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MACROECONOMIC VOLATILITY AND WELFARE LOSS UNDER FREE-TRADE IN TWO-COUNTRY MODELS

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Abstract: *This paper investigates the interlinkage in the business cycles based on sunspot fluctuations of large-country economies in a free-trade equilibrium. We consider a two-country, two-good, two-factor general equilibrium model with Cobb-Douglas technologies, sector-specific externalities and linear preferences. We also assume constant social returns in the investment good sector but decreasing social returns in the consumption good sector. We first identify the determinants of each country's accumulation pattern in autarky equilibrium, and second we show that some country's sunspot fluctuations may spread throughout the world once trade opens even if the other country has determinacy under autarky. We thus prove that under free-trade, globalization and market integration may have destabilizing effects on a country's competitive equilibrium. Finally, we characterize a configuration in which opening to international trade improves the stationary welfare at the world level but deteriorates the stationary welfare of the country which imports investment goods and exports consumption goods. We thus show that in opposition to the standard belief, international trade may not be beneficial to all trading partners in the long run. Moreover, we prove that for some country, international trade may have contrasted consequences as it may at the same time improve the stationary welfare and have a destabilizing effect.*

Keywords: *Two-country general equilibrium model, free-trade, local indeterminacy, sunspot fluctuations, capital intensities, decreasing social returns*

Journal of Economic Literature Classification Numbers: C62, E32, F11, F43, O41.

1 Introduction

Globalization and market integration are two important characteristics of modern developed economies. While these features are often seen as beneficial to growth, one may wonder about the stabilizing or destabilizing effects of an increased international mobility of produced goods and productive factors. International trade interlinks the business cycles of trading countries, as it relates economic activities of agents in one country to those in another. As a result, a country's business cycles may be spread throughout the world or erased.

This paper investigates the dynamic behavior of multiple countries' economic activities in a two-good (consumption and investment), two-factor (capital and labor) model in which capital is freely mobile between countries once trade opens whereas labor is internationally immobile. We consider a discrete-time perfect foresight general equilibrium model with two countries characterized by asymmetric Cobb-Douglas technologies containing sector-specific externalities and linear preferences. As well-known since Woodford [14], local indeterminacy of equilibria, derived for instance from external effects in production, is a sufficient condition for the occurrence of sunspot fluctuations. Our aim in then is to study how macroeconomic volatility based on sunspot fluctuations may spread all over the world through international trade.

The recent macroeconomics literature has shown that local indeterminacy of equilibria and sunspot fluctuations easily arise in closed economy two-sector models with sector specific externalities.[1] Under certain conditions in terms of factor intensities and provided the elasticity of intertemporal substitution in consumption is large enough, there are indeed multiple equilibrium paths starting from the same initial stock of capital. Considering the same type of model but under the assumption of a small-open economy, it has been shown that local indeterminacy arises under the same basic conditions on the technological side but without any restriction on preferences.[2] However, the consideration of closed economies or small-open economies, i.e. with a constant exogenous interest rate, prevents from taking into account the effects of final goods and inputs international mobility on

[1] See for instance Benhabib and Nishimura [2], Benhabib, Nishimura and Venditti [3].
[2] See for instance Meng and Velasco [7], Weder [13].

the stability properties of competitive equilibria. Our aim is thus to understand under what conditions perfect foresight equilibrium path may exhibit sunspot fluctuations in a large-country trade model.

It is well-known since the contributions of Becker [1], Bewley [4], Epstein [5] and Yano [15, 16] that in a perfect foresight model with many consumers or countries, a competitive equilibrium path behaves like an optimal growth path. Building on this property, Nishimura and Yano [11] have shown that a country's business cycles (i.e. two-period cycles) may spread throughout the world once trade opens. Considering a pecialization of this model to the case of Cobb-Douglas technologies, Nishimura, Venditti and Yano [10] have recently provided factor intensities conditions for this result to hold.

In this paper we extend the formulation of Nishimura, Venditti and Yano [10] by considering Cobb-Douglas technologies augmented to include sector-specific externalities in both countries. We consider a market integration in which international trade concerns consumption and investment goods. In order to characterize the stability properties of free-trade equilibrium paths, we assume decreasing social returns to scale in the consumption good sector of each country. Such an assumption is indeed necessary to get a non-degenerate social production function at the world level. Our main objective is to identify the capital intensities restrictions in each country to get sunspot fluctuations along free-trade competitive equilibria.

Assuming a linear utility function in each country, we first prove that although we consider sector-specific externalities, the free-trade equilibrium path behaves like the solution of a pseudo planner problem in which the external effects are considered as given. We then show that for a stationary capital stock at the world level, two types of stationary distributions across countries may occur: an autarky distribution in which each country exactly produces in the long run the amount of capital necessary to produce both goods but trades with the other country along the transition path, and a free-trade distribution in which one country is characterized by net exports in capital and net imports in consumption while the other is characterized by net imports in capital and net exports in consumption.

Then we analyze the local stability properties of each type of stationary distribution. We start by providing factor intensities conditions for the occurence of local indeterminacy in a closed-economy under decreasing social returns. Once international trade opens, focussing first on the autarky distribution, we prove that if both countries have locally indeterminate equilibria under autarky, then the equilibrium under free-trade is also locally indeterminate. In this case, the existence of international sunspot fluctuations is derived from the existence of sunspot fluctuations in each country. Second, building on the same type of arguments, we give factor intensities conditions for the existence of local indeterminacy along the free-trade distribution. However, we prove that a continuum of equilibria may occur at the world level once trade opens even though the capital importing country is characterized by a saddle-point stable steady state under autarky. In this case, opening to international trade has a destabilizing effect on the capital importing country.

Considering finally both the autarky and free-trade distributions, we assume that there is an asymmetry across countries concerning the returns to scale at the social level and we confirm the potential destabilizing role of market integration. We show indeed that if in one country the returns to scale of the consumption good sector are almost constant at the social level and local indeterminacy holds under autarky, then sunspot fluctuations occur at the world level even if the equilibrium under autarky in the second country is locally determinate. The sunspot fluctuations of one country then spread throughout the world. For the free-trade distribution, assuming that the capital importing country is characterized by almost constant social returns to scale, we then derive, in opposition to the previous case, that opening to international trade has a destabilizing effect on the capital exporting country.

We also provide a welfare analysis at the steady state by comparing the stationary amount of consumption obtained in each country under free-trade with the one obtained under autarky. We characterize a configuration in which opening to international trade improves the stationary welfare at the world level but deteriorates the stationary welfare of the country which imports investment goods and exports consumption goods. We thus show that in opposition to the standard belief, international trade may not be beneficial to all trading partners in the long run. Moreover, we prove that for some country, international trade may have contrasted consequences as it may at the same time improve the stationary welfare and have a destabilizing effect.

Our main conclusions can be compared to three recent contributions dealing with the existence of local indeterminacy in large-country trade models. In Nishimura and Shimomura [9], sector-specific externalities are introduced in a continuous-time version of the Hecksher-Ohlin two-country dynamic general equilibrium model with Cobb-Douglas technologies. They show that if in both countries indeterminacy of the equilibrium path holds under autarky then local indeterminacy also holds in the world market once trade opens. The same basic framework is also used by Sim and Ho [12] except that they break the symmetry in which externalities enter the production function in the two countries. Assuming that under autarky indeterminacy holds in one country but determinacy in the other, they show that trade can easily overturn indeterminacy. A limitation of these two papers is that the factors of production are assumed to be internationally immobile.

Ghiglino [6] considers a two-country, two-sector model with labor-augmenting global externalities. He assumes that both countries have the same sectoral production functions, the consumption good being produced with a Cobb-Douglas technology while the investment good is produced with a Leontief technology. He studies the consequences of market integration through international trade on the occurrence of sunspot fluctuations. He shows that provided the inverse of relative risk aversion is not a linear or concave function, the equilibrium under free-trade may be locally indeterminate even if the equilibrium under full autarky is determinate. A limitation of this paper, beside the restriction to a zero elasticity of capital-labor substitution in the investment good sector, is that as soon as standard CES preferences are considered (a case in which the inverse of relative risk aversion is linear), market integration plays no role on the occurrence of indeterminacy.

This paper is organized as follows: The next Section sets up the basic model. In Section 3 we study the stability properties of the competitive equilibrium path in closed economies under decreasing social returns. Section 4 provides the main results on the existence of sunspot fluctuations within open economies under free-trade. Section 5 contains concluding comments. All the proofs are gathered in a final Appendix.

2 The model

We consider a perfect foresight trade model with two countries, two factors and two goods. Each country $i = A, B$ is characterized by an infinitely-lived representative agent with single period linear utility function:

Assumption 1. *$u(c^i) = c^i$ with c^i the consumption level of country $i = A, B$.*

We assume that the labor supply is inelastic. The consumption good, x^i, and the investment good, k^i, are assumed to be produced with Cobb-Douglas technologies which contain some positive sector-specific externalities. We denote by x^i and y^i the outputs of sectors c^i and k^i, and by e^i_c and e^i_y the corresponding external effects. The *private production functions* are thus:

$$x^i = \mathcal{E}^i_c(K^i_c)^{\alpha^i_1}(L^i_c)^{\alpha^i_2}e^i_c,\ y^i = \mathcal{E}^i_y(K^i_y)^{\beta^i_1}(L^i_y)^{\beta^i_2}e^i_y$$

with $\mathcal{E}^i_c, \mathcal{E}^i_y > 0$ some normalization constants. The externalities e^i_c and e^i_y depend on $\bar{K}^i_j$ and $\bar{L}^i_j$, which denote the average use of capital and labor in sector $j = c, y$ of country $i = A, B$, and will be equal to

$$e^i_c = (\bar{K}^i_c)^{a^i_1}(\bar{L}^i_c)^{a^i_2}, \quad e^i_y = (\bar{K}^i_y)^{b^i_1}(\bar{L}^i_y)^{b^i_2} \tag{1}$$

with $a^i_n, b^i_n \geq 0$, $n = 1, 2$. We assume that these economy-wide averages are taken as given by individual firms. At the equilibrium, all firms of sector $i = c, y$ being identical, we have $\bar{K}^i_j = K^i_j$ and $\bar{L}^i_j = L^i_j$. Denoting $\hat{\alpha}^i_n = \alpha^i_n + a^i_n$, $\hat{\beta}^i_n = \beta^i_n + b^i_n$, the *social production functions* are defined as

$$x^i = \mathcal{E}^i_c(K^i_c)^{\hat{\alpha}^i_1}(L^i_c)^{\hat{\alpha}^i_2},\ y^i = \mathcal{E}^i_y(K^i_y)^{\hat{\beta}^i_1}(L^i_y)^{\hat{\beta}^i_2} \tag{2}$$

We assume constant social returns to scale in the investment good sector but decreasing social returns to scale in the consumption good sector, i.e. $\hat{\beta}^i_1 + \hat{\beta}^i_2 = 1$, $\hat{\alpha}^i_1 + \hat{\alpha}^i_2 < 1$. The returns to scale are therefore decreasing at the private level.[3]

[3] A possible interpretation of decreasing private returns would be to assume the existence of a factor in fixed supply such as land in the technology, namely

$$x^i = \mathcal{E}^i_c(K^i_c)^{\alpha^i_1}(L^i_c)^{\alpha^i_2}e^i_c(\mathcal{L}^i_c)^{1-\alpha^i_1-\alpha^i_2},\ y^i = \mathcal{E}^i_y(K^i_y)^{\beta^i_1}(L^i_y)^{\beta^i_2}e^i_y(\mathcal{L}^i_y)^{1-\beta^i_1-\beta^i_2}$$

Private returns to scale are therefore constant when considering this factor but decreasing with respect to capital and labor. In such a case, the income of the representative consumer is increased by the rental rate of land. Our formulation implicitely assumes a normalization $\mathcal{L}^i_c = \mathcal{L}^i_y = 1$.

Labor is normalized to one, $L_c^i + L_y^i = 1$, and the total stock of capital in country i is given by $K_c^i + K_y^i = k^i$. Goods x^i and k^i are assumed to be freely mobile between countries once trade opens, whereas labor is internationally immobile both before and after the opening of trade. Assuming that capital fully depreciates at each period, it follows that along a free-trade equilibrium, the market clearing conditions for goods x^i and k^i are as:

$$c_t^A + c_t^B = x_t^A + x_t^B, \quad k_{t+1}^A + k_{t+1}^B = y_t^A + y_t^B \tag{3}$$

On the contrary along an autarky equilibrium, the market clearing conditions become

$$c_t^i = x_t^i, \quad k_{t+1}^i = y_t^i \tag{4}$$

In each country, the optimal allocation of factors across sectors is obtained by solving the following program:

$$\begin{array}{rl} \max\limits_{K_{ct}^i, L_{ct}^i, K_{yt}^i, L_{yt}^i} & \mathcal{E}_c^i (K_{ct}^i)^{\alpha_1^i} (L_{ct}^i)^{\alpha_2^i} e_{ct}^i \\ s.t. & y_t^i = \mathcal{E}_y^i (K_{yt}^i)^{\beta_1^i} (L_{yt}^i)^{\beta_2^i} e_{yt}^i \\ & 1 = L_{ct}^i + L_{yt}^i \\ & k_t^i = K_{ct}^i + K_{yt}^i \\ & e_{ct}^i, e_{yt}^i \text{ given} \end{array} \tag{5}$$

Denote by q_t^i, p_t^i, ω_t^i and r_t^i respectively the prices of the consumption good and the capital good, the wage rate of labor and the rental rate of the capital good at time t. In the case in which the countries do not trade with each other (autarky case), the prices are generally different between countries. On the contrary, in a free-trade equilibrium, prices must be equated between countries, so that $q_t^A = q_t^B$, $p_t^A = p_t^B$ and $r_t^A = r_t^B$ must hold. However, because labor is immobile across countries, ω_t^i might differ between countries even in the free-trade case. In the following we will choose the consumption good as numeraire and thus adopt the normalization $q_t^A = q_t^B = 1$. The Lagrangian corresponding to program (5) is:

$$\begin{aligned} \mathcal{L}_t &= \mathcal{E}_c^i (K_{ct}^i)^{\alpha_1^i} (L_{ct}^i)^{\alpha_2^i} e_{ct}^i + p_t^i \Big(\mathcal{E}_y^i (K_{yt}^i)^{\beta_1^i} (L_{yt}^i)^{\beta_2^i} e_{yt}^i - y_t^i \Big) \\ &+ \omega_t^i (1 - L_{ct}^i - L_{yt}^i) + r_t^i (k_t^i - K_{ct}^i - K_{yt}^i) \end{aligned} \tag{6}$$

For any given $(k_t^i, y_t^i, e_{ct}^i, e_{yt}^i)$, solving the first order conditions gives input demand functions $\tilde{K}_c^i = K_c^i(k_t^i, y_t^i, e_{ct}^i, e_{yt}^i)$, $\tilde{L}_c^i = L_c^i(k_t^i, y_t^i, e_{ct}^i, e_{yt}^i)$, $\tilde{K}_y^i = K_y^i(k_t^i, y_t^i, e_{ct}^i, e_{yt}^i)$ and $\tilde{L}_y^i = L_y^i(k_t^i, y_t^i, e_{ct}^i, e_{yt}^i)$. We may thus define the social production function of country i as:

$$T^i(k_t^i, y_t^i, e_{ct}^i, e_{yt}^i) = \mathcal{E}_c^i \tilde{K}_c^i(k_t^i, y_t^i, e_{ct}^i, e_{yt}^i)^{\alpha_1^i} \tilde{L}_c^i(k_t^i, y_t^i, e_{ct}^i, e_{yt}^i)^{\alpha_2^i} e_{ct}^i \quad (7)$$

Using the envelope theorem we derive the equilibrium prices:

$$r_t^i = T_1^i(k_t^i, y_t^i, e_{ct}^i, e_{yt}^i), \quad p_t^i = -T_2^i(k_t^i, y_t^i, e_{ct}^i, e_{yt}^i) \quad (8)$$

where $T_1^i = \partial T^i/\partial k^i$ and $T_2^i = \partial T^i/\partial y^i$.[4]

3 Closed economy under decreasing returns

In a closed economy the equilibrium is derived from the following optimization program:

$$\begin{array}{ll} \max\limits_{y_t^i} & \sum\limits_{t=0}^{+\infty} \rho^t T^i(k_t^i, y_t^i, e_{ct}^i, e_{yt}^i) \\ s.t. & k_{t+1}^i = y_t^i \\ & k_0^i, e_{ct}^i, e_{yt}^i \ given \end{array}$$

with $\rho \in (0,1)$ the discount factor. The corresponding Euler equation is thus

$$T_2^i(k_t^i, k_{t+1}^i, e_{ct}^i, e_{yt}^i) + \rho T_1^i(k_{t+1}^i, k_{t+2}^i, e_{ct+1}^i, e_{yt+1}^i) = 0 \quad (9)$$

From the input demand functions together with the external effects (1) considered at the equilibrium we may define the equilibrium factors demand fonctions $\hat{K}_j = \hat{K}_j(k^i, y^i)$, $\hat{L}_j = \hat{L}_j(k^i, y^i)$ so that $\hat{e}_c^i = \hat{e}_c^i(k^i, y^i) = \mathcal{E}_c^i(\hat{K}_c^i)^{a_1^i}(\hat{L}_c^i)^{a_2^i}$ and $\hat{e}_y^i = \hat{e}_y^i(k^i, y^i) = (\hat{K}_y^i)^{b_1^i}(\hat{L}_y^i)^{b_2^i}$. From (8) prices now satisfy

$$\begin{array}{rcl} r^i(k_t^i, k_{t+1}^i) & = & T_1^i(k_t^i, k_{t+1}^i, \hat{e}_c^i(k_t^i, k_{t+1}^i), \hat{e}_y^i(k_t^i, k_{t+1}^i)) \\ p^i(k_t^i, k_{t+1}^i) & = & -T_2^i(k_t^i, k_{t+1}^i, , \hat{e}_c^i(k_t^i, k_{t+1}^i), \hat{e}_y^i(k_t^i, k_{t+1}^i)) \end{array} \quad (10)$$

We then get equation (9) evaluated at $\hat{e}_c^i$ and $\hat{e}_y^i$:

$$-p^i(k_t^i, k_{t+1}^i) + \rho r^i(k_{t+1}^i, k_{t+2}^i) \quad (11)$$

Any solution $\{k_t^i\}_{t=0}^{+\infty}$ which also satisfies the transversality condition

[4] Since the private technologies exhibit decreasing returns to scale, the competitive firms earn positive profits that have to be distributed back to the households who own physical capital. It can be shown as in Mino [8] that solving a peudo-planning problem in which the planner maximizes the discounted sum of utilities, under free-trade or autarky, subject to the social production function (7) and the market clearing conditions (3) or (4), is equivalent to solving a decentralized problem in which the households maximize a discounted sum of utilitites subject to some budget constraint based on given sequences of prices and the distributed profits.

$$\lim_{t \to +\infty} \rho^t p^i(k_t^i, k_{t+1}^i) k_{t+1}^i = 0$$

is called a closed economy equilibrium path.

A closed-economy steady state is defined by $k_t^i = k_{t+1}^i = y_t^i = \bar{k}^i$ and is given by the solving of $-p^i(k^i, k^i) + \rho r^i(k^i, k^i) = 0$.

Proposition 1. *There exists a unique closed-economy steady state $\bar{k}^i$ for country i such that:*

$$\bar{k}^i = \frac{\alpha_1^i \beta_2^i (\mathcal{E}_y^i \rho \beta_1^i)^{1/\hat{\beta}_2^i}}{\alpha_2^i \beta_1^i + (\alpha_1^i \beta_2^i - \alpha_2^i \beta_1^i) \rho \beta_1^i} \tag{12}$$

Moreover, the stationary optimal demand for capital in the investment good sector is given by $K_y^{i*} \equiv g^i = \rho \beta_1^i \bar{y}^i = \rho \beta_1^i \bar{k}^i$

Consider the following notations

$$\begin{aligned} \mathcal{T}_{m1}^i(k^i, y^i) &= \partial T_m(k^i, y^i, \hat{e}_c^i(k^i, y^i), \hat{e}_y^i(k^i, y^i))/\partial k^i \\ \mathcal{T}_{m2}^i(k^i, y^i) &= \partial T_m(k^i, y^i, \hat{e}_c^i(k^i, y^i), \hat{e}_y^i(k^i, y^i))/\partial y^i \end{aligned}$$

for $m = 1, 2$. The linearization of the Euler equation around $\bar{k}^i$ gives the following characteristic polynomial for the closed economy case:

$$\mathcal{P}_c^i(\lambda) = \rho \mathcal{T}_{12}^i(\bar{k}^i, \bar{k}^i)\lambda^2 + \lambda\Big[\mathcal{T}_{22}^i(\bar{k}^i, \bar{k}^i) + \rho \mathcal{T}_{11}^i(\bar{k}^i, \bar{k}^i)\Big] + \mathcal{T}_{21}^i(\bar{k}^i, \bar{k}^i) = 0 \tag{13}$$

As usual with Cobb-Douglas technologies, factor intensities at the private and social levels may be determined by the shares of input into production.[5]

Lemma 1. *The consumption (investment) good sector of country i is capital intensive at the private level if and only if $\alpha_1^i \beta_2^i - \alpha_2^i \beta_1^i > (<) 0$ while it is capital intensive at the social level if and only if $\hat{\alpha}_1^i \hat{\beta}_2^i - \hat{\alpha}_2^i \hat{\beta}_1^i > (<) 0$.*

We may now give conditions for the existence of local indeterminacy.

Proposition 2. *In country i, let the consumption good be capital intensive at the private level with*

$$\alpha_1^i \beta_2^i - \alpha_2^i \beta_1^i > \alpha_2^i + \frac{2(1-\hat{\alpha}_1^i - \hat{\alpha}_2^i)\alpha_1^i \beta_2^i (1+\hat{\beta}_1^i)}{[2(1-\hat{\alpha}_1^i) - \hat{\beta}_2^i](1-\beta_1^i)} \tag{14}$$

If one of the following sets of conditions is satisfied:

i) $\hat{\beta}_1^i - \hat{\alpha}_1^i > 0$,

ii) $1 - \hat{\alpha}_1^i > \hat{\alpha}_1^i - \hat{\beta}_1^i > \frac{(1-\hat{\alpha}_1^i - \hat{\alpha}_2^i)\hat{\beta}_1^i \alpha_1^i \beta_2^i}{\alpha_2^i (1-\beta_1^i)}$,

then there exists $\bar{\rho} \in (0, 1)$ such that the closed-economy steady state $\bar{k}^i$ is locally indeterminate for any $\rho \in (\bar{\rho}, 1)$.

[5] See Benhabib, Nishimura and Venditti [3].

Condition (14) shows that the capital intensity difference at the private level needs to be large enough to compensate the degree of decreasing returns in the consumption good sector. Notice that condition i) implies that the consumption good is labor intensive at the social level while condition ii) can be satisfied even if the consumption good is also capital intensive at the social level.

4 Open economy under free-trade

Denote by λ^i, $i = A, B$, the country i's marginal utility of wealth associated with the free-trade equilibrium. It is standard since Bewley [4] to normalize the price paths so that marginal utilities of wealth add up to a constant value. To simplify the formulation it is convenient to adopt as in Nishimura and Yano [11] the normalization

$$\lambda^A + \lambda^B = 2 \tag{15}$$

Given $\lambda = (\lambda^A, \lambda^B)$, let us define

$$\begin{aligned} W(k_t, y_t, e_{ct}, e_{yt}; \lambda) = \max_{c_t^A, c_t^B, k_t^A, k_t^B, y_t^A, y_t^B} \quad & \frac{u(c_t^A)}{\lambda^A} + \frac{u(c_t^B)}{\lambda^B} \\ s.t. \quad & c_t^A + c_t^B \leq T^A(k_t^A, y_t^A, e_{ct}^A, e_{yt}^A) \\ & \qquad + T^B(k_t^B, y_t^B, e_{ct}^B, e_{yt}^B) \\ & k_t^A + k_t^B \leq k_t \\ & y_t^A + y_t^B \leq y_t \\ & e_{ct}, e_{yt} \text{ given} \end{aligned}$$

with $e_{ct} = (e_{ct}^A, e_{ct}^B)$ and $e_{yt} = (e_{yt}^A, e_{yt}^B)$. Stated that way, the dynamical properties of a free-trade equilibrium path depend on marginal utilities of wealth $\lambda = (\lambda^A, \lambda^B)$ which are endogenous variables. However, under the assumption of linear utility functions, we can show that along a free-trade equilibrium path, the marginal utilities of wealth λ^A and λ^B are equal so that the following result holds:[6]

Proposition 3. *Consider the following value function:*

[6] Although we consider productive externalities, assuming linear utility functions allows to get the same result as in Nishimura and Yano [11].

$$
\begin{array}{rll}
V(k_t, y_t, e_{ct}, e_{yt}) = & \max\limits_{k_t^A, k_t^B, y_t^A, y_t^B} & T^A(k_t^A, y_t^A, e_{ct}^A, e_{yt}^A) + T^B(k_t^B, y_t^B, e_{ct}^B, e_{yt}^B) \\
& s.t. & k_t^A + k_t^B \leq k_t \\
& & y_t^A + y_t^B \leq y_t \\
& & e_{ct}, e_{yt} \text{ } given
\end{array}
$$

Under Assumption 1 and the normalization (15), the marginal utilities of wealth satisfy $\lambda^A = \lambda^B = 1$ *and* $V(k_t, y_t, e_{ct}, e_{yt}) = W(k_t, y_t, e_{ct}, e_{yt}; \lambda)$.

A free-trade equilibrium may then be interpreted as an equilibrium path with respect to the linear world welfare function as defined in Proposition 3. The corresponding first order conditions give:

$$
\begin{array}{rcl}
T_1^A(k_t^A, y_t^A, e_{ct}^A, e_{yt}^A) - T_1^B(k_t^B, y_t^B, e_{ct}^B, e_{yt}^B) & = & 0 \\
T_2^A(k_t^A, y_t^A, e_{ct}^A, e_{yt}^A) - T_2^B(k_t^B, y_t^B, e_{ct}^B, e_{yt}^B) & = & 0
\end{array}
\tag{16}
$$

The intertemporal free-trade equilibrium is finally derived from the following optimization:

$$
\begin{array}{ll}
\max\limits_{y_t} & \sum\limits_{t=0}^{+\infty} \rho^t V(k_t, y_t, e_{ct}, e_{yt}) \\
s.t. & k_{t+1} = y_t \\
& k_0 = k_0^A + k_0^B, e_{ct}, e_{yt} \text{ } given
\end{array}
$$

The corresponding Euler equation is thus

$$
V_2(k_t, k_{t+1}, e_{ct}, e_{yt}) + \rho V_1(k_{t+1}, k_{t+2}, e_{ct+1}, e_{yt+1}) = 0
$$

From the first order conditions (16), the envelope theorem gives

$$
\begin{array}{rcl}
V_1(k_t, y_t, e_{ct}, e_{yt}) & = & T_1^B(k_t^B, y_t^B, e_{ct}^B, e_{yt}^B) = T_1^A(k_t^A, y_t^A, e_{ct}^A, e_{yt}^A) \\
V_2(k_t, y_t, e_{ct}, e_{yt}) & = & T_2^B(k_t^B, y_t^B, e_{ct}^B, e_{yt}^B) = T_2^A(k_t^A, y_t^A, e_{ct}^A, e_{yt}^A)
\end{array}
$$

and the Euler equation becomes

$$
\begin{array}{rl}
& T_2^B(k_t^B, y_t^B, e_{ct}^B, e_{yt}^B) + \rho T_1^B(k_{t+1}^B, y_{t+1}^B, e_{ct+1}^B, e_{yt+1}^B) \\
= & T_2^A(k_t^A, y_t^A, e_{ct}^A, e_{yt}^A) + \rho T_1^A(k_{t+1}^A, y_{t+1}^A, e_{ct+1}^A, e_{yt+1}^A) = 0
\end{array}
\tag{17}
$$

Solving equations (16) with the sector-specific externalities (1) considered at the equilibrium and given in Section 3, namely $\hat{e}_c^i = \hat{e}_c^i(k^i, y^i)$ and $\hat{e}_y^i = \hat{e}_y^i(k^i, y^i)$, we derive $\hat{k}^i = \hat{k}^i(k, y)$, $\hat{y}^i = \hat{y}^i(k, y)$ and thus $\hat{e}_c(k, y) = (\hat{e}_c^A(\hat{k}^A, \hat{y}^A), \hat{e}_c^B(\hat{k}^B, \hat{y}^B))$, $\hat{e}_y(k, y) = (\hat{e}_y^A(\hat{k}^A, \hat{y}^A), \hat{e}_y^B(\hat{k}^B, \hat{y}^B))$. The Euler equation evaluated at the equilibrium finally becomes

$$\begin{aligned} & V_2(k_t, k_{t+1}, \hat{e}_c(k_t, k_{t+1}), \hat{e}_y(k_t, k_{t+1})) \\ + \quad & \rho V_1(k_{t+1}, k_{t+2}, \hat{e}_c(k_{t+1}, k_{t+2}), \hat{e}_y(k_{t+1}, k_{t+2})) = 0 \end{aligned}$$

Any solution $\{k_t\}_{t=0}^{+\infty}$ which also satisfies the transversality condition

$$\lim_{t\to+\infty} \rho^t V_2(k_t, k_{t+1}, \hat{e}_c(k_t, k_{t+1}), \hat{e}_y(k_t, k_{t+1}))k_{t+1} = 0$$

is called a free-trade equilibrium path.

Let us denote k^* the steady state solution of

$$V_2(k, k, \hat{e}_c(k, k), \hat{e}_y(k, k)) + \rho V_1(k, k, \hat{e}_c(k, k), \hat{e}_y(k, k)) = 0 \qquad (18)$$

The steady state k^* gives a total stationary amount of capital at the world level. Contrary to the closed-economy case, an explicit computation of k^* cannot be derived from (18). Moreover, the distribution across the two countries remains to be determined.

4.1 Stationary distributions

The Euler equation along a free-trade equilibrium (17), when compared with the Euler equation along a closed-economy equilibrium (9), clearly shows that different types of distributions are compatible with a total stationary stock of capital at the world level k^*. In particular, an autarky distribution in which each country exactly produces in the long run the amount of capital necessary to produce the consumption and investment goods may occur, i.e. $k^i = y^i$. Consider indeed the closed-economy steady state given in Proposition 1 for each country, i.e. $\bar{k}^A$ and $\bar{k}^B$. Using the normalization constants $\mathcal{E}_c^i$, we can show that $\bar{k} = \bar{k}^A + \bar{k}^B$ is also a steady state of the open economy under free-trade:

Proposition 4. *Let $\mathcal{E}_y^A = \mathcal{E}_c^B = \mathcal{E}_y^B = 1$ and consider $\bar{k}^i$, $i = A, B$, as given in Proposition 1. Then there exists $\bar{\mathcal{E}}_c^A > 0$ such that the autarky distribution $\bar{k} = \bar{k}^A + \bar{k}^B$ is a solution of equation (18), i.e., $\bar{k} = k^*$, if and only if $\mathcal{E}_c^A = \bar{\mathcal{E}}_c^A$.*

The corresponding amount of stationary consumptions under this capital distribution immediately derives from (7) as

$$\bar{c}^i = T^i(\bar{k}^i, \bar{k}^i, \hat{e}_c^i(\bar{k}^i, \bar{k}^i), \hat{e}_y^i(\bar{k}^i, \bar{k}^i)) \equiv \bar{T}^i \qquad (19)$$

Notice that considering the autarky distribution does not imply that countries do not trade. They may actually trade during the transition dynamics while the long run equilibrium is characterized by autarky.

We have also to consider the possible existence of a free-trade distribution such that $k^* = k^{A*} + k^{B*}$ in which one country, say A, is characterized by net imports of capital, i.e. $k^{A*} > y^{A*}$, while country B is characterized by net exports of capital, i.e. $k^{B*} < y^{B*}$. Proceeding as in Proposition 4, we can use the normalization constants $\mathcal{E}_c^i$ and $\mathcal{E}_y^i$ to prove that such a free-trade distribution exists. We actually focus on a particular solution such that $k^{A*} = \theta y^{A*} > y^{A*}$ and $k^{B*} = y^{B*}/\theta < y^{B*}$ with $\theta > 1$ a given constant.

Proposition 5. *Let $\mathcal{E}_c^B = \mathcal{E}_y^B = 1$ and consider a constant $\theta \in (1, 1/\rho\beta_1^B)$. Then there exist $\mathcal{E}_c^{A*} > 0$ and $\mathcal{E}_y^{A*} > 0$ such that the free-trade distribution $k^* = k^{A*} + k^{B*} = \theta y^{A*} + y^{B*}/\theta$ with*

$$k^{A*} = \frac{\theta\alpha_1^A\beta_2^A(\mathcal{E}_y^A\rho\beta_1^A)^{1/\hat{\beta}_2^A}}{\alpha_2^A\beta_1^A\theta + (\alpha_1^A\beta_2^A - \alpha_2^A\beta_1^A)\rho\beta_1^A}, \quad k^{B*} = \frac{\alpha_1^B\beta_2^B(\rho\beta_1^B)^{1/\hat{\beta}_2^B}}{\alpha_2^B\beta_1^B + (\alpha_1^B\beta_2^B - \alpha_2^B\beta_1^A)\rho\beta_1^B\theta} \tag{20}$$

is a solution of equation (18) if and only if $\mathcal{E}_c^A = \mathcal{E}_c^{A*}$ and $\mathcal{E}_y^A = \mathcal{E}_y^{A*}$.

We have now to compute the stationary consumption levels associated with this distribution of capital. At the free-trade steady state with $\theta \in (1, 1/\rho\beta_1^B)$, the country i's production of the consumption good is derived from (7) as:

$$T^{i*} = T^i(k^{i*}, k^{i*}, \hat{e}_c^i(k^{i*}, k^{i*}), \hat{e}_y^i(k^{i*}, k^{i*}))$$

We know that country A imports capital goods while country B exports capital goods, namely

$$\mathcal{M}_y^A = (\theta - 1)y^{A*}, \quad \mathcal{X}_y^B = \left(\frac{\theta-1}{\theta}\right) y^{B*}$$

In order to have a balance of trade in equilibrium, we derive from this that country A has to export consumption goods while country B has to import consumption goods. Let $\eta > 1$ and consider the following distribution of consumption across the two countries

$$c^{A*} = \frac{T^{A*}}{\eta} < T^{A*}, \quad c^{B*} = \eta T^{B*} > T^{B*}$$

It follows that

$$\mathcal{X}_c^A = \left(\frac{\eta-1}{\eta}\right) T^{A*}, \quad \mathcal{M}_c^B = (\eta - 1)T^{B*}$$

Therefore, the balance of trade is in equilibrium in each country if

$$\mathcal{NX}^A = \mathcal{X}_c^A - p\mathcal{M}_y^A = 0, \quad \mathcal{NX}^B = p\mathcal{X}_y^B - \mathcal{M}_c^B = 0$$

or equivalently

$$\begin{aligned}(\theta-1)py^{A*} &= \left(\frac{\eta-1}{\eta}\right)T^{A*} \\ \left(\frac{\theta-1}{\theta}\right)py^{B*} &= (\eta-1)T^{B*}\end{aligned}$$

with p the relative price of the investment good. Taking the ratio of these expressions yields the following corollary:

Corollary 1. *Let $\mathcal{E}_c^B = \mathcal{E}_y^B = 1$ and $\mathcal{E}_c^{A*} > 0$, $\mathcal{E}_y^{A*} > 0$ as given by Proposition 5. Consider $\theta \in (1, 1/\rho\beta_1^B)$ and the free-trade distribution of capital as given by (20), and assume that $\alpha_1^B(\theta - \rho\beta_1^A)/\alpha_1^A(1-\rho\beta_1^B\theta) > 1$. Then the associated free-trade distribution of consumption is $c^* = c^{A*} + c^{B*} = T^{A*}/\eta + \eta T^{B*}$ with $\eta = T^{A*}/T^{B*} = \alpha_1^B(\theta - \rho\beta_1^A)/\alpha_1^A(1-\rho\beta_1^B\theta)$ and*

$$c^{A*} = T^{B*}, \quad c^{B*} = T^{A*} \tag{21}$$

It is worth noticing that the autarky and free-trade distributions cannot co-exist since they are respectively associated with different values for the normalization constants $\mathcal{E}_c^i, \mathcal{E}_y^i$.[7]

4.2 Characteristic polynomial

The linearization of the Euler equation around k^* requires the computations of the partial derivatives of $V_m(k, y, \hat{e}_c(k, y), \hat{e}_y(k, y))$. Let us denote

$$\begin{aligned}\mathcal{V}_{m1}(k,y) &= \partial V_m(k, y, \hat{e}_c(k, y), \hat{e}_y(k, y))/\partial k \\ \mathcal{V}_{m2}(k,y) &= \partial V_m(k, y, \hat{e}_c(k, y), \hat{e}_y(k, y))/\partial y\end{aligned}$$

for $m = 1, 2$. At a steady state under free-trade, we have $y = k$ and the characteristic polynomial for the open economy case may be written as follows:

$$\mathcal{P}_o(\lambda) = \rho\mathcal{V}_{12}(k^*, k^*)\lambda^2 + \lambda\Big[\mathcal{V}_{22}(k^*, k^*) + \rho\mathcal{V}_{11}(k^*, k^*)\Big] + \mathcal{V}_{21}(k^*, k^*) = 0 \tag{22}$$

We may now provide a detailed stability analysis of the two possible distributions of the stationary capital stock k^* across countries.

[7] See Appendix 6.4 and 6.5 for detailed expressions.

4.3 Sunspot fluctuations under free-trade

We focus on local stability results when the consumption good is capital intensive at the private level. As in the closed economy case, such a capital intensity configuration is a necessary condition for the existence of local indeterminacy and sunspot fluctuations.[8] In a first step we study the properties of equilibrium paths at the world level around the autarky distribution.

Proposition 6. *Let* $\mathcal{E}_y^A = \mathcal{E}_c^B = \mathcal{E}_y^B = 1$, $\mathcal{E}_c^A = \bar{\mathcal{E}}_c^A$, *and consider the autarky distribution as defined in Proposition 4. In each country* $i = A, B$, *let the consumption good be capital intensive at the private level with*

$$\alpha_1^i\beta_2^i - \alpha_2^i\beta_1^i > \alpha_2^i + \frac{2(1-\hat{\alpha}_1^i-\hat{\alpha}_2^i)\alpha_1^i\beta_2^i(1+\hat{\beta}_1^i)}{[2(1-\hat{\alpha}_1^i)-\hat{\beta}_2^i](1-\beta_1^i)} \quad (23)$$

If in each country $i = A, B$ *one of the following sets of conditions is satisfied:*

i) $\hat{\beta}_1^i - \hat{\alpha}_1^i > 0$,

ii) $1 - \hat{\alpha}_1^i > \hat{\alpha}_1^i - \hat{\beta}_1^i > \frac{(1-\hat{\alpha}_1^i-\hat{\alpha}_2^i)\hat{\beta}_1^i\alpha_1^i\beta_2^i}{\alpha_2^i(1-\beta_1^i)}$,

then there exists $\hat{\rho} \in (0,1)$ *such that the autarky steady state* $k^* = \bar{k} = \bar{k}^A + \bar{k}^B$ *is locally indeterminate for any* $\rho \in (\hat{\rho}, 1)$.

Considering Proposition 2, Proposition 6 implies that if both countries have locally indeterminate equilibria under autarky, then the equilibrium under free-trade is also locally indeterminate. Put differently, a market integration, in which international trade concerns consumption and investment goods, does not rule out sunspot fluctuations that may exist under autarky. This result is similar to the main conclusion of Nishimura and Shimomura [9] except that we consider a discrete-time model and we assume perfect international mobility of capital across countries.

We may also derive conditions for local indeterminacy of the free-trade steady state as defined by Proposition 5

Proposition 7. *Let* $\mathcal{E}_c^B = \mathcal{E}_y^B = 1$, $\mathcal{E}_c^A = \mathcal{E}_c^{A*}$, $\mathcal{E}_y^A = \mathcal{E}_y^{A*}$, $\theta \in (1, 1/\rho\beta_1^B)$, *and consider the free-trade distribution* $k^* = k^{A*} + k^{B*}$ *as defined by Proposition 5. Assume also that in each country* $i = A, B$, *the consumption good is capital intensive at the private level with*

$$\alpha_1^A\beta_2^A - \alpha_2^A\beta_1^A > \alpha_2^A + \frac{2(1-\hat{\alpha}_1^A-\hat{\alpha}_2^A)\alpha_1^A\beta_2^A\theta(1+\hat{\beta}_1^A)}{[2(1-\hat{\alpha}_1^A)-\hat{\beta}_2^A](\theta-\beta_1^A)} \quad (24)$$

[8] See Benhabib, Nishimura and Venditti [3].

and

$$\alpha_1^B\beta_2^B - \alpha_2^B\beta_1^B > \alpha_2^B + \frac{2(1-\hat{\alpha}_1^B-\hat{\alpha}_2^B)\alpha_1^B\beta_2^B(1+\theta\hat{\beta}_1^B)}{[2(1-\hat{\alpha}_1^B)-\hat{\beta}_2^B](1-\theta\beta_1^B)} \tag{25}$$

If one of the following sets of conditions is satisfied:

i) $\hat{\beta}_1^A - \hat{\alpha}_1^A > 0$ *and* $\hat{\beta}_1^B - \hat{\alpha}_1^B > 0$,

ii) $1 - \hat{\alpha}_1^A > \hat{\alpha}_1^A - \hat{\beta}_1^A > \frac{(1-\hat{\alpha}_1^A-\hat{\alpha}_2^A)\hat{\beta}_1^A\alpha_1^A\beta_2^A}{\alpha_2^A(\theta-\beta_1^A)}$ *and* $1 - \hat{\alpha}_1^B > \hat{\alpha}_1^B - \hat{\beta}_1^B > \frac{(1-\hat{\alpha}_1^B-\hat{\alpha}_2^B)\hat{\beta}_1^B\alpha_1^B\beta_2^B\theta}{\alpha_2^B(1-\theta\beta_1^B)}$,

then there exists $\hat{\rho} \in (0,1)$ *such that the free-trade steady state* $k^* = k^{A*}+k^{B*}$ *is locally indeterminate for any* $\rho \in (\hat{\rho}, 1)$.

Proposition 7 provides conditions on the technologies of both countries for the existence of sunspot fluctuations at the free-trade steady state which are similar to those given in Proposition 6. However, notice that for country A condition (24) in Proposition 7 may hold while condition (23) in Proposition 6 does not, whereas for country B condition (25) implies condition (23). As a result, we show with the following Corollary that opening to free-trade an economy, which is saddle-point stable under autarky, may have a destabilizing effect.

Corollary 2. *Let* $\mathcal{E}_c^B = \mathcal{E}_y^B = 1$, $\mathcal{E}_c^A = \mathcal{E}_c^{A*}$, $\mathcal{E}_y^A = \mathcal{E}_y^{A*}$, $\theta \in (1, 1/\rho\beta_1^B)$, *and consider the free-trade distribution* $k^* = k^{A*} + k^{B*}$ *as defined by Proposition 5. Assume also that in each country* $i = A, B$, *the consumption good is capital intensive at the private level with*

$$\alpha_2^A + \frac{2(1-\hat{\alpha}_1^A-\hat{\alpha}_2^A)\alpha_1^A\beta_2^A(1+\hat{\beta}_1^A)}{[2(1-\hat{\alpha}_1^A)-\hat{\beta}_2^A](1-\beta_1^A)} > \alpha_1^A\beta_2^A - \alpha_2^A\beta_1^A > \alpha_2^A + \frac{2(1-\hat{\alpha}_1^A-\hat{\alpha}_2^A)\alpha_1^A\beta_2^A\theta(1+\hat{\beta}_1^A)}{[2(1-\hat{\alpha}_1^A)-\hat{\beta}_2^A](\theta-\beta_1^A)}$$

and

$$\alpha_1^B\beta_2^B - \alpha_2^B\beta_1^B > \alpha_2^B + \frac{2(1-\hat{\alpha}_1^B-\hat{\alpha}_2^B)\alpha_1^B\beta_2^B(1+\theta\hat{\beta}_1^B)}{[2(1-\hat{\alpha}_1^B)-\hat{\beta}_2^B](1-\theta\beta_1^B)}$$

If one of the following sets of conditions is satisfied:

i) $\hat{\beta}_1^A - \hat{\alpha}_1^A > 0$ *and* $\hat{\beta}_1^B - \hat{\alpha}_1^B > 0$,

ii) $1 - \hat{\alpha}_1^A > \hat{\alpha}_1^A - \hat{\beta}_1^A > \frac{(1-\hat{\alpha}_1^A-\hat{\alpha}_2^A)\hat{\beta}_1^A\alpha_1^A\beta_2^A}{\alpha_2^A(\theta-\beta_1^A)}$ *and* $1 - \hat{\alpha}_1^B > \hat{\alpha}_1^B - \hat{\beta}_1^B > \frac{(1-\hat{\alpha}_1^B-\hat{\alpha}_2^B)\hat{\beta}_1^B\alpha_1^B\beta_2^B\theta}{\alpha_2^B(1-\theta\beta_1^B)}$,

then there exists $\hat{\rho} \in (0,1)$ *such that the free-trade steady state* $k^* = k^{A*} + k^{B*}$ *is locally indeterminate for any* $\rho \in (\hat{\rho}, 1)$ *while the steady state under autarky in economy* A *is saddle-point stable.*

In a last step we study the properties of equilibrium paths around a steady state k^* which may be indifferently characterized by an autarky or free-trade distribution as defined by Propositions 4 and 5. We assume now that there is an asymmetry across sectors concerning the returns to scale at the social level in the final good sector. We will confirm the potential destabilizing role of market integration. Let us introduce for country $i = A, B$ the following parameter

$$\epsilon_c^i = 1 - \hat{\alpha}_1^i - \hat{\alpha}_2^i$$

Constant social returns to scale in the consumption good sector are clearly obtained when $\epsilon_c^i = 0$. On the contrary, decreasing social returns are associated with $\epsilon_c^i > 0$. We say that the consumption good technology of country i has almost constant social returns if ϵ_c^i is sufficiently small.

We may then provide simple conditions for the existence of local indeterminacy if one country, say country A, is characterized by a consumption good technology with almost constant social returns.

Proposition 8. *Let the consumption good in country A be capital intensive at the private level with* $\alpha_1^A\beta_2^A - \alpha_2^A\beta_1^A > \alpha_2^A$*. Then there exist* $\epsilon > 0$ *and* $\rho^* \in (0,1)$ *such that for any* $\epsilon_c^A \in [0, \epsilon)$ *and* $\epsilon_c^B > 0$*, a steady state* $k^* = k^{A*} + k^{B*}$ *is locally indeterminate when* $\rho \in (\rho^*, 1)$ *if one of the following sets of conditions is satisfied:*

i) the consumption good in country A is labor intensive at the social level,

ii) the consumption good in country A is also capital intensive at the social level with $\hat{\alpha}_1^A - \hat{\beta}_1^A \in (0, \hat{\alpha}_2^A)$.

Proposition 8 shows that the occurrence of sunspot fluctuations at the world level is only based on the existence of sunspot fluctuations in country A and may be obtained even if in country B the equilibrium path is locally determinate. A market integration may then have destabilizing effects for some countries if the trade agreement is made with a country characterized by constant returns at the social level and local indeterminacy.

Remark 1: Sim and Ho [12] provide some opposite result to Proposition 8 within a continuous-time model derived from Nishimura and Shimomura [9]. Breaking the symmetry in which externalities enter the production function in the two countries, they show indeed that indeterminacy may arise under autarky while uniqueness is the true outcome given trade. The main difference with our model concerns the fact that they assume internationally immobile capital and labor.

Remark 2: Ghiglino [6] provides results similar to Corollary 2 and Proposition 8 but using a different type of argument. He considers a two-country, two-sector model in which the consumption good is produced with a Cobb-Douglas technology while the investment good is produced with a Leontief technology. Both production functions contain labor-augmenting global externalities. He studies the consequences of an increased size of the market following an international trade agreement on the occurrence of sunspot fluctuations. Indeed, when some country opens to trade, the steady state corresponding to the new integrated market and thus the stationary value of the inverse of relative risk aversion are modified. In the case in which this function is strictly concave, Ghiglino then shows that the equilibrium under free-trade may be locally indeterminate even if the equilibrium under full autarky is determinate. A limitation of this paper is that strict concavity of the inverse of relative risk aversion involves restrictions on the third and fourth derivatives of the utility function which are not limited by standard assumptions on preferences. Moreover, as soon as CES preferences are considered, market integration plays no role on the occurrence of indeterminacy. The main difference with our results is that our methodology is not based on the size of the integrated market per se but on some direct conditions on the technologies.

4.4 Some comments on welfare properties of steady states

Since Ricardo, free-trade and market integration are often seen as beneficial for all trading partners. In order to have a precise evaluation of this claim, the key point is to determine whether or not an economy may increase its welfare by opening to international trade. In our intertemporal general equilibrium model, since we assume linear utility functions in both countries, the stationay welfare of each country can be directly evaluated by looking at the amount of consumption at the steady state. Obviously, if the steady state under free-trade corresponds to the autarky distribution, then opening to international trade does not affect the stationary welfare. However, if the free-trade distribution is reached after the market integration, then one may expect a modification of the stationary welfare of both countries.

As shown by Corollary 1, the stationary amount of consumption associated with the free-trade distribution defined from the parameter $\theta \in (1, 1/\rho\beta_1^B)$ is such that:

$$c^{A*} = T^{B*} \text{ and } c^{B*} = T^{A*}$$

In order to compare the consumption levels under free-trade to those obtained in the closed-economy case we have to consider the normalization constants $\mathcal{E}_c^B = \mathcal{E}_y^B = 1$, $\mathcal{E}_c^A = \mathcal{E}_c^{A*}$ and $\mathcal{E}_y^A = \mathcal{E}_y^{A*}$ as given in Proposition 5. Using these values, we may compute from (19) the corresponding amount of stationary consumptions under autarky

$$\bar{c}^i = \bar{T}^i$$

From these expressions we also compute the welfare at the world level under autarky, *i.e.*, $\bar{\mathcal{W}} \equiv \bar{c}^A + \bar{c}^B$, and under free-trade, *i.e.*, $\mathcal{W}^* \equiv c^{A*} + c^{B*}$. We then provide a configuration in which, in opposition to the standard belief, international trade may not be beneficial to all trading partners. :

Proposition 9. *Let $\mathcal{E}_c^B = \mathcal{E}_y^B = 1$, $\mathcal{E}_c^A = \mathcal{E}_c^{A*}$, $\mathcal{E}_y^A = \mathcal{E}_y^{A*}$ as given in Proposition 5, and $\theta \in (1, 1/\rho\beta_1^B)$. Then there exists $\underline{\theta} \in (1, 1/\rho\beta_1^B)$ such that for any $\theta \in (\underline{\theta}, 1/\rho\beta_1^B)$, $\mathcal{W}^* > \bar{\mathcal{W}}$ while $c^{A*} < \bar{c}^A$, i.e. opening to international trade improves the stationary welfare at the world level but deteriorates the stationary welfare of country A which imports capital and exports consumption.*

The basic intuition for this result is the following: country B, by exporting capital, decreases the production in both sectors, in particular in the consumption good sector. The equilibrium of the balance of trade then implies that country B imports consumption. When θ is large, i.e. the amount of country B's capital exports is large, the corresponding amount of consumption imports increases dramatically. As a consequence, the amount of country A's consumption exports may become larger than the additional production of consumption obtained from the capital imports. As a result, the amount of consumption and thus the stationary welfare of country A are decreased under free-trade.

On the basis of Proposition 9, we may derive relationships between the welfare loss and the destabilizing effects of international trade. Considering first Corollary 2, we conclude that opening to international trade has at the same time a destabilizing effect on country A since sunspot fluctuations are imported from country B and deteriorates its stationary welfare by decreasing the amount of consumption at the steady state. On the contrary, considering Proposition 8 with the free-trade distribution, we conclude that while opening to international trade has a destabilizing effect on country B as sunspot fluctuations are imported from country A, it allows to improve its stationary welfare by increasing the amount of consumption at the steady state. The occurrence of sunspot fluctuations may be thus compatible with a greater welfare.

5 Concluding comments

In a perfect foresight model with two countries characterized by Cobb-Douglas technologies, sector-specific externalities and decreasing returns at the social level, we have investigated the way sunspot fluctuations of countries may spread all over the world through international trade.

We have first identified the factor intensities conditions for the existence of local indeterminacy in a closed economy under decreasing social returns. Second, we have shown how sunspot fluctuations may occur at the world level once trade opens. We have studied two types of stationary distributions across countries compatible with a global stationary capital stock.

Dealing in a first step with the autarky distribution, which is associated with countries that do not trade in the long run but trade along the transition path, we have shown that if both countries have locally indeterminate equilibria under autarky, then local indeterminacy also occurs at the world equilibrium under free-trade.

In a second step, dealing with the free-trade distribution, we have proved that a continuum of equilibria may occur at the world level once trade opens even though the importing country is characterized by a saddle-point stable steady state under autarky. In this case, we have also shown that opening to international trade has at the same time a destabilizing effect and decreases the stationary welfare of the capital importing country.

Considering finally both types of distributions, we have confirmed the potential destabilizing role of market integration by showing that if one country has almost constant social retruns to scale in the consumption good sector and sunspot fluctuations under autarky, then local indeterminacy arises along a free-trade equilibrium even if local determinacy holds in the other country. When applied to the free-trade distribution and assuming that the capital importing country is characterized by almost constant social returns to scale, this conclusion implies, in opposition to the previous case, that while it has a destabilizing effect on the capital exporting country, opening to international trade allows to increase its stationary welfare.

6 Appendix

6.1 Proof of Proposition 1

We start by characterizing the first partial derivatives of $T^i(k^i, y^i, e^i_c, e^i_y)$.

Lemma 6.1. *The first partial derivatives of $T^i(k^i, y^i, e^i_c, e^i_y)$ are given by:*

$$\begin{aligned} T^i_1(k^i, y^i, e^i_c, e^i_y) &= \mathcal{E}^i_c \alpha^i_1 (\alpha^i_2 \beta^i_1/\Delta^i)^{\alpha^i_2} (k^i - g^i)^{\alpha^i_1+\alpha^i_2-1} e^i_c \\ T^i_2(k^i, y^i, e^i_c, e^i_y) &= -\frac{\mathcal{E}^i_c e^i_c \alpha^i_1}{\mathcal{E}^i_y e^i_y \beta^i_1} \frac{(\alpha^i_2\beta^i_1/\Delta^i)^{\alpha^i_2}}{(\alpha^i_1\beta^i_2/\Delta^i)^{\beta^i_2}} (k^i - g^i)^{\alpha^i_1+\alpha^i_2-1} (g^i)^{1-\beta^i_1-\beta^i_2} \\ &= -\frac{T^i_1(k^i, y^i, e^i_c, e^i_y)}{\mathcal{E}^i_y e^i_y \beta^i_1} (\alpha^i_1\beta^i_2/\Delta^i)^{-\beta^i_2} (g^i)^{1-\beta^i_1-\beta^i_2} \end{aligned} \quad (26)$$

where

$\Delta^i = \alpha^i_2\beta^i_1 k^i + (\alpha^i_1\beta^i_2 - \alpha^i_2\beta^i_1) g^i$

$$g^i = g^i(k^i, y^i, e^i_c, e^i_y) = \left\{ K^i_y \in (0, \mathcal{E}^i_y (k^i)^{\beta^i_1} e^i_y) \ / \ y^i = \frac{\mathcal{E}^i_y e^i_y (\alpha^i_1\beta^i_2)^{\beta^i_2} (K^i_y)^{\beta^i_1+\beta^i_2}}{[\alpha^i_2\beta^i_1 k^i + (\alpha^i_1\beta^i_2 - \alpha^i_2\beta^i_1) K^i_y]^{\beta^i_2}} \right\}$$

Proof: From the Lagrangian (6) we derive the first order conditions:

$$\begin{aligned} &\mathcal{E}^i_c \alpha^i_1 (K^i_c)^{\alpha^i_1-1} (L^i_c)^{\alpha^i_2} e^i_c - r^i = 0, \quad \mathcal{E}^i_c \alpha^i_2 (K^i_c)^{\alpha^i_1} (L^i_c)^{\alpha^i_2-1} e^i_c - \omega^i = 0 \\ &p^i \beta^i_1 \mathcal{E}^i_y (K^i_y)^{\beta^i_1-1} (L^i_y)^{\beta^i_2} e^i_y - r^i = 0, \quad p^i \beta^i_2 \mathcal{E}^i_y (K^i_y)^{\beta^i_1} (L^i_y)^{\beta^i_2-1} e^i_y - \omega^i = 0 \end{aligned} \quad (27)$$

Using $K^i_c = k^i - K^i_y$, $L^i_y = 1 - L^i_c$, and merging equations (27) we obtain:

$$L^i_c = \frac{\alpha^i_2\beta^i_1(k^i - K^i_y)}{(\alpha^i_1\beta^i_2 - \alpha^i_2\beta^i_1)K^i_y + \alpha^i_2\beta^i_1 k^i} \quad (28)$$

$$L^i_y = \frac{\alpha^i_1\beta^i_2 K^i_y}{(\alpha^i_1\beta^i_2 - \alpha^i_2\beta^i_1)K^i_y + \alpha^i_2\beta^i_1 k^i} \quad (29)$$

$$K^i_c = k^i - K^i_y \quad (30)$$

$$K^i_y = g^i(k^i, y^i, e^i_c, e^i_y) \equiv g^i \quad (31)$$

where

$$g^i = \left\{ K^i_y \in (0, \mathcal{E}^i_y (k^i)^{\beta^i_1} e^i_y) \ / \ y^i = \frac{\mathcal{E}^i_y e^i_y (\alpha^i_1\beta^i_2)^{\beta^i_2} (K^i_y)^{\beta^i_1+\beta^i_2}}{[\alpha^i_2\beta^i_1 k^i + (\alpha^i_1\beta^i_2 - \alpha^i_2\beta^i_1) K^i_y]^{\beta^i_2}} \right\} \quad (32)$$

To simplify notation let:

$$\Delta^i = \alpha_2^i \beta_1^i k^i + (\alpha_1^i \beta_2^i - \alpha_2^i \beta_1^i) g^i \tag{33}$$

From (27), (28), (30) we obtain $T_1^i(k^i, y^i, e_c^i, e_y^i)$ and from (27), (29), (31) we get $T_2^i(k^i, y^i, e_c^i, e_y^i)$. □

We may now prove Proposition 1. Using (1) with (28)-(31) we obtain the sector-specific externalities evaluated at the equilibrium

$$e_c^i = (k^i - g^i)^{a_1^i + a_2^i} \left(\frac{\alpha_2^i \beta_1^i}{\Delta^i} \right)^{a_2^i}, \quad e_y^i = (g^i)^{b_1^i + b_2^i} \left(\frac{\alpha_1^i \beta_2^i}{\Delta^i} \right)^{b_2^i} \tag{34}$$

Substituting these expressions into (26) and (32) gives:

$$\begin{aligned} r^i(k^i, y^i) &= \mathcal{E}_c^i \alpha_1^i \left(\frac{\alpha_2^i \beta_1^i}{\Delta^i} \right)^{\hat{\alpha}_2^i} (k^i - g^i)^{\hat{\alpha}_1^i + \hat{\alpha}_2^i - 1} \\ p^i(k^i, y^i) &= r^i(k^i, y^i) \frac{g^i}{\beta_1^i y^i} \end{aligned} \tag{35}$$

where g^i is now given by

$$g^i = g^i(k^i, y^i) = \left\{ K_y^i \in (0, \mathcal{E}_y^i (k^i)^{\hat{\beta}_1^i}) \ / \ y^i = \frac{\mathcal{E}_y^i (\alpha_1^i \beta_2^i)^{\hat{\beta}_2^i} K_y^i}{[\alpha_2^i \beta_1^i k^i + (\alpha_1^i \beta_2^i - \alpha_2^i \beta_1^i) K_y^i]^{\hat{\beta}_2^i}} \right\} \tag{36}$$

The steady state is finally obtained by solving $-p^i(k^i, k^i) + \rho r^i(k^i, k^i) = 0$ with $y = k$. □

6.2 Proof of Proposition 2

We first have to characterize the partial derivatives $\mathcal{T}_{mn}^i(k^i, y^i)$, $m, n = 1, 2$:

Lemma 6.2. *The partial derivatives $\mathcal{T}_{mn}^i(k^i, y^i)$ are given by:*

$$\mathcal{T}_{11}^i(k^i, y^i) = -\frac{r^i(k^i, y^i)}{k^i - g^i} \left\{ \frac{(1-\hat{\alpha}_1^i)\alpha_2^i \beta_1^i (k^i - g^i) + (1 - \hat{\alpha}_1^i - \hat{\alpha}_2^i) \hat{\beta}_1^i \alpha_1^i \beta_2^i g^i}{\alpha_2^i \beta_1^i k^i + \hat{\beta}_1^i (\alpha_1^i \beta_2^i - \alpha_2^i \beta_1^i) g^i} \right\}$$

$$\mathcal{T}_{12}^i(k^i, y^i) = \frac{r^i(k^i, y^i) \rho \beta_1^i}{k^i - g^i} \left\{ \frac{(1 - \hat{\alpha}_1^i - \hat{\alpha}_2^i) \alpha_2^i \beta_1^i k^i - (\alpha_1^i \beta_2^i - \alpha_2^i \beta_1^i)[\hat{\alpha}_2^i k^i - (1 - \hat{\alpha}_1^i) g^i]}{\alpha_2^i \beta_1^i k^i + \hat{\beta}_1^i (\alpha_1^i \beta_2^i - \alpha_2^i \beta_1^i) g^i} \right\}$$

$$\mathcal{T}_{21}^i(k^i, y^i) = \frac{r^i(k^i, y^i) \rho}{k^i - g^i} \left\{ \frac{(\hat{\beta}_1^i - \hat{\alpha}_1^i) \alpha_2^i \beta_1^i (k^i - g^i) + (1 - \hat{\alpha}_1^i - \hat{\alpha}_2^i) \hat{\beta}_1^i \alpha_1^i \beta_2^i g^i}{\alpha_2^i \beta_1^i k^i + \hat{\beta}_1^i (\alpha_1^i \beta_2^i - \alpha_2^i \beta_1^i) g^i} \right\}$$

$$\mathcal{T}_{22}^i(k^i, y^i) = -\frac{r^i(k^i, y^i)}{k^i - g^i} \frac{\rho g^i}{y^i \beta_1^i} \left\{ \frac{(1 - \hat{\alpha}_1^i - \hat{\alpha}_2^i) \alpha_1^i \beta_2^i k^i - (\alpha_1^i \beta_2^i - \alpha_2^i \beta_1^i)[(\hat{\alpha}_2^i - \hat{\beta}_2^i) k^i - (\hat{\beta}_1^i - \hat{\alpha}_1^i) g^i]}{\alpha_2^i \beta_1^i k^i + \hat{\beta}_1^i (\alpha_1^i \beta_2^i - \alpha_2^i \beta_1^i) g^i} \right\}$$

and thus

$$\begin{aligned} |H^i(k^i, y^i)| &= \mathcal{T}_{11}^i(k^i, y^i) \mathcal{T}_{22}^i(k^i, y^i) - \mathcal{T}_{12}^i(k^i, y^i) \mathcal{T}_{21}^i(k^i, y^i) \\ &= \frac{r^i(k^i, y^i)^2}{k^i - g^i} \frac{\rho^2 \beta_1^i \hat{\beta}_2^i \alpha_1^i \beta_2^i (1 - \hat{\alpha}_1^i - \hat{\alpha}_2^i)}{\alpha_2^i \beta_1^i k^i + \hat{\beta}_1^i (\alpha_1^i \beta_2^i - \alpha_2^i \beta_1^i) g^i} > 0 \end{aligned}$$

Proof: By definition of g^i as given by (36), we have the identity:

$$y^i[\alpha_2^i\beta_1^ik^i + (\alpha_1^i\beta_2^i - \alpha_2^i\beta_1^i)g^i]^{\hat{\beta}_2^i} = \mathcal{E}_y^i(\alpha_1^i\beta_2^i)^{\hat{\beta}_2^i}g^i \tag{37}$$

Total differentiation gives after simplifications:

$$\frac{y^i}{g^i}dg^i[\alpha_2^i\beta_1^ik^i + \hat{\beta}_1^i(\alpha_1^i\beta_2^i - \alpha_2^i\beta_1^i)g^i] = dy^i\Delta^i + \hat{\beta}_2^i\alpha_2^i\beta_1^iy^idk^i$$

We then get

$$\begin{aligned} g_1^i &= \frac{dg^i}{dk^i} = \frac{\hat{\beta}_2^i\alpha_2^i\beta_1^ig^i}{\alpha_2^i\beta_1^ik^i+\hat{\beta}_1^i(\alpha_1^i\beta_2^i-\alpha_2^i\beta_1^i)g^i} \\ g_2^i &= \frac{dg^i}{dy^i} = \frac{\Delta^ig^i}{y^i\left[\alpha_2^i\beta_1^ik^i+\hat{\beta}_1^i(\alpha_1^i\beta_2^i-\alpha_2^i\beta_1^i)g^i\right]} \end{aligned}$$

The partial derivatives $\mathcal{T}_{mn}^i(k^i, y^i)$ are then obtained by differentiating (35) taking into account the fact that

$$\begin{aligned} g^i - y^ig_2^i &= g^i\frac{\hat{\beta}_2^i(\alpha_2^i\beta_1^i-\alpha_1^i\beta_2^i)g^i}{\alpha_2^i\beta_1^ik^i+\hat{\beta}_1^i(\alpha_1^i\beta_2^i-\alpha_2^i\beta_1^i)g^i} = g^i\frac{\alpha_2^i\beta_1^i-\alpha_1^i\beta_2^i}{\alpha_2^i\beta_1^i}g_1^i \\ 1 - g_1^i &= \frac{\Delta^i-\hat{\beta}_2^i\alpha_1^i\beta_2^ig^i}{\alpha_2^i\beta_1^ik^i+\hat{\beta}_1^i(\alpha_1^i\beta_2^i-\alpha_2^i\beta_1^i)g^i} \end{aligned}$$

We can then compute

$$|H^i(k^i, y^i)| = \mathcal{T}_{11}^i(k^i, y^i)\mathcal{T}_{22}^i(k^i, y^i) - \mathcal{T}_{12}^i(k^i, y^i)\mathcal{T}_{21}^i(k^i, y^i)$$

□

We may now prove Proposition 2. Consider the partial derivatives $\mathcal{T}_{mn}^i(k^i, y^i)$ evaluated at the autarky steady state with $g^i = \rho\beta_1^i\bar{y}^i = \rho\beta_1^i\bar{k}^i$. Straightforward computations give after simplifications:

$$\begin{aligned} \frac{\mathcal{P}_c^i(0)}{\mathcal{A}^i} &= (\hat{\beta}_1^i - \hat{\alpha}_1^i)\alpha_2^i(1-\rho\beta_1^i) + (1-\hat{\alpha}_1^i-\hat{\alpha}_2^i)\hat{\beta}_1^i\alpha_1^i\beta_2^i\rho \\ \frac{\mathcal{P}_c^i(1)}{\mathcal{A}^i} &= -\hat{\beta}_2^i(1-\rho\beta_1^i)\left[\alpha_2^i + \rho(\alpha_1^i\beta_2^i - \alpha_2^i\beta_1^i)\right] < 0 \\ \frac{\mathcal{P}_c^i(-1)}{\mathcal{A}^i} &= 2\rho(1-\hat{\alpha}_1^i-\hat{\alpha}_2^i)\alpha_1^i\beta_2^i(1+\hat{\beta}_1^i) \\ &+ [2(1-\hat{\alpha}_1^i) - \hat{\beta}_2^i](1-\rho\beta_1^i)[\alpha_2^i - \rho(\alpha_1^i\beta_2^i - \alpha_2^i\beta_1^i)] \end{aligned} \tag{38}$$

with

$$\mathcal{A}^i = \frac{T_1^i(\bar{k}^i, \bar{k}^i, \hat{e}_c(\bar{k}^i, \bar{k}^i), \hat{e}_y(\bar{k}^i, \bar{k}^i))\rho}{\bar{k}^i[\alpha_2^i + \rho\hat{\beta}_1^i(\alpha_1^i\beta_2^i - \alpha_2^i\beta_1^i)](1-\rho\beta_1^i)} > 0 \tag{39}$$

Notice first that since $\mathcal{P}_c^i(1) < 0$, a necessary condition for the occurrence of local indeterminacy is

$$\lim_{\lambda\to\pm\infty}\mathcal{P}_c^i(\lambda) = -\infty \quad\Leftrightarrow\quad \mathcal{T}_{12}^i(\bar{k}^i, \bar{k}^i) < 0$$

This property is satisfied if and only if

$$\alpha_1^i\beta_2^i - \alpha_2^i\beta_1^i > \frac{1-\hat{\alpha}_1^i-\hat{\alpha}_2^i}{1-\hat{\alpha}_1^i}\frac{\alpha_1^i\beta_2^i}{1-\rho\beta_1^i}$$

Since the right-hand-side is an increasing function of ρ, we conclude that if

$$\alpha_1^i\beta_2^i - \alpha_2^i\beta_1^i > \frac{(1-\hat{\alpha}_1^i-\hat{\alpha}_2^i)\alpha_1^i\beta_2^i}{(1-\hat{\alpha}_1^i)(1-\beta_1^i)} \equiv \mathcal{Z}_1^i \tag{40}$$

then $\mathcal{T}_{12}^i(\bar{k}^i,\bar{k}^i) < 0$ for any $\rho \in (0,1)$. Taking (40) into account, local indeterminacy may arise in two types of configurations:

i) when $\mathcal{P}_c^i(0) = \mathcal{T}_{21}^i(\bar{k}^i,\bar{k}^i) > 0$ and $\mathcal{P}_c^i(-1) < 0$. In this case the product of characteristic roots $\mathcal{D}_c^i = \mathcal{T}_{21}^i(\bar{k}^i,\bar{k}^i)/\rho\mathcal{T}_{12}^i(\bar{k}^i,\bar{k}^i)$ is negative. We immediately derive from (38):

$$\mathcal{P}_c^i(0) > 0 \Leftarrow \hat{\beta}_1^i - \hat{\alpha}_1^i > 0$$

$$\mathcal{P}_c^i(-1) < 0 \Leftrightarrow \alpha_1^i\beta_2^i - \alpha_2^i\beta_1^i - \frac{\alpha_2^i}{\rho} > \frac{2(1-\hat{\alpha}_1^i-\hat{\alpha}_2^i)\alpha_1^i\beta_2^i(1+\hat{\beta}_1^i)}{[2(1-\hat{\alpha}_1^i)-\hat{\beta}_2^i](1-\rho\beta_1^i)} \tag{41}$$

Notice that $\hat{\beta}_1^i - \hat{\alpha}_1^i > 0$ implies $2(1-\hat{\alpha}_1^i) - \hat{\beta}_2^i > 0$. Moreover, (41) is never satisfied when ρ is close to 0 while it will be satisfied when ρ is close to 1 if

$$\alpha_1^i\beta_2^i - \alpha_2^i\beta_1^i > \alpha_2^i + \frac{2(1-\hat{\alpha}_1^i-\hat{\alpha}_2^i)\alpha_1^i\beta_2^i(1+\hat{\beta}_1^i)}{[2(1-\hat{\alpha}_1^i)-\hat{\beta}_2^i](1-\beta_1^i)} \equiv \mathcal{Z}_2^i \tag{42}$$

Therefore, there exists $\bar{\rho} \in (0,1)$ such that $\mathcal{P}_c^i(-1) < 0$ for any $\rho \in (\bar{\rho},1)$. The rest of the proof is completed by noticing from (40) and (42) that $\mathcal{Z}_2^i > \mathcal{Z}_1^i$.

ii) when $\mathcal{P}_c^i(0) = \mathcal{T}_{21}^i(\bar{k}^i,\bar{k}^i) < 0$, $\mathcal{P}_c^i(-1) < 0$ and $\mathcal{D}_c^i \in (0,1)$. Under (40) indeed, $\mathcal{P}_c^i(0) < 0$ implies $\mathcal{D}_c^i > 0$. We first derive from (38):

$$\mathcal{P}_c^i(0) < 0 \quad \Leftrightarrow \quad \hat{\alpha}_1^i - \hat{\beta}_1^i > \frac{(1-\hat{\alpha}_1^i-\hat{\alpha}_2^i)\hat{\beta}_1^i\alpha_1^i\beta_2^i\rho}{\alpha_2^i(1-\rho\beta_1^i)} > 0$$

Since the right-hand-side is an increasing function of ρ, we conclude that if

$$\hat{\alpha}_1^i - \hat{\beta}_1^i > \frac{(1-\hat{\alpha}_1^i-\hat{\alpha}_2^i)\hat{\beta}_1^i\alpha_1^i\beta_2^i}{\alpha_2^i(1-\beta_1^i)} \equiv \mathcal{Z}_3^i \tag{43}$$

then $\mathcal{P}_c^i(0) < 0$ for any $\rho \in (0,1)$. Notice then that $2(1-\hat{\alpha}_1^i) - \hat{\beta}_2^i = 1 - \hat{\alpha}_1^i - (\hat{\alpha}_1^i - \hat{\beta}_1^i)$. It follows that if (43) holds with $1 - \hat{\alpha}_1^i > \hat{\alpha}_1^i - \hat{\beta}_1^i$, then (41) is satisfied and under (42), there exists $\bar{\rho} \in (0,1)$ such that $\mathcal{P}_c^i(-1) < 0$ for any $\rho \in (\bar{\rho},1)$. Finally, we derive from Lemma 6.2 and (40)

$$\mathcal{D}_c^i < 1 \quad \Leftrightarrow \quad \alpha_1^i\beta_2^i - \alpha_2^i\beta_1^i > \frac{(\hat{\alpha}_1^i-\hat{\beta}_1^i)\alpha_2^i(1-\rho\beta_1^i)+\rho(1-\hat{\alpha}_1^i-\hat{\alpha}_2^i)\hat{\beta}_2^i\alpha_1^i\beta_2^i}{\rho(1-\hat{\alpha}_1^i)(1-\rho\beta_1^i)}$$

When $\rho = 1$ this inequality becomes

$$\alpha_1^i\beta_2^i - \alpha_2^i\beta_1^i > \frac{(\hat{\alpha}_1^i-\hat{\beta}_1^i)\alpha_2^i(1-\beta_1^i)+(1-\hat{\alpha}_1^i-\hat{\alpha}_2^i)\hat{\beta}_2^i\alpha_1^i\beta_2^i}{(1-\hat{\alpha}_1^i)(1-\beta_1^i)} \equiv \mathcal{Z}_4^i \tag{44}$$

Under (44), there exists $\bar{\rho} \in (0,1)$ such that $\mathcal{D}_c^i < 1$ for any $\rho \in (\bar{\rho},1)$. The rest of the proof follows from the fact that if (43) holds with $1-\hat{\alpha}_1^i > \hat{\alpha}_1^i - \hat{\beta}_1^i$, then $\mathcal{Z}_2^i > \mathcal{Z}_4^i$. □

6.3 Proof of Proposition 3

We follow the same procedure as in Nishimura and Yano [11]. We start by stating without proof a standard Lemma which shows that an equilibrium consumption path of a country is associated with a marginal utility of wealth.

Lemma 6.3. *If c_t^i, $t = 0, 1, \ldots$, is country i's equilibrium consumption path (either under free-trade or autarky), then there is $\lambda^i > 0$ such that*

$$\sum_{t=0}^{+\infty} \rho^t u(c_t^i) - \lambda^i \sum_{t=0}^{+\infty} c_t^i \geq \sum_{t=0}^{+\infty} \rho^t u(\xi_t^i) - \lambda^i \sum_{t=0}^{+\infty} \xi_t^i$$

for any $\xi_t^i > 0$, $t = 0, 1, \ldots$.

From this Lemma we derive by setting $\xi_\tau^i = c_\tau^i$ for all $\tau \neq t$ that

$$\rho^t u(c_t^i) - \lambda^i c_t^i \geq \rho^t u(\xi_t^i) - \lambda^i \xi_t^i \tag{45}$$

This implies

$$\rho^t \left[\tfrac{u(c_t^A)}{\lambda^A} + \tfrac{u(c_t^B)}{\lambda^B}\right] - (c_t^A + c_t^B) \geq \rho^t \left[\tfrac{u(\xi_t^A)}{\lambda^A} + \tfrac{u(\xi_t^B)}{\lambda^B}\right] - (\xi_t^A + \xi_t^B) \tag{46}$$

Now consider the social production function of country i, $T^i(k_t^i, y_t^i, e_{ct}^i, e_{yt}^i)$, as the value function of program (5). Taking into account (17), we necessarily have for given externalities (e_{ct}^i, e_{yt}^i):

$$T^i(k_t^i, k_{t+1}^i, e_{ct}^i, e_{yt}^i) + p_{t+1}k_{t+1}^i - p_t k_t^i \geq T^i(\kappa_t^i, \kappa_{t+1}^i, e_{ct}^i, e_{yt}^i) + p_{t+1}\kappa_{t+1}^i - p_t\kappa_t^i$$

for any feasible country i's capital stocks $(\kappa_t^i, \kappa_{t+1}^i)$.[9] Since by definition $T^A(k_t^A, k_{t+1}^A, e_{ct}^A, e_{yt}^A) + T^B(k_t^B, k_{t+1}^B, e_{ct}^B, e_{yt}^B) = c_t^A + c_t^B$ and using (47)-(48), this last inequality becomes

$$\rho^t \left[\tfrac{u(c_t^A)}{\lambda^A} + \tfrac{u(c_t^B)}{\lambda^B}\right] + p_{t+1}k_{t+1} - p_t k_t \geq \rho^t \left[\tfrac{u(\xi_t^A)}{\lambda^A} + \tfrac{u(\xi_t^B)}{\lambda^B}\right] + p_{t+1}\kappa_{t+1} - p_t\kappa_t$$

for any feasible total capital stocks (κ_t, κ_{t+1}) and any $\xi_t^i > 0$ such that $T^A(\kappa_t^A, \kappa_{t+1}^A, e_{ct}^A, e_{yt}^A) + T^B(\kappa_t^B, \kappa_{t+1}^B, e_{ct}^B, e_{yt}^B) = \xi_t^A + \xi_t^B$ and where $k_t = k_t^A + k_t^B$.[10] Thus, by the definition of $W(k_t, y_t, e_{ct}, e_{yt}; \lambda)$ we get

$$\begin{aligned} & \rho^t W(k_t, k_{t+1}, e_{ct}, e_{yt}; \lambda) + p_{t+1}k_{t+1} - p_t k_t \\ \geq \; & \rho^t W(\kappa_t, \kappa_{t+1}, e_{ct}, e_{yt}; \lambda) + p_{t+1}\kappa_{t+1} - p_t\kappa_t \end{aligned}$$

[9] Recall that $y_t \leq \mathcal{E}_y^i (k_t^i)^{\beta_1^i} e_{yt}^i$. A feasible path of country i's capital stocks is thus a pair (k_t^i, k_{t+1}^i) such that $0 \leq k_t^i \leq (\mathcal{E}_y^i e_{yt}^i)^{1/(1-\beta_1^i)}$ and $0 \leq k_{t+1}^i \leq \mathcal{E}_y^i (k_t^i)^{\beta_1^i} e_{yt}^i$.

[10] A feasible path of total capital stocks is a pair (k_t, k_{t+1}) such that $k_t = k_t^A + k_t^B$ and $k_{t+1} = k_{t+1}^A + k_{t+1}^B$, with $(\kappa_t^i, \kappa_{t+1}^i)$ some feasible country i's capital stocks, $i = A, B$.

for any feasible total capital stocks (κ_t, κ_{t+1}). Under Assumption 1, this last inequality implies that along a free-trade equilibrium we have $p_0 > 0$. It follows from inequality (48) considered with $u(c^i) = c^i$ that $\lambda^A = \lambda^B$. The final result follows from the definitions of $W(k_t, y_t, e_{ct}, e_{yt}; \lambda)$ and $V(k_t, y_t, e_{ct}, e_{yt})$. □

6.4 Proof of Proposition 4

A steady state is a solution (k^A, k^B, y^A, y^B, k) of the following system

$$\begin{aligned} T_2^A(k^A, y^A, \hat{e}_c^A, \hat{e}_y^A) + \rho T_1^A(k^A, y^A, \hat{e}_c^A, \hat{e}_y^A) &= 0 \quad (47) \\ T_2^B(k^B, y^B, \hat{e}_c^B, \hat{e}_y^B) + \rho T_1^B(k^B, y^B, \hat{e}_c^B, \hat{e}_y^B) &= 0 \quad (48) \\ T_1^A(k^A, y^A, \hat{e}_c^A, \hat{e}_y^A) - T_1^B(k^B, y^B, \hat{e}_c^B, \hat{e}_y^B) &= 0 \quad (49) \\ T_2^A(k^A, y^A, \hat{e}_c^A, \hat{e}_y^A) - T_2^B(k^B, y^B, \hat{e}_c^B, \hat{e}_y^B) &= 0 \quad (50) \\ k^A + k^B = y^A + y^B &= k \quad (51) \end{aligned}$$

with $c^A + c^B = V(k, k, \hat{e}_c, \hat{e}_y) = T^A(k^A, k^A, \hat{e}_c^A, \hat{e}_y^A) + T^B(k^B, k^B, \hat{e}_c^B, \hat{e}_y^B)$ and $\hat{e}_c^i = \hat{e}_c^i(k^i, y^i)$, $\hat{e}_y^i = \hat{e}_y^i(k^i, y^i)$, $i = A, B$. We get from equations (47)-(48) the following property for a steady state under free-trade:

Lemma 6.4. *At a steady state under free-trade, the following holds:*

$$\frac{g^A}{y^A \beta_1^A} = \frac{g^B}{y^B \beta_1^B} = \rho$$

with $g^i \equiv K_y^{i*}$ *the stationary optimal demand for capital in the investment good sector of country* $i = A, B$.

Proof: The first order conditions (49)-(50) give $T_j^A(k^A, y^A, e_c^A, e_y^A) = T_j^B(k^B, y^B, e_c^B, e_y^B)$, $j = 1, 2$. As $T_2^i(k^i, y^i, e_c^i, e_y^i) = -T_1^i(k^i, y^i, e_c^i, e_y^i) g^i / y^i \beta_1^i$, we derive that $g^A / y^A \beta_1^A = g^B / y^B \beta_1^B$. Consider now the equations (47)-(48). We get $-T_2^i(k^i, y^i, e_c^i, e_y^i) = \rho T_1^i(k^i, y^i, e_c^i, e_y^i)$ and the result follows. □

We may now prove Proposition 4. Using Lemmas 6.4 and 6.1 with (34), equations (47) and (48) may be written as

$$\frac{1}{\beta_1^A} \left(\frac{\alpha_2^A \beta_1^A k^A + (\alpha_1^A \beta_2^A - \alpha_2^A \beta_1^A) \rho \beta_1^A y^A}{\alpha_1^A \beta_2^A} \right)^{\hat{\beta}_2^A} = \frac{1}{\beta_1^B} \left(\frac{\alpha_2^B \beta_1^B k^B + (\alpha_1^B \beta_2^B - \alpha_2^B \beta_1^B) \rho \beta_1^B y^B}{\alpha_1^B \beta_2^B} \right)^{\hat{\beta}_2^B} = \rho$$

It follows that the autarky steady state, i.e. $k^A = y^A = \bar{k}^A$ and $k^B = y^B = \bar{k}^B$, with $\bar{k}^i$ given in Proposition 1, is a solution of the previous equations and satisfies equation (51). Considering $T_1^i(k^i, y^i, e_c^i, e_y^i)$ in Lemma 6.1 with

$\mathcal{E}_y^A = \mathcal{E}_c^B = \mathcal{E}_y^B = 1$ and (34), equation (49) with $k^i = y^i = \bar{k}^i$ is satisfied if and only if $\mathcal{E}_c^A = \bar{\mathcal{E}}_c^A$ with

$$\bar{\mathcal{E}}_c^A = \frac{\left[\alpha_2^A\beta_1^A + (\alpha_1^A\beta_2^A - \alpha_2^A\beta_1^A)\rho\beta_1^A\right]^{\hat{\alpha}_2^A} \alpha_1^B (\alpha_2^B\beta_1^B)^{\hat{\alpha}_2^B} (\bar{k}^B)^{\hat{\alpha}_1^B - 1} (1-\rho\beta_1^B)^{\hat{\alpha}_1^B + \hat{\alpha}_2^B - 1}}{\alpha_1^A (\alpha_2^A\beta_1^A)^{\hat{\alpha}_2^A} (\bar{k}^A)^{\hat{\alpha}_1^A - 1} (1-\rho\beta_1^A)^{\hat{\alpha}_1^A + \hat{\alpha}_2^A - 1} \left[\alpha_2^B\beta_1^B + (\alpha_1^B\beta_2^B - \alpha_2^B\beta_1^B)\rho\beta_1^B\right]^{\hat{\alpha}_2^B}} \tag{52}$$

Then, since from (47) and (48) we have $T_1^i(k^i, y^i, \hat{e}_c^i, \hat{e}_y^i) = T_2^i(k^i, y^i, \hat{e}_c^i, \hat{e}_y^i)/\rho$, $i = A, B$, equation (50) also holds with $k^i = y^i = \bar{k}^i$. □

6.5 Proof of Proposition 5

Consider equations (47)-(51) with $\hat{e}_c^i = \hat{e}_c^i(k^i, y^i)$ and $\hat{e}_y^i = \hat{e}_y^i(k^i, y^i)$, $i = A, B$. We know from Lemma 6.4 that equations (47) and (48) imply $g^i = \rho\beta_1^i y^i$, $i = A, B$. Assume then that $k^A = \theta y^A$ and $k^B = y^B/\theta$ with $\theta > 1$ some constant. We will give conditions on the normalization constants $\mathcal{E}_c^i, \mathcal{E}_y^i$ to get these expressions as solutions of equations (47)-(51). Notice first from (51) that these restrictions imply $k^A = \theta k^B$ and $g^A = \rho\beta_1^A k^A/\theta$, $g^B = \rho\beta_1^B k^B\theta$. Substituting these expressions into (36) with $K_y^i = g^i$ and solving for k^i, $i = A, B$, gives

$$k^{A*} = \frac{\theta\alpha_1^A\beta_2^A(\mathcal{E}_y^A\rho\beta_1^A)^{1/\hat{\beta}_2^A}}{\alpha_2^A\beta_1^A\theta + (\alpha_1^A\beta_2^A - \alpha_2^A\beta_1^A)\rho\beta_1^A}, \quad k^{B*} = \frac{\alpha_1^B\beta_2^B(\mathcal{E}_y^B\rho\beta_1^B)^{1/\hat{\beta}_2^B}}{\alpha_2^B\beta_1^B + (\alpha_1^B\beta_2^B - \alpha_2^B\beta_1^A)\rho\beta_1^B\theta} \tag{53}$$

We may now use the normalization constants $\mathcal{E}_y^A$ and $\mathcal{E}_y^B$ to get $k^{A*} = \theta k^{B*}$. Assuming $\mathcal{E}_y^B = 1$ we derive from (53) that $k^{A*} = \theta k^{B*}$ if and only if $\theta \in (1, 1/\rho\beta_1^B)$ and $\mathcal{E}_y^A = \mathcal{E}_y^{A*}$ with

$$\mathcal{E}_y^{A*} = \frac{1}{\rho\beta_1^A}\left\{\frac{\left[\alpha_2^A\beta_1^A\theta + (\alpha_1^A\beta_2^A - \alpha_2^A\beta_1^A)\rho\beta_1^A\right]\alpha_1^B\beta_2^B(\rho\beta_1^B)^{1/\hat{\beta}_2^B}}{\alpha_1^A\beta_2^A\left[\alpha_2^B\beta_1^B + (\alpha_1^B\beta_2^B - \alpha_2^B\beta_1^A)\rho\beta_1^B\theta\right]}\right\}^{\hat{\beta}_2^A} \tag{54}$$

Considering $T_1^i(k^i, y^i, e_c^i, e_y^i)$ in Lemma 6.1 with $\mathcal{E}_c^B = 1$ and (34), equation (49) with $k^A = \theta y^A$, $k^B = y^B/\theta$, and thus $k^A = \theta k^B$, is satisfied if and only if $\theta \in (1, 1/\rho\beta_1^B)$ and $\mathcal{E}_c^A = \mathcal{E}_c^{A*}$ with

$$\mathcal{E}_c^{A*} = \frac{\left[\alpha_2^A\beta_1^A\theta + (\alpha_1^A\beta_2^A - \alpha_2^A\beta_1^A)\rho\beta_1^A\right]^{\hat{\alpha}_2^A} \alpha_1^B(\alpha_2^B\beta_1^B)^{\hat{\alpha}_2^B}(1-\rho\beta_1^B\theta)^{\hat{\alpha}_1^B + \hat{\alpha}_2^B - 1}(k^{B*})^{\hat{\alpha}_1^B - \hat{\alpha}_1^A}}{\alpha_1^A(\alpha_2^A\beta_1^A)^{\hat{\alpha}_2^A}(\theta - \rho\beta_1^A)^{\hat{\alpha}_1^A + \hat{\alpha}_2^A - 1}\left[\alpha_2^B\beta_1^B + (\alpha_1^B\beta_2^B - \alpha_2^B\beta_1^B)\rho\beta_1^B\theta\right]^{\hat{\alpha}_2^B}} \tag{55}$$

Then, since from (47) and (48) we have $T_1^i(k^i, y^i, \hat{e}_c^i, \hat{e}_y^i) = T_2^i(k^i, y^i, \hat{e}_c^i, \hat{e}_y^i)/\rho$, $i = A, B$, equation (50) also holds with $k^A = \theta y^A$ and $k^B = y^B/\theta$. □

6.6 Proof of Corollary 1

To simplify notation let

$\Phi_\theta^A = \alpha_2^A\beta_1^A\theta + (\alpha_1^A\beta_2^A - \alpha_2^A\beta_1^A)\rho\beta_1^A, \quad \Phi_\theta^B = \alpha_2^B\beta_1^B + (\alpha_1^B\beta_2^B - \alpha_2^B\beta_1^A)\rho\beta_1^B\theta$

Let $\mathcal{E}_c^B = \mathcal{E}_y^B = 1$ and $\mathcal{E}_c^{A*}$, $\mathcal{E}_y^{A*}$ be given by (54) and (55). Substituting these values into the free-trade distribution of capital as given by (20) gives

$$k^{A*} = \theta\frac{\alpha_1^B\beta_2^B(\rho\beta_1^B)^{1/\hat{\beta}_2^B}}{\Phi_\theta^B} = \theta k^{B*} \tag{56}$$

We may then rewrite $\mathcal{E}_c^{A*}$ as follows

$$\mathcal{E}_c^{A*} = \frac{(\Phi_\theta^A)^{\hat{\alpha}_2^A}\alpha_1^B(\alpha_2^B\beta_1^B)^{\hat{\alpha}_2^B}(1-\rho\beta_1^B\theta)^{\hat{\alpha}_1^B+\hat{\alpha}_2^B-1}}{\alpha_1^A(\alpha_2^A\beta_1^A)^{\hat{\alpha}_2^A}(\theta-\rho\beta_1^A)^{\hat{\alpha}_1^A+\hat{\alpha}_2^A-1}(\Phi_\theta^B)^{\hat{\alpha}_2^B}}\left(\frac{\alpha_1^B\beta_2^B(\rho\beta_1^B)^{1/\hat{\beta}_2^B}}{\Phi_\theta^B}\right)^{\hat{\alpha}_1^B-\hat{\alpha}_1^A} \tag{57}$$

Considering $T^i(k^i, y^i, e_c^i, e_y^i)$ as defined by (7) with (28), (30), (34), $\mathcal{E}_c^B = 1$, $k^A = \theta y^A$, $k^B = y^B/\theta$, (56) and (57), we get

$$\begin{aligned} T^{A*} &= \frac{\alpha_1^B(\theta-\rho\beta_1^A)}{\alpha_1^A(1-\rho\beta_1^B\theta)}\frac{(\alpha_1^B\beta_2^B)^{\hat{\alpha}_1^A}(\alpha_2^B\beta_1^B)^{\hat{\alpha}_2^B}(\rho\beta_1^B)^{\hat{\alpha}_1^B/\hat{\beta}_2^B}(1-\rho\beta_1^B\theta)^{\hat{\alpha}_1^B+\hat{\alpha}_2^B}}{(\Phi_\theta^B)^{\hat{\alpha}_1^B+\hat{\alpha}_2^B}} \\ &= \frac{\alpha_1^B(\theta-\rho\beta_1^A)}{\alpha_1^A(1-\rho\beta_1^B\theta)}T^{B*} \equiv \eta T^{B*} \end{aligned} \tag{58}$$

□

6.7 Proof of Proposition 6

We first compute the partial derivatives $\mathcal{V}_{mn}(k, y)$. Simple modifications of the proof of Lemma 7 from Nishimura and Yano [11] allows to get

Lemma 6.5. *Along a free-trade equilibrium, the partial derivatives $\mathcal{V}_{mn}(k, y)$ satisfy the following:*

$$\begin{aligned} \mathcal{V}_{11}(k,y) &= \frac{1}{\Theta}\left[\mathcal{T}_{11}^A(k^A,y^A)|H^B(k^B,y^B)| + \mathcal{T}_{11}^B(k^B,y^B)|H^A(k^A,y^A)|\right] \\ \mathcal{V}_{12}(k,y) &= \frac{1}{\Theta}\left[\mathcal{T}_{12}^A(k^A,y^A)|H^B(k^B,y^B)| + \mathcal{T}_{12}^B(k^B,y^B)|H^A(k^A,y^A)|\right] \\ \mathcal{V}_{21}(k,y) &= \frac{1}{\Theta}\left[\mathcal{T}_{21}^A(k^A,y^A)|H^B(k^B,y^B)| + \mathcal{T}_{21}^B(k^B,y^B)|H^A(k^A,y^A)|\right] \\ \mathcal{V}_{22}(k,y) &= \frac{1}{\Theta}\left[\mathcal{T}_{22}^A(k^A,y^A)|H^B(k^B,y^B)| + \mathcal{T}_{22}^B(k^B,y^B)|H^A(k^A,y^A)|\right] \end{aligned}$$

where $\mathcal{T}_{mn}^i(k^i, y^i)$, $|H^i(k^i, y^i)|$ are given in Lemma 6.2 and

$$\begin{aligned} \Theta &= \mathcal{T}_{11}^A(k^A,y^A)\mathcal{T}_{22}^B(k^B,y^B) + \mathcal{T}_{11}^B(k^B,y^B)\mathcal{T}_{22}^A(k^A,y^A) \\ &- \mathcal{T}_{12}^A(k^A,y^A)\mathcal{T}_{21}^B(k^B,y^B) - \mathcal{T}_{21}^A(k^A,y^A)\mathcal{T}_{12}^B(k^B,y^B) \\ &+ |H^A(k^A,y^A)| + |H^B(k^B,y^B)| \neq 0 \end{aligned}$$

From Lemmas 6.2 and 6.5 we may then derive:

Lemma 6.6. *The partial derivatives $\mathcal{V}_{mn}(k,y)$ are given by:*

$$\frac{\mathcal{V}_{11}(k,y)}{\mathcal{C}} = \bar{\mathcal{V}}_{11}(k,y)$$
$$= -\Big\{(1-\hat\alpha_1^A)(1-\hat\alpha_1^B-\hat\alpha_2^B)\beta_1^A\beta_1^B\beta_2^B\hat\beta_2^B\alpha_1^B\alpha_2^A(k^A-g^A)$$
$$+(1-\hat\alpha_1^B)(1-\hat\alpha_1^A-\hat\alpha_2^A)\beta_1^A\beta_1^B\beta_2^A\hat\beta_2^A\alpha_1^A\alpha_2^B(k^B-g^B)$$
$$+(1-\hat\alpha_1^A-\hat\alpha_2^A)(1-\hat\alpha_1^B-\hat\alpha_2^B)\alpha_1^A\beta_2^A\alpha_1^B\beta_2^B(\beta_1^B\hat\beta_1^A\hat\beta_2^B g^A+\beta_1^A\hat\beta_1^B\hat\beta_2^A g^B)$$

$$\frac{\mathcal{V}_{12}(k,y)}{\mathcal{C}} = \bar{\mathcal{V}}_{12}(k,y)$$
$$= -\rho\beta_1^A\beta_1^B\Big\{(1-\hat\alpha_1^B-\hat\alpha_2^B)(\alpha_1^A\beta_2^A-\alpha_2^A\beta_1^A)\beta_2^B\hat\beta_2^B\alpha_1^B[\hat\alpha_2^A k^A-(1-\hat\alpha_1^A)g^A]$$
$$+(1-\hat\alpha_1^A-\hat\alpha_2^A)(\alpha_1^B\beta_2^B-\alpha_2^B\beta_1^B)\beta_2^A\hat\beta_2^A\alpha_1^A[\hat\alpha_2^B k^B-(1-\hat\alpha_1^B)g^B]$$
$$-(1-\hat\alpha_1^A-\hat\alpha_2^A)(1-\hat\alpha_1^B-\hat\alpha_2^B)(\beta_2^B\hat\beta_2^B\alpha_1^B\alpha_2^A\beta_1^A k^A+\beta_2^A\hat\beta_2^A\alpha_1^A\alpha_2^B\beta_1^B k^B)\Big\}$$

$$\frac{\mathcal{V}_{21}(k,y)}{\mathcal{C}} = \bar{\mathcal{V}}_{21}(k,y)$$
$$= \rho\Big\{(\hat\beta_1^A-\hat\alpha_1^A)(1-\hat\alpha_1^B-\hat\alpha_2^B)\beta_1^B\beta_2^B\hat\beta_2^B\alpha_1^B\alpha_2^A\beta_1^A(k^A-g^A)$$
$$+(\hat\beta_1^B-\hat\alpha_1^B)(1-\hat\alpha_1^A-\hat\alpha_2^A)\beta_1^A\beta_2^A\hat\beta_2^A\alpha_1^A\alpha_2^B\beta_1^B(k^B-g^B)$$
$$+(1-\hat\alpha_1^A-\hat\alpha_2^A)(1-\hat\alpha_1^B-\hat\alpha_2^B)\alpha_1^A\beta_2^A\alpha_1^B\beta_2^B(\beta_1^B\hat\beta_1^A\hat\beta_2^B g^A+\beta_1^A\hat\beta_1^B\hat\beta_2^A g^B)\Big\}$$

$$\frac{\mathcal{V}_{22}(k,y)}{\mathcal{C}} = \bar{\mathcal{V}}_{22}(k,y)$$
$$= \rho^2\beta_1^A\beta_1^B\Big\{(1-\hat\alpha_1^B-\hat\alpha_2^B)(\alpha_1^A\beta_2^A-\alpha_2^A\beta_1^A)\beta_2^B\hat\beta_2^B\alpha_1^B[(\hat\alpha_2^A-\hat\beta_2^A)k^A-(\hat\beta_1^A-\hat\alpha_1^A)g^A]$$
$$+(1-\hat\alpha_1^A-\hat\alpha_2^A)(\alpha_1^B\beta_2^B-\alpha_2^B\beta_1^B)\beta_2^A\hat\beta_2^A\alpha_1^A[(\hat\alpha_2^B-\hat\beta_2^B)k^B-(\hat\beta_1^B-\hat\alpha_1^B)g^B]$$
$$-(1-\hat\alpha_1^A-\hat\alpha_2^A)(1-\hat\alpha_1^B-\hat\alpha_2^B)(\beta_2^B\hat\beta_2^B\alpha_1^B\alpha_2^A\beta_1^A k^A+\beta_2^A\hat\beta_2^A\alpha_1^A\alpha_2^B\beta_1^B k^B)\Big\}$$

with

$$\mathcal{C} = \frac{T_1^A(k^A,y^A)^3\rho^2}{\Theta(k^A-g^A)(k^B-g^B)}\frac{1}{\alpha_2^A\beta_1^A k^A+\hat\beta_1^A(\alpha_1^A\beta_2^A-\alpha_2^A\beta_1^A)g^A}\frac{1}{\alpha_2^B\beta_1^B k^B+\hat\beta_1^B(\alpha_1^B\beta_2^B-\alpha_2^A\beta_1^B)g^B}$$

and $g^i=\rho\beta_1^i y^i$, $i=A,B$, $k=k^A+k^B$, $y=y^A+y^B$.

Proof: The expressions of the partial derivatives $\mathcal{V}_{mn}(k,y)$ follow from Lemmas 6.2, 6.4, 6.5 and the fact that at a steady state under free-trade we have $T_2^A(k^A,y^A,e_c^A,e_y^A)+\rho T_1^A(k^A,y^A,e_c^A,e_y^A) = T_2^B(k^B,y^B,e_c^B,e_y^B)+\rho T_1^B(k^B,y^B,e_c^B,e_y^B)=0$ and that $T_j^A(k^A,y^A,e_c^A,e_y^A)=T_j^B(k^B,y^B,e_c^B,e_y^B)$,

$j = 1, 2$. $\square$

The characteristic polynomial as defined by (6.7) then becomes:

$$\frac{\mathcal{P}_o(\lambda)}{\mathcal{C}} \equiv \bar{\mathcal{P}}_o(\lambda) = \rho\bar{\mathcal{V}}_{12}(k^*, k^*)\lambda^2 + \lambda\Big[\bar{\mathcal{V}}_{22}(k^*, k^*) + \rho\bar{\mathcal{V}}_{11}(k^*, k^*)\Big] + \bar{\mathcal{V}}_{21}(k^*, k^*) = 0$$

The stability analysis is based on the sign of $\bar{\mathcal{P}}_o(0)$, $\bar{\mathcal{P}}_o(1)$ and $\bar{\mathcal{P}}_o(-1)$. The following property holds for the autarky and free-trade distributions:

Lemma 6.7. *Along a steady state $k^* = k^{A*} + k^{B*}$, the characteristic polynomial satisfies $\bar{\mathcal{P}}_o(1) < 0$.*

Proof: Consider the partial derivatives $\bar{\mathcal{V}}_{mn}(k, y)$ given in Lemma 6.6 with Lemma 6.4. Straightforward computations give after simplifications:

$$\begin{aligned}\bar{\mathcal{P}}_o(1) \;=\; & -\rho\hat{\beta}_2^A\hat{\beta}_2^B\beta_1^A\beta_1^B\Big\{(1-\hat{\alpha}_1^B-\hat{\alpha}_2^B)\alpha_1^B\beta_2^B(k^A-g^A)\left[\alpha_2^A+\rho(\alpha_1^A\beta_2^A-\alpha_2^A\beta_1^A)\right] \\ & + (1-\hat{\alpha}_1^A-\hat{\alpha}_2^A)\alpha_1^A\beta_2^A(k^B-g^B)\left[\alpha_2^B+\rho(\alpha_1^B\beta_2^B-\alpha_2^B\beta_1^B)\right]\Big\}\end{aligned}$$

The result derives from the following two facts: firstly, at the autarky distribution we have $k^i - g^i = \bar{k}^i(1-\rho\beta_1^i) > 0$, and secondly, at the free-trade distribution we have $k^A - g^A = k^{A*}(\theta - \rho\beta_1^A) > 0$ and $k^B - g^B = k^{B*}(1-\rho\beta_1^B\theta) > 0$ since $\theta \in (1, 1/\rho\beta_1^B)$. $\square$

We may now prove Proposition 6. Using Lemma 6.6, straightforward computations show that:

$$\begin{aligned}\bar{\mathcal{P}}_o(0) \;=\; & \bar{\mathcal{V}}_{21}(k, y) \\ \bar{\mathcal{P}}_o(-1) \;=\; & \rho(1-\hat{\alpha}_1^B-\hat{\alpha}_2^B)\beta_1^B\beta_2^B\hat{\beta}_2^B\alpha_1^B\Big\{2\rho(1-\hat{\alpha}_1^A-\hat{\alpha}_2^A)\alpha_2^A(\beta_1^A)^2k^A \\ & - \rho\beta_1^A(\alpha_1^A\beta_2^A-\alpha_2^A\beta_1^A)\left[(2\hat{\alpha}_2^A-\hat{\beta}_2^A)k^A-[2(1-\hat{\alpha}_1^A)-\hat{\beta}_2^A]g^A\right] \\ & + [2(1-\hat{\alpha}_1^A)-\hat{\beta}_2^A]\alpha_2^A\beta_1^A(k^A-g^A)+2(1-\hat{\alpha}_1^A-\hat{\alpha}_2^A)\hat{\beta}_1^A\alpha_1^A\beta_2^Ag^A\Big\} \\ & + \rho(1-\hat{\alpha}_1^A-\hat{\alpha}_2^A)\beta_1^A\beta_2^A\hat{\beta}_2^A\alpha_1^A\Big\{2\rho(1-\hat{\alpha}_1^B-\hat{\alpha}_2^B)\alpha_2^B(\beta_1^B)^2k^B \\ & - \rho\beta_1^B(\alpha_1^B\beta_2^B-\alpha_2^B\beta_1^B)\left[(2\hat{\alpha}_2^B-\hat{\beta}_2^B)k^B-[2(1-\hat{\alpha}_1^B)-\hat{\beta}_2^B]g^B\right] \\ & + [2(1-\hat{\alpha}_1^B)-\hat{\beta}_2^B]\alpha_2^B\beta_1^B(k^B-g^B)+2(1-\hat{\alpha}_1^B-\hat{\alpha}_2^B)\hat{\beta}_1^B\alpha_1^B\beta_2^Bg^B\Big\}\end{aligned}$$

Consider the expressions of $\bar{\mathcal{P}}_o(0)$ and $\bar{\mathcal{P}}_o(-1)$ along the autarky steady state with $\mathcal{E}_y^A = \mathcal{E}_y^B = \mathcal{E}_c^B = 1$, $\mathcal{E}_c^A = \bar{\mathcal{E}}_c^A$, $g^i = \rho\beta_1^iy^i$ and $k^i = y^i = \bar{k}^i$. Taking (38) and (39) into account, we find after simplifications

$$
\begin{aligned}
\bar{\mathcal{P}}_o(0) &= \rho(1-\hat{\alpha}_1^B-\hat{\alpha}_2^B)\beta_1^A\beta_1^B\beta_2^B\hat{\beta}_2^B\alpha_1^B\bar{k}^A\frac{\mathcal{P}_c^A(0)}{\mathcal{A}^A} \\
&+ \rho(1-\hat{\alpha}_1^A-\hat{\alpha}_2^A)\beta_1^A\beta_1^B\beta_2^A\hat{\beta}_2^A\alpha_1^A\bar{k}^B\frac{\mathcal{P}_c^B(0)}{\mathcal{A}^B} \\
\bar{\mathcal{P}}_o(-1) &= \rho(1-\hat{\alpha}_1^B-\hat{\alpha}_2^B)\beta_1^A\beta_1^B\beta_2^B\hat{\beta}_2^B\alpha_1^B\bar{k}^A\frac{\mathcal{P}_c^A(-1)}{\mathcal{A}^A} \\
&+ \rho(1-\hat{\alpha}_1^A-\hat{\alpha}_2^A)\beta_1^A\beta_1^B\beta_2^A\hat{\beta}_2^A\alpha_1^A\bar{k}^B\frac{\mathcal{P}_c^B(-1)}{\mathcal{A}^B}
\end{aligned}
$$

We also get from Lemma 6.6

$$
\begin{aligned}
\bar{\mathcal{V}}_{12}(\bar{k},\bar{k}) &= \rho\beta_1^A\beta_1^B\Big\{(1-\hat{\alpha}_1^B-\hat{\alpha}_2^B)\beta_2^B\hat{\beta}_2^B\alpha_1^B\bar{k}^A\Big[(1-\hat{\alpha}_1^A-\hat{\alpha}_2^A)\alpha_1^A\beta_2^A \\
&- (\alpha_1^A\beta_2^A-\alpha_2^A\beta_1^A)(1-\hat{\alpha}_1^A)(1-\rho\beta_1^A)\Big] \\
&+ (1-\hat{\alpha}_1^A-\hat{\alpha}_2^A)\beta_2^A\hat{\beta}_2^A\alpha_1^A\bar{k}^B\Big[(1-\hat{\alpha}_1^B-\hat{\alpha}_1^B)\alpha_1^B\beta_2^B \\
&- (\alpha_1^B\beta_2^B-\alpha_2^B\beta_1^B)(1-\hat{\alpha}_1^B)(1-\rho\beta_1^B)\Big]\Big\}
\end{aligned}
$$

The results follow from the same arguments as in the proof of Proposition 2. □

6.8 Proof of Proposition 7

Consider the free-trade distribution $k = k^A + k^B$ as defined by Proposition 5 with with $\mathcal{E}_c^B = \mathcal{E}_y^B = 1$, $\mathcal{E}_c^A = \mathcal{E}_c^{A*}$, $\mathcal{E}_y^A = \mathcal{E}_y^{A*}$ and $\theta \in (1, 1/\rho\beta_1)$. It follows that $g^A = \rho\beta_1^A k^B$, $k^A - g^A = k^B(\theta - \rho\beta_1^A)$, $g^B = \rho\beta_1^B\theta k^B$, $k^B - g^B = k^B(1-\rho\beta_1^B\theta)$. Using Lemma 6.6, Straightforward computations give after simplifications:

$$
\begin{aligned}
\bar{\mathcal{V}}_{12}(k,k) &= \rho\beta_1^A\beta_1^B k^B\Big\{(1-\hat{\alpha}_1^B-\hat{\alpha}_2^B)\beta_2^B\hat{\beta}_2^B\alpha_1^B\Big[(1-\hat{\alpha}_1^A-\hat{\alpha}_2^A)\alpha_1^A\beta_2^A\theta \\
&- (\alpha_1^A\beta_2^A-\alpha_2^A\beta_1^A)(1-\hat{\alpha}_1^A)(\theta-\rho\beta_1^A)\Big] \\
&+ (1-\hat{\alpha}_1^A-\hat{\alpha}_2^A)\beta_2^A\hat{\beta}_2^A\alpha_1^A\Big[(1-\hat{\alpha}_1^B-\hat{\alpha}_1^B)\alpha_1^B\beta_2^B \\
&- (\alpha_1^B\beta_2^B-\alpha_2^B\beta_1^B)(1-\hat{\alpha}_1^B)(1-\rho\beta_1^B\theta)\Big]\Big\} \\
\bar{\mathcal{P}}_o(0) &= \rho\beta_1^A\beta_1^B k^B\Big\{(1-\hat{\alpha}_1^B-\hat{\alpha}_2^B)\beta_2^B\hat{\beta}_2^B\alpha_1^B\Big[(\hat{\beta}_1^A-\hat{\alpha}_1^A)\alpha_2^A(\theta-\rho\beta_1^A) \\
&+ (1-\hat{\alpha}_1^A-\hat{\alpha}_2^A)\alpha_1^A\beta_2^A\hat{\beta}_1^A\rho\Big] \\
&+ (1-\hat{\alpha}_1^A-\hat{\alpha}_2^A)\beta_2^A\hat{\beta}_2^A\alpha_1^A\Big[(\hat{\beta}_1^B-\hat{\alpha}_1^B)\alpha_2^B(1-\rho\beta_1^B\theta) \\
&+ (1-\hat{\alpha}_1^B-\hat{\alpha}_1^B)\alpha_1^B\beta_2^B\hat{\beta}_1^B\rho\theta\Big]
\end{aligned}
$$

$$\begin{aligned}\bar{\mathcal{P}}_o(-1) \; = \; & \rho(1-\hat{\alpha}_1^B-\hat{\alpha}_2^B)\beta_1^B\beta_2^B\hat{\beta}_2^B\alpha_1^B\beta_1^A k^A\Big\{2\rho(1-\hat{\alpha}_1^A-\hat{\alpha}_2^A)\alpha_1^A\beta_2^A\theta(1+\hat{\beta}_1^A) \\ & + [2(1-\hat{\alpha}_1^A)-\hat{\beta}_2^A](\theta-\rho\beta_1^A)\left[\alpha_2^A-\rho(\alpha_1^A\beta_2^A-\alpha_2^A\beta_1^A)\right]\Big\} \\ & + \rho(1-\hat{\alpha}_1^A-\hat{\alpha}_2^A)\beta_1^A\beta_2^A\hat{\beta}_2^A\alpha_1^A\beta_1^B k^B\Big\{2\rho(1-\hat{\alpha}_1^B-\hat{\alpha}_2^B)\alpha_1^B\beta_2^B(1+\theta\hat{\beta}_1^B) \\ & + [2(1-\hat{\alpha}_1^B)-\hat{\beta}_2^B](1-\rho\beta_1^B\theta)\left[\alpha_2^B-\rho(\alpha_1^B\beta_2^B-\alpha_2^B\beta_1^B)\right]\Big\}\end{aligned}$$

From Lemma 6.7 we derive that a necessary condition for the occurrence of local indeterminacy is

$$\lim_{\lambda\to\pm\infty}\bar{\mathcal{P}}_o(\lambda) = -\infty \quad \Leftrightarrow \quad \bar{\mathcal{V}}_{12}(k,k) < 0$$

This property is satisfied for any $\rho \in (0,1)$ if

$$\alpha_1^A\beta_2^A - \alpha_2^A\beta_1^A > \tfrac{1-\hat{\alpha}_1^A-\hat{\alpha}_2^A}{1-\hat{\alpha}_1^A}\tfrac{\alpha_1^A\beta_2^A\theta}{\theta-\beta_1^A} \equiv \mathcal{Z}_5^A \tag{59}$$

and

$$\alpha_1^B\beta_2^B - \alpha_2^B\beta_1^B > \tfrac{1-\hat{\alpha}_1^B-\hat{\alpha}_2^B}{1-\hat{\alpha}_1^B}\tfrac{\alpha_1^B\beta_2^B}{1-\theta\beta_1^B} \equiv \mathcal{Z}_5^B \tag{60}$$

Under these conditions, local indeterminacy may arise in two types of configurations:

i) when $\bar{\mathcal{P}}_o(0) > 0$ and $\bar{\mathcal{P}}_o(-1) < 0$. In this case the product of characteristic roots $\mathcal{D}_o = \bar{\mathcal{V}}_{21}(k,k)/\rho\bar{\mathcal{V}}_{12}(k,k)$ is negative. We immediately derive that $\bar{\mathcal{P}}_o(0) > 0$ for any $\rho \in (0,1)$ if

$$\hat{\beta}_1^A - \hat{\alpha}_1^A > 0 \text{ and } \hat{\beta}_1^B - \hat{\alpha}_1^B > 0$$

Notice then that $\hat{\beta}_1^i - \hat{\alpha}_1^i > 0$ implies $2(1-\hat{\alpha}_1^i) - \hat{\beta}_2^i > 0$. Hence, there exists $\bar{\rho} \in (0,1)$ such that $\bar{\mathcal{P}}_o(-1) < 0$ for any $\rho \in (\bar{\rho},1)$ if

$$\alpha_1^A\beta_2^A - \alpha_2^A\beta_1^A > \alpha_2^A + \tfrac{2(1-\hat{\alpha}_1^A-\hat{\alpha}_2^A)\alpha_1^A\beta_2^A\theta(1+\hat{\beta}_1^A)}{[2(1-\hat{\alpha}_1^A)-\hat{\beta}_2^A](\theta-\beta_1^A)} \equiv \mathcal{Z}_6^A \tag{61}$$

and

$$\alpha_1^B\beta_2^B - \alpha_2^B\beta_1^B > \alpha_2^B + \tfrac{2(1-\hat{\alpha}_1^B-\hat{\alpha}_2^B)\alpha_1^B\beta_2^B(1+\theta\hat{\beta}_1^B)}{[2(1-\hat{\alpha}_1^B)-\hat{\beta}_2^B](1-\theta\beta_1^B)} \equiv \mathcal{Z}_6^B \tag{62}$$

The rest of the proof is completed by noticing from (59)-(62) that $\mathcal{Z}_6^A > \mathcal{Z}_5^A$ and $\mathcal{Z}_6^B > \mathcal{Z}_5^B$.

ii) when $\bar{\mathcal{P}}_o(0) < 0$, $\bar{\mathcal{P}}_o(-1) < 0$ and $\mathcal{D}_o \in (0,1)$. Under (59)-(60), we know indeed that $\mathcal{P}_o(0) < 0$ implies $\mathcal{D}_o > 0$. We first derive that $\bar{\mathcal{P}}_o(0) < 0$ for any $\rho \in (0,1)$ if:

$$\hat{\alpha}_1^A - \hat{\beta}_1^A > \frac{(1-\hat{\alpha}_1^A-\hat{\alpha}_2^A)\hat{\beta}_1^A\alpha_1^A\beta_2^A}{\alpha_2^A(\theta-\beta_1^A)} \equiv \mathcal{Z}_7^A \tag{63}$$

and

$$\hat{\alpha}_1^B - \hat{\beta}_1^B > \frac{(1-\hat{\alpha}_1^B-\hat{\alpha}_2^B)\hat{\beta}_1^B\alpha_1^B\beta_2^B\theta}{\alpha_2^B(1-\theta\beta_1^B)} \equiv \mathcal{Z}_7^B \tag{64}$$

Notice then that $2(1-\hat{\alpha}_1^i) - \hat{\beta}_2^i = 1 - \hat{\alpha}_1^i - (\hat{\alpha}_1^i - \hat{\beta}_1^i)$. It follows that if (63)-(64) hold with $1-\hat{\alpha}_1^A > \hat{\alpha}_1^A - \hat{\beta}_1^A$ and $1-\hat{\alpha}_1^B > \hat{\alpha}_1^B - \hat{\beta}_1^B$, then (61)-(62) are satisfied and under (59)-(60), there exists $\bar{\rho} \in (0,1)$ such that $\bar{\mathcal{P}}_o(-1) < 0$ for any $\rho \in (\bar{\rho}, 1)$. Finally, when $\rho = 1$, we derive that $\mathcal{D}_o < 1$ if

$$\begin{aligned}
& (1-\hat{\alpha}_1^B-\hat{\alpha}_2^B)\beta_2^B\hat{\beta}_2^B\alpha_1^B\Big[(\alpha_1^A\beta_2^A - \alpha_2^A\beta_1^A)(1-\hat{\alpha}_1^A)(\theta-\beta_1^A) \\
+ \; & (\hat{\beta}_1^A - \hat{\alpha}_1^A)\alpha_2^A(\theta-\beta_1^A) + (1-\hat{\alpha}_1^A-\hat{\alpha}_2^A)\alpha_1^A\beta_2^A(\hat{\beta}_1^A-\theta)\Big] \\
+ \; & (1-\hat{\alpha}_1^A-\hat{\alpha}_2^A)\beta_2^A\hat{\beta}_2^A\alpha_1^A\Big[(\alpha_1^B\beta_2^B - \alpha_2^B\beta_1^B)(1-\hat{\alpha}_1^B)(1-\beta_1^B\theta) \\
+ \; & (\hat{\beta}_1^B - \hat{\alpha}_1^B)\alpha_2^B(1-\beta_1^B\theta) + (1-\hat{\alpha}_1^B-\hat{\alpha}_1^B)\alpha_1^B\beta_2^B(\hat{\beta}_1^B\theta-1)\Big] > 0
\end{aligned}$$

This inequality holds if

$$\alpha_1^A\beta_2^A - \alpha_2^A\beta_1^A > \frac{(\hat{\alpha}_1^A-\hat{\beta}_1^A)\alpha_2^A(\theta-\beta_1^A)+(1-\hat{\alpha}_1^A-\hat{\alpha}_2^A)\alpha_1^A\beta_2^A(\theta-\hat{\beta}_1^A)}{(1-\hat{\alpha}_1^A)(\theta-\beta_1^A)} \equiv \mathcal{Z}_8^A \tag{65}$$

and

$$\alpha_1^B\beta_2^B - \alpha_2^B\beta_1^B > \frac{(\hat{\alpha}_1^B-\hat{\beta}_1^B)\alpha_2^B(1-\theta\beta_1^B)+(1-\hat{\alpha}_1^B-\hat{\alpha}_2^B)\alpha_1^B\beta_2^B(1-\theta\hat{\beta}_1^B)}{(1-\hat{\alpha}_1^B)(1-\theta\beta_1^B)} \equiv \mathcal{Z}_8^B \tag{66}$$

Under (65)-(66), there exists $\bar{\rho} \in (0,1)$ such that $\mathcal{D}_o < 1$ for any $\rho \in (\bar{\rho}, 1)$. The rest of the proof follows from the fact that if (63)-(64) hold with $1-\hat{\alpha}_1^A > \hat{\alpha}_1^A - \hat{\beta}_1^A$ and $1-\hat{\alpha}_1^B > \hat{\alpha}_1^B - \hat{\beta}_1^B$, then $\mathcal{Z}_6^A > \mathcal{Z}_8^A$ and $\mathcal{Z}_6^B > \mathcal{Z}_8^B$. □

6.9 Proof of Corollary 2

In the proof of Proposition 2, assume that inequality (42) applied to country A is not satisfied, i.e.

$$\alpha_1^A\beta_2^A - \alpha_2^A\beta_1^A < \alpha_2^A + \frac{2(1-\hat{\alpha}_1^A-\hat{\alpha}_2^A)\alpha_1^A\beta_2^A(1+\hat{\beta}_1^A)}{[2(1-\hat{\alpha}_1^A)-\hat{\beta}_2^A](1-\beta_1^A)} \equiv \mathcal{Z}_2^A$$

It follows that the autarky steady state of country A is saddle-point stable since $\mathcal{P}_c^A(-1) > 0$ and $\mathcal{P}_c^A(1) < 0$. Consider then condition (62) in the proof of Proposition 7. Straightforward computations give $\mathcal{Z}_2^A > \mathcal{Z}_6^A$. It follows that all the conditions of Proposition 7 for country A may be satisfied for the free-trade steady state while the steady state under autarky is locally determinate. At the same time, we get for country B that $\mathcal{Z}_5^B > \mathcal{Z}_1^B$,

$\mathcal{Z}_6^B > \mathcal{Z}_2^B$, $\mathcal{Z}_7^B > \mathcal{Z}_3^B$ and $\mathcal{Z}_8^B < \mathcal{Z}_4^B$. Since, as shown in the proof of Proposition 2, $\mathcal{Z}_2^B > \mathcal{Z}_4^B$, it follows that if all the conditions of Proposition 7 for country B are satisfied along the free-trade steady state then the steady state under autarky is also locally indeterminate. As a result local indeterminacy arises at the world level while country A is locally determinate under autarky. □

6.10 Proof of Proposition 8

Assume that $\hat{\alpha}_1^A + \hat{\alpha}_2^A = 1$. We get

$$\begin{aligned}
\bar{\mathcal{V}}_{12}(k,k) &= -\rho\beta_1^A\beta_1^B\beta_2^B\hat{\beta}_2^B\alpha_1^B\hat{\alpha}_2^A(1-\hat{\alpha}_1^B-\hat{\alpha}_2^B)(\alpha_1^A\beta_2^A-\alpha_2^A\beta_1^A)(k^A-g^A)\\
\bar{\mathcal{P}}_o(0) &= \rho\beta_1^B\beta_2^B\hat{\beta}_2^B\alpha_1^B\alpha_2^A\beta_1^A(1-\hat{\alpha}_1^B-\hat{\alpha}_2^B)(\hat{\beta}_1^A-\hat{\alpha}_1^A)(k^A-g^A)\\
\bar{\mathcal{P}}_o(-1) &= -\rho\beta_1^A\beta_1^B\beta_2^B\hat{\beta}_2^B\alpha_1^B(1-\hat{\alpha}_1^B-\hat{\alpha}_2^B)(2\hat{\alpha}_2^A-\hat{\beta}_2^A)(k^A-g^A)\\
&\quad\times\left[\rho(\alpha_1^A\beta_2^A-\alpha_2^A\beta_1^A)-\alpha_2^A\right]\\
\mathcal{D}_o &= \frac{\bar{\mathcal{V}}_{21}(k,k)}{\rho\bar{\mathcal{V}}_{12}(k,k)} = \frac{(\hat{\alpha}_1^A-\hat{\beta}_1^A)\alpha_2^A}{\rho\hat{\alpha}_2^A(\alpha_1^A\beta_2^A-\alpha_2^A\beta_1^A)}
\end{aligned}$$

As shown in the proof of Lemma 6.7, along the autarky or free-trade distribution, as defined by Propositions 4 and 5, the differences $k^i - g^i$, $i = A, B$, are always positive. Since $\bar{\mathcal{P}}_o(1) < 0$, a necessary condition for the occurrence of local indeterminacy is

$$\lim_{\lambda\to\pm\infty}\bar{\mathcal{P}}_o(\lambda) = -\infty \quad\Leftrightarrow\quad \bar{\mathcal{V}}_{12}(k,k) < 0$$

This property is satisfied for any $\rho \in (0,1)$ if and only if the consumption good of country A is capital intensive at the private level. Local indeterminacy then arises in two types of configurations:

i) when $\bar{\mathcal{P}}_o(0) > 0$ and $\bar{\mathcal{P}}_o(-1) < 0$. In this case the product of characteristic roots $\mathcal{D}_o$ is negative. If the consumption good of country A is labor intensive at the social level, i.e. $\hat{\beta}_1^A - \hat{\alpha}_1^A > 0$, then $\bar{\mathcal{P}}_o(0) > 0$ for any $\rho \in (0,1)$. Notice also that $\hat{\beta}_1^A - \hat{\alpha}_1^A > 0$ implies $2\hat{\alpha}_2^A - \hat{\beta}_2^A > 0$. Consider finally $\bar{\mathcal{P}}_o(-1)$. When ρ is close to zero $\bar{\mathcal{P}}_o(-1) > 0$ but when ρ is close to 1, $\bar{\mathcal{P}}_o(-1) < 0$ if and only if

$$\alpha_1^A\beta_2^A - \alpha_2^A\beta_1^A > \alpha_2^A \qquad (67)$$

ii) when $\bar{\mathcal{P}}_o(0) < 0$, $\bar{\mathcal{P}}_c(-1) < 0$ and $\mathcal{D}_c \in (0,1)$. We immediately get that $\bar{\mathcal{P}}_o(0) < 0$ if and only if the consumption good is capital intensive at the social level, i.e. $\hat{\alpha}_1^A - \hat{\beta}_1^A > 0$. Moreover under (67), $\bar{\mathcal{P}}_o(-1) < 0$ if $2\hat{\alpha}_2^A - \hat{\beta}_2^A > 0$ or equivalently

$$0 < \hat{\alpha}_1^A - \hat{\beta}_1^A < \hat{\alpha}_2^A \tag{68}$$

Therefore, under (67) and (68), there exists $\rho^* \in (0,1)$ such that $\bar{\mathcal{P}}_o(-1) < 0$ and $\mathcal{D}_o \in (0,1)$ for any $\rho \in (\rho^*, 1)$.

Now denote $\epsilon_c^i = 1 - \alpha_1^i - \alpha_2^i$. By continuity, there exists $\epsilon > 0$ such that all these results are preserved for any $\epsilon_c^A \in [0, \epsilon)$. □

6.11 Proof of Proposition 9

Consider the autarky steady state (20) with $\mathcal{E}_c^B = \mathcal{E}_y^B = 1$, $\mathcal{E}_c^{A*}$, $\mathcal{E}_y^{A*}$ as given by (54), (57) and η as defined by (58). Using (28)-(31), the corresponding amount of stationary consumption is

$$\bar{c}^A = \left(\frac{(1-\rho\beta_1^A)\Phi_\theta^A}{(\theta-\rho\beta_1^A)\Phi_1^A}\right)^{\hat{\alpha}_1^A+\hat{\alpha}_2^A} \eta T^{B*}, \quad \bar{c}^B = \left(\frac{(1-\rho\beta_1^B)\Phi_\theta^B}{(\theta-\rho\beta_1^B)\Phi_1^B}\right)^{\hat{\alpha}_1^B+\hat{\alpha}_2^B} T^{B*}$$

We then derive from Corollary 1:

$$\begin{aligned}
\bar{c}^A - c^{A*} &= T^{B*}\left[\left(\frac{(1-\rho\beta_1^A)\Phi_\theta^A}{(\theta-\rho\beta_1^A)\Phi_1^A}\right)^{\hat{\alpha}_1^A+\hat{\alpha}_2^A}\eta - 1\right] \\
&= \frac{T^{B*}}{1-\rho\beta_1^B\theta}\left[\left(\frac{(1-\rho\beta_1^A)\Phi_\theta^A}{(\theta-\rho\beta_1^A)\Phi_1^A}\right)^{\hat{\alpha}_1^A+\hat{\alpha}_2^A}\frac{\alpha_1^B(\theta-\rho\beta_1^A)}{\alpha_1^A} - (1-\rho\beta_1^B\theta)\right] \\
\bar{c}^B - c^{B*} &= T^{B*}\left[\left(\frac{(1-\rho\beta_1^B)\Phi_\theta^B}{(1-\rho\beta_1^B\theta)\Phi_1^B}\right)^{\hat{\alpha}_1^B+\hat{\alpha}_2^B} - \eta\right] \\
&= \frac{T^{B*}}{(1-\rho\beta_1^B\theta)^{\hat{\alpha}_1^B+\hat{\alpha}_2^B}}\left[\left(\frac{(1-\rho\beta_1^B)\Phi_\theta^B}{\Phi_1^B}\right)^{\hat{\alpha}_1^B+\hat{\alpha}_2^B} - \frac{\alpha_1^B(\theta-\rho\beta_1^A)}{\alpha_1^A(1-\rho\beta_1^B\theta)^{1-\hat{\alpha}_1^B-\hat{\alpha}_2^B}}\right]
\end{aligned}$$

with

$$\begin{aligned}
\frac{T^{B*}}{1-\rho\beta_1^B\theta} &= \frac{(\alpha_1^B\beta_2^B)^{\hat{\alpha}_1^A}(\alpha_2^B\beta_1^B)^{\hat{\alpha}_2^B}(\rho\beta_1^B)^{\hat{\alpha}_1^B/\hat{\beta}_2^B}(1-\rho\beta_1^B\theta)^{\hat{\alpha}_1^B+\hat{\alpha}_2^B-1}}{(\Phi_\theta^B)^{\hat{\alpha}_1^B+\hat{\alpha}_2^B}} \\
\frac{T^{B*}}{(1-\rho\beta_1^B\theta)^{\hat{\alpha}_1^B+\hat{\alpha}_2^B}} &= \frac{(\alpha_1^B\beta_2^B)^{\hat{\alpha}_1^A}(\alpha_2^B\beta_1^B)^{\hat{\alpha}_2^B}(\rho\beta_1^B)^{\hat{\alpha}_1^B/\hat{\beta}_2^B}}{(\Phi_\theta^B)^{\hat{\alpha}_1^B+\hat{\alpha}_2^B}}
\end{aligned}$$

Moreover, denoting by $\bar{\mathcal{W}} \equiv \bar{c}^A + \bar{c}^B$ and $\mathcal{W}^* \equiv c^{A*} + c^{B*}$ respectively the welfare at the world level under autarky and under free-trade, we get using (58):

$$\bar{\mathcal{W}} - \mathcal{W}^* = \frac{T^{B*}}{(1-\rho\beta_1^B\theta)^{\hat{\alpha}_1^B+\hat{\alpha}_2^B}} \left[\left(\frac{(1-\rho\beta_1^B)\Phi_\theta^B}{\Phi_1^B} \right)^{\hat{\alpha}_1^B+\hat{\alpha}_2^B} - (1-\rho\beta_1^B\theta)^{\hat{\alpha}_1^B+\hat{\alpha}_2^B} \right]$$
$$\times \left[1 - \frac{\alpha_1^B(\theta-\rho\beta_1^A)}{\alpha_1^A(1-\rho\beta_1^B\theta)^{1-\hat{\alpha}_1^B-\hat{\alpha}_2^B}} \frac{1-\left(\frac{(1-\rho\beta_1^A)\Phi_\theta^A}{(\theta-\rho\beta_1^A)\Phi_1^A}\right)^{\hat{\alpha}_1^A+\hat{\alpha}_2^A}}{\left(\frac{(1-\rho\beta_1^B)\Phi_\theta^B}{\Phi_1^B}\right)^{\hat{\alpha}_1^B+\hat{\alpha}_2^B} - (1-\rho\beta_1^B\theta)^{\hat{\alpha}_1^B+\hat{\alpha}_2^B}} \right]$$

Notice then that since $\theta > 1$, we get

$$\frac{(1-\rho\beta_1^A)\Phi_\theta^A}{(\theta-\rho\beta_1^A)\Phi_1^A} < 1, \quad \frac{(1-\rho\beta_1^B)\Phi_\theta^B}{(1-\rho\beta_1^B\theta)\Phi_1^B} > 1$$

We derive from all this that

$$\lim_{\theta\to 1/\rho\beta_1^B} \bar{c}^A - c^{A*} = +\infty, \quad \lim_{\theta\to 1/\rho\beta_1^B} \bar{c}^B - c^{B*} = -\infty, \quad \lim_{\theta\to 1/\rho\beta_1^B} \bar{\mathcal{W}} - \mathcal{W}^* = -\infty$$

The result follows. □

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