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Abstract: In most economics textbooks there is a gap between the non-existence of utility functions and the existence of continuous utility functions, although upper semi-continuity is sufficient for many purposes. Starting from a simple constructive approach for countable domains and combining this with basic measure theory, we obtain necessary and sufficient conditions for the existence of upper semi-continuous utility functions on a wide class of domains. Although links between utility theory and measure theory have been pointed out before, to the best of our knowledge this is the first time that the present route has been taken**.**

Key Words : preferences, utility theory, measure theory, outer measure

JEL Classification : C60, D01

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1. Introduction

In most economics textbooks there is a gap between the potential non-existence of utility functions for complete and transitive preference relations on non-trivial connected Euclidean domains — usually illustrated by lexicographic preferences (Debreu, 1954) — and the existence of continuous utility functions for complete, transitive and continuous preferences on connected Euclidean domains; see, e.g. Mas-Colell, Whinston, and Green (1995). Yet, for many purposes, in particular for the existence of a best alternative in a compact set of alternatives, a weaker property — upper semi-continuity — suffices. Hence, the reader of such a textbook treatment might wonder if there exist upper semi-continuous utility functions, and whether this is true even if the domain is not connected. We here fill this gap providing *necessary and sufficient* conditions for the existence of upper semi-continuous utility functions on arbitrary domains; see Theorem 3.1 and Remark 3.2. Our approach is intuitive and constructive: we start by constructing utilities on a countable domain and then generalize the approach to arbitrary domains by way of basic measure theory.

Measure theory is the branch of mathematics that deals with the question of how to define the "size" (area/volume) of sets. We here formalize a direct intuitive link with utility theory: given a binary preference relation on a set of alternatives, the "better" an alternative is, the "larger" is its set of worse alternatives. So if one can measure the "size" of the set of worse elements, for each given alternative, one obtains a utility function.

To be a bit more precise, measure theory starts out by first defining the "size" — measure — of a class of "simple" sets, such as bounded intervals on the real line or rectangles in the plane, and then extends this definition to other sets by way of approximation in terms of simple sets. The outer measure is the best such approximation "from above". This is illustrated in Figure 1, where a set S in the plane is covered by rectangles. The outer measure S is the infimum, over

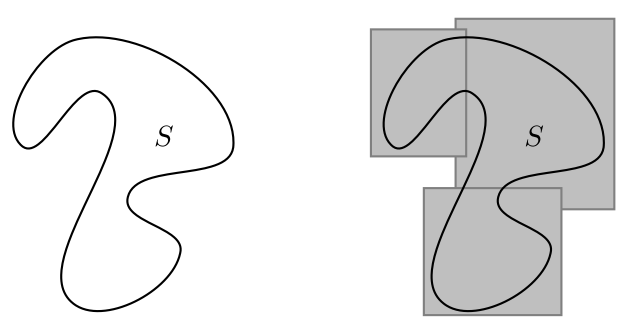

Figure 1: A set S and an approximation of its size using a covering.

all coverings by a countable number of rectangles, of the sum of the rectangles' areas. In more general settings, the outer measure is defined likewise as the infimum over coverings whose sizes have been defined; see, for instance, Rudin (1976, p. 304), Royden (1988, Sec. 3.2), Billingsley (1995, Sec. 3), Ash (2000, p. 14).

We follow this approach by way of defining the utility of an alternative as the outer measure of its set of worse alternatives. We start by doing this for a countable set of alternatives, where this is relatively simple and then proceed to arbitrary sets.

Our paper is not the first to use tools from measure theory to address the question of utility representation: pioneering papers are Neuefeind (1972) and Sondermann (1980). See Bridges and Mehta (1995, sections 2.2 and 4.3) for a textbook treatment. However, our approach differs fundamentally from these precursors. Firstly, we only use the basic notion of outer measure, while the mentioned studies impose additional topological and/or measure-theoretic constraints.[2] To the best of our knowledge, the logical connection between outer measure and utility has never been made before. This link between utility theory and measure theory is more explicit, intuitive and mathematically elementary than the above-mentioned approaches. Secondly, our approach applies to a wider range of domains, in fact to *all* preference relations that admit real-valued representations.

The rest of the paper is organized as follows. Section 2 recalls definitions and provides notation. Our general representation theorem is given in Section 3. Its proof is in the appendix.

2. Definitions and notation

Preferences. Let preferences on an arbitrary set X be defined in terms of a binary relation $\succsim$ ("weakly preferred to") which is:

complete: for all $x, y \in X : x \succsim y, y \succsim x$, or both;

transitive: for all $x, y, z \in X$: if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

As usual, $x \succ y$ means $x \succsim y$, but not $y \succsim x$, whereas $x \sim y$ means that both $x \succsim y$ and $y \succsim x$. The sets of elements strictly worse and strictly better than $y \in X$ are denoted

$$W(y) = \{x \in X : x \prec y\} \text{ and } B(y) = \{x \in X : x \succ y\}.$$

[2] Neuefeind (1972) restricts attention to finite-dimensional Euclidean spaces and assumes that indifference sets have Lebesgue measure zero. Sondermann (1980) assumes that preferences are defined on a probability space or a second countable topological space; see also Corollary 3.4 below.

For $x, y \in X$ with $x \prec y$, the "open interval" of alternatives better than x but worse than y is denoted

$$(x, y) \quad = \quad \{z \in X : x \prec z \prec y\}.$$

Topology. Given a topology on X, preferences $\succsim$ are:

continuous if for each $y \in X$, $W(y)$ and $B(y)$ are open;

upper semi-continuous (usc) if for each $y \in X$, $W(y)$ is open.

Similarly, a function $u : X \to \mathbb{R}$ is usc if for each $r \in \mathbb{R}$, $\{x \in X : u(x) < r\}$ is open.

Three important topologies are, firstly, the *order topology*, generated by (i.e., the smallest topology containing) the collections $\{W(y) : y \in X\}$ and $\{B(y) : y \in X\}$; secondly, the *lower order topology*, generated by the collection $\{W(y) : y \in X\}$, and thirdly, for any subset $D \subseteq X$, the *D-lower order topology*, generated by the collection $\{W(y) : y \in D\}$. By definition, the order topology is the coarsest topology in which $\succsim$ is continuous; the lower order topology is the coarsest topology in which $\succsim$ is usc.

As mentioned in the introduction, although one often appeals to continuity to establish existence of most preferred alternatives, the weaker requirement of upper semi-continuity suffices. A short proof: consider a complete, transitive, usc binary relation $\succsim$ over a compact set X. If X has no most preferred element, then for each $x \in X$, there is a $y \in X$ with $y \succ x$, i.e., the collection $\{W(y) : y \in X\}$ is a covering of X with (by usc) open sets. By compactness, there are finitely many $y^1, \ldots, y^k \in X$ such that $W(y^1), \ldots, W(y^k)$ cover X. Let y^j be the most preferred element of $\{y^1, \ldots, y^k\}$. Then $W(y^j)$ covers the entire set X, a contradiction.

Utility. A preference relation $\succsim$ is *represented* by a utility function $u : X \to \mathbb{R}$ if

$$\forall x, y \in X : \qquad \begin{cases} x \sim y & \Rightarrow \quad u(x) = u(y), \\ x \succ y & \Rightarrow \quad u(x) > u(y). \end{cases} \tag{1}$$

3. Upper semi-continuous utility via outer measures

A complete, transitive binary relation $\succsim$ on a set X can be represented by a utility function if and only if it is *Jaffray order separable*[3] (Jaffray, 1975): there is a countable set $D \subseteq X$ such that for all $x, y \in X$:

$$x \succ y \qquad \Rightarrow \qquad \exists d, d' \in D : x \succsim d \succ d' \succsim y. \tag{2}$$

[3]See Fishburn (1970, Sec. 3.1) or Bridges and Mehta (1995, Sec. 1.4) for alternative separability conditions.

Roughly speaking, countably many alternatives suffice to keep all pairs $x, y \in X$ with $x \succ y$ apart: x lies on one side of d and d', whereas y lies on the other. To make our search for a (usc) utility representation at all meaningful, we will henceforth focus on preference relations that are Jaffray order separable.

The set D in the definition of Jaffray order separability is countable, so let $n : D \to \mathbb{N}$ be an injection. Finding a utility function on D is easy. Give each element d of D a positive weight such that the weights have a finite sum and use the total weight of the elements weakly worse than d as the utility of d. For instance, give weight $\frac{1}{2}$ to the alternative d with label $n(d) = 1$, weight $\frac{1}{4}$ to the alternative d with label $n(d) = 2$, and inductively, weight $w(d) = 2^{-k}$ to the alternative d with label $n(d) = k$. In general, let $(\varepsilon_k)_{k=1}^{\infty}$ be a summable sequence of positive weights; w.l.o.g. its sum $\sum_{k=1}^{\infty} \varepsilon_k$ is one. Assign to each $d \in D$ weight $w(d) = \varepsilon_{n(d)}$.[4] Define $u_0 : D \to \mathbb{R}$ for each $d \in D$ by $u_0(d) = \sum_{d' \precsim d} w(d')$. Clearly, (1) is satisfied.

We can extend this procedure from D to X as follows. Let $\mathcal{W} = \{W(d) : d \in D\} \cup \{\emptyset, X\}$ and define $\rho : \mathcal{W} \to [0, 1]$ as follows: $\rho(\emptyset) = 0, \rho(X) = 1$ and for $d \in D$:

$$\rho(W(d)) = \sum_{d' \in D : d' \precsim d} w(d'). \tag{3}$$

Notice that $\mathcal{W}$ is countable and that it is a covering of X. Extend ρ to an *outer measure* μ^* on X in the usual way (recall Figure 1): for each set $A \subseteq X$, define $\mu^*(A)$ as the smallest total size of sets in $\mathcal{W}$ covering A:

$$\mu^*(A) = \inf\{\textstyle\sum_{i \in \mathbb{N}} \rho(W_i) : \{W_i\}_{i \in \mathbb{N}} \subseteq \mathcal{W}, A \subseteq \cup_{i \in \mathbb{N}} W_i\}.$$

Define $u : X \to \mathbb{R}$ for each $x \in X$ as the outer measure of the set of elements worse than x:

$$u(x) = \mu^*(W(x)). \tag{4}$$

This outer measure gives the desired utility representation:

Theorem 3.1 *Consider a complete, transitive, Jaffray order separable binary relation $\succsim$ on an arbitrary set X. The outer-measure utility function u in (4) represents $\succsim$ and is usc in the D-lower order topology.*

[4] If there is a worst element in X (an $x^0 \in X$ with $x^0 \precsim x$ for all $x \in X$), one may assume w.l.o.g. that D contains one such element, say $\underline{d}$. Its weight can be normalized to zero: $w(\underline{d}) = 0$. This will assure that $\rho(W(\underline{d})) = \rho(\emptyset) = 0$ in (3).

Remark 3.2 We chose to formulate the theorem in its present form, as it stresses that the outer-measure utility function represents preferences whenever a utility representation is at all feasible, even if a usc representation is impossible. Yet it gives *necessary and sufficient* conditions for representation in terms of a usc utility function: there is a usc utility function representing preferences $\succsim$ if and only if they are complete, transitive, Jaffray order separable, and usc in any topology equal to or finer than the D-lower order topology. $\triangleleft$

Corollaries 3.3 and 3.4 below provide applications of our general theorem. Consider preferences $\succsim$ over a commodity space $X = \mathbb{R}^n_+$ $(n \in \mathbb{N})$ with its standard Euclidean topology.[5] If $\succsim$ is usc in this topology, it is Jaffray order separable (Rader, 1963). By assumption, $W(y)$ is open for each $y \in \mathbb{R}^n_+$, so the Euclidean topology is finer than the D-lower order topology. Hence, Theorem 3.1 applies:

Corollary 3.3 *If $\succsim$ is a complete, transitive binary relation over $\mathbb{R}^n_+$ $(n \in \mathbb{N})$ and usc in the Euclidean topology, the outer-measure utility function in* (4) *represents $\succsim$ and is usc in the Euclidean topology.*

The closest result we could find to Theorem 3.1 is a result in Sondermann (1980), which is a special case. Call a preference relation $\succsim$ on a set X *perfectly separable* if there is a countable set $C \subseteq X$ such that for all $x, y \in X$, with $x \not\sim c$ and $y \not\sim c$ for all $c \in C$, the following holds:

$$x \succ y \qquad \Rightarrow \qquad \exists c \in C : x \succ c \succ y.$$

Perfect separability implies Jaffray order separability (Jaffray, 1975), so:

Corollary 3.4 [Sondermann, 1980, Corollary 2] *Consider a complete, transitive, perfectly separable binary relation $\succsim$ on a set X. Then there is a utility function representing $\succsim$, usc in any topology equal to or finer than the lower order topology.*

Appendix: Proof of Theorem 3.1

Preliminaries. By definition,

$$\forall d \in D : \quad u(d) = \mu^*(W(d)) = \rho(W(d)) = \sum_{d' \in D : d' \precsim d} w(d'), \tag{5}$$

[5] Or any other topological space with a countable base.

and the outer measure μ^* is monotonic: if $A \subseteq B \subseteq X$, then $\mu^*(A) \leq \mu^*(B)$.

Representation. We prove (1). Let $x, y \in X$. If $x \sim y$, then $W(x) = W(y)$ by transitivity of $\succsim$, so $u(x) = u(y)$. If $x \succ y$, there are $d, d' \in D$ with $x \succsim d \succ d' \succsim y$ by (2). By monotonicity of μ^* and (5): $u(x) = \mu^*(W(x)) \geq \mu^*(W(d)) > \mu^*(W(d')) \geq \mu^*(W(y)) = u(y)$.

Semi-continuity. Let $r \in \mathbb{R}$. We show that $\{x \in X : u(x) < r\}$ is open. To avoid trivialities, let $\{x \in X : u(x) < r\} \neq \emptyset, X$. Hence, there is a $y^* \in X$ with $r \leq u(y^*) \leq 1$. Let $x \in X$ have $u(x) < r$. It suffices to show that there is an open neighborhood V of x with $u(v) < r$ for each $v \in V$.

Case 1: There is no $d \in D$ with $d \sim x$. As D may be assumed to contain a worst element of X, if such exists (see footnote 4): $W(x) \neq \emptyset$. By definition of μ^*, there are $\{W_i\}_{i\in\mathbb{N}} \subseteq \mathcal{W}$ with $W(x) \subseteq \cup_{i\in\mathbb{N}} W_i$ and $\mu^*(W(x)) \leq \sum_{i\in\mathbb{N}} \rho(W_i) < r \leq 1$. As $W(x) \neq \emptyset$, the set $J = \{i \in \mathbb{N} : W_i \neq \emptyset\}$ is nonempty. As $\rho(X) = 1$ and $\sum_{i\in\mathbb{N}} \rho(W_i) < 1$, $W_i \neq X$ for each $i \in J$. So for each $i \in J$ there is a $d_i \in D$ with $W_i = W(d_i)$. We show that $d_i \succ x$ for some $i \in J$. Suppose, to the contrary, that $d_i \prec x$ for each $i \in J$. For each $j \in J$, the set $\{d_i \in D : i \in J, d_i \succsim d_j\}$ is infinite: otherwise, it has a best element d^*, but then $\cup_{i\in\mathbb{N}} W_i = \cup_{i\in J} W(d_i) = W(d^*)$ is a proper subset of $W(x)$ by Jaffray order separability, contradicting $W(x) \subseteq \cup_{i\in\mathbb{N}} W_i$. Let $j \in J$ with $\rho(W(d_j)) := \varepsilon > 0$. By the above, there are infinitely many $i \in J$ with $\rho(W_i) = \rho(W(d_i)) \geq \rho(W(d_j)) = \varepsilon$, contradicting that $\sum_{i\in\mathbb{N}} \rho(W_i) < 1$. We conclude that $d_i \succ x$ for some $i \in J$. So $x \in W(d_i)$, an open set in the D-lower order topology, and for each $v \in W(d_i)$: $u(v) < u(d_i) = \rho(W(d_i)) < r$.

Case 2: There is a $d \in D$ with $d \sim x$. By (2) and $y^* \succ x$: $B(d) \cap D = \{d' \in D : d' \succ d\} \neq \emptyset$.

Case 2A: There is a $d' \in B(d) \cap D$ with $(d, d') = \emptyset$. Then $\{z \in X : z \precsim d\} = \{z \in X : z \prec d'\} = W(d')$ is open in the D-lower order topology, contains x, and for each $z \in W(d') : u(z) \leq u(d) = u(x) < r$.

Case 2B: For each $d' \in B(d) \cap D$, $(d, d') \neq \emptyset$. Then by (2), there is, for each $d' \in B(d) \cap D$, a $d'' \in B(d) \cap D$ that is strictly worse: $d'' \prec d'$. So $B(d) \cap D$ is infinite. Since the sequence of weights $(\varepsilon_k)_{k=1}^{\infty}$ is summable, there is a $k \in \mathbb{N}$ such that $\sum_{\ell=k}^{\infty} \varepsilon_\ell < r - u(x)$. Since there are only finitely many $d' \in D$ with $n(d') < k$, there is a sufficiently bad $d^* \in B(d) \cap D$ such that $n(d') \geq k$ for each $d' \in B(d) \cap D$ with $d' \precsim d^*$.

Since $d^* \in B(d) \cap D$, $x \in W(d^*)$, which is open in the D-lower order topology. By $x \sim d$ and the construction of d^*:

$$u(x) = \sum_{d' \in D : d' \precsim d} w(d')$$

and

$$\sum_{d' \in B(d) \cap D : d' \precsim d^*} w(d') = \sum_{d' \in B(d) \cap D : d' \precsim d^*} \varepsilon_{n(d')} \leq \sum_{\ell=k}^{\infty} \varepsilon_\ell < r - u(x).$$

Hence, for each $v \in W(d^*)$:

$$u(v) < u(d^*) = \rho(W(d^*)) = \sum_{d' \in D : d' \precsim d} w(d') + \sum_{d' \in B(d) \cap D : d' \precsim d^*} w(d') < u(x) + r - u(x) = r.$$

References

Ash, R.B., 2000. Probability and Measure Theory, 2nd edition. London/San Diego: Academic Press.

Billingsley, P., 1995. Probability and Measure, 3rd edition. New York: John Wiley and Sons.

Bridges, D.S., Mehta, G.B., 1995. Representations of Preference Orderings. Berlin: Springer-Verlag.

Debreu, G., 1954. Representation of a preference ordering by a numerical function. In: Decision Processes. Thrall, Davis, Coombs (eds.), New York: John Wiley and Sons, pp. 159-165.

Fishburn, P.C., 1970. Utility Theory for Decision Making. New York: John Wiley and Sons.

Jaffray, J.-Y., 1975. Existence of a continuous utility function: An elementary proof. Econometrica 43, 981-983.

Mas-Colell, A., Whinston, M.D, Green, J.R., 1995. Microeconomic Theory. Oxford: Oxford University Press.

Neuefeind, W., 1972. On continuous utility. Journal of Economic Theory 5, 174-176.

Rader, T., 1963. The existence of a utility function to represent preferences. Review of Economic Studies 30, 229-232.

Royden, H.L., 1988. Real Analysis, 3rd edition. New Jersey: Prentice-Hall.

Rudin, W., 1976. Principles of Mathematical Analysis, 3rd edition. McGraw-Hill.

Sondermann, D., 1980. Utility representations for partial orders. Journal of Economic Theory 23, 183-188.