x_ℓ : when the net indemnity provided by C_ℓ increases, insurers are incited to perform an audit for a lower proportion of h-types among C_ℓ-claimants, hence less cross-subsidization and a positive externality for ℓ-types. This is illustrated in Figure 2. The locus PP' is the zero profit line in the (W_1, W_2) plane, with equation $W_1 = W_N - \phi(W_2 - W_A)$[11].

[11] EL is an aymptote for PP'. Furthermore, PP' crosses the 45° line if A/c is large enough. For A/c small, PP' is entirely above the 45° line. See the proof of Proposition 2 in the Appendix for

We denote $\widehat{C}_\ell = (\widehat{k}_\ell, \widehat{x}_\ell)$ the C_ℓ contract at an optimal solution to $\mathbb{P}_4(\lambda, c, U_h^*, 0)$. $\widehat{C}_\ell$ maximizes $U_\ell(k_\ell, x_\ell)$ subject to $k_\ell = \phi(x_\ell)$. In the (W_1, W_2) plane, $\widehat{C}_\ell$ is located at the tangency of PP' and a ℓ-type indifference curve, above the 45° line. As expected, the lower the audit cost c, the lower the loading factor σ. PP' shifts rightwards when c is decreasing and it goes to the low risk fair-odds line $k_\ell = \pi_\ell x_\ell/(1 - \pi_\ell)$ when c goes to zero. Hence $\widehat{C}_\ell$ goes to C_ℓ^* when c goes to zero.

Figure 2

Finally, Proposition 4-*iv* states that, if the ℓ-types expected utility is lower than $\lneqq_4(\lambda, c, u_0, \Pi_0)$ and h-types reach expected utility u_0, then it possible to improve the welfare of everybody while increasing profit, through a pair of contracts with auditing where ℓ-types choose C_ℓ and h-types randomize between C_h and C_ℓ.

We will show later that the insurance market equilibrium can be characterized by comparing $\Phi_3(\lambda, c, U_h^*, 0)$, $\Phi_4(\lambda, c, U_h^*, 0) = U_\ell(\widehat{C}_\ell)$ and $U_\ell(C_\ell^{**})$. This comparison depends on λ and c and it follows from some simple properties of Φ_3 and Φ_4[12]. First, of course $\partial\Phi_3(\lambda, c, U_h^*, 0)/\partial c \equiv 0$ because there is no auditing in $\mathbb{P}_3(\lambda, c, U_h^*, 0)$. Second, as shown in Corollary 1, when $\lambda < \lambda^*$ then the optimal solution to $\mathbb{P}_3(\lambda, c, U_h^*, 0)$ requires cross-subsidization between risk types and, as expected, the larger the proportion of high risks in the population, the lower the expected utility that ℓ-types may reach in this problem. This gives $\partial\Phi_3(\lambda, c, U_h^*, 0)/\partial\lambda < 0$ if $\lambda < \lambda^*$ and $\Phi_3(\lambda, c, U_h^*, 0) = U_\ell(C_\ell^{**})$ if $\lambda \geq \lambda^*$. Third, when $\Phi_4(\lambda, c, U_h^*, 0) \geq \Phi_3(\lambda, c, U_h^*, 0)$ then at an optimal solution to $\mathbb{P}_4(\lambda, c, U_h^*, 0)$, the proportion of h-types among the individuals who choose C_ℓ is independent from λ: it depends on x_ℓ in such a way that insurers are indifferent between auditing and no auditing[13]. In such a case any increase in λ does not affect the ℓ-types optimal expected utility in $\mathbb{P}_4(\lambda, c, U_h^*, 0)$, which gives $\partial\Phi_4(\lambda, c, U_h^*, 0)/\partial\lambda = 0$. Conversely, an increase in the cost of auditing increases the h-types proportion among the individuals who choose C_ℓ, hence an adverse effect on the ℓ-types optimal expected utility, which implies $\partial\Phi_4(\lambda, c, U_h^*, 0)/\partial c < 0$. Let $c^* > 0$ such that $\Phi_4(\lambda, c^*, U_h^*, 0) = U_\ell(C_\ell^{**})$. c^* is the audit cost for which the ℓ-types optimal expected utility with auditing is equal to the ℓ-types expected utility at the Rothshild-Stiglitz allocation. We can check thet c^* is uniquely defined because $\Phi_4(\lambda, 0, U_h^*, 0) = U_\ell(C_\ell^*) > U_\ell(C_\ell^{**})$, $\Phi_4(\lambda, c, U_h^*, 0) < U_\ell(C_\ell^{**})$ for c large enough and $\Phi_4(\lambda, c^*, U_h^*, 0)$ is continuously decreasing in c and is independent from λ. Let us also define $\widetilde{c}(\lambda)$ by $\Phi_3(\lambda, \widetilde{c}(\lambda), U_h^*, 0) = \Phi_4(\lambda, \widetilde{c}(\lambda), U_h^*, 0)$ for $\lambda \in [0, \lambda^*]$ with $\widetilde{c}'(\lambda) > 0$. Using $\Phi_3(\lambda^*, c, U_h^*, 0) = U_\ell(C_\ell^{**})$ gives $\widetilde{c}(\lambda^*) = c^*$ and $\Phi_3(0, c, U_h^*, 0) = U(C_\ell^*) = \Phi_4(\lambda, 0, U_h^*, 0)$ gives $\widetilde{c}(0) = 0$. Let $\Phi_2(\lambda, c, u_0, \Pi_0)$ be the

computational details.

[12]See Corollaries 2 and 3 in Appendix for details.

[13]Equation (31) shows that this proportion is equal to $c\pi_\ell/[c\pi_\ell + \pi_h(x_\ell - c)]$.

value function to $\mathbb{P}_2(\lambda, c, u_0, \Pi_0)$, with

$$\Phi_2(\lambda, c, u_0, \Pi_0) = \max\{\Phi_3(\lambda, c, u_0, \Pi_0), \Phi_4(\lambda, c, u_0, \Pi_0)\}.$$

We thus have

$$\begin{aligned}
\Phi_2(\lambda, c, U_h^*, 0) &= \Phi_3(\lambda, c, U_h^*, 0) = U_\ell(C_\ell^{**}) > \Phi_4(\lambda, c, U_h^*, 0) \text{ if } \lambda > \lambda^*, c > c^*, \\
\Phi_2(\lambda, c, U_h^*, 0) &= \Phi_4(\lambda, c, U_h^*, 0) = U_\ell(\widehat{C}_\ell) > \Phi_3(\lambda, c, U_h^*, 0) \text{ if } \lambda > \lambda^*, c < c^* \text{ or } \lambda \leq \lambda^*, c < \widetilde{c}(\lambda), \\
\Phi_2(\lambda, c, U_h^*, 0) &= \Phi_3(\lambda, c, U_h^*, 0) > \max\{U_\ell(C_\ell^{**}), U_\ell(\widehat{C}_\ell)\} \text{ if } \lambda < \lambda^*, c > \widetilde{c}(\lambda).
\end{aligned}$$

Using these results as well as Propositions 1 and 2, we are now able to fully characterize the optimal solution to $P_2(\lambda, c, U_h^*, 0)$. This is done in Proposition 3 which is a straightforward consequence of our previous results.

Proposition 3. *An optimal solution to $\mathbb{P}_2(\lambda, c, U_h^*, 0)$ is characterized by*

(i) $C_h = C_h^*, C_\ell = C_\ell^{**}, \widehat{\sigma}_{hh} = 1, \widehat{\sigma}_{h\ell} = 0, \widehat{\sigma}_{\ell h} = 0, \widehat{\sigma}_{\ell\ell} = 1$ *if* $\lambda \geq \lambda^*, c \geq c^*$,

(ii) $C_h = C_h^*, C_\ell = \widehat{C}_\ell, \widehat{\sigma}_{h\ell} = K(\widehat{x}_\ell, \lambda, c) \in (0,1), \widehat{\sigma}_{hh} = 1 - \widehat{\sigma}_{h\ell}, \widehat{\sigma}_{\ell h} = 0, \widehat{\sigma}_{\ell\ell} = 1$ *if* $c \leq c^*, \lambda \geq \lambda^*$ *or* $c \leq \widetilde{c}(\lambda), \lambda < \lambda^*$,

(iii) $C_h = (k_h, x_h), C_\ell = (k_\ell, x_\ell)$ *with* $k_h + x_h = A, U_h(C_h) > U_h^*, k_\ell + x_\ell < A$, *and* $\widehat{\sigma}_{hh} = 1, \widehat{\sigma}_{h\ell} = 0, \widehat{\sigma}_{\ell h} = 0, \widehat{\sigma}_{\ell\ell} = 1$ *if* $c \geq \widetilde{c}(\lambda), \lambda \leq \lambda^*$,

where $\widehat{C}_\ell = (\widehat{k}_\ell, \widehat{x}_\ell)$ *is such that* $\widehat{k}_\ell + \widehat{x}_\ell > A$ *and maximizes* $U_\ell(C_\ell)$ *subject to*

$$\Pi_\ell(C_\ell) + \frac{c\pi_\ell}{\pi_h(x_\ell - c)}\Pi_h(C_\ell) \geq 0.$$

If $\lambda \geq \lambda^*, c \geq c^*$, then the optimal solution to $\mathbb{P}_2(\lambda, c, U_h^*, 0)$ is reached in $\mathbb{P}_3(\lambda, c, U_h^*, 0)$: it coincides with the Rothschild-Stiglitz allocation $C_h = C_h^*$ and $C_\ell = C_\ell^{**}$ without auditing. This is area I in Figure 3. When $\lambda > \lambda^*, c \leq c^*$ or $\lambda \leq \lambda^*, c \leq \widetilde{c}(\lambda)$, which corresponds to area II, then the optimal solution to $\mathbb{P}_2(\lambda, c, U_h^*, 0)$ is reached in $P_4(\lambda, c, U_h^*, 0)$ with $C_h = C_h^*$,$C_\ell = \widehat{C}_\ell$ and random auditing $\widehat{p} \in (0,1)$. In areas I and II, each contract C_h and C_ℓ breaks even. Lastly, when $\lambda \leq \lambda^*, c \geq \widetilde{c}(\lambda)$, the optimal solution to $\mathbb{P}_2(\lambda, c, U_h^*, 0)$ is reached in $\mathbb{P}_3(\lambda, c, U_h^*, 0)$, hence without any auditing, but it involves cross-subsidization between C_h and C_ℓ. This is area III.

Figure 3

We are now in position to analyse the optimal solution to $\mathbb{P}_1(\lambda, c, u_0, \Pi_0^1, ..., \Pi_0^n)$ by using our results about $\mathbb{P}_2(\lambda, c, u_0, \Pi_0)$. It is convenient at this stage to define the following variables

$$\begin{aligned} N^i &= \sigma^i_{hh} + \sigma^i_{h\ell} + \sigma^i_{\ell h} + \sigma^i_{\ell\ell}, \quad \lambda^i = \frac{\sigma^i_{hh} + \sigma^i_{h\ell}}{N^i} \text{ if } N^i > 0, \\ \widehat{\sigma}^i_{hh} &= \frac{\lambda\sigma^i_{hh}}{\lambda^i N^i}, \ \widehat{\sigma}^i_{h\ell} = \frac{\lambda\sigma^i_{h\ell}}{\lambda^i N^i} \quad \text{if} \quad \lambda^i N^i > 0, \\ \widehat{\sigma}^i_{\ell h} &= \frac{(1-\lambda)\sigma^i_{\ell h}}{(1-\lambda^i)N^i}, \ \widehat{\sigma}^i_{\ell\ell} = \frac{(1-\lambda)\sigma^i_{\ell\ell}}{(1-\lambda^i)N^i} \quad \text{if} \quad (1-\lambda^i)N^i > 0. \end{aligned}$$

$N^i \in [0,1]$ is the fraction of individuals who purchase insurance from insurer i and $\lambda^i \in [0,1]$ is the fraction of h-types among these policyholders. $\widehat{\sigma}^i_{hh}$, $\widehat{\sigma}^i_{h\ell}, \widehat{\sigma}^i_{\ell h}$, $\widehat{\sigma}^i_{\ell\ell}$ specify the contract choices of insurer i policyholders. $\mathbb{P}_1(\lambda, c, u_0, \Pi^1_0, ..., \Pi^n_0)$ may then be rewritten as

$$\text{Maximize} \sum_{i=0}^{n} \frac{N^i(1-\lambda^i)}{1-\lambda}[\widehat{\sigma}^i_{\ell h}U_\ell(C^i_h) + \widehat{\sigma}^i_{\ell\ell}U_\ell(C^i_\ell)]$$

with respect to C^i_h, C^i_ℓ, p^i, $i = 1, ..., n$ and $N^i, \lambda^i, \widehat{\sigma}^i_{hh}, \widehat{\sigma}^i_{h\ell}, \widehat{\sigma}^i_{\ell h}$, $\widehat{\sigma}^i_{\ell\ell}$, $i = 0, ..., n$, subject to

$$N^i\lambda^i[\widehat{\sigma}^i_{hh}\Pi_h(C^i_h) + \widehat{\sigma}^i_{h\ell}\Pi_h(C^i_\ell)] + N^i(1-\lambda^i)[\widehat{\sigma}^i_{\ell h}\Pi_\ell(C^i_h) + \widehat{\sigma}^i_{\ell\ell}\Pi_\ell(C^i_\ell)] \geq \Pi^i_0 \quad \text{for all } i = 1, ..., n, \tag{37}$$

$$\lambda^i\pi_h\widehat{\sigma}^i_{h\ell}x^i_\ell = c[\lambda^i\pi_h\widehat{\sigma}^i_{h\ell} + (1-\lambda^i)\pi_\ell\widehat{\sigma}^i_{\ell\ell}] \quad \text{if} \quad p^i > 0 \text{ and } N^i > 0, i > 0, \tag{38}$$

$$\lambda^i\pi_h\widehat{\sigma}^i_{h\ell}x^i_\ell \leq c[\lambda^i\pi_h\widehat{\sigma}^i_{h\ell} + (1-\lambda^i)\pi_\ell\widehat{\sigma}^i_{\ell\ell}] \quad \text{if} \quad p^i = 0 \text{ and } N^i > 0, i > 0, \tag{39}$$

$$\sum_{i=0}^{n} \frac{N^i\lambda^i}{\lambda}[\widehat{\sigma}^i_{hh}U_h(C^i_h) + \widehat{\sigma}^i_{h\ell}U_{h\ell}(C^i_\ell, p^i)] = \max\{U_h(C^i_h), U_{h\ell}(C^i_\ell, p^i), i = 1, ..., n\}, \tag{40}$$

$$\sum_{i=0}^{n} \frac{N^i(1-\lambda^i)}{1-\lambda}[\widehat{\sigma}^i_{\ell h}U_\ell(C^i_h) + \widehat{\sigma}^i_{\ell\ell}U_\ell(C^i_\ell)] = \max\{U_\ell(C^i_h), U_\ell(C^i_\ell), i = 1, ..., n\}, \tag{41}$$

$$\sum_{i=0}^{n} \frac{N^i\lambda^i}{\lambda}[\widehat{\sigma}^i_{hh}U_h(C^i_h) + \widehat{\sigma}^i_{h\ell}U_{h\ell}(C^i_\ell, p^i)] \geq u_0, \tag{42}$$

$$\sum_{i=0}^{n} N^i = 1, \sum_{i=0}^{n} \lambda^i N^i = \lambda \text{ and } N^i \geq 0, \lambda^i \geq 0 \text{ if } i \geq 0, \tag{43}$$

$$C^i \in \mathbb{R}^4_+, p^i \in [0,1] \text{ if } i > 0, (\widehat{\sigma}^i_{hh}, \widehat{\sigma}^i_{h\ell}) \in S^1, (\widehat{\sigma}^i_{\ell h}, \widehat{\sigma}^i_{\ell\ell}) \in S^1 \text{ if } i \geq 0. \tag{44}$$

Let $\widehat{\mathbb{P}}_1(\lambda, c, u_0, \Pi^1_0, ..., \Pi^n_0)$ denote this new way of writing problem $\mathbb{P}_1(\lambda, c, u_0, \Pi^1_0, ..., \Pi^n_0)$.

Proposition 4. $\widehat{\mathbb{P}}_1(\lambda, c, u_0, 0, ..., 0)$ *has feasible solutions if and only if* $u_0 \in [\overline{U}_h, \widehat{u}_0(\lambda, c)]$ *where* $\widehat{u}_0(\lambda, c) \equiv \sup\{u_0 \text{ s.t. } \Phi_2(\lambda, c, u_0, 0) \geq \overline{U}_\ell\} \in (U^*_h, U^*_\ell)$. *Let* $\{C^i \equiv (C^i_h, C^i_\ell), p^i, N^i, \lambda^i, \widehat{\sigma}^i_{hh}, \widehat{\sigma}^i_{h\ell}, \widehat{\sigma}^i_{\ell h}, \widehat{\sigma}^i_{\ell\ell}, i = 0, ..., n\}$ *be an optimal solution to* $\widehat{\mathbb{P}}_1(\lambda, c, u_0, 0, ..., 0)$ *when* $u_0 \in [\overline{U}_h, \widehat{u}_0(\lambda, c)]$. *Let* $u^i_0 = \max\{U_h(C^i_h),\ U_{h\ell}(C^i_\ell, p^i)\}$ *for* $i = 0, ..., n$ *and* $u'_0 = \max\{u^0_0, ..., u^n_0\}$. *Then* $N^0 = 0$ *and* $\{C^i, p^i, \widehat{\sigma}^i_{hh}, \widehat{\sigma}^i_{h\ell}, \widehat{\sigma}^i_{\ell h}, \widehat{\sigma}^i_{\ell\ell}\}$ *is an optimal solution to* $\mathbb{P}_2(\lambda, c, u_0, 0)$ *for all* $i > 0$ *such that* $N^i > 0$ *. Furthermore if* $u'_0 \neq U^*_h$, *then* $\lambda^i = \lambda$ *for all* i. *If* $u'_0 = U^*_h$

and $U_\ell(C_\ell^{**}) > U_\ell(\widehat{C}_\ell)$*, then*

$$\begin{aligned} C_h^i &= C_h^*, \widehat{\sigma}_{hh}^i = 1, \widehat{\sigma}_{h\ell}^i = 1 \text{ if } N^i\lambda^i > 0, \\ C_\ell^i &= C_\ell^{**}, \widehat{\sigma}_{\ell h}^i = 0, \widehat{\sigma}_{\ell\ell}^i = 1,\ p^i = 0 \text{ if } N^i(1-\lambda^i) > 0. \end{aligned}$$

If $u_0' = U_h^*$ *and* $U_\ell(C_\ell^{**}) < U_\ell(\widehat{C}_\ell)$*, then* $\lambda^i > 0$ *for all* i *such that* $N^i > 0$ *and*

$$\begin{aligned} C_h^i &= C_h^*, \widehat{\sigma}_{hh}^i = 1 \text{ if } \lambda^i = 1, N^i > 0 \\ C_h^i &= C_h^*, \widehat{\sigma}_{hh}^i = 1 - K(x_\ell, \lambda^i, c) \text{ if } \lambda^i < 1, K(x_\ell, \lambda^i, c) < 1, N^i > 0 \\ C_\ell^i &= \widehat{C}_\ell, U_{h\ell}(\widehat{C}_\ell, p^i) = U_h^*, \widehat{\sigma}_{h\ell}^i = K(x_\ell, \lambda^i, c) \in (0,1], \widehat{\sigma}_{\ell h}^i = 0, \widehat{\sigma}_{\ell\ell}^i = 1 \text{ if } \lambda^i < 1, N^i > 0. \end{aligned}$$

When u_0 is larger than the threshold $\widehat{u}_0$, then it is impossible to simultaneously provide expected utility larger than $\overline{U}_\ell$ to ℓ-types and larger than u_0 to h-types, and in such a case the set of feasible solutions to $\widehat{\mathbb{P}}_1(\lambda, c, u_0, 0, ..., 0)$ is empty. When $u_0 \in [\overline{U}_h, \widehat{u}_0]$ the optimal solution to $\widehat{\mathbb{P}}_1(\lambda, c, u_0, 0, ..., 0)$ is symmetric: generically, all insurers (at least those with customers) offer the same menu of contracts and they have the same auditing strategy as at the optimal solution to $\mathbb{P}_2(\lambda, c, u_0, 0)$. No individual remains uninsured. In Proposition 4, u_0' denotes the h-types' expected utility: it is the left-hand side in (42), with $u_0' \geq u_0$. When $u_0' \neq U_h^*$, then all insurers get the same proportions of high risks and low risks among their customers (i.e. $\lambda^i = \lambda$ for all i such that $N^i > 0$) and $\{C^i, p^i, \widehat{\sigma}_{hh}^i, \widehat{\sigma}_{h\ell}^i, \widehat{\sigma}_{\ell h}^i, \widehat{\sigma}_{\ell\ell}^i\}$ is an optimal solution to $\mathbb{P}_2(\lambda, c, u_0, 0)$. To get the intuition of this result, consider the case $u_0' > U_h^*$, which implies that insurers make losses on h-types. If $0 < \lambda^i < \lambda^j < 1$ and $N^i, N^j > 0$, then ℓ-types and h-types reach the same expected utility from insurer i than from insurer j, while the burden of high risk individuals is larger (per insured) for insurer j than for insurer i. The offer of insurer i would be inefficient in such case since she could make a more advantageous offer to ℓ-types while providing the same expected utility to h-types and making non-negative profits. The proof of the Proposition elaborates on this intuitive argument (extended to the case $u_0' < U_h^*$) to establish that the optimal proportion of high risk individuals is the same for all insurers and consequently all insurers offer the same contracts, with the same auditing strategy. When $u_0' = U_h^*$, then the distribution of h-types and ℓ-types is arbitrary (we may have $\lambda^i \neq \lambda^j$, $N^i > 0, N^j > 0$) and $\{C^i, p^i, \widehat{\sigma}_{hh}^i, \widehat{\sigma}_{h\ell}^i, \widehat{\sigma}_{\ell h}^i, \widehat{\sigma}_{\ell\ell}^i\}$ is an optimal solution to $\mathbb{P}_2(\lambda^i, c, u_0, 0)$. However, this optimal solution does not depend on λ^i when $u_0' = U_h^*$ (indeed in such a case, it is neither a burden nor an advantage to have a large proportion of high risk individuals among the insureds) : we have $C_h^i = C_h^*$ and either $C_\ell^i = C_\ell^{**}, p^i = 0$ if $U_\ell(C_\ell^{**}) > U_\ell(\widehat{C}_\ell)$ or $C_\ell^i = \widehat{C}_\ell, p^i = \widehat{p}$ such that $U_{h\ell}(\widehat{C}_\ell, \widehat{p}) = U_h^*$ otherwise. Hence, whatever the value of u_0', all insurers should offer the same menu of contracts, with the same auditing strategy. Since a second–best Pareto-optimal allocation $\{C, p, \sigma_h, \sigma_\ell\}$ is an optimal solution to $\widehat{\mathbb{P}}_1(\lambda, c, u_0, 0, ..., 0)$, with

$u_0 = \sum_{i=0}^{n}[\sigma^i_{hh}U_h(C^i_h)+\sigma^i_{h\ell}U_{h\ell}(C^i_\ell, p^i)]$, Proposition 4 allows us to conclude that such an allocation is symmetric with the same contracts and audit probability as at the optimal solution to $\mathbb{P}_2(\lambda, c, u_0, 0)$. Finally, Propositions 1,2 and 4 jointly show that there is cross-subsidization between contracts when $u'_0 \neq U^*_h$: when $u'_0 > U^*_h$, then C^i_h is in deficit and C^i_ℓ is profitable for all i such that $N^i > 0$ and the situation is reversed when $u'_0 < U^*_h$. On the contrary, each contract breaks even when $u'_0 = U^*_h$.

5 Existence and characterization of market equilibrium

Let us consider the conditions under which an equilibrium allocation is second-best Pareto-optimal. Intuitively, if this were not the case, then a deviant insurer - say insurer j - could offer a menu of contracts C^j_h, C^j_ℓ that would be advantageous to all individuals while making positive profit, hence a contradiction with the definition of an equilibrium. In the Rothschild-Stiglitz model, this kind of argument directly shows that $\lambda \geq \lambda^*$ is a necessary condition for C^*_h, C^{**}_ℓ to be an equilibrium offer and, as we know, it is also a sufficient condition[14]. The matter is less trivial here since the auditing probability of any insurer i may be changed if insurer j deviates from her equilibrium contract offer. Formally, we may have $\overline{p}^i \equiv \widetilde{p}^i(\widetilde{C}) \neq \widetilde{p}^i(C^j, \widetilde{C}^{-j})$ if $C^j \neq \widetilde{C}^j$. In such a case, we may conceive that an inefficient feasible allocation cannot be destabilized by insurer j because the change in p^i makes the deviation unprofitable. More explicitly, if $\widetilde{p}^i(C^j, \widetilde{C}^{-j}) < \overline{p}^i$, then h-types may decide to choose $\widetilde{C}^i$ after C^j is offered in deviation, even if the new offer (provided that it attracts everybody) Pareto-dominates the equilibrium allocation. This may make the deviation unprofitable. More explicitly, consider a contract $\widetilde{C}^i_\ell = (\widetilde{k}^i_\ell, \widetilde{x}^i_\ell)$ not chosen on the equilibrium path (i.e. $\overline{\sigma}^i_{h\ell} = \overline{\sigma}^i_{\ell\ell} = 0$) and such that $\overline{p}^i > 0$ and $U_{h\ell}(\widetilde{C}^i_\ell, \overline{p}^i) < U^e_h < U_h(\widetilde{C}^i_\ell)$. In words, at equilibrium h-types are detered from choosing $\widetilde{C}^i_\ell$ because they fear they may be audited. In some circumstances, $\widetilde{C}^i_\ell$ may act as an implicit threat to prevent deviant insurer j to attract h-types. Let us focus on the case where $U^e_h < U_h(C^j_h) < U^*_h, U_\ell(C^j_h) < U^e_\ell, \Pi_h(C^j_h) > 0$ and $C^j_\ell = (0,0)$, where U^e_ℓ denotes the ℓ-type equilibrium expected utility. In words, insurer j aims at making profit by attracting h-types and her offer is strictly dominated for ℓ-types[15].

Suppose first that $U_\ell(\widetilde{C}^i_\ell) < U^e_\ell$. In that case only h-types may choose $\widetilde{C}^i_\ell$ after insurer j's deviation. We know from Lemma 1 that $\overline{p}^i > 0$ requires $\widetilde{x}^i_\ell \geq c$. If $\widetilde{x}^i_\ell > c$ and if a h-type individual chooses $\widetilde{C}^i_\ell$ after the deviation (i.e. if $\widetilde{\sigma}^i_{h\ell}(C^j, \widetilde{C}^{-j}) > 0$), we would have $\widetilde{\mu}^i(C^j, \widetilde{C}^{-j}) = 1$ and thus $\widetilde{p}^i(C^j, \widetilde{C}^{-j}) = 1$, which makes $\widetilde{C}^i_\ell$ unattractive to h-types. However if $\widetilde{x}^i_\ell = c$, then insurer i is indifferent between auditing and not auditing when $\widetilde{C}^i_\ell$ is chosen by h-types only and (for instance) $\widetilde{p}^i(C^j, \widetilde{C}^{-j}) = 0, \widetilde{\sigma}^i_{h\ell}(C^j, \widetilde{C}^{-j}) = 1$ is a

[14] Of course, we here consider the version of the Rothschild-Stiglitz model where each insurer offers a menu of contracts. If each insurer can only offer one contract, then a Rothshild-Stiglitz equilibrium exists but is not second-best Pareto-optimal when $\widehat{\lambda} < \lambda < \lambda^*$.

[15] We here assume that contracts offered by other insurers $j' \neq j$ allow ℓ-types to still reach U^e_ℓ after the deviation.

continuation equilibrium strategy where insurer j does not attract h-types. Note however that in such a case, insurer i would play a weakly dominated strategy on the equilibrium path: indeed when $\widetilde{x}_\ell^i = c$ then $\overline{p}^i > 0$ is an optimal strategy of insurer i only if $\overline{\mu}^i = 1$, i.e. if $\widetilde{C}_\ell^i$ is chosen by h-types only. If ℓ-types may unintentionally choose $\widetilde{C}_\ell^i$ with a positive probability, then $\overline{p}^i = 0$ would be the only equilibrium strategy. In other words, errors in the ℓ-types' decisions jeopardize the use of auditing as an implicit threat to prevent deviations at the contract offer stage. Assuming that insurers play weakly dominated strategy is probably not very convincing and in our main results this possibility is ruled out by resorting to the trembling hand perfection criterion of Selten (1975)[16]. In what follows, we say that an equilibrium $\mathcal{E} = \{\widetilde{C}, \widetilde{p}(.), \widetilde{\sigma}_h(.), \widetilde{\sigma}_\ell(.), \widetilde{\mu}(.)\}$ satisfies the *THP* condition if $\{\overline{p}, \overline{\sigma}_h, \overline{\sigma}_\ell\}$ is a trembling hand perfect Bayesian equilibrium of the continuation subgame that follows the equilibrium offer $\widetilde{C}$, hence the following Lemma.

Lemma 3. *At any equilibrium $\mathcal{E} = \{\widetilde{C}, \widetilde{p}(.), \widetilde{\sigma}_h(.), \widetilde{\sigma}_\ell(.), \widetilde{\mu}(.)\}$ that satisfies THP, we have $\overline{p}^i = 0$ for all i such that $\widetilde{x}_\ell^i = c$*

Under the *THP* condition, if $\widetilde{x}_\ell^i = c$ then decreasing the audit probability of $\widetilde{C}_\ell^i$ cannot act as an implicit threat to prevent a deviant insurer to attract h-types. Any equilibrium allocation is then second-best Pareto-optimal (see Proposition 5). As we shall see later, this is not necessarily the case if the *THP* condition is not satisfied.

Suppose now $U_\ell(\widetilde{C}_\ell^i) = U_\ell^e$. Hence $\widetilde{C}_\ell^i$ belongs to the set of equilibrium contracts that are optimal for ℓ-type individuals. These individuals may conceivably change their contract choice following the new offer by insurer j, and in particular we may have $\widetilde{\sigma}_{\ell\ell}^i(C^j, \widetilde{C}^{-j}) > \overline{\sigma}_{\ell\ell}^i = 0$. Given this change in the way ℓ-types randomize between contracts, the equilibrium audit probabilities may also change. In particular, we may have $\widetilde{p}^i(C^j, \widetilde{C}^{-j}) < \overline{p}^i$ and $U_{h\ell}(\widetilde{C}_\ell^i, \widetilde{p}^i(C^j, \widetilde{C}^{-j})) > U_h(C_h^j)$. In such a case, h-types would not choose C_h^j - i.e. $\widetilde{\sigma}_{hh}^j(C^j, \widetilde{C}^{-j}) = 0$ - and insurer j wouldn't make any profit in the deviation. In fact, in this scenario, a change in strictly dominated contracts (i.e. the deviation from $\widetilde{C}^j$ to C^j) acts as a sunspot for ℓ-types: they modify the way they randomize between contracts although there is no change in the set of their optimal contracts. This is conceptually possible but not very convincing from the realism standpoint. In our main characterization of the equilibrium, we will dismiss this possibility by appealing to a Markov-type restriction on the ℓ-types strategy. We will say that a ℓ-type strategy $\sigma_\ell(.)$ is stable with respect to changes in strictly dominated strategies if $\sigma_\ell(C) = \sigma_\ell(C')$ when C and C' only differ through contracts that are strictly dominated for ℓ-types and we say that $\mathcal{E}$ satisfies *SDS* if $\widetilde{\sigma}_\ell(.)$ is stable with respect to changes in strictly dominated strategies. Restricting attention to the case where $\mathcal{E}$ satisfies *SDS* is in the spirit of the Markov Perfect Equilibrium (*MPE*) concept[17]. In an extensive-form game, a *MPE* is a profile of strategies that are a perfect equilibrium

[16]The definition of trembling hand perfection is reminded in the proof of Lemma 3 in the Appendix.
[17]See Fudenberg and Tirole (1991, Ch.13) and Maskin-Tirole (2001).

and that are measurable with respect to the payoff-relevant history: in other words, only changes in payoff-relevant past events can affect the players' strategy. The *SDS* condition is thus a variation on the *MPE* concept where the restriction on strategies only concerns ℓ-type individuals: their choices are not affected by changes in the offer of strictly dominated contracts.

Proposition 5. *Any equilibrium allocation that satisfies THP is second-best Pareto-optimal and it is an optimal solution to* $\mathbb{P}_1(\lambda, U_h^e, 0, ..., 0)$ *with* $U_h^e \leq U_h^*$. *If the equilibrium also satisfies SDS , then* $U_h^e = U_h^*$.

Propositions 4 with $u_0 = u_0' = U_h^e$ and Proposition 5 jointly show that under *THP* all insurers (at least those who attract customers) offer the same equilibrium contracts and they play the same auditing strategy as at the optimal solution to $\mathbb{P}_2(\lambda, c, U_h^e, 0)$. If *SDS* is postulated in addition, then $U_h^e = U_h^*$ and the equilibrium allocation is characterized as in Proposition 4 with $u_0' = U_h^*$, which means that the equilibrium is second-best Pareto optimal without cross-subsidization between contracts. In particular, since $U_h^* \in [\overline{U}_h, \widehat{u}_h(\lambda, c)]$, all individuals purchase insurance at equilibrium.

Under *THP* and *SDS*, only two types of equilibrium may thus generically occur. When $U_\ell(C_\ell^{**}) > U_\ell(\widehat{C}_\ell)$, a candidate equilibrium is such that h-types choose C_h^* and ℓ-types choose C_ℓ^{**}: it is a separating equilibrium without auditing. When $U_\ell(C_\ell^{**}) < U_\ell(\widehat{C}_\ell)$, a candidate equilibrium is such that h-types randomize between C_h^* and $\widehat{C}_\ell$ while ℓ-types only choose $\widehat{C}_\ell$: it is a semi-separating equilibrium. The risk type of $\widehat{C}_\ell$-claimants is then randomly audited and the audit probability makes h-types indifferent between C_h^* and $\widehat{C}_\ell$.

For all i, let $\overline{N}^i = \overline{\sigma}_{hh}^i + \overline{\sigma}_{h\ell}^i + \overline{\sigma}_{\ell h}^i + \overline{\sigma}_{\ell\ell}^i$ with $\overline{N}^0 = 0$ and $\overline{\lambda}^i = (\overline{\sigma}_{hh}^i + \overline{\sigma}_{h\ell}^i)/\overline{N}^i$ if $\overline{N}^i > 0$, with $\sum_{i=1}^n \overline{N}^i = 1$ and $\sum_{i=1}^n \overline{N}^i \overline{\lambda}^i = \lambda$. Propositions 6 and 7 provide necessary and sufficient conditions for a separating equilibrium and for a semi-separating equilibrium to exist.

Proposition 6. *Under THP and SDS, there exists a separating equilibrium* $\mathcal{E} = \{\widetilde{C}, \widetilde{p}(.), \widetilde{\sigma}_h(.), \widetilde{\sigma}_\ell(.), \widetilde{\mu}(.)\}$ *if and only if* $c \geq c^*$ *and* $\lambda \geq \lambda^*$. *The separating equilibrium allocation coincides with the Rothschild-Stiglitz allocation and there is no type verification on the equilibrium path, i.e.* $\widetilde{C}_h^i = C_h^*, \widehat{\sigma}_{hh}^i = 1, \widehat{\sigma}_{h\ell}^i = 0$ *if* $\overline{N}^i \overline{\lambda}^i > 0$ *and* $\widetilde{C}_\ell^i = C_\ell^{**}, \widehat{\sigma}_{\ell h}^i = 0, \widehat{\sigma}_{\ell\ell}^i = 1$ *if* $\overline{N}^i (1 - \overline{\lambda}^i) > 0$.

We know from Propositions 4 and 5 that under *THP* and *SDS* a separating equilibrium is such that $\widetilde{C}_h^i = C_h^*$ and $\widetilde{C}_\ell^i = C_\ell^{**}$ for any insurer i that attracts h-types and ℓ-types. In other words, the separating equilibrium contracts coincide with the Rothschild-Stiglitz pair of contracts. A separating equilibrium requires $U_\ell(C_\ell^{**}) \geq U_\ell(\widehat{C}_\ell)$, or equivalently $U_\ell(C_\ell^{**}) \geq \Phi_4(\lambda, c, U_h^*, 0)$. Under this inequality, a deviant insurer i cannot make positive profit by offering a menu C^i with auditing in the continuation equilibrium, i.e. with $\widetilde{p}^i(C^i, \widetilde{C}^{-i}) > 0$. For a separating equilibrium to exist it should also be impossible for insurer i to make profit by attracting all individuals without auditing, i.e. with $\widetilde{p}^i(C^i, \widetilde{C}^{-i}) = 0$, which requires $U_\ell(C_\ell^{**}) = \Phi_3(\lambda, c, U_h^*, 0)$. Hence a necessary condition for a separating equilibrium is

$$U_\ell(C_\ell^{**}) = \Phi_3(\lambda, c, U_h^*, 0) \geq \Phi_4(\lambda, c, U_h^*, 0),$$

or equivalently $\lambda \geq \lambda^*, c \geq c^*$. Conversely, as shown in the proof of Proposition 6, under this condition, any deviation from $\widetilde{C}^i = (C_h^*, C_\ell^{**})$ to another menu C^i is unprofitable at a continuation equilibrium. Hence a separating equilibrium exists if and only if (λ, c) is in the area I of Figure 5, boundary line included.

Proposition 7. *Under THP and SDS, there exists a semi-separating equilibrium* $\mathcal{E} = \{\widetilde{C}, \widetilde{p}(.),\ \widetilde{\sigma}_h(.), \widetilde{\sigma}_\ell(.), \widetilde{\mu}(.)\}$ *if and only if* $c \leq c^*$, $\lambda \geq \lambda^*$ *or* $c \leq \widetilde{c}(\lambda)$, $\lambda < \lambda^*$. *At a semi-separating allocation,* ℓ*-types choose* $\widehat{C}_\ell$ *while* h*-types randomize between* C_h^* *and* $\widehat{C}_\ell$ *and the risk type of* $\widehat{C}_\ell$*-claimants is audited with positive probability. For all* i *such that* $\overline{N}^i > 0$*, we have*

$$\begin{aligned}
\widetilde{C}_\ell^i &= \widehat{C}_\ell, U_{h\ell}(\widehat{C}_\ell, \overline{p}^i) = U_h^*, \widehat{\sigma}_{h\ell}^i = K(\widehat{x}_\ell, \overline{\lambda}^i, c) \in (0,1], \widehat{\sigma}_{\ell h}^i = 0, \widehat{\sigma}_{\ell\ell}^i = 1 \quad \text{if } \overline{\lambda}^i < 1, \\
\widetilde{C}_h^i &= C_h^*, \widehat{\sigma}_{hh}^i = 1 - K(x_\ell, \overline{\lambda}^i, c) \quad \text{if } \overline{\lambda}^i < 1, K(x_\ell, \overline{\lambda}^i, c) < 1 \\
\widetilde{C}_h^i &= C_h^*, \widehat{\sigma}_{hh}^i = 1, \widehat{\sigma}_{h\ell}^i = 0 \quad \text{if } \overline{\lambda}^i = 1.
\end{aligned}$$

Any semi-separating equilibrium is such that $\widetilde{C}_h^i = C_h^*$ and $\widetilde{C}_\ell^i = \widehat{C}_\ell$ for any contract that attracts customers. The proportion of h-types among the individuals who choose a C_ℓ^i contract is the same for all i: it is equal to $c\pi_\ell/[c\pi_\ell + \pi_h(\widehat{x}_\ell - c)]$. When $\overline{\lambda}^i$ is larger than this proportion (i.e. when $K(x_\ell, \overline{\lambda}^i, c) < 1$) then other h-type customers of insurer i choose $\widetilde{C}_h^i$[18]. As shown in Proposition 3, $\widehat{C}_\ell$ involves overinsurance, which is in sharp contrast with the Rothschild-Stiglitz separating equilibrium[19]. A semi-separating equilibrium requires $U_\ell(\widehat{C}_\ell) \geq U_\ell(C_\ell^{**})$, or equivalently $\Phi_4(\lambda, c, U_h^*, 0) \geq U_\ell(C_\ell^{**})$. In that case, a deviant insurer cannot make profit by offering a pair of incentive compatible contracts without cross-subsidization. The existence of a semi-separating equilibrium also requires that a deviant insurer cannot make profit by cross-subsidizing incentive compatible contracts, which may be written as

$$\Phi_4(\lambda, c, U_h^*, 0) \geq \Phi_3(\lambda, c, U_h^*, 0),$$

[18] Since $K(\widehat{x}_\ell, \lambda, c) < 1$, there is an infinite number of possible distributions of individuals among insurers.

[19] In practice, the insureds' moral hazard may make insurers reluctant to offer such overinsurance contracts. The optimal contract would then trade off the incentives to costly risk verification and the mitigation of insureds' moral hazard. For instance, if we simply impose that claims shouldn't be overpaid, the semi-separating equilibrium is at point F on Figure 2, with full coverage of losses.

and equivalently $\lambda > \lambda^*, c \leq c^*$ or $\lambda \leq \lambda^*, c \leq \widetilde{c}(\lambda)$ as stated in Proposition 7. Conversely, under this condition, any deviation from $\widetilde{C}^i = (C_h^*, \widehat{C}_\ell)$ to C^i is unprofitable at a continuation equilibrium. Hence a semi-separating equilibrium exists when (λ, c) is in the area II of Figure 5, with its boundary line. Finally, no equilibrium exists in area III. An equilibrium allocation is thus a second-best Pareto optimal allocation that breaks even.

We may conclude these comments on Propositions 6 and 7 with some straightforward but important remarks. Firstly, if the insurers were not allowed to void the contract when misrepresentation is established, then an equilibrium would exist only if $\lambda \geq \lambda^*$ as in the standard Rothschild-Stiglitz model. Hence allowing the insurers to void the contract enlarges the set of parameters for which an equilibrium exists. The smaller the verification cost c, the smaller the threshold for λ above which an equilibrium exists. Equivalently, for any λ, an equilibrium always exists if c is small enough. If c were equal to zero, uncertainty on the insureds' risk type would vanish and competition on the insurance market would lead to type separation and full insurance at fair price. When c goes to zero, then $\widehat{C}_\ell$ goes to C_ℓ^* without any discontinuity at $c = 0$: the equilibrium semi-separating allocation then converges to the full information solution. On the contrary, there is a discontinuity in insurance coverage and premium when c reaches the threshold c^* since we go from partial coverage in the separating equilibrium area I to overinsurance in the semi-separating equilibrium area II. Last but not least, the semi-separating equilibrium (when it exists) Pareto-dominates the Rothschild-Stiglitz equilibrium since the welfare of high risk individuals is increased while the low risks' expected utility is unchanged. All things considered, although insurers cannot commit on their verification strategy, allowing them to void the contract improves efficiency in the market and makes existence of equilibrium more likely.

Proposition 5 suggests that an equilibrium that does not satisfy *THP* may not be second-best Pareto efficient. This is actually the case as shown by the following example. Assume $n = 4, \lambda > \lambda^*$ and $c > c^*$. For $i = 1$ or 2, let $\widetilde{C}_h^i = (\widehat{\pi}_h \widehat{A}, \widehat{A} - \widehat{\pi}_h \widehat{A})$ with $\widehat{\pi}_h > \pi_h, \widehat{A} \neq A$ and $\widetilde{C}_\ell^i$ is such that

$$U_h(\widetilde{C}_\ell^i) = U_h(\widetilde{C}_h^i) \quad \text{and } (1-\lambda)\Pi_\ell(\widetilde{C}_\ell^i) + \lambda(\widehat{\pi}_h - \pi_h)\widehat{A} = 0.$$

When $\widehat{\pi}_h - \pi_h$ and $\widehat{A} - A$ goes to 0, $(\widetilde{C}_h^1, \widetilde{C}_\ell^1)$ and $(\widetilde{C}_h^2, \widetilde{C}_\ell^2)$ converge to (C_h^*, C_ℓ^{**}) which is the optimal solution to $\mathbb{P}_2(\lambda, c, U_h^*, 0)$. For $i = 3$ or 4, let $\widetilde{C}_h^i = (0, 0)$ and $\widetilde{C}_\ell^i = (\widetilde{k}_\ell^i, \widetilde{x}_\ell^i) = (\widetilde{k}_\ell^i, c)$, where $\widetilde{k}_\ell^i$ is such that

$$U_h(\widetilde{C}_\ell^i) > U_h^* \quad \text{and} \quad U_\ell(\widetilde{C}_\ell^i) < U_\ell(\widetilde{C}_\ell^1) = U_\ell(\widetilde{C}_\ell^2).$$

Let $\overline{\sigma}_{hh}^i = \overline{\sigma}_{\ell\ell}^i = 1/2$ for $i = 1$ or $2, \overline{p}^1 = \overline{p}^2 = 0, \overline{p}^3 = \overline{p}^4 = 1, \overline{\mu}^1 = \overline{\mu}^2 = 0$ and $\overline{\mu}^3 = \overline{\mu}^4 = 1$. In words, on the equilibrium path h -types choose $\widetilde{C}_h^1$ or $\widetilde{C}_h^2$ and ℓ -types choose $\widetilde{C}_\ell^1$ or $\widetilde{C}_\ell^2$ and nobody chooses the contracts offered by insurers $i = 3$ or 4. h-type individuals are detered from choosing $\widetilde{C}_\ell^3$ or $\widetilde{C}_\ell^4$ because $\overline{p}^3 = \overline{p}^4 = 1$ and this auditing strategy is optimal for insurers 3 and 4 given the out of equilibrium beliefs $\overline{\mu}^3 = \overline{\mu}^4 = 1$. The couples of contracts $\widetilde{C}_h^1$, $\widetilde{C}_\ell^1$ and $\widetilde{C}_h^2$, $\widetilde{C}_\ell^2$ are incentive compatible and no auditing is performed on the equilibrium path. Furthermore there is cross-subsidization between $\widetilde{C}_h^1$ and $\widetilde{C}_\ell^1$: insurer 1 makes profit with $\widetilde{C}_h^1$ and losses with $\widetilde{C}_\ell^1$. Likewise for insurer 2. When $\widehat{\pi}_h - \pi_h$ is not too large, then $\mathcal{A} \equiv \{\widetilde{C}_h^i, \widetilde{C}_\ell^i, \overline{p}^i, \overline{\sigma}_{hh}^i, \overline{\sigma}_{h\ell}^i, \overline{\sigma}_{\ell h}^i, \overline{\sigma}_{\ell\ell}^i; i = 1, ..., 4\}$ is not second-best Pareto-optimal since $\widehat{A} \neq A$. However, for any deviation to a pair of contracts that Pareto-dominates $\mathcal{A}$, there exists continuous equilibrium strategies that make it non-profitable. Consider for example the case where insurer 1 deviates from $\widetilde{C}^1 = (\widetilde{C}_h^1, \widetilde{C}_\ell^1)$ to $C^1 = (C_h^1, C_\ell^1)$ with $C_h^1 = (\widehat{\pi}_h' A, A - \widehat{\pi}_h' A)$, $\pi_h < \widehat{\pi}_h' < \widehat{\pi}_h$ and C_ℓ^1 is such that $U_h(C_\ell^1) = U_h(C_h^1)$ and $(1-\lambda)\Pi_\ell(C_\ell^1) + \lambda(\widehat{\pi}_h' - \pi_h)A > 0$. For $\widehat{\pi}_h' - \pi_h$ small enough, we have $U_h(C_h^1) > U_h^e$ and $U_\ell(C_\ell^1) > U_\ell^e$, where U_h^e and U_ℓ^e are the expected utility of h-types and ℓ-types at $\mathcal{A}$. Intuitively, insurer 1 aims at attracting h-types through C_h^1 and ℓ-types through C_ℓ^1. Consider the strategy $\widetilde{p}^3(C^1, \widetilde{C}^{-1}) = 0$ and beliefs $\widetilde{\mu}^3(C^1, \widetilde{C}^{-1}) = 1$ for all $C^1 \neq \widetilde{C}^1$. Note that these beliefs are consistent with the strategy of the individuals since $\widetilde{C}_\ell^3$ is not chosen by ℓ-types (they prefer $\widetilde{C}_\ell^1$ or $\widetilde{C}_\ell^2$ to $\widetilde{C}_\ell^3$): $\widetilde{C}_\ell^3$ can only be chosen by h-types. Given these beliefs, $\widetilde{p}^3(C^1, \widetilde{C}^{-1}) = 0$ is an optimal strategy of insurer 3 because $\widetilde{x}_\ell^i = c$. $\widetilde{\sigma}_{h\ell}^3(C^1, \widetilde{C}^{-1}) = 1$ is then an optimal strategy of h-types: they do not choose C_h^1 and consequently the deviation is unprofitable. Symmetrically, there exists continuation equilibrium strategies that make any deviation by insurer 2 non profitable. In case of deviation by insurer 3, $\widetilde{p}^4(C^3, \widetilde{C}^{-3}) = 0$ is a continuation equilibrium strategy of insurer 4, and here also the deviation cannot attract h-types. Likewise, $\widetilde{p}^3(C^4, \widetilde{C}^{-4}) = 0$ is a continuation equilibrium strategy of insurer 3 that make any deviation by insurer 4 non profitable. In this example, insurers 3 and 4 play a weakly dominated strategy on the equilibrium path: they choose $\overline{p}^3 = \overline{p}^4 = 1$, which is an optimal strategy for their out-of-equilibrium beliefs $\overline{\mu}^3 = \overline{\mu}^4 = 1$. Decreasing p^3 from 1 to 0, after any deviation by insurer $i \neq 3$ is an implicit threat that prevents the deviation to be profitable. Such an equilibrium would vanish if insurers do not play weakly dominated strategies, which is the case under the *THP* condition.

Proposition 5 also suggests that we may have $U_h^e < U_h^*$ (and thus positive profit on h-types) at an equilibrium that does not satisfy *SDS*. This is true, as shown by the following example. Assume $n = 4$ and either $c < c^*, \lambda \geq \lambda^*$ or $c < \widetilde{c}(\lambda), \lambda < \lambda^*$. For $i = 1$ or 2, let $(\widetilde{C}_h^i, \widetilde{C}_\ell^i)$ be the optimal solution to $\mathbb{P}_4(\lambda, c, u(W_N - \widehat{\pi}_h A), 0)$ with $\widehat{\pi}_h > \pi_h$. When $\widehat{\pi}_h - \pi_h$ goes to 0, $(\widetilde{C}_h^1, \widetilde{C}_\ell^1)$ and $(\widetilde{C}_h^2, \widetilde{C}_\ell^2)$ converge to $(C_h^*, \widehat{C}_\ell)$ which is the optimal solution to $\mathbb{P}_2(\lambda, c, U_h^*, 0)$. For $i = 3$ or 4, let $\widetilde{C}_h^i = (0, 0)$ and $\widetilde{C}_\ell^i = (\widetilde{k}_\ell^i, \widetilde{x}_\ell^i)$ such that

$$U_\ell(\widetilde{C}_\ell^i) = U_\ell(\widetilde{C}_\ell^1) = U_\ell(\widetilde{C}_\ell^2), U_h(\widetilde{C}_\ell^i) > U_h^* \text{ and } c < \widetilde{x}_\ell^i < \frac{c[(1-\lambda)\pi_\ell + \lambda\pi_h]}{\lambda\pi_h}$$

For $i = 1$ or 2, let $\overline{\sigma}^i_{h\ell} = K(\widetilde{x}^i_\ell, \lambda, c)/2$, $\overline{\sigma}^i_{hh} = [1 - K(\widetilde{x}^i_\ell, \lambda, c)]/2$, $\overline{\sigma}^i_{\ell\ell} = 1/2, \overline{\sigma}^i_{\ell h} = 0$ and $U_{h\ell}(\widetilde{C}^i_\ell, \overline{p}^i) = U_h(\widetilde{C}^i_h)$. For $i = 3$ or 4, let $\overline{\sigma}^i_{h\ell} = \overline{\sigma}^i_{hh} = \overline{\sigma}^i_{\ell\ell} = \overline{\sigma}^i_{\ell h} = 0, \overline{\mu}^i = 1$ and $\overline{p}^i = 1$.

$\mathcal{A} \equiv \{\widetilde{C}^i_h, \widetilde{C}^i_\ell, \overline{p}^i, \overline{\sigma}^i_{hh}, \overline{\sigma}^i_{h\ell}, \overline{\sigma}^i_{\ell h}, \overline{\sigma}^i_{\ell\ell}; i = 1, ..., 4\}$ is second-best Pareto-optimal if $\widehat{\pi}_h - \pi_h$ is not too large. The only way to make a profitable deviation is to attract only h-types. Consider for example the case where insurer 1 deviates from $\widetilde{C}^1 = (\widetilde{C}^1_h, \widetilde{C}^1_\ell)$ to another pair of contracts $C^1 = (C^1_h, C^1_\ell)$ that attracts h-types, but not ℓ-types. We may assume w.l.o.g. that $C^1_\ell = (0, 0)$. A necessary condition for this deviation to be profitable is $U_h(C^1_h) < U^*_h$. Let $\widetilde{\sigma}^i_{h\ell} \equiv \widetilde{\sigma}^i_{h\ell}(C^1, \widetilde{C}^{-1})$, $\widetilde{\sigma}^i_{\ell\ell} \equiv \widetilde{\sigma}^i_{h\ell}(C^1, \widetilde{C}^{-1})$ and $\widetilde{p}^i \equiv \widetilde{p}^i(C^1, \widetilde{C}^{-1})$ for $i = 1, ..., 4$. This continuation equilibrium strategy profile (and the corresponding beliefs $\widetilde{\mu}^i \equiv \widetilde{\mu}^i(C^1, \widetilde{C}^{-1})$) can be chosen in such a way that h-types and ℓ-types randomize between $\widetilde{C}^2_\ell$ or $\widetilde{C}^3_\ell$ and they choose neither C^1 nor $\widetilde{C}^4_\ell$. Indeed, let $\widetilde{\sigma}^2_{\ell\ell} + \widetilde{\sigma}^3_{\ell\ell} = 1$. Choose $\widetilde{p}^2$ and $\widetilde{p}^3$ such that $U_h(\widetilde{C}^i_\ell, \widetilde{p}^i) = U^*_h$ for $i = 2$ and 3 and $\widetilde{p}^1 = \widetilde{p}^4 = 1$. Given the ℓ-types' contract choice strategy, the insurers' auditing strategy and the h-types' contract choice strategy are mutual best responses when $\widetilde{\sigma}^2_{h\ell} + \widetilde{\sigma}^3_{h\ell} = 1, \widetilde{\sigma}^2_{h\ell} = \widetilde{\sigma}^2_{\ell\ell}\widetilde{K}^2$ and $\widetilde{\sigma}^3_{h\ell} = \widetilde{\sigma}^3_{\ell\ell}\widetilde{K}^3$, where $\widetilde{K}^i \equiv K(\widetilde{x}^i_\ell, \lambda, c)$ for $i = 2$ and 3 and $\widetilde{K}^2 < 1 < \widetilde{K}^3$. These conditions are fulfilled when $\widetilde{\sigma}^2_{h\ell} = \widetilde{K}^2(\widetilde{K}^3 - 1)/(\widetilde{K}^3 - \widetilde{K}^2)$, $\widetilde{\sigma}^3_{h\ell} = \widetilde{K}^3(1 - \widetilde{K}^2)/(\widetilde{K}^3 - \widetilde{K}^2)$, $\widetilde{\sigma}^2_{\ell\ell} = (\widetilde{K}^3 - 1)/(\widetilde{K}^3 - \widetilde{K}^2)$ and $\widetilde{\sigma}^3_{\ell\ell} = (1 - \widetilde{K}^2)/(\widetilde{K}^3 - \widetilde{K}^2)$. The same kind of continuation equilibrium exists in case of a deviation by insurers 2, 3 or 4. In this equilibrium, $\sigma_\ell(.)$ is not stable with respect to changes in strongly dominated strategies because once C_1 is proposed by insurer 1 in deviation from equilibrium, then ℓ-types choose $\widetilde{C}^2_\ell$ or $\widetilde{C}^3_\ell$ while they chose $\widetilde{C}^1_\ell$ or $\widetilde{C}^2_\ell$ before the deviation. This equilibrium is Pareto-optimal but $U^e_h < U^*_h$ and insurers 1 and 2 cross-subsidize their contracts. Such an equilibrium would vanish under the *SDS* condition.

Note finally that in these two examples, $\widehat{\pi}_h$ may be chosen such that $U_h(C^i_h) = \overline{U}_h$ for $i = 1, 2$, so that h-types do not draw any surplus from insurance. Simple variations on the examples would lead to market equilibria where h-types randomize between $\widetilde{C}^1_h$ or $\widetilde{C}^2_h$ and no-insurance. However, given the lack of robustness of the underlying strategy, such equilibria should probably be considered a theoretical curiosity rather than a realistic view of the insurance market.

6 Conclusion

The good faith principle is a major pillar of the law of insurance contracts. It states that insureds have a duty of good faith and it allows insurers to rescind contracts *ex post* when intentional misrepresentation of risk is established. Thereby it contributes to more efficient risk sharing in insurance markets under asymmetric information. However the effects of the good faith principle may conceivably be weakened or even cancelled by a credibility constraint on the verification strategy.

In order to better understand the effects of this credibility constraint, we have analyzed the equilibrium of an insurance market where applicants for insurance have a duty of good faith when revealing their risk type and insurers cannot precommit to their risk verification policy. Three main results have been reached. Firstly, the equilibrium qualitatively differs from the one that prevails in the standard Rothschild-Stiglitz model : here it may be either separating or semi-separating. At a semi-separating equilibrium, there is some degree of bad faith from high risk individuals : they do not always reveal their risk type truthfully. Furthermore, low risk individuals get overinsurance at a semi-separating equilibrium, contrary to the main prediction of the standard Rothschild-Stiglitz model. Secondly, the possibility of cancealing the contract when bad faith is established extends the set of parameters for which a competitive equilibrium exists. In particular, an equilibrium always exists if the verification cost is low enough. Thirdly, the good faith principle remains Pareto-improving in comparison with the Rothschild - Stiglitz equilibrium, although insurers are deprived of any possibility of precommitment in their risk verification strategy.

We have approached these issues in two stages. The first stage consisted in characterizing second-best Pareto optimal allocations and, in a second stage, we have shown that, under adequate assumptions, the equilibrium allocation is second-best Pareto-optimal without cross-subsidization between contracts. In a sense, this is a very natural result. Intuitively, an equilibrium allocation is necessarily second-best Pareto optimal for otherwise it would be possible to offer a profitable menu of contracts that would attract all the individuals. Furthermore, an equilibrium allocation does not cross-subsidize contracts for otherwise it would be to the insurers' advantage to delete the contract in deficit. Although the general principle of this argument is true, it requires careful attention. The two main difficulties were firstly to establish the symmetry of second-best Pareto-optimal allocations and secondly to characterize the precise conditions under which an equilibrium allocation is second-best Pareto optimal without cross-subsidization. This roundabout way through second-best Pareto-optimality is not trivial and one may find it somewhat tedious, but we think it is an adequate way to characterize the equilibrium of a market under adverse selection. Hopefully a similar approach may be useful for the analysis of other markets with adverse selection where agents interact after the contract offer stage, such as the credit market or the labour market.

Appendix

This Appendix gathers the proofs of the Lemmas, Propositions and Corollary stated in the paper. Lemmas 4 to 8 and Corollaries 2 to 4 are intermediate stages of the proofs.

Proof of Lemma 1: Consider an equilibrium $\mathcal{E}$. (5) gives

$$\widetilde{p}^i(C) = 0 \text{ (resp. } \in [0,1], = 1) \text{ if } \widetilde{\mu}^i(C)x^i_\ell < c \text{ (resp. } = c, > c) \text{ for all } i \text{ and all } C. \quad (45)$$

Assume that $\widetilde{\sigma}^i_{h\ell}(C) + \widetilde{\sigma}^i_{\ell\ell}(C) > 0$. If $\widetilde{p}^i(C) = 1$, then $U_{h\ell}(C^i_\ell, \widetilde{p}^i(C)) < \overline{U}_h$ and thus $\widetilde{\sigma}^i_{h\ell}(C) = 0$. We get $\widetilde{\mu}^i(C) = 0$ from (7) and then (45) gives $\widetilde{p}^i(C) = 0$, hence a contradiction. We thus have $\widetilde{p}^i(C) < 1$. Using (45) then gives (8). When $\widetilde{\sigma}^i_{h\ell}(C) = 0$ and $\widetilde{\sigma}^i_{\ell\ell}(C) > 0$, we have $\widetilde{\mu}^i(C) = 0$ from (7) and then(45) gives $\widetilde{p}^i(C) = 0$.

Proof of Lemma 2: Using (8) yields

$$\begin{aligned} & \lambda\widetilde{\sigma}^i_{h\ell}(C)\Pi_{\ell h}(C^i_\ell, \widetilde{p}^i(C)) + (1-\lambda)\widetilde{\sigma}^i_{\ell\ell}(C)\Pi_{\ell\ell}(C^i_\ell, \widetilde{p}^i(C)) \\ = \; & \lambda\widetilde{\sigma}^i_{h\ell}(C)\Pi_{\ell h}(C^i_\ell, 0) + (1-\lambda)\widetilde{\sigma}^i_{\ell\ell}(C)\Pi_{\ell\ell}(C^i_\ell, 0). \end{aligned}$$

(9) then follows from $\Pi_{\ell h}(C^i_\ell, 0) = \Pi_h(C^i_\ell)$ and $\Pi_{\ell\ell}(C^i_\ell, 0) = \Pi_\ell(C^i_\ell)$.

Proof of Proposition 1: (i) and (ii). $\mathbb{P}_3(\lambda, c, u_0, \Pi_0)$ is obtained by imposing $\widehat{p} = 0$ and deleting (20) in $\mathbb{P}_2(\lambda, c, u_0, \Pi_0)$. Assume $\widehat{\sigma}_{hh} > 0$ and $\widehat{\sigma}_{h\ell} > 0$ in $\mathbb{P}_3(\lambda, c, u_0, \Pi_0)$. (22), (24) and $\widehat{p} = 0$ then give $U_h(C_h) = U_{h\ell}(C_\ell, 0) = U_h(C_\ell) \geq u_0$. Indifference curves of h-types and ℓ-types cross only once. Hence we have either $U_\ell(C_\ell) > U_\ell(C_h)$ or $U_\ell(C_\ell) < U_\ell(C_h)$. Assume $U_\ell(C_\ell) > U_\ell(C_h)$. (23) then gives $\widehat{\sigma}_{\ell\ell} = 1$ and $\widehat{\sigma}_{\ell h} = 0$. Let $C'_h = (k'_h, x'_h)$ be defined by $C'_h = \widehat{\sigma}_{h\ell}C_\ell + \widehat{\sigma}_{hh}C_h$. Using $u'' < 0$ gives

$$U_h(C'_h) > U_h(C_h) = U_h(C_\ell) \geq u_0. \quad (46)$$

Furthermore

$$\Pi_h(C'_h) = \widehat{\sigma}_{hh}\Pi_h(C_h) + \widehat{\sigma}_{h\ell}\Pi_h(C_\ell). \quad (47)$$

Hence there exists $C''_h = (k''_h, x''_h), k''_h > k'_h, x''_h < x'_h$ such that[20]

$$\begin{aligned} \Pi_h(C''_h) &> \widehat{\sigma}_{hh}\Pi_h(C_h) + \widehat{\sigma}_{h\ell}\Pi_h(C_\ell), & (48)\\ U_h(C''_h) &= U_h(C_h) = U_h(C_\ell) \geq u_0, & (49)\\ U_\ell(C_h) &< U_\ell(C''_h) < U_\ell(C_\ell). & (50) \end{aligned}$$

[20]This can be checked by drawing the h-type and ℓ-type indifference curves going through C_ℓ in the (W_1, W_2) plane with $U_\ell(C_\ell) > U_\ell(C_h)$.

$\widehat{\sigma}_{\ell\ell} = 1$, (19) and (48) show that there exists C'_ℓ in a neighbourhood of C_ℓ such that

$$\lambda\Pi_h(C''_h) + (1-\lambda)\Pi_\ell(C'_\ell) \geq \Pi_0, \tag{51}$$

$$U_\ell(C'_\ell) > U_\ell(C_\ell), \tag{52}$$

$$U_h(C_\ell) > U_h(C'_\ell). \tag{53}$$

We deduce from (50) and (52) that

$$U_\ell(C'_\ell) > U_\ell(C''_h). \tag{54}$$

Lastly (49) and (53) give

$$U_h(C''_h) \geq u_0 \text{ and } U_h(C''_h) > U_h(C'_\ell). \tag{55}$$

Hence there exist C'_ℓ, C''_h such that (51), (54) and (55) are satisfied which shows that C_ℓ, C_h is dominated in $\mathbb{P}_3(\lambda, c, u_0, \Pi_0)$ by a feasible solution with $\widehat{\sigma}_{\ell\ell} = 1$. A similar conclusion is obtained when $U_\ell(C_\ell) < U_\ell(C_h)$ by inverting the roles of C_ℓ and C_h. Hence any feasible solution to $\mathbb{P}_3(\lambda, c, u_0, \Pi_0)$ where h-types randomize between contracts is dominated by a solution where they don't. We can thus restrict attention to solutions such that either $\widehat{\sigma}_{hh} = 1, \widehat{\sigma}_{h\ell} = 0$ or $\widehat{\sigma}_{h\ell} = 1$, $\widehat{\sigma}_{hh} = 0$.

Assume $\widehat{\sigma}_{hh} = 1$, $\widehat{\sigma}_{h\ell} = 0$: (22) and $\widehat{p} = 0$ then give $U_h(C_h) \geq U_h(C_\ell)$. Suppose $\widehat{\sigma}_{\ell h} > 0$ and $\widehat{\sigma}_{\ell\ell} > 0$. (23) then gives $U_\ell(C_\ell) = U_\ell(C_h)$. Let $C''_\ell = \widehat{\sigma}_{\ell\ell}C_\ell + \widehat{\sigma}_{\ell h}C_h$ with $U_h(C_h) \geq U_h(C''_\ell)$,$U_\ell(C''_\ell) > U_\ell(C_h)$ and $\Pi_\ell(C''_\ell) = \widehat{\sigma}_{\ell\ell}\Pi_\ell(C_\ell) + \widehat{\sigma}_{\ell h}\Pi_\ell(C_h)$. C''_ℓ, C_h is feasible in $\mathbb{P}_3(\lambda, c, u_0, \Pi_0)$, with ℓ-types choosing C''_ℓ and h-types choosing C_h. Since $U_\ell(C''_\ell) > U_\ell(C_\ell)$ we get a contradiction. The same argument is valid when $\widehat{\sigma}_{h\ell} = 1$. Hence neither ℓ-types nor h-types randomize at an optimal solution to $\mathbb{P}_3(\lambda, c, u_0, \Pi_0)$. This problem is then written as in part (i) of the Proposition by calling C_h the contract chosen by h-types and C_ℓ the contract chosen by ℓ-types: we then have $\widehat{\sigma}_{hh} = 1$, $\widehat{\sigma}_{h\ell} = 0, \widehat{\sigma}_{\ell\ell} = 1$ and $\widehat{\sigma}_{\ell h} = 0$. Finally we check that (21) is satisfied in that case.

Let $k^0_j = k_j - \Pi_0, x^0_j = x_j + \Pi_0$ and $C^0_j = (k^0_j, x^0_j)$ for $j = h$ or ℓ. Let $u^0(W) \equiv u(W - \Pi_0)$ with $\overline{u}_0 = u^0(W_N - \overline{\pi}A)$ and let $U^0_j(k,x) \equiv (1-\pi_j)u^0(W_N - k) + \pi_j u^0(W_A + x)$ for $j = h$ or ℓ. $\mathbb{P}_3(\lambda, c, u_0, \Pi_0)$ is then written as : choosing $(C^0_h, C^0_\ell) \in \mathbb{R}^4_+$ so as to maximize $U^0_\ell(C^0_\ell)$ subject to $(1-\lambda)\Pi_\ell(C^0_\ell) + \lambda\Pi_h(C^0_h) \geq 0, U^0_\ell(C^0_\ell) \geq U^0_\ell(C^0_h), U^0_h(C^0_\ell) \leq U^0_h(C^0_h)$ and $U^0_h(C^0_h) \geq u_0$. This problem has a unique optimal solution which is characterized as in part (ii) of the Proposition[21].

(iii) is a consequence of the continuity of $\Phi_3(\lambda, c, u_0, \Pi_0)$ - which itself follows from the Maximum Theorem - and of the fact that given (27) and (28), there exists (C'_h, C'_ℓ)

[21]See Crocker and Snow (1985). The proof runs as follows. When $u_0 < \overline{u}_0$, we delete the ℓ-type incentive constraint (27). Maximizing $U_\ell(C_\ell)$ subject to the other constraints then gives an optimal solution characterized as in the part (ii) of the Proposition, and (27) is satisfied for this solution. Similarly, when $u_0 > \overline{u}_0$, (28) is deleted and there is an optimal solution to the relaxed problem : it is specified as in part (ii) of the Proposition and it satisfies (28).

in a neighbourhood of (C_h, C_ℓ) such that $U_\ell(C'_\ell) > U_\ell(C'_h)$ and $U_h(C'_h) > U_h(C'_\ell)$.

Proof of Corollary 1: We have $\Phi_3(\lambda, c, U_h^*, 0) \geq U_\ell(C_\ell^{**})$ for all λ because C_h^*, C_ℓ^{**} is feasible in $\mathbb{P}_3(\lambda, c, U_h^*, 0)$. Note that Φ_3 is continuous in λ from the Maximum Theorem. Assume that λ is such that $\Phi_3(\lambda, c, U_h^*, 0) > U_\ell(C_\ell^{**})$. We then have $U_h(C_h) > U_h^*$ and thus $\Pi_h(C_h) < 0$. (26) then gives $\Pi_\ell(C_\ell) > 0$ which implies $\partial\Phi_3(\lambda, c, u_0, 0)/\partial\lambda = \mu[\Pi_h(C_h) - \Pi_\ell(C_\ell)] < 0$ at any point of differentiability, where $\mu > 0$ is a Kuhn-Tucker multiplier associated with (26). We also have $\Phi_3(0, c, U_h^*, 0) = U_\ell(C_\ell^*) > U_\ell(C_\ell^{**})$. Hence there exists λ^* in $(0, 1]$ such that $\Phi_3(\lambda, c, U_h^*, 0) > U_\ell(C_\ell^{**})$ if $0 \leq \lambda < \lambda^*$ and $\Phi_3(\lambda, c, U_h^*, 0) = U_\ell(C_\ell^{**})$ if $\lambda^* \leq \lambda \leq 1$.

It remains to show that $\lambda^* < 1$. Assume that $v \equiv \Phi_3(\lambda, c, U_h^*, 0) - U_\ell(C_\ell^{**}) > 0$ and let $C_\ell = (k_\ell, x_\ell), C_h = (k_h, x_h)$ be an optimal solution to $\mathbb{P}_3(\lambda, c, U_h^*, 0)$ with $k_\ell + x_\ell < A, k_h + x_h = A$ and $k_h = \widehat{\pi}_h A$ with $\widehat{\pi}_h \leq \pi_h$. Condition $U_\ell(C_\ell) = U_\ell(C_\ell^{**}) + v$ may be equivalently written as $k_\ell = f(x_\ell, v)$ where function f is such that $\partial f/\partial x_\ell > 0$, $\partial^2 f/\partial x_\ell^2 < 0$ and $\partial f/\partial v < 0$. Condition $x_\ell + k_\ell \leq A$ is equivalent to $x_\ell \leq \overline{x}_\ell(v)$ where $\overline{x}_\ell(v)$ is defined by $\overline{x}_\ell(v) + f(\overline{x}_\ell(v)) = A$, with $\overline{x}'_\ell(v) > 0$ and we have $x_\ell^{**} \leq x_\ell \leq x_\ell^*$. Condition (28) is binding and gives $\widehat{\pi}_h = g(k_\ell, x_\ell)$ with $g'_1 > 0, g'_2 < 0$. Let $h(x_\ell, v) \equiv g(f(x_\ell, v), x_\ell)$ and let $\widetilde{\Pi}(x_\ell, v)$ be the expected profit after having substituted $k_\ell = f(x_\ell, v)$ and $\widehat{\pi}_h = h(x_\ell, v)$ into the LHS of (26):

$$\widetilde{\Pi}(x_\ell, v) = (1 - \lambda)[(1 - \pi_\ell) f(x_\ell, v) - \pi_\ell x_\ell] + \lambda[h(x_\ell, v) - \pi_h]A < \widetilde{\Pi}(x_\ell, 0),$$

with $\widetilde{\Pi}(x_\ell^{**}, 0) = 0$. We have $\partial\widetilde{\Pi}(x_\ell, 0)/\partial x_\ell < 0$ for all x_ℓ in $[x_\ell^{**}, x_\ell^*]$ if

$$\lambda > \overline{\lambda} \equiv \frac{(1 - \pi_\ell)\partial f(x_\ell^{**}, 0)/\partial x_\ell - \pi_\ell}{(1 - \pi_\ell)\partial f(x_\ell^{**}, 0)/\partial x_\ell - \pi_\ell - \overline{h}' A} \in (0, 1),$$

where $\overline{h}' = \max\{\partial h(x_\ell, 0)/\partial x_\ell \mid x_\ell \in [x_\ell^{**}, x_\ell^*]\} < 0$. Hence $\widetilde{\Pi}(x_\ell, v) < 0$ for all x_ℓ in $[x_\ell^{**}, x_\ell^*]$ if $v > 0$ and $\lambda > \overline{\lambda}$. Consequently $v > 0$ implies $\lambda \leq \overline{\lambda}$, which gives $\lambda^* \leq \overline{\lambda} < 1$.

Corollary 2. $\Phi_3(\lambda, c, u_0, \Pi_0)$ *is decreasing in* Π_0*, stationary in* c *and non-increasing in* u_0*. It is locally decreasing (respect. locally non-increasing) in* λ *if* $u_0 > U_h^0$*(respect.=* U_h^0*).*

Proof: Let C_h, C_ℓ be the optimal solution to $\mathbb{P}_3(\lambda, c, u_0, \Pi_0)$: it is characterized in Proposition 1. For $\varepsilon > 0$, small enough, there exists C'_ℓ in a neighbourhood of C_ℓ such that $U_\ell(C'_\ell) > U_\ell(C_\ell), U_h(C'_\ell) < U_h(C_\ell)$ and $(1 - \lambda)\Pi_\ell(C'_\ell) + \lambda\Pi_h(C_h) \geq \Pi_0 - \varepsilon$. Hence C_h, C'_ℓ is feasible in $\mathbb{P}_3(\lambda, c, u_0, \Pi_0 - \varepsilon)$, which shows that $\Phi_3(\lambda, c, u_0, \Pi_0 - \varepsilon) > \Phi_3(\lambda, c, u_0, \Pi_0)$. Hence Φ_3 is increasing in Π_0. Obviously $\Phi_3(\lambda, c, u_0, \Pi_0)$ is independent from c and non-increasing in u_0. Finally, at any point of differentiability, using Proposition 1 and the Envelope Theorem gives $\partial\Phi_3/\partial\lambda = \mu[\Pi_h(C_h) - \Pi_\ell(C_\ell)]$ where $\mu > 0$ is the Kuhn-Tucker multiplier associated with (26). (ii) in Proposition 1 gives

$\Pi_h(C_h) < (\leq)\Pi_0$ when $u_0 > (=)U_h^0$. Using (26) shows that $\Pi_\ell(C_\ell) > (=)\Pi_0$ when $\Pi_h(C_h) < (=)\Pi_0$ and $0 < \lambda < 1$, hence the last result of the Corollary.

Definition 3. *An allocation $\mathcal{A}$ feasible in $\mathbb{P}_i(\lambda, c, u_0, \Pi_0)$ is dominated in $\mathbb{P}_j(\lambda, c, u_0, \Pi_0)$ by an allocation $\mathcal{A}'$, for $i, j \in \{2, 3, 4\}$, if $\mathcal{A}'$ is feasible in $\mathbb{P}_j(\lambda, c, u_0, \Pi_0)$ and if ℓ-types reach a higher expected utility at $\mathcal{A}'$ than at $\mathcal{A}$.*

Lemma 4. *Any allocation $\mathcal{A} = \{C_h, C_\ell, \widehat{p}, , \widehat{\sigma}_{hh}, \widehat{\sigma}_{h\ell}, \widehat{\sigma}_{\ell h}, \widehat{\sigma}_{\ell\ell}\}$, $\widehat{\sigma}_{\ell\ell} < 1$, feasible in $\mathbb{P}_4(\lambda, c, u_0, \Pi_0)$ is dominated in $\mathbb{P}_2(\lambda, c, u_0, \Pi_0)$ by another allocation $\mathcal{A}' = \{C_h', C_\ell', \widehat{p}', , \widehat{\sigma}_{hh}', \widehat{\sigma}_{h\ell}', \widehat{\sigma}_{\ell h}', \widehat{\sigma}_{\ell\ell}'\}$ such that $\widehat{\sigma}_{\ell\ell}' = 1$.*

Proof: Let $\mathcal{A}$ be a feasible allocation in $\mathbb{P}_4(\lambda, c, u_0, \Pi_0)$ such that $\widehat{\sigma}_{\ell\ell} < 1$.

1. Suppose first $\widehat{\sigma}_{\ell\ell} = 0, \widehat{\sigma}_{\ell h} = 1$, and thus $U_\ell(C_h) \geq U_\ell(C_\ell)$. Using (20) gives $x_\ell = c$. Let $k_\ell' > k_\ell$ such that $U_h(k_\ell', c) = u_0 \leq U_{h\ell}(k_\ell, c, \widehat{p})$ and let $C_\ell' = (k_\ell', c)$.

If $\widehat{\sigma}_{h\ell} = 1$ or if $\widehat{\sigma}_{h\ell} \in (0, 1)$ and $\Pi_h(C_\ell') \geq \Pi_h(C_h)$, then $\mathcal{A}^1 = \{C_h^1 = C_h, C_\ell^1 = C_\ell', \widehat{p}^1 = 0, \widehat{\sigma}_{hh}^1 = 0, \widehat{\sigma}_{h\ell}^1 = 1, \widehat{\sigma}_{\ell h}^1 = 1, \widehat{\sigma}_{\ell\ell}^1 = 0\}$ is feasible in $\mathbb{P}_3(\lambda, c, u_0, \Pi_0)$, with an expected profit larger than Π_0. Hence $\mathcal{A}^1$ is not optimal in $\mathbb{P}_3(\lambda, c, u_0, \Pi_0)$, which implies $\Phi_3(\lambda, c, u_0, \Pi_0) > U_\ell(C_h) \geq \Phi_4(\lambda, c, u_0, \Pi_0)$. Proposition 1 then implies that there exists a feasible allocation $\mathcal{A}'$ with $\widehat{\sigma}_{\ell\ell}' = 1$ that dominates $\mathcal{A}$ in $\mathbb{P}_2(\lambda, c, u_0, \Pi_0)$.

If $\widehat{\sigma}_{h\ell} \in (0, 1)$ and $\Pi_h(C_\ell) < \Pi_h(C_h)$, then $\mathcal{A}^2 = \{C_h^2 = C_h, C_\ell^2 = (0, 0), \widehat{p}^2 = 0, \widehat{\sigma}_{hh}^2 = 1, \widehat{\sigma}_{h\ell}^2 = 0, \widehat{\sigma}_{\ell h}^2 = 1, \widehat{\sigma}_{\ell\ell}^2 = 0\}$ is feasible in $\mathbb{P}_3(\lambda, c, u_0, \Pi_0)$, with an expected profit larger than Π_0, hence the same conclusion.

2. Suppose now $\widehat{\sigma}_{\ell\ell}, \widehat{\sigma}_{\ell h} \in (0, 1)$. (20) gives $x_\ell > c$, $\widehat{\sigma}_{h\ell} = K(x_\ell, \lambda, c)\widehat{\sigma}_{\ell\ell}$ where $K(x, \lambda, c)$ is given by (31), which allows us to write (19) as

$$(1 - \lambda)\Pi_\ell(C_h) + \lambda\Pi_h(C_h) + (1 - \lambda)\widehat{\sigma}_{\ell\ell}\Delta(C_h, C_\ell) \geq \Pi_0, \tag{56}$$

$$\text{where } \Delta(C_h, C_\ell) = \Pi_\ell(C_\ell) - \Pi_\ell(C_h) + \frac{c\pi_\ell}{\pi_h(x_\ell - c)}[\Pi_h(C_\ell) - \Pi_h(C_h)]. \tag{57}$$

Consider first the case $\Delta(C_h, C_\ell) > 0$. If $K(x_\ell, \lambda, c) \leq 1$,then $\mathcal{A}^3 = \{C_h^3 = C_h, C_\ell^3 = C_\ell, \widehat{p}^3 = \widehat{p}, \widehat{\sigma}_{hh}^3 = 1 - \widehat{\sigma}_{h\ell}^3, \widehat{\sigma}_{h\ell}^3 = K(x_\ell, \lambda, c), \widehat{\sigma}_{\ell h}^3 = 0, \widehat{\sigma}_{\ell\ell}^3 = 1\}$ is feasible in $\mathbb{P}_4(\lambda, c, u_0, \Pi_0)$ with the same expected utility for ℓ-types and a larger expected profit, which shows that $\mathcal{A}$ is dominated in $\mathbb{P}_4(\lambda, c, u_0, \Pi_0)$ by an allocation such that $\widehat{\sigma}_{\ell\ell} = 1$. If $K(x_\ell, \lambda, c) > 1$, then $\mathcal{A}^4 = \{C_h^4 = C_\ell, C_\ell^4 = C_h, \widehat{p}^4 = 0, \widehat{\sigma}_{hh}^4 = 1, \widehat{\sigma}_{h\ell}^4 = 0, \widehat{\sigma}_{\ell h}^4 = 1 - \widehat{\sigma}_{\ell\ell}^4, \widehat{\sigma}_{\ell\ell}^4 = 1/K(x_\ell, \lambda, c)]\}$ is feasible in $\mathbb{P}_3(\lambda, c, u_0, \Pi_0)$ with the same expected utility for ℓ-types. We know from Proposition 1 that ℓ-type individuals do not randomize at the optimum of this problem. $\mathcal{A}^4$ is thus dominated in $\mathbb{P}_3(\lambda, c, u_0, \Pi_0)$, hence the conclusion of the Lemma.

Consider now the case $\Delta(C_h, C_\ell) \leq 0$. Let us show that $\mathcal{A}$ is dominated in $\mathbb{P}_3(\lambda, c, u_0, \Pi_0)$. Assume first $\widehat{\sigma}_{h\ell} < 1$, which implies $U_h(C_h) \geq u_0$. (56) gives $(1-\lambda)\Pi_\ell(C_h) + \lambda\Pi_h(C_h) \geq \Pi_0$ which shows that the allocation $\mathcal{A}^5$ where all individuals choose C_h is feasible in $\mathbb{P}_3(\lambda, c, u_0, \Pi_0)$ with the same expected utility for ℓ-types. When $u_0 \neq \overline{u}_0$ or $u_0 = \overline{u}_0$

and $C_h \neq \overline{C}_0$, then Proposition 1 shows that $\mathcal{A}^5$ is dominated in $\mathbb{P}_3(\lambda, c, u_0, \Pi_0)$ by a separating allocation $\mathcal{A}'$ with $\widehat{\sigma}'_{\ell\ell} = 1$. When $u_0 = \overline{u}_0$ and $C_h = \overline{C}_0$, we have $\lambda\Pi_h(C_h) + (1-\lambda)\Pi_\ell(C_h) = \Pi_0$ and (22), (23), $0 < \widehat{\sigma}_{\ell\ell} < 1$, $\widehat{\sigma}_{h\ell} > 0$ and $\widehat{p} > 0$ show that $U_\ell(C_\ell) = U_\ell(C_h)$ and $U_h(C_\ell) > U_h(C_h)$. Using $C_h = \overline{C}_0$ then gives $\Pi_\ell(C_h) > \Pi_\ell(C_\ell)$ and $\Pi_h(C_h) > \Pi_h(C_\ell)$, which implies

$$\lambda[\widehat{\sigma}_{hh}\Pi_h(C_h) + \widehat{\sigma}_{h\ell}\Pi_h(C_\ell)] + (1-\lambda)[\widehat{\sigma}_{\ell h}\Pi_\ell(C_h) + \widehat{\sigma}_{\ell\ell}\Pi_\ell(C_\ell)] < \Pi_0,$$

which contradicts the fact that $\mathcal{A}$ is feasible in $\mathbb{P}_4(\lambda, c, u_0, \Pi_0)$.

Assume now $\widehat{\sigma}_{h\ell} = 1$. If $\Pi_\ell(C_\ell) \geq \Pi_\ell(C_h)$, then $\mathcal{A}^6 = \{C_h^6 = C_h, C_\ell^6 = C_\ell, \widehat{p}^6 = 0, \widehat{\sigma}_{hh}^6 = 0, \widehat{\sigma}_{h\ell}^6 = 1, \widehat{\sigma}_{\ell h}^6 = 0, \widehat{\sigma}_{\ell\ell}^6 = 1\}$ is feasible in $\mathbb{P}_3(\lambda, c, u_0, \Pi_0)$ with unchanged expected utility for ℓ-types. If $C_\ell = \overline{C}_0$, then (19) gives

$$\lambda\Pi_h(\overline{C}_0) + (1-\lambda)[\widehat{\sigma}_{\ell h}\Pi_\ell(C_h) + \widehat{\sigma}_{\ell\ell}\Pi_\ell(\overline{C}_0)] \geq \Pi_0,$$

which contradicts $\widehat{\sigma}_{\ell h} > 0$ and $C_h \neq C_\ell$. Hence $C_\ell \neq \overline{C}_0$. Proposition 1-*ii* shows that the pooling allocation $\mathcal{A}^6$ is not optimal in $\mathbb{P}_3(\lambda, c, u_0, \Pi_0)$, hence the result. If $\Pi_\ell(C_\ell) < \Pi_\ell(C_h)$ then $\mathcal{A}^7 = \{C_h^7 = C_h, C_\ell^7 = C_\ell, \widehat{p}^7 = 0, \widehat{\sigma}_{hh}^7 = 0, \widehat{\sigma}_{h\ell}^7 = 1, \widehat{\sigma}_{\ell h}^7 = 1, \widehat{\sigma}_{\ell\ell}^7 = 0\}$ is feasible in $\mathbb{P}_3(\lambda, c, u_0, \Pi_0)$ with profit larger Π_0: thus $\mathcal{A}^7$ is not an optimal solution to this problem and the same result follows.

Proof of Proposition 2: Assume that $\mathcal{A}$ is feasible in $\mathbb{P}_4(\lambda, c, u_0, \Pi_0)$ and not dominated in $\mathbb{P}_2(\lambda, c, u_0, \Pi_0)$. We know from Lemma 4 that $\widehat{\sigma}_{\ell\ell} = 1$ which gives (32). Furthermore (19)-(20) gives (30)-(31). (33) follows from $\widehat{\sigma}_{h\ell} > 0$ and $\widehat{p} > 0$ and (34) holds if $\widehat{\sigma}_{hh} > 0$. C_h is undetermined when $\widehat{\sigma}_{hh} = 0$. W.l.o.g. $\mathcal{A}$ can be chosen such that (34) hold. Conversely, if $\widehat{\sigma}_{\ell\ell} = 1$ and $\widehat{\sigma}_{h\ell}, C_h, C_\ell$ satisfy conditions (30) to (34), then $\mathcal{A}$ is feasible in $\mathbb{P}_4(\lambda, c, u_0, \Pi_0)$ with $\widehat{p} > 0$ given by $U_{h\ell}(C_\ell, \widehat{p}) = U_h(C_h)$.

(i) Suppose that $u_0 \geq \overline{u}_0$. We know that $\Pi_h(C) \leq$ (respect. $<$)$(\overline{\pi} - \pi_h)A + \Pi_0$ for all C such that $U_h(C) \geq$ (respect. $>$)$\overline{u}_0$, with similar inequalities for ℓ-types. Hence $\widehat{\sigma}_{hh}\Pi_h(C_h) + \widehat{\sigma}_{h\ell}\Pi_h(C_\ell) < (\overline{\pi} - \pi_h)A + \Pi_0$. (19) then gives $\Pi_\ell(C_\ell) > (\overline{\pi} - \pi_\ell)A + \Pi_0$, which implies $U_\ell(C_\ell) < \overline{u}_0$. One checks that $U_h(C_\ell) > \overline{u}_0 > U_\ell(C_\ell)$ implies $k_\ell + x_\ell > A$. Let $C'_\ell = (k'_\ell, x'_\ell)$ such that $k'_\ell + x'_\ell = A, U_\ell(C'_\ell) = U_\ell(C_\ell)$, which implies $\Pi_\ell(C'_\ell) > \Pi_\ell(C_\ell)$ and $U_h(C_\ell) > U_h(C'_\ell)$. Let $C'_h = C_\ell$ if $\widehat{\sigma}_{hh} = 0$ or if $\widehat{\sigma}_{hh} > 0$ and $\Pi_h(C_h) \leq \Pi_h(C_\ell)$ and $C'_h = C_h$ otherwise. Hence $U_\ell(C'_\ell) \geq U_\ell(C'_h), U_h(C'_h) > U_h(C'_\ell)$ and $U_h(C'_h) > u_0$. Finally (19) yields

$$\lambda\Pi_h(C'_h) + (1-\lambda)\Pi_\ell(C'_\ell) > \lambda[\widehat{\sigma}_{hh}\Pi_h(C_h) + \widehat{\sigma}_{h\ell}\Pi_h(C_\ell)] + (1-\lambda)\Pi_\ell(C_\ell) = \Pi_0.$$

Thus $\mathcal{A}^1 = \{C_h^1 = C'_h, C_\ell^1 = C'_\ell, \widehat{p}^1 = 0, \widehat{\sigma}_{hh}^1 = 1, \widehat{\sigma}_{h\ell}^1 = 0, \widehat{\sigma}_{\ell h}^1 = 0, \widehat{\sigma}_{\ell\ell}^1 = 1\}$ is feasible in $\mathbb{P}_3(\lambda, c, u_0, \Pi_0)$ with expected utility $U_\ell(C'_\ell) = U_\ell(C_\ell)$ and expected profit larger than Π_0. We deduce that $\mathcal{A}$ is dominated in $\mathbb{P}_2(\lambda, c, u_0, \Pi_0)$, hence a contradiction.

(ii) Proposition 1 gives $\Phi_3(\lambda, c, u_0, \Pi_0) > \overline{u}_0$ when $u_0 < \overline{u}_0$. Given that $\Phi_4(\lambda, c, u_0, \Pi_0) \geq \Phi_3(\lambda, c, u_0, \Pi_0)$,we may thus restrict the constraint set of $\mathbb{P}_4(\lambda, c, u_0, \Pi_0)$ by assuming $U_\ell(C_\ell) \geq \overline{u}_0$. Let us consider the maximization of $U_\ell(C_\ell)$ with respect to $C_h, C_\ell, \widehat{\sigma}_{h\ell}$ subject to (30), (31),(34),$U_\ell(C_\ell) \geq \overline{u}_0$ and $U_h(C_\ell) \geq U_h(C_h)$. In words $\mathbb{P}_4(\lambda, c, u_0, \Pi_0)$ is modified by deleting (32), by writing (33) as a weak inequality and by adding $U_\ell(C_\ell) \geq \overline{u}_0$. This maximization is denoted $\overline{\mathbb{P}}_4(\lambda, c, u_0, \Pi_0)$. It gives $k_h = A - x_h = W_N - u^{-1}(u_0)$ and $\Pi_h(C_\ell) \leq \Pi_h(C_h) = k_h - \pi_h A$. (30) then gives $\Pi_\ell(C_\ell) \geq [\Pi_0 - \lambda(k_h - \pi_h A)]/(1-\lambda) \equiv \underline{\Pi}_\ell$. Let $S_\ell \equiv \{C_\ell \in \mathbb{R}^2_+ \mid \Pi_\ell(C_\ell) \geq \underline{\Pi}_\ell, U_\ell(C_\ell) \geq \overline{u}_0\}$. Note that S_ℓ is bounded. Furthermore, given $C_h = (W_N - u^{-1}(u_0), A - W_N + u^{-1}(u_0))$, we have $C_\ell \in S_\ell$ if $C_\ell, \widehat{\sigma}_{h\ell}$ is feasible in $\overline{\mathbb{P}}_4(\lambda, c, u_0, \Pi_0)$. The constraint set for $C_\ell, \widehat{\sigma}_{h\ell}$ in $\overline{\mathbb{P}}_4(\lambda, c, u_0, \Pi_0)$ is thus bounded when C_h is optimally chosen. Since this set is closed, we deduce that it is compact and, given that U_ℓ is continuous, $\overline{\mathbb{P}}_4(\lambda, c, u_0, \Pi_0)$ has an optimal solution $C_h, C_\ell, \widehat{\sigma}_{h\ell}$ with $C_h \neq C_\ell$. (32) is necessarily satisfied for this optimal solution, for otherwise we would have $U_\ell(C_\ell) < U_\ell(C_h) = u_0$, hence a contradiction. Furthermore, (33) is not binding for otherwise $\mathcal{A}^2 = \{C^2_h = C_h, C^2_\ell = C_\ell, \widehat{p}^2 = 0, \widehat{\sigma}^2_{hh} = 1, \widehat{\sigma}^2_{h\ell} = 0, \widehat{\sigma}^2_{\ell h} = 0, \widehat{\sigma}^2_{\ell\ell} = 1\}$ would be feasible in $\mathbb{P}_3(\lambda, c, u_0, \Pi_0)$ with positive profit (because $\Pi_h(C_\ell) < \Pi_h(C_h)$ from (33) and $C_h \neq C_\ell$). $\mathcal{A}^2$ would not be optimal in in $\mathbb{P}_3(\lambda, c, u_0, \Pi_0)$, which would contradict $\Phi_4(\lambda, c, u_0, \Pi_0) \geq \Phi_3(\lambda, c, u_0, \Pi_0)$. We thus conclude that this optimal solution is also an optimal solution to $\mathbb{P}_4(\lambda, c, u_0, \Pi_0)$. It is such that $\widehat{\sigma}_{h\ell} < 1$, for otherwise $\mathcal{A}^3 = \{C^3_h = C_h, C^3_\ell = C_\ell, \widehat{p}^3 = 0, \widehat{\sigma}^3_{hh} = 0, \widehat{\sigma}^3_{h\ell} = 1, \widehat{\sigma}^3_{\ell h} = 0, \widehat{\sigma}^3_{\ell\ell} = 1\}$ would be feasible, but not optimal in $\mathbb{P}_3(\lambda, c, u_0, \Pi_0)$ since $u_0 \neq \overline{u}_0$, hence once again a contradiction.

(iii) Using $0 < \widehat{\sigma}_{h\ell} < 1$ gives $U_{h\ell}(C_\ell, \widehat{p}) = U_h(C_h)$, with $\widehat{p} \in (0,1)$ from (33). The fact that $U_h(C_h) = u_0 < \overline{u}_0 < U_\ell(C_\ell)$ and $k_h + x_h = A$ has been established in the proof of part (ii). Assume $\Pi_0 = 0$ and $u_0 = U^*_h$. We directly obtain $C_h = C^*_h$. Furthermore, substituting $\widehat{\sigma}_{h\ell}$ given by (31) into (30) gives $k_\ell \geq \phi(x_\ell)$ with $\phi(x_\ell)$ given by (35). $\phi(x_\ell)$ has a minimum at $x_\ell = 2c(\pi_h - \pi_\ell)/\pi_h(1-\pi_\ell) \equiv x^m$, with $\phi(x_\ell) \to \infty$ when $x_\ell \to x^m/2 < c$, $\phi(x_\ell) \simeq \pi_\ell x_\ell/(1-\pi_\ell)$ when $x_\ell \to \infty$. and $\phi''(x_\ell) > 0$ for all $x_\ell > x^m/2$. Note also that $\phi'(x_\ell) < \pi_\ell/(1-\pi_\ell)$ for all $x_\ell > x^m/2$. $\phi(x_\ell)$ is drawn in Figure 4. When A/c is large enough, as represented in Figure 4, the loci $k_\ell = \phi(x_\ell)$ and $k_\ell + x_\ell = A$ cross twice. In that case, PP' and the 45° degree line cross twice in the W_1, W_2 plane, as represented in Figure 2. For small values of A/c, we have $k_\ell + x_\ell > A$ on the whole locus $k_\ell = \phi(x_\ell)$ and PP' is entirely above the 45° degree line. Maximizing $U_\ell(k_\ell, x_\ell)$ subject to $k_\ell = \phi(x_\ell)$ gives

$$-\frac{\partial U_\ell(k_\ell, x_\ell)/\partial x_\ell}{\partial U_\ell(k_\ell, x_\ell)/\partial k_\ell} = \frac{\pi_\ell u'(W_A + x_\ell)}{(1-\pi_\ell)u'(W_N - k_\ell)} = \phi'(x_\ell) < \frac{\pi_\ell}{1-\pi_\ell},$$

which implies $u'(W_A + x_\ell) < u'(W_N - k_\ell)$. Using $u'' < 0$ then gives $W_A + x_\ell > W_N - k_\ell$ or $k_\ell + x_\ell > W_N - W_A = A$.

Figure 4

(iv) follows from the continuity of $\lneqq_4(\lambda, c, u_0, \Pi_0)$ in u_0 and Π_0 and from the fact that given (33) and (34), there exists (C'_h, C'_ℓ) in a neighbourhood of (C_h, C_ℓ) such that $U_\ell(C'_\ell) > U_\ell(C'_h)$ and $U_h(C'_\ell) > U_h(C'_h)$.

Corollary 3. *At any point where $\Phi_4(\lambda, c, u_0, \Pi_0) > \Phi_3(\lambda, c, u_0, \Pi_0)$, Φ_4 is locally decreasing in Π_0, c and u_0. Furthermore Φ_4 is locally decreasing (respect. stationary, locally increasing) in λ if $u_0 > U_h^0$ (respect $= U_h^0, < U_h^0$).*

Proof: Let λ, c, u_0, Π_0 such that $\Phi_4(\lambda, c, u_0, \Pi_0) > \Phi_3(\lambda, c, u_0, \Pi_0)$ and let C_h, C_ℓ be a pair of optimal contracts in $\mathbb{P}_4(\lambda, c, u_0, \Pi_0)$. The fact that Φ_4 is locally decreasing in Π_0 follows from Proposition 2 with the same kind of argument as for $\Phi_3$in Corollary 1. Furthermore, we know from Proposition 2 that $0 < \widehat{\sigma}_{h\ell} < 1$ and $U_h(C_h) = u_0$ at an optimal solution to $\mathbb{P}_4(\lambda, c, u_0, \Pi_0)$, which gives $\partial\Phi_4/\partial u_0 < 0$ at any point of differentiability. Using the Enveloppe Theorem gives

$$\frac{\partial\Phi_4(\lambda, c, u_0, \Pi_0)}{\partial c} = \mu[\Pi_h(C_\ell) - \Pi_h(C_h)]\frac{(1-\lambda)c\pi_\ell}{\pi_h(x_\ell - c)^2} < 0,$$

$$\frac{\partial\Phi_4(\lambda, c, u_0, \Pi_0)}{\partial\lambda} = -\mu[\Pi_\ell(C_\ell) + \frac{c\pi_\ell}{\pi_h(x_\ell - c)}\Pi_h(C_\ell)] + \mu\frac{\pi_h x_\ell - c(\pi_h - \pi_\ell)}{\pi_h(x_\ell - c)}\Pi_h(C_h), \quad (58)$$

at any point of differentiability, with $\mu > 0$ a Kuhn-Tucker multiplier. Proposition 2-*iii* gives $\Pi_h(C_h) < (=, >)\Pi_0$ when $u_0 > (=, <)U_h^0$. (30), (31) and (58) then yield $\partial\Phi_4(\lambda, c, u_0, \Pi_0)/\partial\lambda < (=, >)0$ when $u_0 > (=, <)U_h^0$.

Corollary 4. *Φ_2 is decreasing in Π_0, non-increasing in u_0 and c and locally decreasing (respect. locally non-increasing) in λ if $u_0 > U_h^0$ (respect. $u_0 \geq U_h^0$).*

Proof: Corollary 4 is a direct consequence of Corollaries 2 and 3.

Let $\widetilde{\mathbb{P}}_j(\lambda, c, u_0, \Pi_0)$ be the same maximization problem as $\mathbb{P}_j(\lambda, c, u_0, \Pi_0)$ with $j = 2, 3$ or 4, up to the difference that (24) is written as an equality instead of an inequality and let $\widetilde{\Phi}_j(\lambda, c, u_0, \Pi_0)$ be the corresponding value function with $\widetilde{\Phi}_j \leq \Phi_j$.

Lemma 5. (i) *$\widetilde{\Phi}_2$ and $\widetilde{\Phi}_3$ are decreasing in Π_0,*

(ii) *$\widetilde{\Phi}_2$ and $\widetilde{\Phi}_3$ are decreasing (respect. stationary, increasing) in λ if and only if $u_0 > U_h^0$ (respect. $u_0 = U_h^0, u_0 < U_h^0$),*

(iii) *$\widetilde{\Phi}_3(0, c, u_0, \Pi_0) <$ (respect.$=$)$\widetilde{\Phi}_3(\lambda, c, u_0, \Pi_0)$ for all $\lambda \in (0, 1]$ if $u_0 < U_h^0$(respect.$= U_h^0$),*

(iv) *$\widetilde{\Phi}_3(0, c, u_0, \Pi_0)$ is locally increasing in u_0 if $u_0 < u(W_N - \pi_\ell A)$.*

Proof: (i) can be proved in the same way as the equivalent property for Φ_3 in Corollary 2. When $u_0 < \overline{u}_0$ the optimal solution to $\widetilde{\mathbb{P}}_3(\lambda, c, u_0, \Pi_0)$ is such that $k_h + x_h = A$ with $\Pi_h(C_h) >$ (respect.$=, <$)Π_0 if $u_0 <$ (respect.$=, >$)U_h^0. (ii) can then be proved in the same way as the equivalent property for Φ_3 in Corollary 2. (iii) and (iv) are obvious.

Lemma 6. *An optimal solution to* $\widetilde{\mathbb{P}}_2(\lambda, c, U_h^*, 0)$ *is characterized by* $C_h = C_h^*, C_\ell = C_\ell^{**}, \widehat{\sigma}_{hh} = 1, \widehat{\sigma}_{h\ell} = 0, \widehat{\sigma}_{\ell h} = 0, \widehat{\sigma}_{\ell\ell} = 1$ *if* $U_\ell(C_\ell^{**}) \geq U_\ell(\widehat{C}_\ell)$ *and* $C_h = C_h^*, C_\ell = \widehat{C}_\ell, \widehat{\sigma}_{h\ell} = K(\widehat{x}_\ell, \lambda, c) \in (0,1], \widehat{\sigma}_{hh} = 1 - \widehat{\sigma}_{h\ell}, \widehat{\sigma}_{\ell h} = 0, \widehat{\sigma}_{\ell\ell} = 1$ *if* $U_\ell(C_\ell^{**}) \leq U_\ell(\widehat{C}_\ell)$, *where* $\widehat{C}_\ell = (\widehat{k}_\ell, \widehat{x}_\ell)$ *is defined as in Proposition 3.*

Proof: The Lemma straighforwardly follows from $\widetilde{\Phi}_2 = \inf\{\widetilde{\Phi}_3, \widetilde{\Phi}_4\}$ and from the fact that $C_h = C_h^*, C_\ell = C_\ell^{**}, \widehat{\sigma}_{hh} = 1, \widehat{\sigma}_{h\ell} = 0, \widehat{\sigma}_{\ell h} = 0, \widehat{\sigma}_{\ell\ell} = 1$ is an optimal solution to $\widetilde{\mathbb{P}}_3(\lambda, c, U_h^*, 0)$ and $C_h = C_h^*, C_\ell = \widehat{C}_\ell, \widehat{\sigma}_{h\ell} = K(\widehat{x}_\ell, \lambda, c) \in (0,1], \widehat{\sigma}_{hh} = 1 - \widehat{\sigma}_{h\ell}, \widehat{\sigma}_{\ell h} = 0, \widehat{\sigma}_{\ell\ell} = 1$ is an optimal solution to $\widetilde{\mathbb{P}}_4(\lambda, c, U_h^*, 0)$.

Let us consider problem $\widetilde{\mathbb{P}}_1(\lambda, c, u_0, \Pi_0^0, ..., \Pi_0^n)$, with value function $\widetilde{\Phi}_1(\lambda, c, u_0, \Pi_0^0, ..., \Pi_0^n)$, which is analogous to $\widehat{\mathbb{P}}_1(\lambda, c, u_0, \Pi_0^1, ..., \Pi_0^n)$ except that C_h^0, C_ℓ^0 and p^0 can be freely chosen (C_h^0 and C_ℓ^0 are no more necessarily equal to $(0,0)$ and p^0 is no more necessarily equal to 0) and (37), (38) and (39) should hold also for $i = 0$. Of course, $C_h^0 = C_\ell^0 = (0,0), p^0 = 0$ is possible in $\widetilde{\mathbb{P}}_1$, and we thus have $\widetilde{\Phi}_1(\lambda, c, u_0, \Pi_0^0, ..., \Pi_0^n) \geq \Phi_1(\lambda, c, u_0, \Pi_0^1, ..., \Pi_0^n)$, where Φ_1 is the value function of $\widehat{\mathbb{P}}_1$.

Lemma 7. (i) $\widetilde{\Phi}_1(\lambda, c, u_0, 0, ..., 0) = \Phi_2(\lambda, c, u_0, 0)$ *for all* λ, c, u_0.
(ii) $\widetilde{\Phi}_1(\lambda, c, u_0, \Pi_0, ..., \Pi_0) < \Phi_2(\lambda, c, u_0, \Pi_0)$ *for all* $\Pi_0 > 0$ *and all* λ, c, u_0.

Proof: Let $\{C_h, C_\ell, \widehat{p}, , \widehat{\sigma}_{hh}, \widehat{\sigma}_{h\ell}, \widehat{\sigma}_{\ell h}, \widehat{\sigma}_{\ell\ell}\}$ be an optimal solution to $\mathbb{P}_2(\lambda, c, u_0, 0)$. Then $\{C^i = (C_h, C_\ell), p^i = \widehat{p}, N^i = 1/(n+1), \lambda^i = \lambda, \widehat{\sigma}_{hh}^i = \widehat{\sigma}_{hh}, \widehat{\sigma}_{h\ell}^i = \widehat{\sigma}_{h\ell}, \widehat{\sigma}_{\ell h}^i = \widehat{\sigma}_{\ell h}, \sigma_{\ell\ell}^i = \widehat{\sigma}_{\ell\ell}$ for all $i = 0, ..., n\}$ is feasible in $\widetilde{\mathbb{P}}_1(\lambda, c, u_0, 0, ..., 0)$, which implies

$$\widetilde{\Phi}_1(\lambda, c, u_0, 0, ..., 0) \geq \Phi_2(\lambda, c, u_0, 0). \tag{59}$$

Let $\mathcal{A} = (\mathcal{A}^0, ..., \mathcal{A}^n)$ with $\mathcal{A}^i = \{C_h^i, C_\ell^i, p^i, N^i, \lambda^i, \widehat{\sigma}_{hh}^i, \widehat{\sigma}_{h\ell}^i, \widehat{\sigma}_{\ell h}^i, \widehat{\sigma}_{\ell\ell}^i\}$ be an optimal solution to $\widetilde{\mathbb{P}}_1(\lambda, c, u_0, \Pi_0, ..., \Pi_0)$ with $\Pi_0 > 0$ and let $u_0^i = \max\{U_h(C_h^i), U_{h\ell}(C_\ell^i, p^i)\}$ for $i = 0, ..., n$, with $u_0^i \geq u_0$ if $N^i\lambda^i > 0$. Note that $\widehat{\sigma}_{hh}^i$ and $\widehat{\sigma}_{h\ell}^i$ are indeterminate if $N^i\lambda^i = 0$: in such a case, we choose $\widehat{\sigma}_{hh}^i$ and $\widehat{\sigma}_{h\ell}^i$ such that $\widehat{\sigma}_{hh}^i U_h(C_h^i) + \widehat{\sigma}_{h\ell}^i U_{h\ell}(C_\ell^i, p^i) = u_0^i$. Note also that $\Pi_0 > 0$ gives $0 < N^i < 1$ for all $i = 0, ..., n$.

Case 1: $\lambda^i \in \{0,1\}$ for all $i = 0, ..., n$.

For all i such that $\lambda^i = 1$, we necessarily have $u_0 \leq u_0^i \leq u(W - \pi_h A - \frac{\Pi_0}{N^i})$ since otherwise (37),(40) and (42) would be incompatible. Using $\Pi_0 > 0$ and $N^i < 1$ then implies $u_0 < U_h^0$.

For all i such that $\lambda^i = 0$,we have $\widetilde{\Phi}_1(\lambda, c, u_0, \Pi_0, ..., \Pi_0) = \widehat{\sigma}_{\ell h}^i U_\ell(C_h^i) + \widehat{\sigma}_{\ell\ell}^i U_\ell(C_\ell^i)$ and (38) gives $p^i = 0$. Hence $\mathcal{A}^i$ is feasible in $\widetilde{\mathbb{P}}_3(0, c, u_0^i, \frac{\Pi_0}{N^i})$, which gives

$$\widetilde{\Phi}_3(0, c, u_0^i, \frac{\Pi_0}{N^i}) \geq \widetilde{\Phi}_1(\lambda, c, u_0, \Pi_0, ..., \Pi_0). \tag{60}$$

We also have $u_0^i \leq u_0^j$ if $\lambda^j = 1$ and thus $u_0^i \leq u(W_N - \pi_h A - \frac{\Pi_0}{N^j}) < U_h^0$.

If $u_0^i \leq u_0$, we can write

$$\begin{aligned}
\widetilde{\Phi}_3(0,c,u_0^i,\frac{\Pi_0}{N^i}) &< \widetilde{\Phi}_3(0,c,u_0^i,\Pi_0) \text{ from } \Pi_0>0,\ 0<N^i<1 \text{ and Lemma 5-}i,\\
&\leq \widetilde{\Phi}_3(0,c,u_0,\Pi_0) \text{ from } u_0^i \leq u_0 \leq U_h^0 < u(W_N-\pi_\ell A) \text{ and Lemma 5-}iv,\\
&\leq \widetilde{\Phi}_3(\lambda,c,u_0,\Pi_0) \text{ from Lemma 5-}iii,\\
&\leq \Phi_2(\lambda,c,u_0,\Pi_0) \text{ from the definition of } \Phi_2 \text{ and } \widetilde{\Phi}_3.
\end{aligned}$$

and (60) finally yields

$$\Phi_2(\lambda,c,u_0,\Pi_0) > \widetilde{\Phi}_1(\lambda,c,u_0,\Pi_0,...,\Pi_0). \tag{61}$$

If $u_0^i > u_0$, we can write

$$\begin{aligned}
\widetilde{\Phi}_3(0,c,u_0^i,\frac{\Pi_0}{N^i}) &< \widetilde{\Phi}_3(0,c,u_0^i,\Pi_0) \text{ from } \Pi_0>0,\ 0<N^i<1 \text{ and Lemma 5-}i,\\
&\leq \widetilde{\Phi}_3(\lambda,c,u_0^i,\Pi_0) \text{ from } u_0^i \leq U_h^0 \text{ and Lemma 5-}iii,\\
&\leq \Phi_3(\lambda,c,u_0^i,\Pi_0) \text{ from the definitions of } \Phi_3 \text{ and } \widetilde{\Phi}_3,\\
&\leq \Phi_3(\lambda,c,u_0,\Pi_0) \text{ from } u_0^i > u_0,\\
&\leq \Phi_2(\lambda,c,u_0,\Pi_0) \text{ from the definitions of } \Phi_2 \text{ and } \Phi_3,
\end{aligned}$$

which also leads to (61).

Case 2: There exists $i \in \{0,...,n\}$ such that $0<\lambda^i<1$ and $N^i>0$.

Let $u_0' = \max\{u_0^0,...,u_0^n\}$, with $u_0' \geq u_0$. For all i such that $0<\lambda^i<1$ and $N^i>0$, we have $u_0^i = u_0'$ and

$$\widetilde{\Phi}_2(\lambda^i,c,u_0',\Pi_0) > \widetilde{\Phi}_2(\lambda^i,c,u_0',\frac{\Pi_0}{N^i}) \geq \widetilde{\Phi}_1(\lambda,c,u_0,\Pi_0,...,\Pi_0). \tag{62}$$

Note that the second inequality in (62) is a consequence of $\mathcal{A}^i$ beeing feasible in $\widetilde{\mathbb{P}}_2(\lambda^i,c,u_0^i,\frac{\Pi_0}{N^i})$ and $\widetilde{\Phi}_1(\lambda,c,u_0,\Pi_0,...,\Pi_0) = \widehat{\sigma}_{\ell h}^i U_\ell(C_h^i) + \widehat{\sigma}_{\ell\ell}^i U_\ell(C_\ell^i)$ when $N^i>0$ and $0<\lambda^i<1$.

For all i such that $\lambda^i=1$ and $N^i>0$, we have $u_0^i = u_0' \leq u(W-\pi_h A - \frac{\Pi_0}{N^i}) < U_h^0$. Hence, if there exists i such that $\lambda^i=1$ and $N^i>0$, then Lemma 5-ii gives

$$\widetilde{\Phi}_2(1,c,u_0',\Pi_0) > \widetilde{\Phi}_2(\lambda^j,c,u_0',\Pi_0) \text{ for all } j \text{ such that } 0<\lambda^j<1,$$

and (62) yields

$$\widetilde{\Phi}_2(1,c,u_0',\Pi_0) > \widetilde{\Phi}_1(\lambda,c,u_0,\Pi_0,...,\Pi_0). \tag{63}$$

Suppose that

$$\widetilde{\Phi}_2(0,c,u_0',\Pi_0) \leq \widetilde{\Phi}_1(\lambda,c,u_0,\Pi_0,...,\Pi_0). \tag{64}$$

(62) and (64) give $\widetilde{\Phi}_2(0,c,u_0',\Pi_0) < \widetilde{\Phi}_2(\lambda^i,c,u_0',\Pi_0)$ for all i such that $0<\lambda^i<1, N^i>0$ and Lemma 5-ii gives $u_0' < U_h^0$. Suppose in addition that there exists i such that $\lambda^i=0$

and $N^i > 0$. If $\widehat{\sigma}^i_{\ell\ell} > 0$, we have $p^i = 0$ from (38) and thus $\mathcal{A}^i$ is feasible in $\widetilde{\mathbb{P}}_3(0, c, u^i_0, \frac{\Pi_0}{N^i})$. Furthermore $\Phi_1(\lambda, c, u_0, \Pi_0, ..., \Pi_0) = \widehat{\sigma}^i_{\ell h} U_\ell(C^i_h) + \widehat{\sigma}^i_{\ell\ell} U_\ell(C^i_\ell)$ when $N^i > 0$ and $\lambda^i < 1$. Hence

$$\widetilde{\Phi}_3(0, c, u^i_0, \Pi_0) > \widetilde{\Phi}_3(0, c, u^i_0, \frac{\Pi_0}{N^i}) \geq \widetilde{\Phi}_1(\lambda, c, u_0, \Pi_0, ..., \Pi_0), \quad (65)$$

and

$$\begin{aligned} \widetilde{\Phi}_3(0, c, u^i_0, \Pi_0) &\leq \widetilde{\Phi}_3(0, c, u'_0, \Pi_0) \text{ from } u^i_0 \leq u'_0 < U^0_h < u(W_N - \pi_\ell A) \text{ and Lemma 5-}iv, \\ &\leq \widetilde{\Phi}_2(0, c, u'_0, \Pi_0) \text{ from the definitions of } \Phi_2 \text{ and } \widetilde{\Phi}_3. \end{aligned} \quad (66)$$

(65) and (66) imply $\widetilde{\Phi}_1(\lambda, c, u_0, \Pi_0, ..., \Pi_0) < \widetilde{\Phi}_2(0, c, u'_0, \Pi_0)$, which contradicts (64). Hence if there exists i such that $\lambda^i = 0$ and $N^i > 0$, we have

$$\widetilde{\Phi}_2(0, c, u'_0, \Pi_0) > \widetilde{\Phi}_1(\lambda, c, u_0, \Pi_0, ..., \Pi_0). \quad (67)$$

(62),(63) and (67) show that $\widetilde{\Phi}_2(\lambda^i, c, u'_0, \Pi_0) \geq \widetilde{\Phi}_1(\lambda, c, u_0, \Pi_0, ..., \Pi_0)$ for all i such that $N^i > 0$. Using (43) and the fact that $\widetilde{\Phi}_2$ is monotonic in λ (as shown in Lemma 5-ii) gives

$$\widetilde{\Phi}_2(\lambda, c, u'_0, \Pi_0) > \widetilde{\Phi}_1(\lambda, c, u_0, \Pi_0, ..., \Pi_0). \quad (68)$$

Since $u'_0 \geq u_0$ and Φ_2 is non-increasing in u_0, we can write

$$\Phi_2(\lambda, c, u_0, \Pi_0) \geq \Phi_2(\lambda, c, u'_0, \Pi_0) \geq \widetilde{\Phi}_2(\lambda, c, u'_0, \Pi_0), \quad (69)$$

and (68) and (69) imply (61), this inequality beeing valid when $\Pi_0 > 0$. This is part (ii) of the Lemma. The computations are unchanged when $\Pi_0 = 0$ with weak inequalities instead of strong inequalities and (61) is still valid with $\geq$ instead of $>$. (59) and (61) written as a weak inequality together give part (i) of the Lemma.

Proof of Proposition 4:

Consider problem $\widetilde{\mathbb{P}}_1(\lambda, c, u_0, 0, ..., 0)$ and let u^i_0 and u'_0 be defined as in the Proposition.

1. Assume first $u'_0 > U^*_h$. Let i such that $N^i > 0$. In that case (37) and (42) imply $\lambda^i < 1$, which gives $\widetilde{\Phi}_1(\lambda, c, u'_0, 0, ..., 0) = \widehat{\sigma}^i_{\ell h} U_\ell(C^i_h) + \widehat{\sigma}^i_{\ell\ell} U_\ell(C^i_\ell)$. Furthermore, if $\lambda^i > 0$ then$\{C^i, p^i, \widehat{\sigma}^i_{hh}, \widehat{\sigma}^i_{h\ell}, \widehat{\sigma}^i_{\ell h}, \widehat{\sigma}^i_{\ell\ell}\}$ is feasible in $\widetilde{\mathbb{P}}_2(\lambda^i, c, u'_0, 0)$. Hence if $\lambda^i > 0$, we have

$$\begin{aligned} \widetilde{\Phi}_2(\lambda^i, c, u'_0, 0) &\geq \widetilde{\Phi}_1(\lambda, c, u'_0, 0, ..., 0), \\ &= \Phi_2(\lambda, c, u'_0, 0) \text{ from Lemma 7-}i, \\ &\geq \widetilde{\Phi}_2(\lambda, c, u'_0, 0) \text{ from the definition of } \Phi_2 \text{ and } \widetilde{\Phi}_2. \end{aligned}$$

Since $u'_0 > U^*_h$, Lemma 5-ii then implies $\lambda^i \leq \lambda$. Since this inequality should hold for all i such that $N^i > 0$, (43) allows us to deduce that $\lambda^i = \lambda$ for all i such that $N^i > 0$.

2. Assume now $u_0' < U_h^*$ (which implies $u_0 < U_h^*$). Let i such that $N^i > 0$. The same argument as in the case $u_0' > U_h^*$ yields $\lambda^i \geq \lambda$ if $0 < \lambda^i < 1$. Suppose that $\lambda^i = 0$. We have

$$\begin{aligned} \widetilde{\Phi}_3(0, c, u_0', 0) &\geq \widetilde{\Phi}_3(0, c, u_0^i, 0) \text{ from } u_0^i \leq u_0' < U_h^* < u(W_N - \pi_\ell A) \text{ and Lemma 5-}iv, \\ &\geq \widetilde{\Phi}_1(\lambda, c, u_0, 0, ..., 0), \end{aligned} \tag{70}$$

because $\{C^i, p^i, \widehat{\sigma}_{hh}^i, \widehat{\sigma}_{h\ell}^i, \widehat{\sigma}_{\ell h}^i, \widehat{\sigma}_{\ell\ell}^i\}$, is feasible in $\widetilde{\mathbb{P}}_3(0, c, u_0^i, 0)$ and $\widetilde{\Phi}_1(\lambda, c, u_0, 0, ..., 0) = \widehat{\sigma}_{\ell h}^i U_\ell(C_h^i) \quad + \widehat{\sigma}_{\ell\ell}^i U_\ell(C_\ell^i)$ when $N^i > 0$ and $\lambda^i = 0$. Using

$$\begin{aligned} \widetilde{\Phi}_1(\lambda, c, u_0, 0, ..., 0) &= \Phi_2(\lambda, c, u_0, 0) \text{ from Lemma 7-}i, \\ &\geq \widetilde{\Phi}_2(\lambda, c, u_0', 0) \text{ from the definition of } \widetilde{\Phi}_2 \text{ and } u_0', \\ &\geq \widetilde{\Phi}_3(\lambda, c, u_0', 0) \text{ from the definition of } \widetilde{\Phi}_3, \end{aligned} \tag{71}$$

(70) and (71) give $\widetilde{\Phi}_3(0, c, u_0', 0) \geq \widetilde{\Phi}_3(\lambda, c, u_0', 0)$ which contradicts Lemma 5-*iii* because $u_0' < U_h^*$. Hence, we have $\lambda^i \geq \lambda$ if $N^i > 0$. Using (43) gives $\lambda^i = \lambda$ for all i such that $N^i > 0$.

3. When $u_0' \neq U_h^*$, using $\lambda^i = \lambda$ for all i such that $N^i > 0$ allows us to write

$$\widehat{\sigma}_{\ell h}^i U_\ell(C_h^i) + \widehat{\sigma}_{\ell\ell}^i U_\ell(C_\ell^i) = \widetilde{\Phi}_1(\lambda, c, u_0, 0, ..., 0) = \Phi_2(\lambda, c, u_0, 0),$$

if $N^i > 0$ and since $\{C^i, p^i, \widehat{\sigma}_{hh}^i, \widehat{\sigma}_{h\ell}^i, \widehat{\sigma}_{\ell h}^i, \widehat{\sigma}_{\ell\ell}^i\}$ is feasible in $\mathbb{P}_2(\lambda, c, u_0, 0)$, we conclude that it is an optimal solution to this problem.

4. Assume $u_0' = U_h^*$. If $0 < \lambda^i < 1$ and $N^i > 0$, we have $\widetilde{\Phi}_1(\lambda, c, U_h^*, 0, ..., 0) = \widehat{\sigma}_{\ell h}^i U_\ell(C_h^i) + \widehat{\sigma}_{\ell\ell}^i U_\ell(C_\ell^i)$ and $U_h^* = u_0^i$. Thus $\{C^i, p^i, \widehat{\sigma}_{hh}^i, \widehat{\sigma}_{h\ell}^i, \widehat{\sigma}_{\ell h}^i, \widehat{\sigma}_{\ell\ell}^i\}$ is feasible in $\widetilde{\mathbb{P}}_2(\lambda^i, c, U_h^*, 0)$.

$$\begin{aligned} \widetilde{\Phi}_1(\lambda, c, U_h^*, 0, ..., 0) &= \Phi_2(\lambda, c, U_h^*, 0) \text{ from Lemma 7-}i, \\ &\geq \widetilde{\Phi}_2(\lambda, c, U_h^*, 0) \text{ from the definition of } \widetilde{\Phi}_2, \\ &= \widetilde{\Phi}_2(\lambda^i, c, U_h^*, 0) \text{ from Lemma 5-}ii. \end{aligned}$$

Hence $\{C^i, p^i, \widehat{\sigma}_{hh}^i, \widehat{\sigma}_{h\ell}^i, \widehat{\sigma}_{\ell h}^i, \widehat{\sigma}_{\ell\ell}^i\}$ is the optimal solution to $\widetilde{\mathbb{P}}_2(\lambda^i, c, U_h^*, 0)$, which is characterized in Lemma 6. If $\lambda^i = 0$ and $N^i > 0$, then $\{C^i, p^i, \widehat{\sigma}_{hh}^i, \widehat{\sigma}_{h\ell}^i, \widehat{\sigma}_{\ell h}^i, \widehat{\sigma}_{\ell\ell}^i\}$ is feasible in $\widetilde{\mathbb{P}}_2(0, c, u_0^i, 0)$, with $u_0^i \leq U_h^*$. Suppose $u_0^i < U_h^*$. Then

$$\begin{aligned} \widetilde{\Phi}_2(0, c, u_0^i, 0) &\geq \widehat{\sigma}_{\ell h}^i U_\ell(C_h^i) + \widehat{\sigma}_{\ell\ell}^i U_\ell(C_\ell^i), \\ &= \widetilde{\Phi}_1(\lambda, c, U_h^*, 0, ..., 0) \text{ because } N^i > 0, \lambda^i = 0, \\ &= \Phi_2(\lambda, c, U_h^*, 0) \text{ from Lemma 7-}i, \\ &\geq \widetilde{\Phi}_2(\lambda, c, U_h^*, 0) \text{ from the definition of } \widetilde{\Phi}_2, \\ &= \widetilde{\Phi}_2(0, c, U_h^*, 0) \text{ from Lemma 5-}ii, \end{aligned}$$

which contradicts Lemma 5-*iv*. Using $u_0' = U_h^*$ and $u_0^i \geq U_h^*$ then gives $u_0^i = U_h^*$. Hence $\{C^i, p^i, \widehat{\sigma}_{hh}^i, \widehat{\sigma}_{h\ell}^i, \widehat{\sigma}_{\ell h}^i, \widehat{\sigma}_{\ell\ell}^i\}$ is optimal in $\widetilde{\mathbb{P}}_2(0, c, U_h^*, 0)$ and is characterized as in Lemma 6. Finally if $N^i > 0$ and $\lambda^i = 1$, then the feasibility constraints in $\widetilde{\mathbb{P}}_1(1, c, U_h^*, 0, ..., 0)$ imply $C_h^i = C_h^*$ and $\widehat{\sigma}_{hh}^i = 1$.

5. Thus far we have characterized the optimal solution to $\widetilde{\mathbb{P}}_1(\lambda, c, u_0, 0, ..., 0)$. For this optimal solution the ℓ-types'expected utility is larger or equal to $\overline{U}_\ell$ if $\Phi_2(\lambda, c, u_0, 0) \geq \overline{U}_\ell$ or equivalently if $u_0 \leq \widehat{u}_0(\lambda, c)$. In such a case, the optimal solution to $\widehat{\mathbb{P}}_1(\lambda, c, u_0, 0, ..., 0)$ involves $N^0 = 0$ and it is characterized as in the Proposition, since problem $\widehat{\mathbb{P}}_1$ with $N^0 = 0$ is equivalent to problem $\widetilde{\mathbb{P}}_1$ (there are n insurers in $\widehat{\mathbb{P}}_1$ and $n+1$ insurers in $\widetilde{\mathbb{P}}_1$, but the number of insurers do not affect the characterization of the optimal solution). When $u_0 > \widehat{u}_0(\lambda, c)$, then $\widehat{\mathbb{P}}_1(\lambda, c, u_0, 0, ..., 0)$ has no feasible solution. Indeed assume $\Phi_2(\lambda, c, u_0, 0) < \overline{U}_\ell$. If $\widehat{\mathbb{P}}_1(\lambda, c, u_0, 0, ..., 0)$ has a feasible solution then $\Phi_1(\lambda, c, u_0, 0, ..., 0) \geq \overline{U}_\ell$, which implies $\widetilde{\Phi}_1(\lambda, c, u_0, 0, ..., 0) \geq \overline{U}_\ell$. Lemma 7-*i* then gives $\Phi_2(\lambda, c, u_0, 0) \geq \overline{U}_\ell$, hence a contradiction.

Proof of Lemma 3 : Let Γ be the continuation subgame after $\widetilde{C}$ was offered at stage 2. Consider a perturbed continuation subgame Γ_ε, where mixed strategies are constrained by $p^i \in [\varepsilon, 1-\varepsilon]$ for all i, $\sigma_h \in \widetilde{S}_\varepsilon^{2n+1}$, $\sigma_\ell \in \widetilde{S}_\varepsilon^{2n+1}$, with $\widetilde{S}_\varepsilon^{2n+1} = \{t = (t_1, ..., t_{2n+2}) \in S^{2n+1}, t_j \geq \varepsilon$ for all $j = 1, ..., 2n+2\}$ and $\varepsilon > 0$. Γ_ε is derived from Γ by requiring that each player chooses each pure strategy with at least some minimal positive probability ε[22], with $\Gamma \equiv \Gamma_0$. $\{p, \sigma_h, \sigma_\ell, \mu\}$ is a trembling hand perfect Bayesian equilibrium of Γ if it is a perfect Bayesian equilibrium of Γ and if there is some sequence of perturbed games $\{\Gamma_{\varepsilon_m}\}_{m=1}^\infty$ that converges to Γ [in the sense that $\varepsilon_m > 0$ for all m and $\lim_{m\to\infty} \varepsilon_m = 0$] for which there is some associated sequence of perfect Bayesian equilibria $\{p_m, \sigma_{hm}, \sigma_{\ell m}, \mu_m\}$ such that $\{p_m, \sigma_{hm}, \sigma_{\ell m}\}$ converges to $\{p, \sigma_h, \sigma_\ell\}$ when $m \to \infty$.

Let's get to the proof of the Lemma. $\{\overline{p}, \overline{\sigma}_h, \overline{\sigma}_\ell\}$ is a perfect Bayesian equilibrium of Γ. Assume it is trembling hand perfect and consider a sequence of perturbed games $\{\Gamma_{\varepsilon^m}\}_{m=1}^\infty$ that converges to Γ and an associated sequence of perfect Bayesian equilibria $\{p_m, \sigma_{hm}, \sigma_{\ell m}, \mu_m\}$ such that $\{p_m, \sigma_{hm}, \sigma_{\ell m}\}$ converges to $\{\overline{p}, \overline{\sigma}_h, \overline{\sigma}_\ell\}$. Bayes law gives gives

$$\mu_m^i \equiv \frac{\lambda \pi_h \sigma_{h\ell m}^i}{\lambda \pi_h \sigma_{h\ell m}^i + (1-\lambda)\pi_\ell \sigma_{\ell\ell m}^i} \in (0,1) \quad \text{for all } m.$$

Minimizing $p^i(\mu_m^i \widetilde{x}_\ell^i - c)$ with respect to $p^i \in [\varepsilon_m, 1-\varepsilon_m]$ implies $p_m^i = \varepsilon_m$ for all m if $\widetilde{x}_\ell^i = c$. Using $\lim_{m\to\infty} \varepsilon_m = 0$ then gives $\lim_{m\to\infty} \varepsilon_m = 0\overline{p}^i = 0$ if $\widetilde{x}_\ell^i = c$.

Lemma 8. *For any equilibrium* $\mathcal{E} = \{\widetilde{C}, \widetilde{p}(.), \widetilde{\sigma}_h(.), \widetilde{\sigma}_\ell(.), \widetilde{\mu}(.)\}$ *and for all* C *such that*

$$U_h(C_\ell^i) > \max\{U_h(C_h^j), j = 1, ..., n; U_{h\ell}(C_\ell^j, \widetilde{p}^j(C)), j = 1, ..., n \text{ and } j \neq i\} > \overline{U}_h,$$

[22] For notational simplicity, we suppose that this minimal probability ε is the same for all players and all pure strategies.

(i) If $x_\ell^i > c$ and $\widetilde{\sigma}_{\ell\ell}^i(C) > 0$, then

$$\widetilde{\sigma}_{h\ell}^i(C) = \min\{1, \widetilde{\sigma}_{\ell\ell}^i(C)K(x_\ell^i, \lambda, c)\},$$
$$\widetilde{p}^i(C) = \phi^i(C) \text{ (resp.} = 0, \in [0, \phi^i(C))) \text{ if } \widetilde{\sigma}_{\ell\ell}^i(C)K(x_\ell^i, \lambda, c) < 1 \text{(resp.} > 1, = 1).$$

(ii) If $x_\ell^i > c$ and $\widetilde{\sigma}_{\ell\ell}^i(C) = 0$, then $\widetilde{\sigma}_{h\ell}^i(C) = 0$ and $\widetilde{p}^i(C) \in [\phi^i(C), 1]$.

(iii) If $x_\ell^i = c$ and $\widetilde{\sigma}_{\ell\ell}^i(C) > 0$, then $\widetilde{\sigma}_{h\ell}^i(C) = 1$ and $\widetilde{p}^i(C) = 0$.

(iv) If $x_\ell^i = c$, and $\widetilde{\sigma}_{\ell\ell}^i(C) = 0$, then $\widetilde{\sigma}_{h\ell}^i(C) = 1, \widetilde{p}^i(C) < \phi^i(C)$ or $\widetilde{\sigma}_{h\ell}^i(C) \in [0,1], \widetilde{p}^i(C) = \phi^i(C)$ or $\widetilde{\sigma}_{h\ell}^i(C) = 0, \widetilde{p}^i(C) > \phi^i(C)$.

(v) If $x_\ell^i < c$ then $\widetilde{\sigma}_{h\ell}^i(C) = 1$ and $\widetilde{p}^i(C) = 0$,

where $K(x, \lambda, c)$ and $\phi^i(C) \in (0,1)$ are respectively given by (31) and by

$$U_{h\ell}(C_\ell^i, \phi^i(C)) = \max\{U_h(C_h^j), j = 1, ..., n; U_{h\ell}(C_\ell^j, \widetilde{p}^j(C)), j = 1, ..., n \text{ and } j \neq i\}.$$

Proof : Consider the case where $x_\ell^i > c$ and $\widetilde{\sigma}_{\ell\ell}^i(C) > 0$. (3), (5) and (7) show that $\sigma_{h\ell}^i \equiv \widetilde{\sigma}_{h\ell}^i(C)$, $\sigma_{\ell\ell}^i \equiv \widetilde{\sigma}_{\ell\ell}^i(C)$and $p^i \equiv \widetilde{p}^i(C)$ are linked by the reaction functions of the h-type individuals and of insurer i, which are $\sigma_{h\ell}^i = 1$ (resp.$\in [0,1], = 0$) if $p^i < \phi^i(C)$ (resp. $p^i = \phi^i(C), p^i > \phi^i(C)$) and $p^i = 1$ (resp.$\in [0,1], = 0$) if $\sigma_{h\ell}^i > \sigma_{\ell\ell}^i K(x_\ell^i, \lambda, c)$ (resp. $\sigma_{h\ell}^i = \sigma_{\ell\ell}^i K(x_\ell^i, \lambda, c), \sigma_{h\ell}^i < \sigma_{\ell\ell}^i K(x_\ell^i, \lambda, c)$). These reaction functions are illustrated in Figure 5, with $\widetilde{\sigma}_{h\ell}^i(C) = \widetilde{\sigma}_{\ell\ell}^i(C)K(x_\ell^i, \lambda, c)$ and $\widetilde{p}^i(C) = \phi^i(C)$. The results are similarly obtained in the other cases.

Figure 5

Proof of Proposition 5

Step 1 of the proof shows that under *THP* any equilibrium allocation is an optimal solution to $\mathbb{P}_1(\lambda, U_h^e, 0, ..., 0)$ and Step 2 establishes that $U_h^e \leq U_h^*$, with $U_h^e = U_h^*$ if *SDS* is supposed in addition.

Step 1

Let $\{\widetilde{C}, \overline{p}, \overline{\sigma}_h, \overline{\sigma}_\ell\}$ be an equilibrium allocation that satisfies *THP*, with $\widetilde{C}, \widetilde{p}(.), \widetilde{\sigma}_h(.), \widetilde{\sigma}_\ell(.)$ the profile of strategies and $\widetilde{\mu}(.)$ the system of beliefs. $\{\widetilde{C}, \overline{p}, \overline{\sigma}_h, \overline{\sigma}_\ell\}$is feasible in $\mathbb{P}_1(\lambda, c, U_h^e, \overline{\Pi}^1, ..., \overline{\Pi}^n)$. Because $\overline{\Pi}^i \geq 0$, we deduce that $\{\widetilde{C}, \overline{p}, \overline{\sigma}_h, \overline{\sigma}_\ell\}$is feasible in $\mathbb{P}_1(\lambda, c, U_h^e, 0, ..., 0)$. Assume that this equibrium allocation is **not** an optimal solution to $\mathbb{P}_1(\lambda, c, U_h^e, 0, ..., 0)$. Then using Lemma 7-$i$ allows us to write

$$\max\{U_\ell(\widetilde{C}_h^i), U_\ell(\widetilde{C}_\ell^i), i = 1, ..., n\} < \widetilde{\Phi}_1(\lambda, c, U_h^e, 0, ..., 0) = \Phi_2(\lambda, c, U_h^e, 0).$$

Consider a deviation by an insurer i_0 from $\widetilde{C}^{i_0} = (\widetilde{C}_h^{i_0}, \widetilde{C}_\ell^{i_0})$ to $C^{i_0} = (C_h^{i_0}, C_\ell^{i_0})$ and let $\widetilde{p}^i \equiv \widetilde{p}^i(C^{i_0}, \widetilde{C}^{-i_0}), \widetilde{\mu}^i \equiv \widetilde{\mu}^i(C^{i_0}, \widetilde{C}^{-i_0}), \widetilde{\sigma}_{jk}^i \equiv \widetilde{\sigma}_{jk}^i(C^{i_0}, \widetilde{C}^{-i_0})$ for $(j,k) \in \{h, \ell\}^2$. Let also $\widetilde{\Pi}^{i_0}$ be the profit of insurer i_0 at the continuation equilibrium following the deviation. Hence $\overline{\Pi}^{i_0}$ and $\widetilde{\Pi}^{i_0}$ are respectively the LHS and the RHS of (6) for $i = i_0$.

Case 1: There exists $i_0 \in \{1, ..., n\}$ such that $\overline{\Pi}^{i_0} = 0$.

Propositions 1-*iii* and 2-*iv* show that there exists $\{C^{i_0}, \widehat{p}, \widehat{\sigma}_{hh}, \widehat{\sigma}_{h\ell}, \widehat{\sigma}_{\ell h}, \widehat{\sigma}_{\ell\ell}\}$ feasible in $\mathbb{P}_2(\lambda, c, U_h^e+\varepsilon, \varepsilon')$, with $\varepsilon, \varepsilon' > 0$, and such that $\widehat{\sigma}_{\ell\ell} = 1$, $U_\ell(C_\ell^{i_0}) > U_\ell(C_h^{i_0}), U_h(C_h^{i_0}) \neq U_h(C_\ell^{i_0}), U_h(C_h^{i_0}) \geq U_h^e + \varepsilon, U_\ell(C_\ell^{i_0}) \geq \Phi_2(\lambda, c, U_h^e, 0) - \eta$ and

$$0 < \eta < \Phi_2(\lambda, c, U_h^e, 0) - \max\{U_\ell(\widetilde{C}_h^i), U_\ell(\widetilde{C}_\ell^i), i = 1, ..., n\}.$$

Hence $U_\ell(C_\ell^{i_0}) > \max\{U_\ell(\widetilde{C}_h^i), U_\ell(\widetilde{C}_\ell^i), i = 1, ..., n; i \neq i_0\}$ and thus $\widetilde{\sigma}_{\ell\ell}^{i_0} = 1$. Let $i \neq i_0$ such that $\overline{p}^i > 0$. Lemmas 1 and 3 then imply $\widetilde{x}_\ell^i > c$. When $\widetilde{\sigma}_{h\ell}^i > 0$, we have $\widetilde{\mu}^i = 1$ and (5) gives $\widetilde{p}^i = 1$. We thus have $\widetilde{p}^i \geq \overline{p}^i$ and $\max\{U_h(\widetilde{C}_h^i), U_{h\ell}(\widetilde{C}_\ell^i, \widetilde{p}^i), i = 1, ..., n; i \neq i_0\} \leq U_h^e$ for all $i \neq i_0$ such that $\widetilde{\sigma}_{h\ell}^i > 0$. $U_h(C_h^{i_0}) \geq U_h^e + \varepsilon$ then implies $\widetilde{\sigma}_{hh}^{i_0} + \widetilde{\sigma}_{h\ell}^{i_0} = 1$.

Case 1.1: $U_h(C_\ell^{i_0}) > U_h(C_h^{i_0})$. Let $\varphi \in (0, 1)$ such that $U_{h\ell}(C_\ell^{i_0}, \varphi) = U_h(C_h^{i_0})$.

Suppose $x_\ell^{i_0} \leq c$. Then (20) and $\widehat{\sigma}_{\ell\ell} = 1$ show that $\widehat{p} = 0$ and (22) gives $\widehat{\sigma}_{h\ell} = 1$. Hence the profit of insurer i_0 is larger or equal to ε' when all individuals choose $C_\ell^{i_0}$. Furthermore, using $\widetilde{\sigma}_{\ell\ell}^{i_0} = 1$ and Lemma 8 gives $\widetilde{p}^{i_0} = 0$, $\widetilde{\sigma}_{h\ell}^{i_0} = 1$. We get $\widetilde{\Pi}^{i_0} \geq \varepsilon' > \overline{\Pi}^{i_0}$, hence a contradiction with (6).

Suppose now $x_\ell^{i_0} > c$ and let $K_0 \equiv K(x_\ell^{i_0}, \lambda, c) > 0$.

If $K_0 < 1$: Suppose $\widehat{p} = 0$. Using $\widehat{\sigma}_{\ell\ell} = 1$ and (21) then gives $\widehat{\sigma}_{h\ell} \leq K_0$ and thus $\widehat{\sigma}_{h\ell} < 1$. (22) then implies $\widehat{p} \geq \varphi > 0$, which is a contradiction. Hence $\widehat{p} > 0$ and (20) yields $\widehat{\sigma}_{h\ell} = K_0 \in (0, 1)$ and (22) gives $\widehat{p} = \varphi$. Lemma 8 then yields $\widetilde{\sigma}_{h\ell}^{i_0} = \widehat{\sigma}_{h\ell}$ and $\widetilde{\sigma}_{hh}^{i_0} = \widehat{\sigma}_{hh}$.

If $K_0 > 1$: A similar argument gives $\widetilde{p}^{i_0} = \widehat{p} = 0$ and $\widetilde{\sigma}_{h\ell}^{i_0} = \widehat{\sigma}_{h\ell} = 1$.

If $K_0 = 1$: We obtain $\widehat{\sigma}_{h\ell} = 1$ and $\widehat{p} \in [0, \varphi]$. There is a continuum of continuation equilibria defined by $\widetilde{\sigma}_{h\ell}^{i_0} = 1$, $\widetilde{p}^{i_0} \in [0, \varphi]$.

Case 1.2: In the case $U_h(C_\ell^{i_0}) < U_h(C_h^{i_0})$, we have $\widetilde{p}^{i_0} = \widehat{p} = 0, \widetilde{\sigma}_{hh}^{i_0} = \widehat{\sigma}_{hh} = 1$ and $\widetilde{\sigma}_{\ell\ell}^{i_0} = \widehat{\sigma}_{\ell\ell} = 1$.

In Cases 1.1 and 1.2, we get $\widetilde{\Pi}^{i_0} \geq \varepsilon' > \overline{\Pi}^{i_0}$, hence a contradiction with (6).

Case 2: $\overline{\Pi}_i > 0$ for all $i = 1, ..., n$.

Let $i_0 \in \{1, ..., n\}$ such that $0 < \overline{\Pi}_{i_0} \leq \overline{\Pi}_i$ for all i. Since$\{\widetilde{C}, \overline{p}, \overline{\sigma}_h, \overline{\sigma}_\ell\}$ is feasible in $\mathbb{P}_1(\lambda, c, U_h^e, \overline{\Pi}^1, ..., \overline{\Pi}^n)$, we have

$$\max\{U_\ell(\widetilde{C}_h^i), U_\ell(\widetilde{C}_\ell^i), i = 1, ..., n\} \leq \widetilde{\Phi}_1(\lambda, c, U_h^e, \overline{\Pi}^1, ..., \overline{\Pi}^n).$$

$\partial\widetilde{\Phi}_1/\partial\Pi_0^i \leq 0, \partial\Phi_2/\partial\Pi_0 > 0$ and Lemma 7-*ii* then show that there exists $\delta > 0$ such that

$$\max\{U_\ell(\widetilde{C}_h^i), U_\ell(\widetilde{C}_\ell^i), i = 1, ..., n\} < \Phi_2(\lambda, c, U_h^e, \overline{\Pi}^{i_0} + \delta).$$

Note that $\overline{\sigma}_{hh}^{i_0} + \overline{\sigma}_{h\ell}^{i_0} < 1$ and/or $\overline{\sigma}_{\ell h}^{i_0} + \overline{\sigma}_{\ell\ell}^{i_0} < 1$ for otherwise we would have $\overline{\Pi}_i = 0$ for all $i \neq i_0$. Propositions 1-*iii* and 2-*iv* show that for all $\eta > 0$, we can find $\varepsilon > 0$ such that $\{C^{i_0}, \widehat{p}, \widehat{\sigma}_{hh}, \widehat{\sigma}_{h\ell}, \widehat{\sigma}_{\ell h}, \widehat{\sigma}_{\ell\ell}\}$ is an optimal solution to $\mathbb{P}_2(\lambda, c, U_h^e + \varepsilon, \overline{\Pi}^{i_0} + \delta), U_\ell(C_\ell^{i_0}) >$

$U_\ell(C_h^{i_0}), U_\ell(C_\ell^{i_0}) \geq \Phi_2(\lambda, c, U_h^e + \varepsilon, \overline{\Pi}^{i_0} + \delta) - \eta$. Let η such that

$$0 < \eta < \Phi_2(\lambda, c, U_h^e, \overline{\Pi}^{i_0} + \delta) - \max\{U_\ell(\widetilde{C}_h^i), U_\ell(\widetilde{C}_\ell^i), i = 1, ..., n\}.$$

The same argument as in Case 1 shows $\widetilde{\sigma}_{\ell\ell}^{i_0} = \widehat{\sigma}_{\ell\ell} = 1, \widetilde{\sigma}_{hh}^{i_0} = \widehat{\sigma}_{hh}$ and $\widetilde{\sigma}_{h\ell}^{i_0} = \widehat{\sigma}_{h\ell}$, which gives $\widetilde{\Pi}^{i_0} = \overline{\Pi}^{i_0} + \delta$, hence a contradiction with (6).

Step 2

Suppose $U_h^e \neq U_h^*$. Since $\{\widetilde{C}, \overline{p}, \overline{\sigma}_h, \overline{\sigma}_\ell\}$ is an optimal solution to $\mathbb{P}_1(\lambda, c, U_h^e, 0, .., 0)$, we know from Proposition 4 (with $u_0 = u_0' = U_h^e$) that for all i such that $\overline{N}^i > 0$, then $\mathcal{A}^i \equiv \{\widetilde{C}^i, \overline{p}^i, \widehat{\sigma}_{hh}^i, \widehat{\sigma}_{h\ell}^i, \widehat{\sigma}_{\ell h}^i, \widehat{\sigma}_{\ell\ell}^i\}$ is an optimal solution to $\mathbb{P}_2(\lambda, c, U_h^e, 0)$, where $\widehat{\sigma}_{hh}^i, \widehat{\sigma}_{h\ell}^i$ ... are deduced from $\overline{\sigma}_h, \overline{\sigma}_\ell$ as in $\widehat{\mathbb{P}}_1(\lambda, c, U_h^e, 0, .., 0)$. We also know that $\overline{\Pi}_i = 0$ for all i because (19) is binding at an optimal solution to $\mathbb{P}_2(\lambda, c, U_h^e, 0)$. Finally, Propositions 1, 2 and 4 show that $\overline{\sigma}_{\ell h}^i = 0$ and that $\overline{\sigma}_{\ell\ell}^i = \overline{\sigma}_{hh}^i + \overline{\sigma}_{h\ell}^i$ for all i such that $\overline{N}^i > 0$.

Case 1′: $U_h^e > U_h^*$.

Case 1′.1: $\overline{\sigma}_{hh}^i > 0$ for all i such that $\overline{N}^i > 0$.

Let i_0 such that $\overline{N}^{i_0} > 0$. Using $U_h^e > U_h^*$ gives $\Pi_h(\widetilde{C}_h^{i_0}) < 0$. Then (19) - written for $\mathcal{A}^{i_0}$ with $\Pi_0 = 0$ - gives $\Pi_\ell(\widetilde{C}_\ell^{i_0}) > 0$.

Suppose first that $\overline{p}^{i_0} = 0$, which implies that $\mathcal{A}^{i_0}$ is an optimal solution to $\mathbb{P}_3(\lambda, c, U_h^e, 0)$ and thus $\overline{\sigma}_{hh}^{i_0} = \overline{\sigma}_{\ell\ell}^i = 1$. If there exists $i_1 \neq i_0$ such that $\overline{N}^{i_1} > 0$, let $C_h^{i_0} = (0, 0)$ and $C_\ell^{i_0}$ such that $U_\ell(C_\ell^{i_0}) > U_\ell(\widetilde{C}_\ell^{i_0})$ and $U_h(C_\ell^{i_0}) < U_h(\widetilde{C}_\ell^{i_0})$, which implies $\widetilde{\sigma}_{\ell\ell}^{i_0} = 1$ and $\widetilde{\sigma}_{h\ell}^{i_0} = 0$[23]. For $C_\ell^{i_0}$ close enough to $\widetilde{C}_\ell^{i_0}$, we have

$$\widetilde{\Pi}^{i_0} = (1 - \lambda)\Pi_\ell(C_\ell^{i_0}) > \lambda\Pi_h(\widetilde{C}_h^{i_0}) + (1 - \lambda)\Pi_\ell(\widetilde{C}_\ell^{i_0}) = \overline{\Pi}_{i_0} = 0,$$

which contradicts (6). If $\overline{N}^i = 0$ for all $i \neq i_0$, choose the same deviation as above but for $i_1 \neq i_0$, which gives $\widetilde{\Pi}^{i_1} > \overline{\Pi}^{i_1} = 0$, hence a contradiction once again.

Suppose now $\overline{p}^{i_0} > 0$, i.e. $\mathcal{A}^{i_0}$ is an optimal solution to $\mathbb{P}_4(\lambda, c, U_h^e, 0)$, which implies $\widetilde{x}_\ell^{i_0} > c$. Choose $C_h^{i_0} = (0, 0)$ and $C_\ell^{i_0}$ such that $x_\ell^{i_0} > \widetilde{x}_\ell^{i_0}$ and $U_\ell(C_\ell^{i_0}) > U_\ell(\widetilde{C}_\ell^{i_0})$, which gives $\widetilde{\sigma}_{\ell\ell}^{i_0} = 1$. Using (31) gives $K(x_\ell^{i_0}, \lambda, c) < K(\widetilde{x}_\ell^{i_0}, \lambda, c) = \widehat{\sigma}_{h\ell}^{i_0} \leq 1$. Lemma 8 then gives $\widetilde{\sigma}_{h\ell}^{i_0} = K(x_\ell^{i_0}, \lambda, c)$. We have

$$\lambda[\widehat{\sigma}_{hh}^{i_0}\Pi_h(\widetilde{C}_h^{i_0}) + \widehat{\sigma}_{h\ell}^{i_0}\Pi_h(\widetilde{C}_\ell^{i_0})] + (1 - \lambda)\Pi_\ell(\widetilde{C}_\ell^{i_0}) = \overline{\Pi}^{i_0}/\overline{N}^{i_0} = 0. \tag{72}$$

Using $U_h^e > U_h^*$ and $\widehat{\sigma}_{hh}^{i_0} > 0$ gives $\Pi_h(\widetilde{C}_h^{i_0}) < 0$. Hence (72) and $\widehat{\sigma}_{h\ell}^{i_0} = K(\widetilde{x}_\ell^{i_0}, \lambda, c)$ yield $\lambda K(\widetilde{x}_\ell^{i_0}, \lambda, c)\Pi_h(\widetilde{C}_\ell^{i_0}) + (1 - \lambda)\Pi_\ell(\widetilde{C}_\ell^{i_0}) > 0$. Thus

$$\widetilde{\Pi}^{i_0} = \lambda\widetilde{\sigma}_{h\ell}^{i_0}\Pi_h(C_\ell^{i_0}) + (1-\lambda)\widetilde{\sigma}_{\ell\ell}^{i_0}\Pi_\ell(C_\ell^{i_0}) = \lambda K(x_\ell^{i_0}, \lambda, c)\Pi_h(C_\ell^{i_0}) + (1-\lambda)\Pi_\ell(C_\ell^{i_0}) > 0 = \overline{\Pi}^{i_0},$$

for $C_\ell^{i_0}$ close enough to $\widetilde{C}_\ell^{i_0}$, hence a contradiction.

[23]Note that $\overline{\sigma}_{hh}^{i_0} > 0$ and $\overline{p}^{i_0} = 0$ give $U_h(\widetilde{C}_h^{i_0}) \geq U_h(\widetilde{C}_\ell^{i_0})$. Hence $\widetilde{\sigma}_{h\ell}^{i_0} = 0$ follows from $U_h(C_\ell^{i_0}) < U_h(\widetilde{C}_\ell^{i_0}) \leq U_h(\widetilde{C}_h^{i_0}) = U_h(\widetilde{C}_h^{i_1})$.

Case 1′.2: $\overline{\sigma}^{i_0}_{hh} = 0$ and $\overline{N}^{i_0} > 0$ for some $i_0 \in \{1, ..., n\}$.

In that case we have $\overline{\sigma}^{i_0}_{h\ell} = \overline{\sigma}^{i_0}_{\ell\ell} = \overline{N}^{i_0}$ and $\widehat{\sigma}^{i_0}_{h\ell} = \widehat{\sigma}^{i_0}_{\ell\ell} = 1$. The optimal solution to $\mathbb{P}_2(\lambda, c, U^e_h, 0)$ is thus a pooling allocation where both types choose $\widetilde{C}^{i_0}_\ell$. Propositions 1

and 2 that this is possible only when $U^e_h = u(W_N - \overline{\pi}A)$ and $\widetilde{C}^{i_0}_\ell = (\overline{\pi}A, A - \overline{\pi}A)$. Let $C^{i_0}_h$, close to $\widetilde{C}^{i_0}_\ell$, such that $U_\ell(C^{i_0}_h) > U_\ell(\widetilde{C}^{i_0}_\ell)$ and $\lambda\Pi_h(C^{i_0}_h) + (1-\lambda)\Pi_\ell(C^{i_0}_h) > 0$. Let $C^{i_0}_\ell = (0, 0)$. We obtain a contradiction since

$$\widetilde{\Pi}^{i_0} = \lambda\widetilde{\sigma}^{i_0}_{hh}\Pi_h(C^{i_0}_h) + (1-\lambda)\Pi_\ell(C^{i_0}_h) \geq \lambda\Pi_h(C^{i_0}_h) + (1-\lambda)\Pi_\ell(C^{i_0}_h) > 0 = \overline{\Pi}^{i_0}.$$

Case 2′: $U^e_h < U^*_h$.

Assume in addition that the equilibrium satisfies *SDS*. When $U^e_h < U^*_h$, we know from Propositions 1, 2 and 4 that $\widetilde{C}^i_h = (\widehat{\pi}_h A, (1-\widehat{\pi}_h)A)$ with $\widehat{\pi}_h > \pi_h$ such that $u(W - \widehat{\pi}_h A) = U^e_h$, for all i such that $\overline{\sigma}^i_{hh} > 0$. Let i_0 such that $\overline{N}^{i_0} > 0$ and suppose that there exists $i_1 \neq i_0$ such that $\overline{\sigma}^{i_1}_{hh} > 0$. Let $C^{i_0}_h = (\widehat{\pi}'_h A, (1-\widehat{\pi}'_h)A)$ with $\pi_h < \widehat{\pi}'_h < \widehat{\pi}_h$ and $C^{i_0}_\ell = \widetilde{C}^{i_0}_\ell$. For all $i \neq i_0$, we have $U_h(C^{i_0}_h) > U_h(\widetilde{C}^i_h)$ and thus $\widetilde{\sigma}^i_{hh} = 0$. Hence $\widetilde{\sigma}^{i_0}_{hh} = 1 - \sum_{i=1}^n \widetilde{\sigma}^i_{h\ell}$. *SDS* gives $\widetilde{\sigma}^i_{\ell\ell} = \overline{\sigma}^i_{\ell\ell}$ for all i. Furthermore *THP* and Lemma 3 imply $\widetilde{x}^i_\ell > c$ if $\overline{p}^i > 0$. Hence we have $\widetilde{\sigma}^i_{h\ell} = \overline{\sigma}^i_{h\ell}$ if $\overline{p}^i > 0$ from Lemma 8. Furthermore when $\overline{p}^i = 0$ we have $U_h(\widetilde{C}^i_\ell) \leq \overline{U}^e$ and thus $U_h(\widetilde{C}^i_\ell) < U_h(C^{i_0}_h)$, which gives $\widetilde{\sigma}^i_{h\ell} = 0$. Hence, $\widetilde{\sigma}^i_{h\ell} \leq \overline{\sigma}^i_{h\ell}$ for all i. Consequently

$$\overline{\sigma}^{i_0}_{hh} = 1 - \sum_{i=1}^n \overline{\sigma}^i_{h\ell} - \sum_{i\neq i_0} \overline{\sigma}^i_{hh} \leq 1 - \sum_{i=1}^n \widetilde{\sigma}^i_{h\ell} - \sum_{i\neq i_0} \overline{\sigma}^i_{hh}.$$

Hence $\widetilde{\sigma}^{i_0}_{hh} \geq \overline{\sigma}^{i_0}_{hh} + \sum_{i\neq i_0} \overline{\sigma}^i_{hh} > \overline{\sigma}^{i_0}_{hh}$. We have

$$\overline{\Pi}^{i_0} = \lambda[\overline{\sigma}^{i_0}_{hh}\Pi_h(\widetilde{C}^{i_0}_h) + \overline{\sigma}^{i_0}_{h\ell}\Pi_h(\widetilde{C}^{i_0}_\ell)] + (1-\lambda)\overline{\sigma}^{i_0}_{\ell\ell}\Pi_\ell(\widetilde{C}^{i_0}_\ell).$$

We have $\Pi_h(\widetilde{C}^{i_0}_h) > 0$ if $\overline{\sigma}^{i_0}_{hh} > 0$ from $U^e_h < U^*_h$ and we also have $\Pi_h(C^{i_0}_h) > 0$. Since $\Pi_h(\widetilde{C}^{i_0}_\ell) < \Pi_\ell(\widetilde{C}^{i_0}_\ell)$ and $\overline{\Pi}^{i_0} = 0$, we deduce $\Pi_h(\widetilde{C}^{i_0}_\ell) < 0$ if $\overline{\sigma}^{i_0}_{h\ell} > 0$. Hence

$$\overline{\Pi}^{i_0} \leq \lambda[(\widetilde{\sigma}^{i_0}_{hh} - \sum_{i\neq i_0} \overline{\sigma}^i_{hh})\Pi_h(C^{i_0}_h) + \widetilde{\sigma}^{i_0}_{h\ell}\Pi_h(C^{i_0}_\ell)] + (1-\lambda)\widetilde{\sigma}^{i_0}_{\ell\ell}\Pi_\ell(C^{i_0}_\ell) < \widetilde{\Pi}^{i_0},$$

for $\widehat{\pi}_h - \pi'_h$ small enough, hence a contradiction. If $\overline{\sigma}^i_{hh} = 0$ for all $i \neq i_0$, we may consider a similar deviation by any insurer $i \neq i_0$ leading to the same kind of contradiction.

Proof of Proposition 6

Let $\mathcal{E}$ be a separating equilibrium that satisfies *THP* and *SDS*. Propositions 4 and 5 yield the characterization of the equilibrium allocation given in Proposition 6. They also show that $U_\ell(C^{**}_\ell) = \Phi_2(\lambda, c, U^*_h, 0)$ is a necessary condition for a separating equilibrium to exist. Let us show that it is also a sufficient condition. Hence assume $U_\ell(C^{**}_\ell) = \Phi_2(\lambda, c, U^*_h, 0)$ and let $\widetilde{C}, \overline{p}, \overline{\sigma}_h, \overline{\sigma}_\ell$ be defined as in Proposition 6, with $\overline{N}^i = 1/n$, $\overline{\lambda}^i = \lambda$

and $\overline{\mu}^i \equiv \widetilde{\mu}^i(\widetilde{C}) = 0$ for all i. Equations (3) to (6) are satisfied for $C = \widetilde{C}$. Insurers and insureds play mixed strategy in the continuation subgame following C and consequently for all $C \neq \widetilde{C}$, $\widetilde{p}(C), \widetilde{\sigma}_h(C), \widetilde{\sigma}_\ell(C), \widetilde{\mu}(C)$ can be chosen in such a way that (3),(4), (5) and (7) are satisfied, i.e. it is a continuation equilibrium. We have to show that for all $C^i \neq \widetilde{C}^i$ and all i, $\widetilde{p} \equiv \widetilde{p}(C^i, \widetilde{C}^{-i}), \widetilde{\sigma}_h \equiv \widetilde{\sigma}_h(C^i, \widetilde{C}^{-i}), \widetilde{\sigma}_\ell \equiv \widetilde{\sigma}_\ell(C^i, \widetilde{C}^{-i}), \widetilde{\mu}^i \equiv \widetilde{\mu}^i(C^i, \widetilde{C}^{-i})$ can be chosen such that equation (6) is also satisfied. Let $\widetilde{\Pi}^i$ be the RHS in (6). The LHS is $\overline{\Pi}^i = 0$ and thus (6) is written as $\widetilde{\Pi}^i \leq 0$. Note that at any continuation equilibrium following the deviation by insurer i, we have $\widetilde{\sigma}^i_{hh}\Pi_h(C^i_h) + \widetilde{\sigma}^i_{h\ell}\Pi_\ell(C^i_\ell) \leq 0$ because $\widetilde{\sigma}^i_{hk} = 0$ if $U_h(C^i_k) < U^*_h$ for $k = h, \ell$.

If $\max\{U_\ell(C^i_h), U_\ell(C^i_\ell)\} \leq U_\ell(C^{**}_\ell)$, we may choose the continuation equilibrium such that $\widetilde{\sigma}^i_{\ell h} + \widetilde{\sigma}^i_{\ell\ell} = 0$ which implies $\widetilde{\Pi}^i \leq 0$.

If $\max\{U_\ell(C^i_h), U_\ell(C^i_\ell)\} > U_\ell(C^{**}_\ell)$, let $\widetilde{\sigma}^i_{\ell h}$ and $\widetilde{\sigma}^i_{\ell\ell}$ in $[0, 1]$ such that $\widetilde{\sigma}^i_{\ell h} + \widetilde{\sigma}^i_{\ell\ell} = 1$ and $\widetilde{\sigma}^i_{\ell h}U_\ell(C^i_h) + \widetilde{\sigma}^i_{\ell\ell}U_\ell(C^i_\ell) = \max\{U_\ell(C^i_h), U_\ell(C^i_\ell)\}$. If $\widetilde{\sigma}^i_{\ell h}\Pi_\ell(C^i_h) + \widetilde{\sigma}^i_{\ell\ell}\Pi_\ell(C^i_\ell) \leq 0$, then $\widetilde{\Pi}^i \leq 0$. If $\widetilde{\sigma}^i_{\ell h}\Pi_\ell(C^i_h) + \widetilde{\sigma}^i_{\ell\ell}\Pi_\ell(C^i_\ell) > 0$ (which is assumed in what follows), we have $\max\{U_h(C^i_h), U_h(C^i_\ell)\} > U^*_h$. Consider the two following cases.

Case 1: $U_h(C^i_h) \geq U^*_h$. If $U_h(C^i_\ell) > U_h(C^i_h), x^i_\ell > c$ and $K(x^i_\ell, \lambda, c)\widetilde{\sigma}^i_{\ell\ell} \leq 1$, let $\widetilde{p}^i \in [0, 1]$ such that $U_{h\ell}(C^i_\ell, \widetilde{p}^i) = U_h(C^i_h), \widetilde{\sigma}^i_{h\ell} = K(x^i_\ell, \lambda, c)\widetilde{\sigma}^i_{\ell\ell}, \widetilde{\sigma}^i_{hh} = 1 - \widetilde{\sigma}^i_{h\ell}$. Otherwise, let $\widetilde{p}^i = 0$ and let $\widetilde{\sigma}^i_{h\ell}$ and $\widetilde{\sigma}^i_{hh}$ such that $\widetilde{\sigma}^i_{h\ell} + \widetilde{\sigma}^i_{hh} = 1, \widetilde{\sigma}^i_{hh}U_h(C^i_h) + \widetilde{\sigma}^i_{h\ell}U_h(C^i_\ell) = \max\{U_h(C^i_h), U_h(C^i_\ell)\}$. Let also $\widetilde{\mu}^i$ given by (7) and $\widetilde{\sigma}^{i'}_{jk} = 0$, $\widetilde{p}^{i'} = 0$, $\widetilde{\mu}^{i'} = 1$ for all $i' \neq i$ and $(j, k) \in \{h, \ell\}^2$. This is a continuation equilibrium, i.e. (3),(4), (5) and (7) are satisfied. Furthermore $\mathcal{A} \equiv \{C_h = C^i_h, C_\ell = C^i_\ell, \widehat{p} = \widetilde{p}^i, \widehat{\sigma}_{hh} = \widetilde{\sigma}^i_{hh}, \widehat{\sigma}_{h\ell} = \widetilde{\sigma}^i_{h\ell}, \widehat{\sigma}_{\ell h} = \widetilde{\sigma}^i_{\ell h}, \widehat{\sigma}_{\ell\ell} = \widetilde{\sigma}^i_{\ell\ell}\}$ satisfies conditions (20) to (25). If (19) also holds, then $\mathcal{A}$ is feasible in $\mathbb{P}_2(\lambda, c, U^*_h, 0)$ and thus $\max\{U_\ell(C^i_h), U_\ell(C^i_\ell)\} \leq \Phi_2(\lambda, c, U^*_h, 0) = U_\ell(C^{**}_\ell)$, hence a contradiction. (19) thus does not hold, which gives $\widetilde{\Pi}^i < 0$.

Case 2: $U_h(C^i_h) < U^*_h$. In that case, we necessary have $U_h(C^i_\ell)\} > U^*_h$. If $x^i_\ell > c$ and $K(x^i_\ell, \lambda, c)\widetilde{\sigma}^i_{\ell\ell} \leq 1$ (Case 2.1), let $\widetilde{p}^i \in (0, 1]$ such that $U_{h\ell}(C^i_\ell, \widetilde{p}^i) = U^*_h, \widetilde{\sigma}^i_{h\ell} = K(x^i_\ell, \lambda, c)\widetilde{\sigma}^i_{\ell\ell}, \widetilde{\sigma}^i_{hh} = 0$. Otherwise (Case 2.2), let $\widetilde{p}^i = 0, \widetilde{\sigma}^i_{h\ell} = 1, \widetilde{\sigma}^i_{hh} = 0$. Let also $\widetilde{\mu}^i$ given by (7), and $\widetilde{\sigma}^{i'}_{hh} = (1 - \widetilde{\sigma}^i_{h\ell})/(n - 1)$, $\widetilde{\sigma}^{i'}_{j\ell} = 0$, $\widetilde{p}^{i'} = 0$, $\widetilde{\mu}^{i'} = 1$ for all $i' \neq i$ and $j \in \{h, \ell\}$. This is a continuation equilibrium and $\mathcal{A}$ (defined as in Case 1) satisfies (20) to (25). In Case 2.1, if (19) also holds, then $\mathcal{A}$ is feasible in $\mathbb{P}_4(\widetilde{\lambda}^i, c, U^*_h, 0)$ where $\widetilde{\lambda}^i = \widetilde{\sigma}^i_{\ell\ell}c\pi_\ell/[\widetilde{\sigma}^i_{\ell\ell}c\pi_\ell + \pi_h(x^i_\ell - c)] \in (0, 1)$ and thus $\max\{U_\ell(C^i_h), U_\ell(C^i_\ell)\} \leq \Phi_4(\widetilde{\lambda}^i, c, U^*_h, 0) = \Phi_4(\lambda, c, U^*_h, 0) \leq \Phi_2(\lambda, c, U^*_h, 0) = U_\ell(C^{**}_\ell)$, hence the same contradiction as in Case 1, which gives $\widetilde{\Pi}^i < 0$. In Case 2.2, if (19) also holds, then $\mathcal{A}$ is feasible in $\mathbb{P}_3(\lambda, c, U^*_h, 0)$, which also leads to $\widetilde{\Pi}^i < 0$.

Proof of Proposition 7

Proposition 7 can be proved in almost the same way as Proposition 6. Propositions 4 and 5 provide the characterization of a semi-separating equilibrium and they show that $U_\ell(\widehat{C}_\ell) = \Phi_2(\lambda, c, U^*_h, 0)$ is a necessary condition for a semi-separating equilibrium to exist. The same reasoning as in the proof of Proposition 7 (by substituting $\widehat{C}_\ell$ to

C_ℓ^{**}) shows that it is also a sufficient condition.

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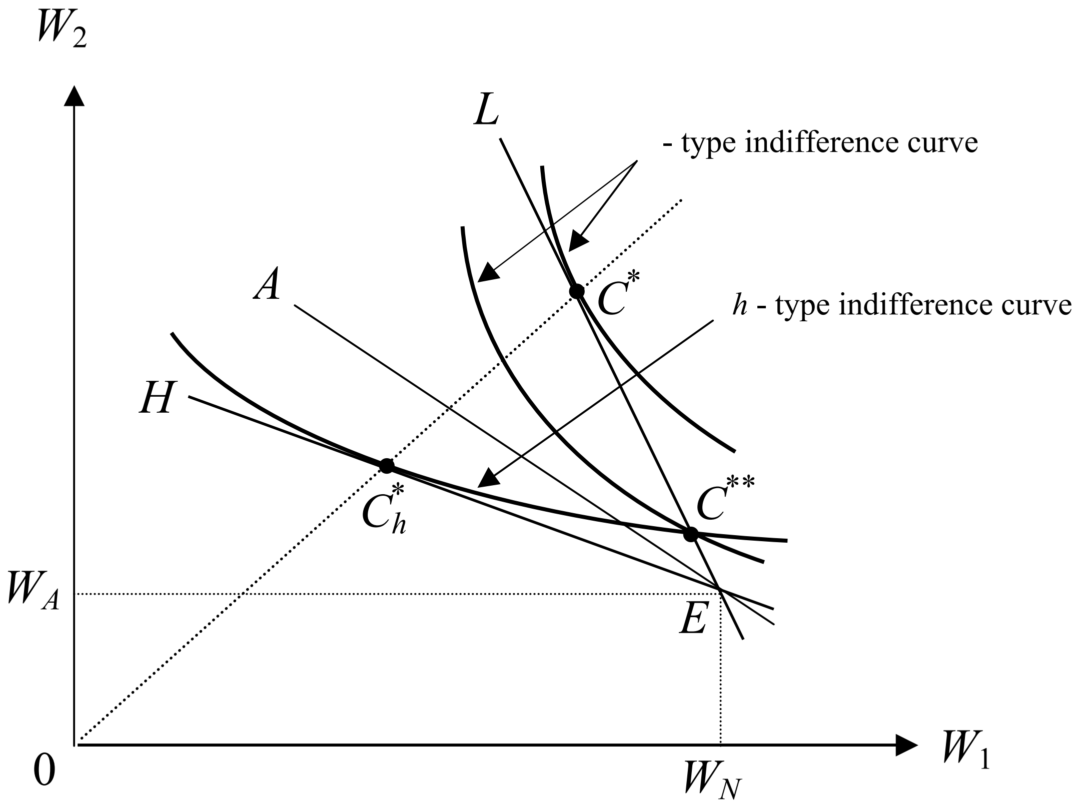


W_2
L
- type indifference curve
A
C^*
h - type indifference curve
H
C_h^*
C^{**}
W_A
E
0
W_1
W_N

Figure 1

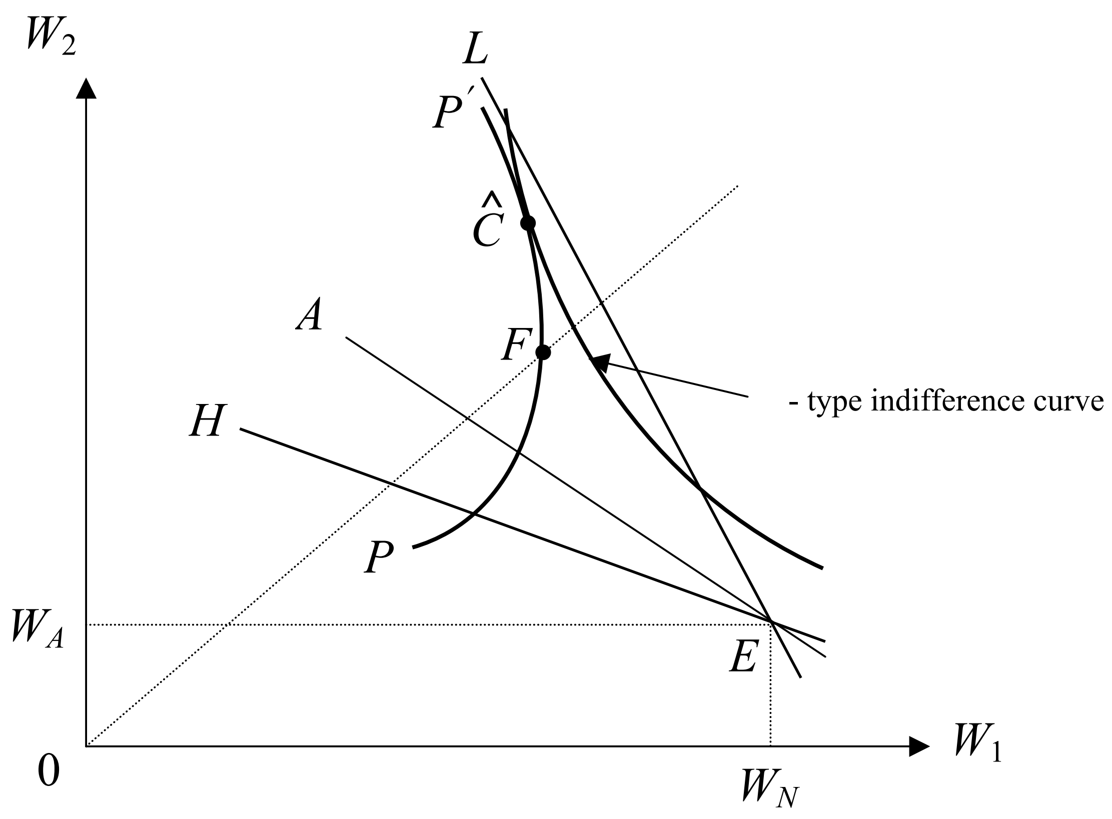


Figure 2

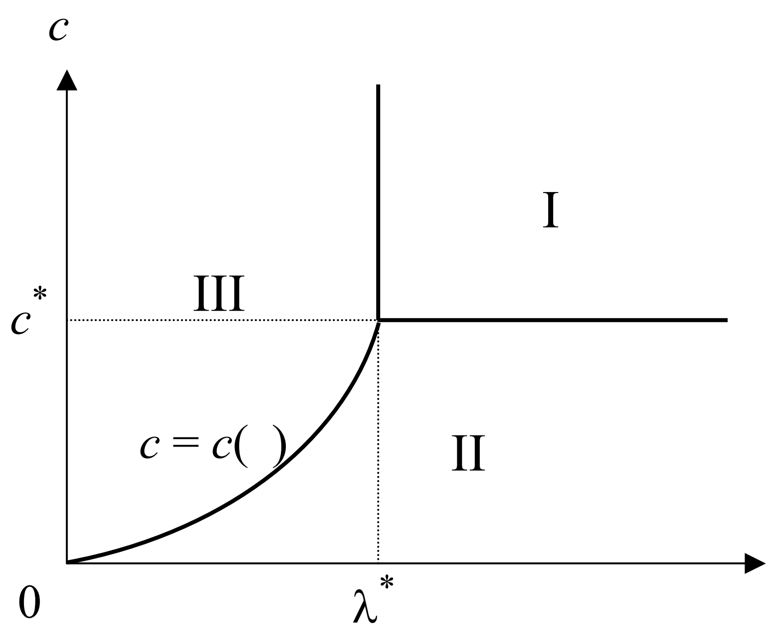
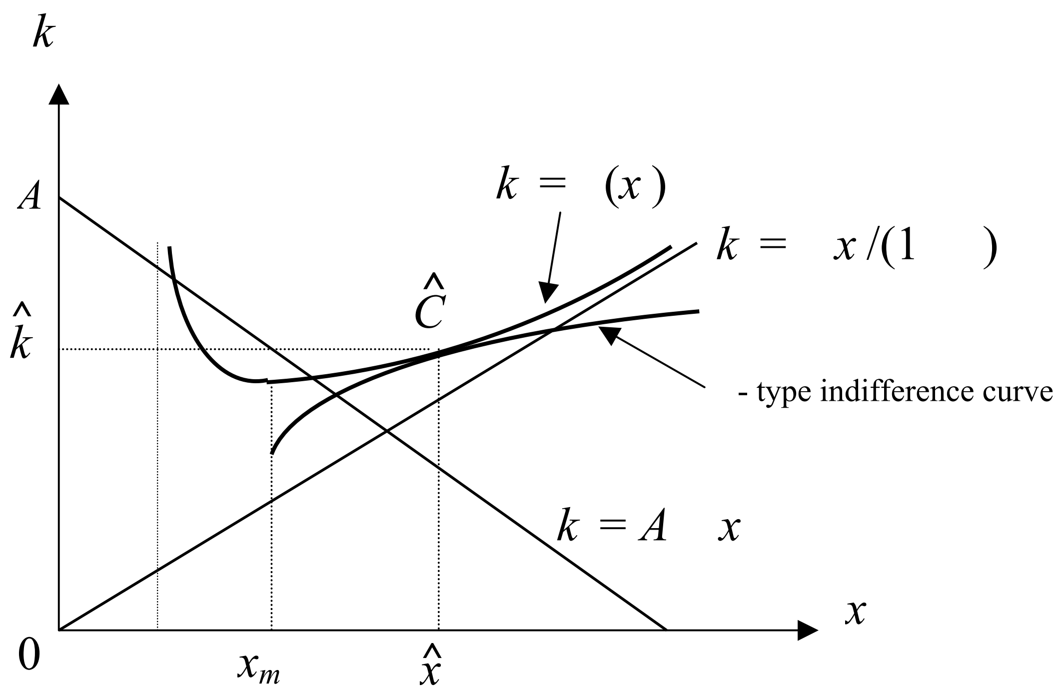


Figure 3



k
A
$\hat{k}$
$k = (x)$
$k = x/(1)$
$\hat{C}$
- type indifference curve
$k = A \quad x$
x
0
x_m
$\hat{x}$

Figure 4

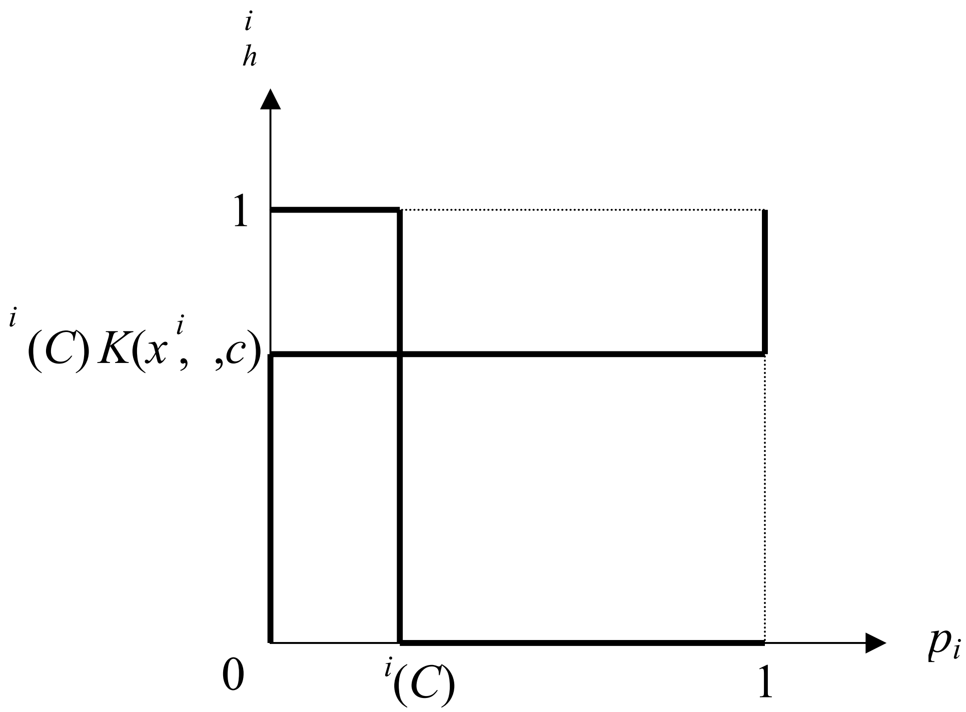


Figure 5