Sharing the cost of a public good without subsidies

Marc Fleurbaey*and Yves Sprumont†

August 8, 2006

Abstract

We study the construction of a social ordering function for the case of a public good financed by contributions from the population, and we extend the analysis of Maniquet and Sprumont (2004) to the case when contributions cannot be negative, i.e., agents cannot receive subsidies from others.

Keywords: social ordering, public good, maximin.

JEL Classification: D63, D71, H41.

1 Introduction

In an economy where a private good can be used to produce a public good, we reconsider the problem of sharing the production cost when negative cost shares are ruled out. This constraint is often imposed in practice; it is unobjectionable if the private good is leisure.

Because first-best allocations may not be achievable due to informational or institutional constraints, a complete social ordering of all allocations is desirable: maximizing such an ordering under the relevant constraints delivers a second best solution. Following Maniquet and Sprumont (2004), we recommend to use the following "public good welfare maximin ordering". Consider an allocation consisting of a level of production of the public good and a list

*CNRS-CERSES, Paris, and IDEP. Email: marc.fleurbaey@univ-paris5.fr.

†Département de sciences économiques and CIREQ, Université de Montréal. Email: yves.sprumont@umontreal.ca.

of cost shares paid by the agents. The "public good welfare" of a given agent at that allocation is the quantity of the public good that, consumed for free, would leave her indifferent to that allocation (Moulin, 1987). The ordering we advocate ranks allocations by applying the maximin criterion to the distributions of public good welfare levels they generate.

Suppose that negative cost shares are allowed and consider a two-agent allocation where agent 1's cost share x_1 is positive while agent 2's cost share x_2 is negative. In the spirit of the Pigou-Dalton principle, a transfer of private good from 2 to 1 that does not reverse the signs of their cost shares should be regarded as a social improvement. Figure 1 shows that the public good welfare maximin ordering satisfies this "Free Lunch Aversion" property. In the figure, a transfer of private good from 2 to 1 increases the lowest intersection of the indifference curves with the vertical axis. Maniquet and Sprumont (2004) proved that Free Lunch Aversion combined with two other standard requirements known as Pareto Indifference and Responsiveness force us to use an ordering compatible with the public good welfare maximin principle.

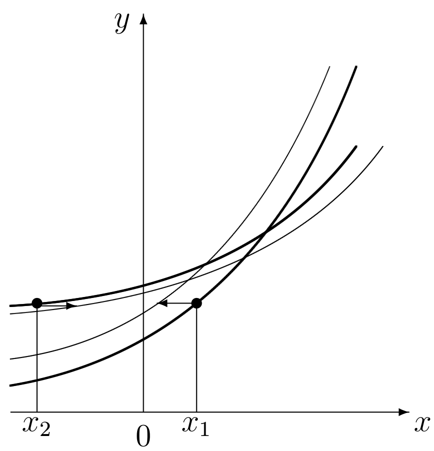

Figure 1

When negative cost shares are forbidden, Free Lunch Aversion does not apply. Yet, Figure 2 shows that the public good welfare maximin ordering satisfies the following local version of the axiom. In a situation where agent 1's cost share is positive while agent 2's cost share is zero, a *small enough* transfer of private good from 2 to 1 is a social improvement. (Note, however, that a transfer leaving 2's cost share smaller than 1's need not yield a better allocation: the transfer must be smaller than δ in Figure 2.). We propose a variant of Maniquet and Sprumont's characterization of the public good welfare maximin ordering based on this weak version of Free Lunch Aversion.

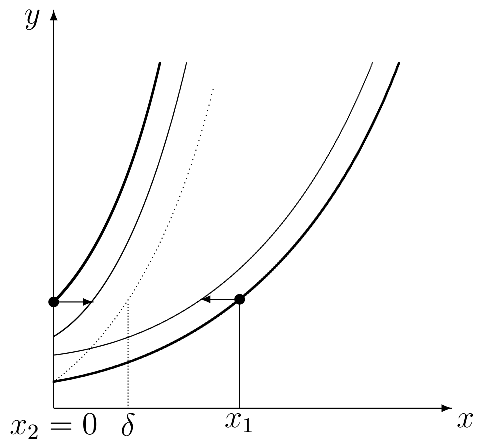

Figure 2

To fix ideas, we present our result for a pure public good. The straightforward extension to an excludable public good is briefly discussed in the last section.

2 Setup

There is a fixed finite set of agents, $N = \{1, ..., n\}$, with $n \geq 2$. There are two goods: one public good and one private good. We denote by $z_i = (x_i, y) \in \mathbb{R}^2_+$ agent i's consumption bundle: y is the consumption of the public good, and x_i is agent i's contribution to its cost, measured in terms of the private good. All agents consume the same quantity of the public good: exclusion is ruled out.

A *preference* for agent i is a binary relation R_i over $\mathbb{R}^2_+$ which is complete, transitive, continuous, strictly decreasing in the private good contribution level x_i, strictly increasing in the public good level y, and convex. The indifference and strict preference relations corresponding to R_i are denoted by I_i and P_i. The set of all preferences is denoted by $\mathcal{R}$. A (*preference*) *profile* is a list $R \in \mathcal{R}^N$.

The set of admissible consumption bundles for an agent with preferences R_i is $Z_i(R_i) = \{z_i \in \mathbb{R}^2_+ \mid z_i R_i (0, 0)\}$. Given a preference profile R, an (*admissible*) *allocation* $\mathbf{z} = (z_1, ..., z_n)$ specifies an admissible consumption bundle $z_i = (x_i, y) \in Z_i(R_i)$ for each agent $i \in N$. Writing $\mathbf{z} = (\mathbf{x}, y) = (x_1, ..., x_n, y)$, we denote the set of admissible allocations by $Z(R) = \{(\mathbf{x}, y) | (x_i, y) \in Z_i(R_i)$ for each $i \in N\}$.

A *social ordering for* R is a complete and transitive binary relation defined over $Z(R)$, the set of all admissible allocations for R. A *social ordering function* $\mathbf{R}$ assigns to each preference profile $R \in \mathcal{R}^N$ a social ordering $\mathbf{R}(R)$ for R. Thus, $\mathbf{zR}(R)\mathbf{z}'$ means that the allocation $\mathbf{z}$ is at least as desirable as $\mathbf{z}'$ from a social viewpoint if the preference profile is R. Similarly, $\mathbf{I}(R)$ and $\mathbf{P}(R)$ denote social indifference and strict social preference.

3 Axioms for social ordering functions

We impose four conditions on the function $\mathbf{R}$. The first one is a weak version of the Pareto principle.

Weak Pareto. Let $R \in \mathcal{R}^N$ and $\mathbf{z}, \mathbf{z}' \in Z(R)$. If $z_i P_i z_i'$ for all $i \in N$, then $\mathbf{zP}(R)\mathbf{z}'$.

The stronger axiom asking that $\mathbf{zP}(R)\mathbf{z}'$ if $z_i R_i z_i'$ for all i and $z_i P_i z_i'$ for some i, is not satisfied by the public good welfare maximin function discussed in the Introduction.

Our second condition is a well known informational simplicity requirement due to Hansson (1973). The condition weakens Arrow's binary independence: it states that the social ranking of two allocations is insensitive to changes in individual preferences that do not affect the indifference curves at those allocations. Given a preference $R_i \in \mathcal{R}$ and a consumption bundle $z_i \in \mathbb{R}^2_+$, let $I(R_i, z_i) = \{z_i' \in \mathbb{R}^2_+ | z_i' I_i z_i\}$ denote the indifference curve of R_i through z_i.

Hansson Independence. Let $R, R' \in \mathcal{R}^N$ and $\mathbf{z}, \mathbf{z}' \in Z(R) \cap Z(R')$. If $I(R_i, z_i) = I(R_i', z_i)$ and $I(R_i, z_i') = I(R_i', z_i')$ for all $i \in N$, then $\mathbf{zR}(R)\mathbf{z}' \Leftrightarrow \mathbf{zR}(R')\mathbf{z}'$.

The third condition is our central axiom. It says that a small enough transfer of the private good from an agent paying nothing to an agent paying a positive cost share is socially desirable. Given two agents $i, j \in N$, define $\mathbf{t}(i, j) \in \mathbb{R}^N$ by $t_i(i, j) = 1$, $t_j(i, j) = -1$, and $t_k(i, j) = 0$ for all $k \in N \backslash \{i, j\}$. The vector $\mathbf{t}(i, j)$ represents a transfer of one unit of the private good from j to i.

Weak Free Lunch Aversion. Let $R \in \mathcal{R}^N$, $(\mathbf{x}, y) \in Z(R)$, and $i, j \in N$ be such that $x_i = 0$ and $x_j > 0$. For all $\varepsilon > 0$ there exists δ, $0 < \delta \leq \varepsilon$, such that $(\mathbf{x}+\delta\mathbf{t}(i, j), y)\mathbf{R}(R)(\mathbf{x},y)$.

A slightly stronger condition would require, under the same premise, that there exist $\varepsilon > 0$ such that $(\mathbf{x}+\delta\mathbf{t}(i,j),y)\mathbf{R}(R)(\mathbf{x},y)$ whenever $0 < \delta \leq \varepsilon$. Both versions are satisfied by the public good welfare maximin function.

Our last condition requires that social preferences be continuous.

Continuity. For all $R \in \mathcal{R}^N$ and $\mathbf{z} \in Z(R)$, the sets $\{\mathbf{z}' \in Z(R) \mid \mathbf{z}'\mathbf{R}(R)\mathbf{z}\}$ and $\{\mathbf{z}' \in Z(R) \mid \mathbf{z}\mathbf{R}(R)\mathbf{z}'\}$ are closed.

4 Public good welfare maximin

For each $R_i \in \mathcal{R}$ and $z_i \in Z_i(R_i)$, there is a unique level of the public good $y \in \mathbb{R}_+$ such that $z_i I_i(0,y)$. We may therefore define the numerical welfare representation function $u(R_i,.) : Z_i(R_i) \to \mathbb{R}_+$ by

$$u(R_i,z_i) = y \Leftrightarrow z_i I_i(0,y).$$

The number $u(R_i,z_i)$ is agent i's *public good welfare level* at bundle z_i. The *public good welfare maximin* (social ordering) function $\mathbf{R}^*$ is defined as follows. For any $R \in \mathcal{R}^N$ and $\mathbf{z},\mathbf{z}' \in Z(R)$,

$$\min_{i\in N} u(R_i,z_i) \geq \min_{i\in N} u(R_i,z_i') \Leftrightarrow \mathbf{z}\mathbf{R}^*(R)\mathbf{z}'.$$

Theorem 1 *The public good welfare maximin function* $\mathbf{R}^*$ *is the only social ordering function satisfying Weak Pareto, Hansson Independence, Weak Free Lunch Aversion, and Continuity.*

Proof. It is routine to check that $\mathbf{R}^*$ satisfies Weak Pareto, Hansson Independence, Weak Free Lunch Aversion, and Continuity. Conversely, let $\mathbf{R}$ be a social ordering function satisfying these axioms. We prove that for all $R \in \mathcal{R}^N$ and $\mathbf{z},\mathbf{z}' \in Z(R)$,

$$\min_{i\in N} u(R_i,z_i) > \min_{i\in N} u(R_i,z_i') \Rightarrow z\mathbf{P}(R)z'. \tag{1}$$

Continuity then implies that

$$\min_{i\in N} u(R_i,z_i) \geq \min_{i\in N} u(R_i,z_i') \Leftrightarrow z\mathbf{R}(R)z',$$

hence $\mathbf{R} = \mathbf{R}^*$.

To prove (1), we rely on the fact that Weak Pareto and Continuity imply the following two properties.

Pareto Indifference. Let $R \in \mathcal{R}^N$ and $\mathbf{z}, \mathbf{z}' \in Z(R)$. If $z_i I_i z_i'$ for all $i \in N$, then $\mathbf{z}\mathbf{I}(R)\mathbf{z}'$.

Pareto. Let $R \in \mathcal{R}^N$ and $\mathbf{z}, \mathbf{z}' \in Z(R)$. If $z_i R_i z_i'$ for all $i \in N$, then $\mathbf{z}\mathbf{R}(R)\mathbf{z}'$.

Let $R \in \mathcal{R}^N$ and $\mathbf{z}, \mathbf{z}' \in Z(R)$ be such that $\min_{i \in N} u(R_i, z_i) > \min_{i \in N} u(R_i, z_i')$. Assume, contrary to the desired result, that

$$\mathbf{z}'\mathbf{R}(R)\mathbf{z}. \tag{2}$$

Without loss of generality, suppose $\min_{i \in N} u(R_i, z_i') = u(R_n, z_n')$.

Step 1. We derive from the social preference (2) a social preference between two allocations related to $\mathbf{z}, \mathbf{z}'$ at a profile related to R. Pareto and Hansson Independence are used repeatedly in this part of the proof. Figure 3 is provided to illustrate the argument in the case $n = 2$ (to keep the notations of the figure close to the text, one agent is called n and the other i).

Step 1.1. Let $\mathbf{z}^a = (\mathbf{x}^a, y^a)$, $\mathbf{z}^b = (\mathbf{x}^b, y^b) \in Z(R)$ be such that

(i) $x_i^a = x_i^b = 0$ for all $i \in N \backslash n$,
(ii) $z_i P_i z_i^a$, $z_i^b P_i z_i'$ for all $i \in N$,
(iii) $u(R_n, z_n^b) < u(R_n, z_n^a) < y^a < y^b$.

Define $\alpha = y^b - y^a$.

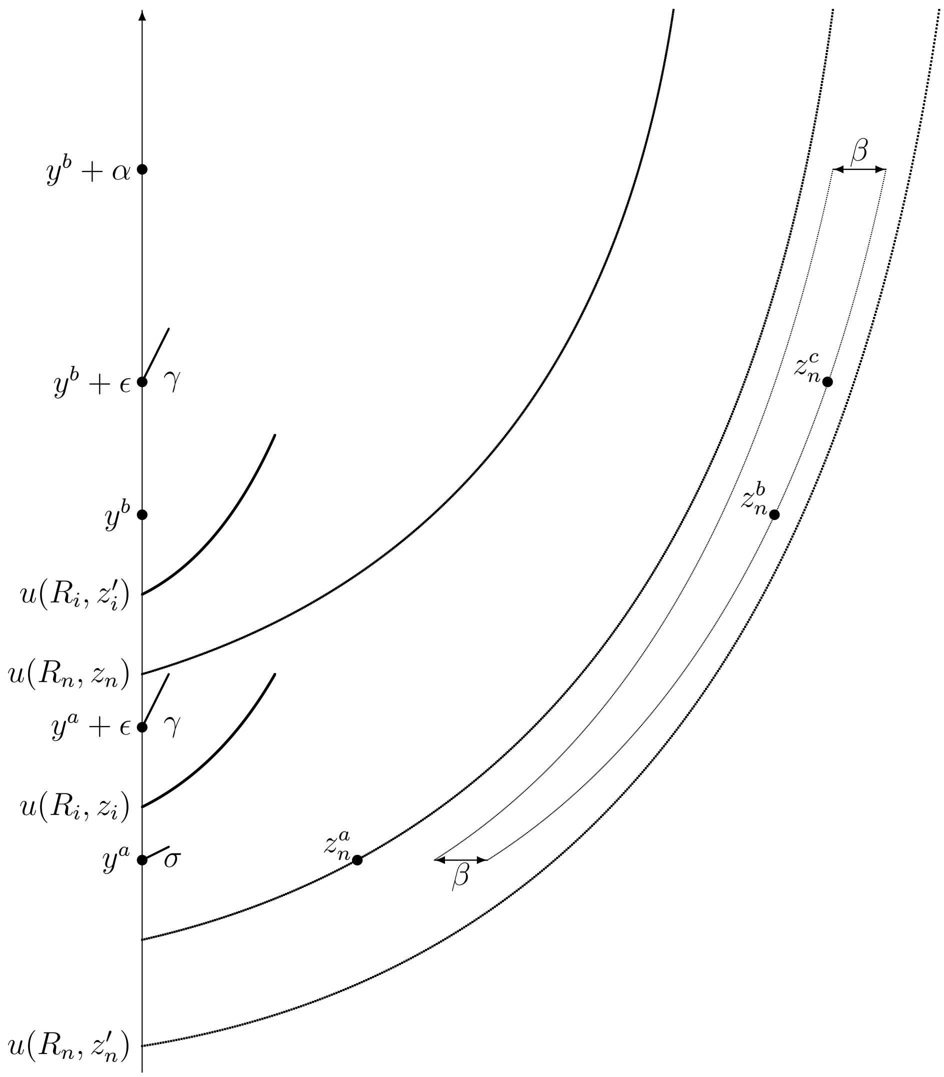

Figure 3

Modify R to obtain a profile $R' \in \mathcal{R}^N$ satisfying the following conditions. First, the indifference curves at the allocations $\mathbf{z}$, $\mathbf{z}'$ are unchanged:

$$I(R_i', z_i) = I(R_i, z_i) \text{ and } I(R_i', z_i') = I(R_i, z_i')$$

for all $i \in N$. Second, the indifference curves of R_n' restricted to public good levels between y^a and $y^b + \alpha$, are parallel to the curve $I(R_n, z_n^b)$ within a small distance to the left of that curve. The purpose of this construction will become clear in Step 2, where we perform a sequence of private good transfers from agent n to the others. Formally: for each $(x, y) \in \mathbb{R}_+ \times [y^a, y^b + \alpha]$, let $d(x, y)$ be the number such that $(x + d(x, y), y) \in I(R_n, z_n^b)$, define $Z^\beta = \{(x, y) \in \mathbb{R}_+ \times [y^a, y^b + \alpha] \mid 0 \leq d(x, y) \leq \beta\}$, and choose $\beta > 0$ small enough that $I(R_n, z_n^a) \cap Z^\beta = \emptyset$. For all $(x, y), (x', y') \in Z^\beta$, let

$$(x, y) R'_n (x', y') \Leftrightarrow d(x, y) \geq d(x', y').$$

Observe that $I(R_n, z_n^a) \cap Z^\beta = \emptyset$ implies that $z_n^a P'_n (x, y)$ for all $(x, y) \in Z^\beta$. Third, the indifference curves of all other preferences have a positive slope at $(0, y^a)$. Using straightforward notation,

$$s(R'_i, (0, y^a)) > 0$$

for all $i \in N \backslash n$.

By (2) and Hansson Independence, $\mathbf{z}'\mathbf{R}(R')\mathbf{z}$. By Weak Pareto, $\mathbf{z}\mathbf{P}(R')\mathbf{z}^a$ and $\mathbf{z}^b\mathbf{P}(R')\mathbf{z}'$. Since $\mathbf{R}(R')$ is transitive,

$$\mathbf{z}^b \mathbf{P}(R') \mathbf{z}^a. \tag{3}$$

Step 1.2. Let $\sigma = \min_{i \in N \backslash n} s(R'_i, (0, y^a))$: this smallest slope is positive by construction of R'. Choose a positive number ε such that

$$\varepsilon \leq \min\{\alpha, \frac{\beta\sigma}{2(n-1)}\}$$

and choose

$$\gamma \geq \max\{\sigma, \frac{2\alpha(n-1)}{\beta}\}.$$

Note that $y^a + \varepsilon \leq y^b$.

Let $R'' \in \mathcal{R}^N$ be a profile such that $R''_n = R'_n$ and the indifference curves of all other agents are unchanged at $(0, y^a)$ and $(0, y^b)$, and steep enough at $(0, y^b + \varepsilon)$:

$$\begin{aligned} I(R''_i, (0, y^a)) &= I(R'_i, (0, y^a)), \\ I(R''_i, (0, y^b)) &= I(R'_i, (0, y^b)), \\ s(R''_i, (0, y^b + \varepsilon)) &\geq \gamma \end{aligned}$$

for all $i \in N \backslash n$.

By (3) and Hansson Independence, $\mathbf{z}^b\mathbf{P}(R'')\mathbf{z}^a$. Next, consider the allocation $\mathbf{z}^c$ given by $z_i^c = (0, y^b + \varepsilon)$ for all $i \in N\backslash n$ and $z_n^c = (x_n^c, y^b + \varepsilon)I_n'' z_n^b$. By Pareto, $\mathbf{z}^c\mathbf{R}(R'')\mathbf{z}^b$, hence

$$\mathbf{z}^c\mathbf{P}(R'')\mathbf{z}^a. \tag{4}$$

Next, let $R''' \in \mathcal{R}^N$ be a profile such that $R_n''' = R_n'' = R_n'$ and the indifference curves of all other agents are steep enough for levels of the public good between y^a and $y^b + \varepsilon$:

$$\begin{aligned} s(R_i''', (0, y)) &\geq \sigma \text{ if } y^a \leq y < y^a + \varepsilon, \\ s(R_i''', (0, y)) &\geq \gamma \text{ if } y^a + \varepsilon \leq y \leq y^b + \varepsilon \end{aligned}$$

for all $i \in N\backslash n$. These requirements are consistent because $\sigma \leq \gamma$.

By (4) and Hansson Independence,

$$\mathbf{z}^c\mathbf{P}(R''')\mathbf{z}^a. \tag{5}$$

Step 2. We use Weak Free Lunch Aversion, together with Pareto and Continuity, to derive a contradiction to (5).

Step 2.1. Let Z^c denote the set of allocations $\mathbf{z} = (\mathbf{x}, y) \in Z(R''')$ such that

(i) $\mathbf{z}\mathbf{R}(R''')\mathbf{z}^c$,
(ii) $y^a + \varepsilon \leq y \leq y^b + \varepsilon$,
(iii) $(x_i, y)R_i'''(0, y^a + \varepsilon)$ for all $i \in N\backslash n$ and $x_i = 0$ for some $i \in N\backslash n$,
(iv) $d(x_n, y) \leq \sum_{i\in N\backslash n} x_i + (n-1)\frac{y^b+\varepsilon-y}{\gamma}$.

The set Z^c is nonempty since it contains $\mathbf{z}^c$. It is compact by Continuity. Let $\mathbf{z}^* = (\mathbf{x}^*, y^*) \in Z^c$ be such that $y^* \leq y$ for all $(\mathbf{x}, y) \in Z^c$. We claim that $y^* = y^a + \varepsilon$.

Suppose that $y^* > y^a + \varepsilon$. By (iii), $x_i^* = 0$ for at least one agent $i \in N\backslash n$. Applying Weak Free Lunch Aversion as many times as there are such agents, there exists $\mathbf{x}$ such that

$$x_i > 0 \text{ and } (x_i, y^*)R_i'''(0, y^a + \varepsilon) \text{ for all } i \in N\backslash n, \tag{6}$$

$\sum_{i\in N} x_i = \sum_{i\in N} x_i^*$, and $(\mathbf{x},y^*)\mathbf{R}(R''')(\mathbf{x}^*, y^*)$.

Construct $(\mathbf{x}', y')$ such that $(x_i', y')I_i'''(x_i, y^*)$ for all $i \in N$ and $x_i' = 0$ for some $i \in N\backslash n$. Since preferences are strictly monotonic, $y' < y^*$. We claim that $(\mathbf{x}', y') \in Z^c$, contradicting the definition of $(\mathbf{x}^*, y^*)$.

By Pareto Indifference, $(\mathbf{x}',y')\mathbf{I}(R''')(\mathbf{x},y^*)$. It follows that $(\mathbf{x}',y')$ $\mathbf{R}(R''')$ $(\mathbf{x}^*,y^*)$ $\mathbf{R}(R''')$ $\mathbf{z^c}$, hence $(\mathbf{x}',y')$ satisfies (i). In view of (6), $(\mathbf{x}',y')$ satisfies (ii) and (iii). Since $(\mathbf{x}^*,y^*)\in Z^c$,

$$d(x_n^*,y^*)\leq\sum_{i\in N\backslash n}x_i^*+(n-1)\frac{y^b+\varepsilon-y^*}{\gamma},\tag{7}$$

and since $s(R_i''',(0,y'))\geq\gamma$ for all $i\in N\backslash n$,

$$\sum_{i\in N\backslash n}(x_i-x_i')\leq(n-1)\frac{y^*-y'}{\gamma}.\tag{8}$$

Because the indifference curves of R_n''' are parallel over the relevant region, $d(x_n',y')-d(x_n^*,y^*)=x_n^*-x_n=\sum_{i\in N\backslash n}(x_i-x_i^*)$. Combining this equality with inequalities (7), (8),

$$\begin{aligned}d(x_n',y') &= d(x_n^*,y^*)+\sum_{i\in N\backslash n}(x_i-x_i^*)\\ &\leq \sum_{i\in N\backslash n}x_i+(n-1)\frac{y^b+\varepsilon-y^*}{\gamma}\\ &\leq \sum_{i\in N\backslash n}x_i'+(n-1)\frac{y^b+\varepsilon-y'}{\gamma},\end{aligned}$$

that is, $(\mathbf{x}',y')$ satisfies (iv), completing the proof that $(\mathbf{x}',y')\in Z^c$. This is the announced contradiction; we conclude that $y^*=y^a+\varepsilon$.

Since $y^*=y^a+\varepsilon$ and $y^b-y^a=\alpha$, conditions (iii) and (iv) applied to $\mathbf{z}^*=(\mathbf{x}^*,y^*)$ give

$$x_i^*=0\text{ for all }i\in N\backslash n\text{ and }d(x_n^*,y^*)\leq(n-1)\frac{\alpha}{\gamma}.$$

Step 2.2. Let Z^* denote the set of allocations $\mathbf{z}=(\mathbf{x},y)\in Z(R''')$ such that

(i) $\mathbf{z}\mathbf{R}(R''')\mathbf{z}^*$,
(ii) $y^a\leq y\leq y^a+\varepsilon$,
(iii) $(x_i,y)R_i'''(0,y^a)$ for all $i\in N\backslash n$ and $x_i=0$ for some $i\in N\backslash n$,
(iv) $d(x_n,y)\leq\sum_{i\in N\backslash n}x_i+(n-1)(\frac{\alpha}{\gamma}+\frac{y^a+\varepsilon-y}{\sigma})$.

Again, this set is nonempty and compact. Letting $\mathbf{z}^{**}=(\mathbf{x}^{**},y^{**})\in Z^*$ be such that $y\geq y^{**}$ for all $\mathbf{z}=(\mathbf{x},y)\in Z^*$, it is straightforward to mimic the argument in Step 2.1 (this time, using the fact that $s(R_i''', (0, y)) \geq \sigma$ for all $i \in N\backslash n$ and $y^a \leq y < y^a + \varepsilon$) to show that $y^{**} = y^a$.

Conditions (iii) and (iv) applied to $\mathbf{z}^{**} = (\mathbf{x}^{**}, y^{**})$ give

$$x_i^* = 0 \text{ for all } i \in N\backslash n \text{ and } d(x_n^{**}, y^{**}) \leq (n-1)(\frac{\alpha}{\gamma} + \frac{\varepsilon}{\sigma}).$$

Since $\gamma \geq \frac{2\alpha(n-1)}{\beta}$ and $\varepsilon \leq \frac{\beta\sigma}{2(n-1)}$, we get $d(x_n^{**}, y^{**}) \leq \beta$.

Recall from Step 1.1 that this implies $z_n^a P_n' z_n^{**}$, and since $R_n''' = R_n'$, we have $z_n^a P_n''' z_n^{**}$. Pareto implies

$$\mathbf{z}^a \mathbf{R}(R''') \mathbf{z}^{**},$$

and, since $\mathbf{z}^* \mathbf{R}(R''') \mathbf{z}^c$ and $\mathbf{z}^{**} \mathbf{R}(R''') \mathbf{z}^*$, by transitivity, $\mathbf{z}^a \mathbf{R}(R''') \mathbf{z}^c$, the announced contradiction to (5). ■

5 Discussion

1) The following examples show that our axioms are independent.

i) The social ordering function recommending indifference between all allocations at all preference profiles satisfies all axioms in Section 3, except Weak Pareto.

ii) Let $\mathcal{L} \subset \mathcal{R}$ be the subset of linear preferences: each $R_i \in \mathcal{L}$ is characterized by a positive number $\alpha(R_i)$ such that $(x_i, y) R_i (x_i', y')$ if and only if $y - \alpha(R_i) x_i \geq y' - \alpha(R_i) x_i'$. Define $\mathbf{R}$ as follows. If $R \in \mathcal{L}^N$ and $\mathbf{z}, \mathbf{z}' \in Z(R)$,

$$z\mathbf{R}(R)z' \Leftrightarrow \sum_{i \in N} \frac{u(R_i, z_i)}{\alpha(R_i)} \geq \sum_{i \in N} \frac{u(R_i, z_i')}{\alpha(R_i)}.$$

If $R \in \mathcal{R}^N \backslash \mathcal{L}^N$ and $\mathbf{z}, \mathbf{z}' \in Z(R)$, then $z\mathbf{R}(R)z' \Leftrightarrow z\mathbf{R}^*(R)z'$. The function $\mathbf{R}$ satisfies all our axioms but Hansson Independence.

iii) To construct a function $\mathbf{R}$ satisfying all axioms but Weak Free Lunch Aversion, we begin with a remark. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly increasing function and define, for each $R_i \in \mathcal{R}$, a welfare representation function $v(R_i, .) : Z_i(R_i) \to \mathbb{R}_+$ by $v(R_i, z_i) = f(y) \Leftrightarrow z_i I_i (0, y)$. We say that the welfare representation procedure v is *equivalent to* the welfare representation procedure u. Clearly, the social ordering function $\mathbf{R}$ defined by $\mathbf{z}\mathbf{R}(R)\mathbf{z}' \Leftrightarrow \min_{i \in N} v(R_i, z_i) \geq \min_{i \in N} v(R_i, z_i')$ for all $R \in \mathcal{R}^N$ and $\mathbf{z}, \mathbf{z}' \in Z(R)$ *is* the public good welfare maximin function $\mathbf{R}^*$.

To obtain the desired example, choose a continuous welfare representation procedure v that is *not* equivalent to u, and apply the maximin criterion to the welfare distributions generated by the allocations. For instance, let $v(R_i, z_i) = y \Leftrightarrow z_i I_i(1/y, y)$ (i.e. v measures the vertical coordinate of the intersection between the indifference curve and the hyperbola of equation $x = 1/y$), and let $\mathbf{z}\mathbf{R}(R)\mathbf{z}' \Leftrightarrow \min_{i\in N} v(R_i, z_i) \geq \min_{i\in N} v(R_i, z_i')$ for all $R \in \mathcal{R}^N$ and $\mathbf{z}, \mathbf{z}' \in Z(R)$.

iv) Finally, the public good welfare leximin function (Maniquet and Sprumont, 2004) satisfies all our axioms but Continuity. That social ordering function agrees with the maximin function whenever the latter expresses a strict preference between two allocations. For a more radically different example, consider the following function $\mathbf{R}$. For all $R \in \mathcal{R}^N$ and $\mathbf{z}, \mathbf{z}' \in Z(R)$, let $z\mathbf{R}(R)z'$ if and only if either

$$|\{i \in N \mid u(R_i, z_i) > 1\}| > |\{i \in N \mid u(R_i, z_i') > 1\}|$$

or

$$|\{i \in N \mid u(R_i, z_i) > 1\}| = |\{i \in N \mid u(R_i, z_i') > 1\}| \text{ and } \min_{i\in N} u(R_i, z_i) \geq \min_{i\in N} u(R_i, z_i').$$

2) Contrary to Maniquet and Sprumont (2004), we use Weak Pareto and Hansson Independence instead of Pareto Indifference (defined in the proof above) and Responsiveness (which says that the relative ranking of an allocation with respect to another is not worsened if the upper contour sets of the agents shrink at the bundles of this allocation and expand at the other). This difference is inessential and Maniquet and Sprumont's result holds with either combination of axioms. Hansson Independence is logically weaker than Responsiveness, and Weak Pareto is usually considered more intuitively compelling than Pareto Indifference, although under Continuity, Weak Pareto implies Pareto Indifference as recalled in our proof.

Another slight difference between the two papers is that our Weak Free Lunch Aversion axiom only requires weak social preference whereas Maniquet and Sprumont's Free Lunch Aversion requires strict preference for the post-transfer allocation. This difference is connected to the other because with an axiom requiring only weak social preference, Pareto-Indifference and Responsiveness do not permit to exclude universal indifference as a possible social preference. Maniquet and Sprumont show that Pareto Indifference, Responsiveness and Free Lunch Aversion imply Weak Pareto. With a variant of our Weak Free Lunch Aversion requiring strict preference, we can prove a similar result (with a similar argument, which we omit here). This implies that with this variant of Weak Free Lunch Aversion, we could replace Weak Pareto and Hansson Independence by Pareto Indifference and Responsiveness and still obtain our result.

3) Finally, we provide a brief explanation of the claim made in the Introduction that our result extends to the case of a public good with exclusion. When a nonrival good is excludable, agents may consume different quantities of the good. By Pareto Indifference, and thanks to monotonicity and convexity of preferences, one can always find an equivalent allocation in which all agents consume the same quantity of the nonrival good. Therefore, with just this additional step, our proof shows that the public good welfare maximin function is the only one satisfying the axioms in this extended setting, even when the Weak Free Lunch Aversion axiom is written so as to apply only to allocations in which all agents consume the same quantity of the nonrival good.

6 References

Hansson, B. 1973, "The independence condition in the theory of social choice," *Theory and Decision* 4: 25-49.

Maniquet, F. and Sprumont, Y. 2004, "Fair production and allocation of an excludable nonrival good," *Econometrica* 72: 627-640.

Moulin, H. 1987, "Egalitarian-equivalent cost sharing of a public good," *Econometrica* 55: 963-976.