A study of Approval voting on Large Poisson Games*

Matías Núñez †

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Abstract

Approval voting features are analysed in a context of large elections with strategic voters: Myerson's Large Poisson Games. I first establish the Magnitude Equivalence Theorem (MET) which substantially reduces the complexity of computing the magnitudes of pivotal events. I also show that the Winner of the election coincides with the Profile Condorcet Winner at equilibrium when preferences are restricted to be single-peaked. This is a positive result that strengthens the positive conclusions some scholars have previously drawn over this voting rule. I finally show that, without the previous restriction over preferences, both concepts do not generally coincide anymore.

KEYWORDS: Approval voting, Poisson Games, magnitude computation, Condorcet Winner.

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1 Introduction

The economic analysis of voting rules has given some insight into the understanding of their properties. However, one can assert that these analyses are "too rich" in the sense that they show that a plethora of equilibria can arise under most voting rules. In particular, there is a controversy over approval voting or AV, a voting rule which has been called "the electoral reform of the twentieth century". This voting rule allows the voter to vote for as many candidates as she wishes and the candidate who gets the most votes wins the election. Its detractors claim that this kind of method enhances strategic voting when compared for instance to majority voting, whereas its proponents consider that it has several advantages as far as strategic voting is concerned. For an extensive discussion on this controversy over AV, the reader can refer to Brams (2008), Brams and Fishburn (1983, 2007), and Weber (1995).

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†Département d' Economie, Ecole Polytechnique, 1 Rue Descartes 75005, Paris, France. E-mail address: matias.nunez@shs.polytechnique.fr

One important feature of AV was characterized by Brams and Fishburn (1981). They show that if a Condorcet Winner exists then the AV game has a Nash equilibrium in undominated strategies that selects the Condorcet Winner. The Condorcet Winner - the candidate who beats all other candidates on pairwise contests - has often been considered to be a good equilibrium solution in voting games. The robustness of this result has been weakened by De Sinopoli, Dutta and Laslier (2006) who show that, in a setting with a fixed number of voters, AV can imply that the Condorcet Winner gets no vote. Therefore, AV does not guarantee what is called Condorcet consistency: the Winner of the election does not always coincide with the Condorcet Winner.

However, the previous works were performed in a basic game theoretical framework. Such a framework faces some criticisms when dealing with elections with a large number of voters. Indeed, it is no longer realistic to assume that voters have no prior beliefs over the expected scores of the candidates. The existence of non-serious candidates[1] might affect voters' behaviour as a voter might not vote for a candidate she prefers if she considers he has no chance of winning the election. Thus, the introduction of priors over the expected scores of the candidates is particularly relevant in a Large election context. Besides, it substantially reduces the set of equilibria that can arise under every voting rule allowing us to compare them through equilibrium outcomes. To my knowledge, there has one main approach to tackle this sort of uncertainty: Myerson (1998)'s Population Uncertainty model ([10]).

In the present work, I show that Condorcet Consistency of approval voting does not hold in general in the Myerson's Population uncertainty framework. This framework, also known as Large Poisson Games ([10]), introduces an uncertainty over the total number of voters in the election. Indeed, it assumes that the total number of voters in the game is not constant and is drawn from a Poisson distribution of a given parameter n, the expected size of the population. Voters are assumed to be instrumentally motivated[2] and consequently, their utility depends only on the candidate who wins the election. This implies that they only care about the influence their ballot can have in pivotally changing the result of the election. Thus, it seems logical that the computation of pivotal probabilities is essential to set up an analysis of voters' beliefs over the possible serious candidates in the election and then to determine the equilibria of the game.

I first provide a result which substantially simplifies these computations: the Magnitude Equivalence Theorem (MET). This result theorem is helpful to set up an ordering among the different probabilities of the pivotal events. Using the MET, I show that when preferences are restricted to be single-peaked, AV ensures Condorcet Consistency and sincere behaviour of the electors for *any* number of candidates. The sincerity of a voter facing approval voting can be defined as follows: if the voter sincerely approves of a candidate

[1] A non-serious candidate is a candidate "who is considered unlikely to be in contention for victory", as defined by Myerson and Weber (1993).

[2] In the literature, some authors have argued that since each single ballot has a negligible weight on the outcome of the election when turnout is large, assuming that voters are instrumental is misleading. However, [4] shows that, counter to the usual perceptions, assuming instrumental voting can reproduce several stylized facts. His work concludes that instrumental voting is not the main cause of the failure of "standard" models in explaining empirical regularities in elections .

x she also approves of any candidate she prefers to x. In Large Poisson Games, given the prior beliefs over the candidates who are the front-runner and the main challenger in the election, the optimal strategy is the following one. First, if the voter prefers the main challenger to the front-runner, she votes for every candidate she prefers to the former. Otherwise, she votes for the front-runner and for every candidate she prefers to him.

Finally, I also prove through a simple voting situation that Condorcet Consistency is not generically satisfied. In this example, the presence of a small group of voters makes the majority of the population behave in a non-conventional way. As will be shown, this is a consequence of the correlation between the scores of the candidates that arise in the Large Poisson Games. Furthermore, a discussion of the stability of the given equilibrium is provided.

This paper is structured as follows. Section 2 introduces the basic model. Section 3 presents the Magnitude Equivalence Theorem *(MET)* and Section 4 introduces the positive result concerning AV when preferences are single-peaked. Finally, Section 5 discusses in detail the situation where the Profile Condorcet Winner does not coincide with the Expected Winner of the election and Section 6 concludes.

2 The basic setting

Given is $K = \{a, b, \ldots, k, \ldots\}$, the finite set of candidates running for the election. The aim of the election is to choose one candidate. The voters are asked to express their preferences among candidates through ballots. Unless otherwise stated, the voting rule is approval voting: every voter can vote for as many candidates as she wishes. The votes are summed and the winning candidate has the most points. Ties are resolved by a fair lottery.

By assumption, the number of voters is a random variable drawn from a Poisson distribution of parameter n, the expected size of the population. Each voter has a type t in the set of types T that determines her preferences over the candidates. A voter's payoff only depends on her type and on the candidate who is elected. Each candidate k is assumed to have some exogenous position in the policy space represented by the real line. The preferences of a voter with a type t is denoted by $u_t = (u_t(k))_{k \in K}$. Thus, for a given t, $u_t(k) > u_t(k')$ implies that t-type voters strictly prefer candidate k to candidate k'. No voter is indifferent among two or more candidates.

Given the preferences of a type-t voter, the sincere ballot can be defined as follows, as stated by Brams (2008).

Definition 2.1 (Sincerity) *An AV ballot is sincere if, given the lowest-preferred candidate x that a voter approves of, she also approves of all candidates she prefers to x.*

The main result of this work is stated under the assumption that preferences are restricted to be single-peaked. That is to say, for each voter with a type $t \in T$, u_t satisfies the following condition:

$$\begin{cases} \text{There exists } t^* \in [0,1] \text{ such that for all } y, z \in [0,1] \\ t^* < y < z \Longrightarrow u_t(x^*) > u_t(y) > u_t(z) \\ z < y < t^* \Longrightarrow u_t(x^*) > u_t(y) > u_t(z) \end{cases}$$

We call t^* the peak of the preference ordering u_t for every $t \in T$.

Each voter's type is independently drawn from T according to the *expected distribution of types*[3] denoted by $r = (r(t))_{t\in T}$. This implies that $r(t)$ represents the probability that a voter randomly drawn from the population has type t.

The Profile Condorcet Winner (*P.C.W.*) of the election depends solely on the expected distribution of types. Indeed, the *P.C.W.* is the candidate who beats all other candidates in pairwise contests and is defined as:

Definition 2.2 *A candidate k is the Profile Condorcet Winner of the election (P.C.W.) of the election if*

$$\sum_{t\in T_{k,j}} r(t) > 1/2 \ \forall\, j \in K, j \neq k$$

where $T_{k,j}$ is the set of preference types where candidate k is preferred to candidate j.

Each voter i must choose a ballot c from a finite set of possible ballots, denoted by C. An AV ballot simply consists of a subset of the set of candidates. Voters are allowed to use mixed strategies. These strategies are determined by the *strategy function*[4] $\sigma(c|\, t)$ which is a function from T into the set of probability distributions over C ([12]). That is, $\sigma(c|\, t)$ determines the probability that a given voter of type t chooses ballot c. Therefore, all the voters with the same type choose the same strategy. Then, taking into account the expected distribution of voters and the strategy function, the *expected vote distribution* $\tau = (\tau(c))_{c\in C}$ can be determined. Formally,

$$\tau(c) = \sum_{t\in T} r(t)\sigma(c|t)$$

As a consequence of the Poisson-Myerson framework, the number of voters in the game who choose ballot c is drawn from a Poisson random variable with parameter $n\tau(c)$. The *ballot profile*, $x = (x_c)_{c\in C}$, is the vector that gives the number of voters who are choosing ballot c, for every $c \in C$. In particular, the random variable $x_{a,b}$ represents the number of voters who are expected to vote jointly for candidates a and b. The set of available ballots where candidate k is approved is denoted by $\mathcal{C}_k$.

The score $s(k)$ of candidate k represents the total number of votes this candidate gets and, formally, is defined by

$$s(k) = \sum_{c\in\mathcal{C}_k} x_c \sim \mathcal{P}(n \sum_{c\in\mathcal{C}_k} \tau(c))$$

Once we have defined the score of a candidate in a Poisson game, a natural concept that arises is the $Winner$ of the election: it is just the candidate with the highest expected score. Formally,

[3]The expected distribution of types satisfies $r(t) > 0 \ \forall\, t \in T$ and $\sum_{t\in T} r(t) = 1$.

[4]The strategy function satisfies $\sigma(c|\, t) \geq 0 \ \forall\, c \in C$ and $\sum_{d\in C} \sigma(c|t) = 1$.

Definition 2.3 *A candidate k is the Winner of the election if*

$$\sum_{c \in \mathcal{C}_k} \tau(c) > \sum_{c \in \mathcal{C}_j} \tau(c) \;\; \forall j \in K, j \neq k$$

The parameters (T, n, r, C, U) define a Poisson game.

Taking into account the *independent-actions* property (in Myerson's notation, an action is equivalent to a ballot), for any ballot profile x, the probability that x is the voters' ballot profile in the game is

$$P[x|\, n\tau] = \prod_{c\,\in\, C} \left(\frac{e^{-n\tau(c)}(n\tau(c))^{x_c}}{x_c!} \right)$$

Strategic behaviour of the voters.

Defining a pivotal event. I assume that each voter determines which ballot she casts by maximizing her expected utility. As voters are instrumentally motivated, they care only about the influence of their own vote in determining the Winner's identity. Thus, a voter needs to estimate the probability that any given set of candidates has of being in a close race for first place where one ballot could pivotally change the result of the election: a *pivot*.

Definition 2.4 *For each subset $Y = \{a, b, \ldots, y\} \subset K$ of candidates, a pivot(Y) is denoted by:*

$$\forall\, y \in Y, \;\; s(y) \geq \max_{k \in K} s(k) - 1$$
$$\forall\, k \notin Y, \;\; s(k) < \max_{k \in K} s(k) - 1$$

The probability of such a sequence of pivotal events, $P[pivot(Y)|\, n\tau]$, is small if n is large, but some sequences are nevertheless much less likely than others. The probability of any pivotal event generally tends to zero as the expected population n becomes large, but the serious races can be identified by comparing the rates at which their probabilities tend to zero. These rates can be measured by a concept of magnitude, defined as follows: given an expected vote distribution $\tau = (\tau(c))_{c \in C}$, the magnitude $\mu(M)$ of a sequence of events $(M_n)_{n \in \mathbb{N}}$ is

$$\mu(M) = \lim_{n \to \infty} \frac{1}{n} \log \mathrm{P}[M_n|\, n\tau]$$

It is noticeable that the magnitude of a sequence of events must be inferior or equal to zero, since the logarithm of a probability is never positive. If one can show that a pivot between one pair of candidates has a magnitude that is strictly greater than the magnitude of a pivot between another pair of candidates, then the latter is not serious. That is to say that, for any pair of pivots Y and $Y^{'}$, if

$$\mu(pivot(Y)) > \mu(pivot(Y^{'})) \Longrightarrow \lim_{n \to \infty} \frac{\mathrm{P}[pivot(Y)^{'}|\, n\tau]}{\mathrm{P}[pivot(Y)|\, n\tau]} = 0$$

then we know that the close race between candidates in $Y^{'}$ is not serious when compared to the close race in candidates in Y.

The Decision Process. Following [7], the decision process of the voters can be described as follows. Let k be a candidate. Let c and c' be two ballots that only differ by one extra candidate k: $c' = c \cup \{k\}$. In order to evaluate which of the ballots the type-t voter casts, she computes the sign of the following expression

$$\Delta = \sum_{Y} \mathrm{P}[pivot(Y) \mid n\tau] E[u_t(c' | \, Y) - u_t(c | \, Y)]$$

where $u_t(c| \, Y)$ represents the utility of a type-t voter when she casts ballot c knowing the event $pivot(Y)$. The sum Δ simply represents the effect of adding candidate k to her ballot in her expected utility.

However, adding this extra candidate to her ballot can only have an impact in the cases where this candidate is involved in a pivot. Therefore, Δ can be rewritten as follows:

$$\Delta = \sum_{k \in Y} \mathrm{P}[pivot(Y) \mid n\tau] E[u_t(c' | \, Y) - u_t(c | \, Y)]$$

Then, if there exists a pivot event (for instance, pivot(A)) where candidate k is involved which has a magnitude higher than the others, one can factor out by this pivot. Thus, in this case, as every pivot where candidate k is involved is such that

$$\lim_{n \to \infty} \frac{\mathrm{P}[pivot(Y) | \, n\tau]}{\mathrm{P}[pivot(A) | \, n\tau]} = 0 \qquad \text{for any } Y \text{such that } k \in Y$$

one can write that

$$\mathrm{sign}(\Delta) = \ \mathrm{sign}\, E[u_t(c' | \, A) - u_t(c | \, A)]$$

Repeating this procedure allows the voter to compare every pair of ballots which differ by a single candidate. The existence of a strict ordering of the magnitudes of the pivots where candidate k is involved and this for *every* candidate $k \in K$ is therefore a sufficient condition to ensure voters have a unique preferred ballot. Then it is easy to see that it also ensures that voters' best responses are Pure strategies. Unfortunately, this is only a sufficient condition as we will see throughout: one can have Pure strategies even without a strict ordering and, in Large Poisson Games, this strict ordering is not guaranteed.

Equilibrium of the game I refer to $\{\sigma_n, \tau_n\}$ as an *equilibrium* of the finite voting game (T, n, r, C, U). However, as this paper deals with elections with a large number of voters, one shall look at the limits of equilibria as the expected number of voters n tends to infinity. Thus, I refer to $\{\sigma, \tau\}$ as the equilibria sequence of the finite voting games, i.e. the limit of $\{\sigma_n, \tau_n\}$ when n tends to infinity. Therefore, $\{\sigma, \tau\}$ is an *equilibrium* of the voting game (T, n, r, C, U) if and only if τ is the expected vote distribution and, for each c in C and each t in T,

$$\sigma(c | \, t) > 0 \text{ implies that } c \in \mathrm{argmax}_{c \in C} E[u_t(.) | n\tau]$$

It should be noticed that only equilibria in which *weakly dominated strategies* have been eliminated are taken into account.

The following section introduces new mathematical results dealing with the subtleties of working with magnitudes. Particular attention is given to the interpretation of the offset-ratio concept.

3 Magnitude equivalence theorem

There exists a certain amount of complexity when working on situations where the number of voters tends towards infinity. For instance, the probability of any pivotal event generally tends to zero as the number of voters becomes larger. Therefore, instead of working with these probabilities, it is more reasonable to work with a measure of the speed at which these probabilities tend towards zero: the magnitude.
Two seminal papers ([11], [12]) give the three results that could be considered as the state-of-the-art techniques in the characterization of magnitudes in this type of games. The *magnitude* theorem states a method to compute such a limit as the solution of a maximization problem with a concave and smooth objective function. The *offset* theorem characterizes the ratios between probabilities of events that differ by a finite additive translation. Finally, the *dual magnitude* theorem (DMT) gives a method of computing events that have the geometrical structure of a cone, in a simpler way than the Magnitude theorem. Though Myerson's seminal contributions solved some of the main problems in order to work in this new kind of framework, some issues still needed to be elucidated.
The Magnitude Equivalence Theorem (MET) is the main technical result provided in the present work. which substantially reduces the computations of the magnitude of a pivotal event. The main advantage of this tool is that it allows us to use the DMT (Dual Magnitude Theorem, see [12]) to compute magnitudes of pivotal events directly. Indeed, the DMT is conceived to compute the magnitude of events that have the geometrical structure of a cone. However, a pivotal event does not have this geometrical structure. The MET shows that the magnitude of a pivotal event is the same one than the magnitude of a cone event which is included in the pivotal event.

Theorem 3.1 (Magnitude Equivalence Theorem) *Let (T,n,r,C,U) be a Poisson game. Let* $Y = \{i_1, i_2, \ldots, i_Y\}$ *be a subset of K. Then :*

$$\mu[pivot(Y)] = \mu[s(i_1) = s(i_2) = \ldots = s(i_Y) \geq s(x) \ \forall x \in K \setminus Y]$$

Proof:
Preliminary lemma: I first provide a Lemma which will be useful throughout. No proof is provided at it is a simple consequence of the definition of magnitude.

Lemma 3.1 *Let A and B be a couple of events with a finite magnitude. Then, if*

$$\lim_{n\to\infty} \frac{P(A \mid n\tau)}{P(B \mid n\tau)} = \varepsilon_n$$

for some $\{\varepsilon_n\}_{n=1}^{+\infty}$*, such that*

$$\lim_{n\to\infty} \frac{\log[\varepsilon_n]}{n} = 0 \quad and\ \varepsilon_n > 0 \ \forall\, n \ \in \mathbb{N}.$$

Then,

$$\mu(A) = \mu(B)$$

By definition, the probability of a pivotal event as

$$\begin{aligned}P[pivot(Y) \mid n\tau] = P[\{ \bigcup_{j_2=-1}^{1} \bigcup_{j_3=-1}^{1} \bigcup_{j_4=-1}^{1} \dots \bigcup_{j_K=-1}^{1} (s(i_1) = s(i_2) + j_2 = s(i_3) + j_3 = \dots \\ \dots = s(i_l) + j_l = \dots = s(i_Y) + j_Y > s(x)) \mid i_1, i_2, \dots, i_Y \in Y \text{and } \forall x \in K \setminus Y\}] \\ = P[\bigcup_{m \in M} A_m]\end{aligned}$$

For some arbitrary finite set M, the sets A_m represent the different cases in which one more ballot can change the outcome of the election. However, it should be noted that this is a disjoint union as $A_m \cap A_j = \emptyset$ if $m \neq j$. That is, the probability of the union is equal to the sum of the probabilities of every event, i.e.

$$P[pivot(Y) \mid n\tau] = \sum_{m \in M} P[A_m \mid n\tau]$$

Besides, the relationship between every A_m and the simple cone event $A_0 = [s(i_1) = s(i_2) = \dots = s(i_Y) \geq s(x)]$ can be expressed as a single positive translation. The vector of translation is denoted by $w_m = (w_m(c))_{c \in C} \in \mathbb{Z}^C$. In set theoretical terms, this can be written as

$$A_m = A_0 - w_m$$

Then, by the offset theorem,

$$\begin{aligned}\lim_{n\to\infty} \frac{P[A_m \mid n\tau]}{P[A_0 \mid n\tau]} &= \lim_{n\to\infty} \frac{P[A_0 - w_m \mid n\tau]}{P[A_0 \mid n\tau]} \\ &= \prod_{c \in C} \alpha(c)^{w_m(c)}\end{aligned}$$

where $\alpha(c)$ represents the offset ratio of ballot c at the event A_0.

The proof proceeds in several steps. In *Step 1*, the result is shown when every offset ratio is different from zero and finite (for regular events). *Step 2* shows the result when there exists one offset ratio that is different from zero (for irregular events). Finally, *Step 3* extends these results for the case where there exists two or more offset ratios that are equal to zero.

Step 1: Let us suppose first that every $\alpha(c)$ is different from zero and finite, where c represents a ballot which gives a point to at least one of the candidates in the set Y. In this case,

$$\lim_{n\to\infty} \frac{\log[\prod_{c \in C} \alpha(c)^{w_m(c)}]}{n} = 0$$

Therefore, by Lemma 3.1 both events have the same magnitude. Denoting for every

$m \in M$, $\rho_m = \prod_{c \in C} \alpha(c)^{w_m(c)}$, the magnitude of the pivotal event is such that

$$\begin{aligned}
\mu[pivot(Y)] &= \lim_{n\to\infty} \frac{1}{n} \log P[\text{pivot(Y)}|\, n\tau] \\
&= \lim_{n\to\infty} \frac{1}{n} \log \left[\sum_{m\in M} \rho_m P[A_0|\, n\tau] \right] \\
&= \lim_{n\to\infty} \frac{1}{n} \log \left[P[A_0|\, n\tau] \sum_{m\in M} \rho_m \right] \\
&= \lim_{n\to\infty} \frac{1}{n} \log P[A_0|\, n\tau] + \lim_{n\to\infty} \frac{1}{n} \log \sum_{m\in M} \rho_m \\
&= \lim_{n\to\infty} \frac{1}{n} \log P[A_0|\, n\tau] \\
&= \mu[s(i_1) = s(i_2) = \ldots = s(i_Y) \geq s(x)]
\end{aligned}$$

Step 2: Now, I suppose that there exists an $\alpha(c)$ which is equal to zero, where c represents a ballot which gives a point to at least one of the candidates in the set Y. Then, the event $A_0 - \{c\}$ (the event of subtracting a ballot $\{c\}$ from the event A_0) is infinitesimal with respect to the event A_0 when n tends to infinity. Conversely, the event $A_0 + \{c\}$ (the event of adding a ballot $\{c\}$ to the event A_0) is infinitely more probable as the event A_0 tends to infinity.

However, the event $A_{-c} = (A_0 - \{c\})$ does not occur with a positive probability. This can be explained as follows. Given that we are at event A_0, no voter who was supposed to vote for ballot $\{c\}$ has done so (as $\alpha(c) = 0$): $x(c) = 0$. The event $(A_0 - \{c\})$ would imply $x(c) = -1$. This is impossible given that $x(c)$ is a Poisson random variable defined only for the positive integers. Therefore, this event is not taken into account in the probability of the pivotal event.

The fact that the probabilities of A_0 and $A_c = (A_0 + \{c\})$ diverge when n tends to infinity, can be explained as follows. At event A_0, we know that $\alpha(c) = 0$ and thus that $x(c) = 0$. So, the event A_c implies that $(x(c) = 1)$. And,

$$\begin{aligned}
\lim_{n\to\infty} \frac{P[A_c|\, n\tau]}{P[A_0|\, n\tau]} &= \lim_{n\to\infty} \frac{P[x(c) = 1|\, n\tau]}{P[x(c) = 0|\, n\tau]} \\
&= \lim_{n\to\infty} \frac{n\tau(c) P[x(c) = 0|\, n\tau]}{P[x(c) = 0|\, n\tau]} \\
&= \lim_{n\to\infty} n\tau(c) = +\infty
\end{aligned}$$

Even if the ratio between these probabilities diverges, it is true that

$$\lim_{n\to\infty} \frac{\log[n\tau(c)]}{n} = 0$$

Thus, by Lemma 3.1, the events A_0 and A_c have the same magnitude.

By assumption, there is only one offset ratio which is equal to zero. Therefore, for every event A_m, $m \in M \setminus \{-c, c\}$, the following limit condition remains true,

$$\lim_{n\to\infty} \frac{P[A_m|\, n\tau]}{P[A_0|\, n\tau]} = \rho_m$$

for some ρ_m which is constant with respect to n.

Thus, for every event A_m, $m \in M \setminus \{-c, c\}$, one can write that

$$\begin{aligned}\lim_{n\to\infty} \frac{P[A_m|\, n\tau]}{P[A_c|\, n\tau]} &= \lim_{n\to\infty} \frac{\rho_m P[A_0|\, n\tau]}{P[A_c|\, n\tau]} \\ &= \lim_{n\to\infty} \frac{\rho_m P[A_0|\, n\tau]}{P[A_c|\, n\tau]} \\ &= \rho_m \lim_{n\to\infty} \frac{P[A_0|\, n\tau]}{P[A_c|\, n\tau]} = 0\end{aligned}$$

Therefore,

$$\begin{aligned}\mu[pivot(Y)] &= \lim_{n\to\infty} \frac{1}{n} \log P[\text{pivot(Y)} \mid n\tau] \\ &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{m\in M} \rho_m P[A_c \mid n\tau]\end{aligned}$$

with

$$\rho_m = \begin{cases} 1 & \text{if } m = c \\ 0 & \text{if not} \end{cases}$$

So,

$$\begin{aligned}\mu[pivot(Y)] =& \mu[A_c] \\ =& \mu[A_0] \\ =& \mu[s(i_1) = s(i_2) = \ldots = s(i_Y) \geq s(x)]\end{aligned}$$

Step 3: Finally, I assume that there exist two ballots $\{c\}$ and $\{c^{'}\}$ such that $\alpha(c) = \alpha(c^{'}) = 0$, where $\{c\}$ and $\{c^{'}\}$ are both ballots that give a point to at least one of the candidates in the set Y. Given this hypothesis, it is easy to see that

$$\lim_{n\to\infty} \frac{P[A_0 + \{c\}|\, n\tau]}{P[A_0|\, n\tau]} = \lim_{n\to\infty} \frac{P[A_0 + \{c^{'}\}|\, n\tau]}{P[A_0|\, n\tau]} = +\infty$$

Unfortunately, one cannot repeat the same process as before as we have two different events that are infinitely more likely. However, given the following limit condition

$$\lim_{n\to\infty} \frac{P[A_0 + \{c\}|\, n\tau]}{P[A_0 + \{c^{'}\}|\, n\tau]} = \lim_{n\to\infty} \frac{P[x(c) = 1|\, n\tau]}{P[x(c^{'}) = 1|\, n\tau]} = \frac{\tau(c)}{\tau(c^{'})}$$

and due to the fact that

$$\mu(A_0) = \mu(A_0 + \{c\}) = \mu(A_0 + \{c^{'}\})$$

the following equality still holds.

$$\mu[pivot(Y)] = \mu[A_0]$$

Similarly, the same result can be proven even if there exists more than two $\alpha(c)$ that are either equal or infinite.[5] $\square$

[5]It should be noted that if there exists an offset ratio $\alpha(c) = 0$, then there always exists another $\alpha(c')$ that is infinite. If $\alpha(c)$ represents the effect of the translation $-\{c\}$ to the event A_0, it suffices to take, for instance, the translation $\{c'\} = +\{c\}$.

4 Single-peakedness: A positive result for approval voting

The intuition of the result is that the kind of paradoxical situations described in Nuñez (2007) solely arise in situations where only some specific preference profiles are present in the election. In particular, the following theorems summarize the strategic behaviour of voters under AV and thus, the Condorcet Consistency of AV on Large Poisson Games.

Theorem 4.1 *If preferences are single-peaked, strategic voters vote sincerely under Approval voting. Besides, the Winner of the election coincides with the Profile Condorcet Winner whenever the latter exists.*

Proof:

There are K candidates running for the election $K = \{a_1, a_2, \ldots, a_K\}$ and the type set T satisfies single-peakedness. As previously discussed, voters have some prior beliefs concerning the expected result of the candidates. In particular, a_1 denotes the front-runner in the election, and a_2 the main challenger. Formally, this can be written as follows

$$E[s(a_1)] > E[s(a_2)] > E[s(a_j)]$$

for every $j \neq \{1, 2\}$

Candidates are assumed to represent some policy represented by a position in the real line. In particular, candidates are ordered in the political axis in the following way: the ones with an *even subscript* are located in the *left* side of the axis (closer to the position of the main challenger) and the ones with an *odd subscript* are located to its *right* (closer to the position of the front runner) (see Figure 1).

Under approval voting, we show that the Profile Condorcet Winner coincides with the Winner of the election at equilibrium. Indeed, we assume that a_1 is the *P.C.W.* and then we show that a_1 is the Winner of the election.

Some notations are now introduced for a matter of clarity. By assumption, the type set T satisfies single-peakedness. We divide it into two disjoint sets L and R, i.e.

$$\begin{aligned} T &= L \cup R \\ &= \bigcup_{i=1}^{K-1} L_i \cup \bigcup_{i=1}^{K-1} R_i \end{aligned}$$

where L stands for the subset of types located to the *left* of the political axis and R stands for the subset of types located to its *right*. In particular, the types L_1 and R_1 are characterized by the following preference profiles:

$$L_1 = \begin{pmatrix} a_2 \\ a_1 \\ a_4 \\ a_3 \\ \vdots \\ a_K \end{pmatrix} \quad R_1 = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_K \end{pmatrix}$$

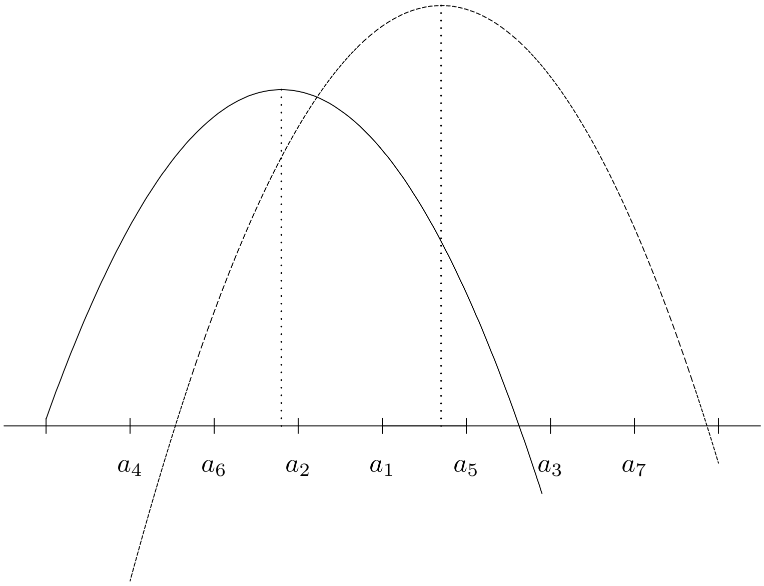

Figure 1: Single-peakedness over a finite set of candidates

In a similar manner, the type L_2 simply represents the preference profile which is located just to the left of L_1 on the policy space, L_3 the preference profile which is located just to the left of L_2 on the policy space, and so on. Similarly, R_2 represents the preference profile which is located just to the right of R_1 in the real line, R_3 the preference profile which is located just to the right of R_1 and so forth. The L-type voters prefer the main challenger a_2 to the front-runner a_1 and conversely, the R-type voters prefer the front-runner a_1 to the main challenger a_2. Figure 2 gives a more intuitive representation of this division. For simplicity of the notation, we have assumed the front-runner a_1 and the main challenger a_2 are located one next to the other. The main result still holds with this assumption. The purpose of this arbitrary division of the type set T is the simplication of notations when characterizing voters'strategies.

The proof is structured as follows. *Step 1* provides a description of voters' strategies, *Step 2* elucidates the informational structure generated by these strategies and *Step 3* checks that the informational structure leads voters to adopt Step 1's strategies and shows that the Winner of the election coincides with the Profile Condorcet Winner under AV.

Step 1: Description of voters' strategies.

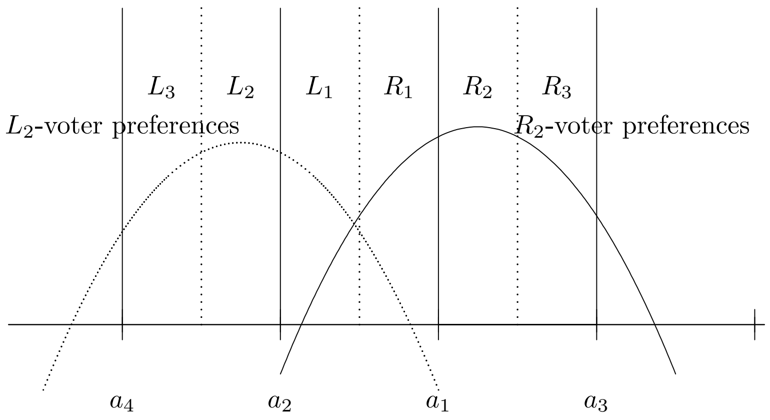

Figure 2: Type division over the political axis.

Let us assume that voters' strategies satisfy the following constraints:

$$\text{If } u_t(a_1) > u_t(a_2) \text{ then, the optimal ballot is } B_t^* = \{a_i \in K : u_t(a_i) \geq u_t(a_1)\}$$
$$\text{If } u_t(a_2) > u_t(a_1) \text{ then, the optimal ballot is } B_t^* = \{a_i \in K : u_t(a_i) \geq u_t(a_2)\}$$

Then, every t-type with $t \in L$, votes for the main challenger and for all her better-ranked candidates. Similarly, for any $t \in R$, a t-type voter votes for the front-runner and for all her better-ranked candidates.

Step 2: Elucidation of the informational structure.

The informational structure, that is the relative probabilities of the pivotal events is key to determining voters' strategies as illustrated in detail in the description of the Decision process.

We now state the following inequalities concerning the pivotal probabilities

$$P[a_1, a_2] >> P[a_1, a_3] >> \ldots >> P[a_1, a_j] >> \ldots \text{ for any } j, l \in \{4, K-1\}$$

$$P[a_2, a_1] >> P[a_2, a_3] >> \ldots >> P[a_2, a_j] >> \ldots \text{ for any } j, l \in \{4, K-1\}$$

and more generically that

$$P[a_2, a_{2j}] >> P[a_1, a_{2j}] >> \ldots >> P[a_l, a_{2j}] >> \ldots \text{ for any } j, l \in \{2, K-1\}$$
$$P[a_1, a_{2j+1}] >> P[a_2, a_{2j+1}] >> \ldots >> P[a_l, a_{2j}] >> \ldots \text{ for any } j, l \in \{1, K-1\}$$

We refer to these inequalities as the informational structure (IS) throughout. *Step 2.A: Pivot between the front-runner and the main challenger* The magnitude of such an event can be written as

$$\mu_{a_1,a_2} = \lim_{n\to\infty} \frac{1}{n} \log P[a_1, a_2]$$

For ease of notation, we refer to $P[x, y \mid n\tau]$ as $P[x, y]$. By definition, we know that

$$[a_1, a_2] = [s(a_1) + 1 = s(a_2) \geq s(a_i)] \cup [s(a_1) = s(a_2) \geq s(a_i)] \cup [s(a_1) - 1 = s(a_2) \geq s(a_i)]$$
$$\text{for every } i \in \{3, 4, \ldots, K-1\}$$

As a consequence of the MET, we know that

$$\mu_{a_1,a_2} = \mu([s(a_1) = s(a_2) \geq s(a_i)]) \text{ for every } i \in \{3, 4, \ldots, K-1\}$$

Yet, one can geometrically describe the event $\{[a_1, a_2]\}$ as the intersection of the following set of cones:

$$\{[a_1, a_2]\} = \{s(a_1) = s(a_2)\} \cap \{s(a_1) \geq s(a_3)\} \cap \{s(a_1) \geq s(a_5)\} \ldots \cap \{s(a_1) \geq s(a_{2j+1})\}$$
$$\ldots \cap \{s(a_2) \geq s(a_4)\} \ldots \cap \{s(a_2) \geq s(a_{2j})\} \ldots$$

However, it is easy to show that

$$\mu(\{s(a_1) \geq s(a_{2j+1})\}) = 0 \text{ for any } j \in \{1, 2, \ldots, K-1\}$$
$$\mu(\{s(a_2) \geq s(a_{2j}))\}) = 0 \text{ for any } j \in \{2, \ldots, K-1\}$$

which implies that

$$\mu(\{[a_1, a_2]\}) = \mu(\{s(a_1) = s(a_2)\})$$

And we know that

$$s(a_1) = \sum_{c\in\mathcal{C}_{a_1}} x_c \text{ and } s(a_2) = \sum_{c\in\mathcal{C}_{a_2}} x_c$$

Therefore, the magnitude of a pivotal event between the front-runner and the main challenger is such that

$$\mu_{a_1,a_2} = \mu(s(a_1) = s(a_2)) = -\left(\sqrt{\mathcal{C}_{a_1}} - \sqrt{\mathcal{C}_{a_2}}\right)^2$$

Step 2.B: Pivot between the front-runner and the second challenger

The magnitude of such an event can be written as

$$\mu_{a_1,a_3} = \lim_{n\to\infty} \frac{1}{n} \log P[a_1, a_3] = \mu([s(a_1) = s(a_3) \geq s(a_i)])$$
$$\text{for every } i \in \{2, 4, 5, \ldots, K-1\}$$

where the second equality is a consequence of the MET.
As previously, one can describe the event $\{[s(a_1) = s(a_3) \geq s(a_i)]\}$ as

$$\begin{aligned}
\{[s(a_1) = s(a_3) \geq s(a_i)]\} =& \{s(a_1) = s(a_3)\} \cap \{s(a_3) \geq s(a_5)\} \cap \{s(a_3) \geq s(a_7)\} \\
& \ldots \cap \{s(a_3) \geq s(a_{2j+1})\} \cap \ldots \cap \{s(a_2) \geq s(a_4)\} \ldots \cap \{s(a_2) \geq s(a_{2j})\} \ldots \\
=& \{x_1 = 0\} \cap \{x_{1,3} \geq x_2\} \cap \{s(a_3) \geq s(a_5)\} \cap \{s(a_3) \geq s(a_7)\} \\
& \ldots \cap \{s(a_3) \geq s(a_{2j+1})\} \cap \ldots \cap \{s(a_2) \geq s(a_4)\} \ldots \cap \{s(a_2) \geq s(a_{2j})\} \ldots
\end{aligned}$$

which implies that

$$\begin{aligned}
\mu_{a_1,a_3} &= \mu(\{x_{1,3} \geq x_2\}) + \mu(\{x_1 = 0\}) \\
&< \mu(\{s(a_1) = s(a_2)\}) = \mu_{a_1,a_2}
\end{aligned}$$

as we know that

$$\mu(\{x_1 = 0\}) < 0$$

Step 2.C: Pivot between the main and the second challenger

The magnitude of such an event can be written as

$$\begin{aligned}
\mu_{a_2,a_3} = \lim_{n\to\infty} \frac{1}{n} \log P[a_2, a_3] &= \mu([s(a_2) = s(a_3) \geq s(a_i)]) \\
& \text{for every } i \in \{1, 4, 5, \ldots, K-1\}
\end{aligned}$$

Geometrically, one can describe the event $\{[s(a_2) = s(a_3) \geq s(a_i)]\}$ as

$$\begin{aligned}
\{[s(a_2) = s(a_3) \geq s(a_i)]\} =& \{s(a_2) = s(a_3)\} \cap \{s(a_3) \geq s(a_1)\} \cap \{s(a_3) \geq s(a_5)\} \\
& \cap \{s(a_3) \geq s(a_7)\} \ldots \cap \{s(a_3) \geq s(a_{2j+1})\} \cap \\
& \ldots \cap \{s(a_2) \geq s(a_4)\} \ldots \cap \{s(a_2) \geq s(a_{2j})\} \ldots \\
=& \{x_1 = 0\} \cap \{x_{1,3} = x_2\} \cap \{s(a_3) \geq s(a_5)\} \cap \{s(a_3) \geq s(a_7)\} \\
& \ldots \cap \{s(a_3) \geq s(a_{2j+1})\} \cap \ldots \cap \{s(a_2) \geq s(a_4)\} \ldots \cap \{s(a_2) \geq s(a_{2j})\} \ldots \\
=& \mu(\{x_1 = 0\}) + \mu(\{x_{1,3} = x_2\})
\end{aligned}$$

which implies that

$$\mu_{a_2,a_3} = \mu_{a_1,a_3}$$

This implies that the probabilities of both events are not "too different" in the sense that they do not diverge when the expected size of the population tends towards infinity.

$$\lim_{n\to\infty} \frac{P(\{s(a_1) = s(a_3) \geq s(a_i)\})}{P(\{s(a_2) = s(a_3) \geq s(a_j)\})} = \varepsilon_n$$

where $i, j \in K$ and ε_n such that

$$\lim_{n\to\infty} \frac{1}{n} \log[\varepsilon_n] = 0$$

and $\varepsilon_n > 0 \, \forall n \in \mathbb{N}$. However, even if the magnitudes of both events have the same value, we can still discern which of the events is more probable. The aim of discerning whether one the events is infinitely more probable than the other is to determine whether voters will cast their vote against the main challenger (if $[a_1, a_3] >> [a_2, a_3]$) or against the front runner (if $[a_2, a_3] >> [a_1, a_3]$). As we want to compare the probabilities of the pivotal events where a_3, $[a_1, a_3]$ consists on the events where a single vote for candidate a_3 can change the result of the election. In other words, the situation $[a_1, a_3]$ consists of every event where a single vote for candidate a_3 can be decisive. Both events, $[a_1, a_3]$ and $[a_2, a_3]$ are characterized by the fact that there is only one offset ratio which is nil. In particular, in both events, the random variable x_1 is equal to zero.

Therefore, using the offset-theorem, we can write that,

$$[a_1, a_3] = \bigcup_{j \in J} A_j \quad , \quad [a_2, a_3] = \bigcup_{j \in L} B_j$$

for some arbitrary sets J and L. Using the offset-theorem, one can write that

$$\lim_{n \to \infty} \frac{P[M \mid n\tau]}{P[N \mid n\tau]} = c \text{ for some finite constant c}$$

for every pair of events M and N which differ by a finite translation of ballots where voting for candidate a_3 is not involved. That is, every event where voting for candidate is decisive where candidates a_1 and a_3 get the same score. However, we can write that

$$\lim_{n \to \infty} \frac{P[L \mid n\tau]}{P[M \mid n\tau]} = \infty$$

whenever the event M belongs to the set where candidates a_1 and a_3 get the same score and the event L is characterized by the fact that candidate a_1 gets exactly one more vote that candidate a_3. In other words, we can set up a relative ordering between the different events where adding candidate a_3 in voter's ballot pivotally changes the result in the election. In this ordering, there is a set of events which is infinitely more probable than the other sets whereas the other sets have finite relationships between them.

Thus, it is infinitely more probable that the close race occurs between the front-runner a_1 and the second challenger a_3 rather than between the main challenger a_2 and the second challenger a_3, that is

$$[a_1, a_3] >> [a_2, a_3]$$

Finally, by similar reasonings, we can write

$$P[a_1, a_2] >> P[a_1, a_3] >> \ldots >> P[a_1, a_j] >> \ldots \text{ for any } j, l \in \{4, K-1\}$$

$$P[a_2, a_1] >> P[a_2, a_3] >> \ldots >> P[a_2, a_j] >> \ldots \text{ for any } j, l \in \{4, K-1\}$$

Step 2.D: General Computations. We first show that

$$P[a_2, a_{2j}] >> P[a_1, a_{2j}] \text{ for any} j \in \{1, 2, \ldots, K-1\}$$

As a consequence of the MET, we know that

$$\mu_{a_1,a_{2j}} = \lim_{n\to\infty} \frac{1}{n} \log P[a_1, a_{2j}] = \mu([s(a_1) = s(a_{2j}) \geq s(a_i)])$$
$$\text{for every } i \in \{1, \ldots, K-1\} \setminus \{1, 2j\}$$

and

$$\mu_{a_2,a_{2j}} = \lim_{n\to\infty} \frac{1}{n} \log P[a_2, a_{2j}] = \mu([s(a_2) = s(a_{2j}) \geq s(a_i)])$$
$$\text{for every } i \in \{1, \ldots, K-1\} \setminus \{2, 2j\}$$

Geometrically, one can describe the event $\{[a_1, a_{2j}]\}$ as

$$\begin{aligned}\{[a_1, a_{2j}]\} = &\{s(a_1) = s(a_{2j})\} \cap \{s(a_{2j}) \geq s(a_2)\} \cap \{s(a_{2j}) \geq s(a_4)\} \ldots \cap \{s(a_{2j}) \geq s(a_{2(j-1)})\} \\ &\ldots \cap \{s(a_1) \geq s(a_3)\} \ldots \cap \{s(a_1) \geq s(a_{2j+1})\} \ldots \\ &= \{x_2 = 0\} \cap \{x_{2,4} = 0\} \cap \{x_{2,4,\ldots,2(j-1)} = 0\} \cap \\ &\{x_{2,4,\ldots,2(j)} + x_{2,4,\ldots,2j,2(j+1)} + \ldots = x_1 + x_{1,3} + x_{1,3,5} + \ldots\} \cap \ldots \\ &\{s(a_{2j}) \geq s(a_{2(j-1)})\} \ldots \cap \{s(a_1) \geq s(a_3)\} \ldots \cap \{s(a_1) \geq s(a_{2j+1})\} \ldots\end{aligned}$$

and the event $\{[a_2, a_{2j}]\}$ as

$$\begin{aligned}\{[a_2, a_{2j}]\} = &\{s(a_2) = s(a_{2j})\} \cap \{s(a_{2j}) \geq s(a_2)\} \cap \{s(a_{2j}) \geq s(a_4)\} \ldots \cap \{s(a_{2j}) \geq s(a_{2(j-1)})\} \\ &\ldots \cap \{s(a_1) \geq s(a_3)\} \ldots \cap \{s(a_1) \geq s(a_{2j+1})\} \ldots \\ &= \{x_2 = 0\} \cap \{x_{2,4} = 0\} \cap \{x_{2,4,\ldots,2(j-1)} = 0\} \cap \\ &\{x_{2,4,\ldots,2(j)} + x_{2,4,\ldots,2j,2(j+1)} + \ldots = x_1 + x_{1,3} + x_{1,3,5} + \ldots\} \cap \ldots \\ &\{s(a_{2j}) \geq s(a_{2(j-1)})\} \ldots \cap \{s(a_1) \geq s(a_3)\} \ldots \cap \{s(a_1) \geq s(a_{2j+1})\} \ldots\end{aligned}$$

which implies that

$$\mu_{a_1,a_{2j}} = \mu_{a_2,a_{2j}}$$

As previously explained, the previous equality implies that

$$\lim_{n\to\infty} \frac{P(\{s(a_1) = s(a_{2j}) \geq s(a_i)\})}{P(\{s(a_2) = s(a_{2j}) \geq s(a_i)\})} = \varepsilon_n$$

with ε_n such that

$$\lim_{n\to\infty} \frac{1}{n} \log[\varepsilon_n] = 0$$

and $\varepsilon_n > 0 \,\forall n \in \mathbb{N}$. However, even if the magnitudes of both events have the same value, we can still discern which of the events is more probable.

$$[a_1, a_{2j}] = [s(a_1)+1 = s(a_{2j}) \geq s(a_i)] \cup [s(a_1) = s(a_{2j}) \geq s(a_i)] \cup [s(a_1)-1 = s(a_{2j}) \geq s(a_i)]$$

and

$$[a_2, a_3] = [s(a_2)+1 = s(a_{2j}) \geq s(a_2)] \cup [s(a_{2j}) = s(a_i) \geq s(a_i)] \cup [s(a_2)-1 = s(a_{2j}) \geq s(a_i)]$$

and we know that, in order to get this pivot event we must have $x_2 = 0$, $x_{2,4} = 0$ and so on until $x_{2,4,\ldots,2(j-1)} = 0$. This implies that in order to get such a pivot event no voter who was supposed to vote only for candidate $a_2, a_4, \dots, a_{2(j-1)}$ actually does it. Formally, this is equivalent to state that the offset-ratio of ballots B_{a_2}, B_{a_2,a_4} and so on are nil and so, applying the Offset Theorem,

$$\lim_{n\to\infty} \frac{P[s(a_2) - 1 = s(a_{2j}) \geq s(a_i)]}{P[s(a_2) = s(a_{2j}) \geq s(a_i)]} = +\infty$$

Similarly,

$$\lim_{n\to\infty} \frac{P[s(a_2) - 1 = s(a_{2j}) \geq s(a_i)]}{P[s(a_1) = s(a_{2j}) \geq s(a_i)]} = \lim_{n\to\infty} \varepsilon_n \frac{P[s(a_2) - 1 = s(a_{2j}) \geq s(a_i)]}{P[s(a_1) = s(a_{2j}) \geq s(a_i)]} = +\infty$$

Thus,

$$\lim_{n\to\infty} \frac{P[a_2, a_{2j}]}{P[a_1, a_{2j}]} = +\infty \Leftrightarrow P[a_2, a_{2j}] >> P[a_1, a_{2j}] \quad \text{for every } j \in \{2, 3, \dots, K-1\}$$

By similar reasonings, it is easy to show that

$$\lim_{n\to\infty} \frac{P[a_1, a_{2j+1}]}{P[a_2, a_{2j+1}]} = +\infty \Leftrightarrow P[a_1, a_{2j+1}] >> P[a_1, a_{2j+1}] \quad \text{for every } j \in \{1, 2, \dots, K-1\}.$$

and so, we can finally state that

$$P[a_2, a_{2j}] >> P[a_1, a_{2j}] >> \dots >> P[a_l, a_{2j}] >> \dots \text{ for any } j, l \in \{2, K-1\}$$
$$P[a_1, a_{2j+1}] >> P[a_2, a_{2j+1}] >> \dots >> P[a_l, a_{2j+1}] >> \dots \text{ for any } j, l \in \{1, K-1\}$$

Step 3: The Winner of the Election coincides with the P.C.W.

Once we have described the informational structure, we need to prove that it actually leads to the strategies described in *Step 1.* The proof is summarized on the next lemma and based in Laslier's (2004) main theorem. The main intuition is that voters cast their optimal ballot taking into account the partial orderings of the different pivotal events.

Lemma 4.1 *Given the informational structure IS, t-type voter has a unique best response ballot B_t^*:*

If $u_t(a_1) > u_t(a_2)$ *then the optimal ballot is* $B_t^* = \{a_i \in K : u_t(a_i) \geq u_t(a_1)\}$
If $u_t(a_2) > u_t(a_1)$ *then the optimal ballot is* $B_t^* = \{a_i \in K : u_t(a_i) \geq u_t(a_2)\}$

Proof:

Let us suppose first that $u_t(a_1) > u_t(a_2)$, that is $t \in L$. In order to show that B_t^* is the unique best response, we prove that any other ballot B is not.
(i) Let us assume that there exits $k \in K$ such that $u_t(k) > u_t(a_1)$ and $k \notin B$. Let then $B^{'} = B \cup k$.

To compare ballots c and $c^{'}$, the voter computes the difference

$$\Delta = \sum_Y P[\text{pivot}(Y)] E[u_t(c^{'}|Y) - u_t(c|Y)]$$

where $E[u_t(c|Y)]$ denotes the expected utility for her of choosing ballot c knowing the event $pivot(Y)$. The events $pivot(Y)$ with $k \notin Y$ have no consequences. Therefore the the sum in Δ can run on the subsets of Y such that $k \in Y$.

Among these subsets there are $\{k, a_1\}$ and $\{k, a_2\}$. Let us suppose first that k is a candidate with an even subscript, located on the left side of the political axis. In this case, $u_t(k) < u_t(a_2) < u_t(a_1)$. This is a contradiction.

It follows that candidate k has an odd subscript and its position is located on the right side of the political axis. Given the event $\{k, a_1\}$, the voter's utility is strictly higher voting for k. Thus $E[u_t(c^{'}|\{k, a_1\}) - u_t(c|\{k, a_1\})] < 0$.

As proved by the informational structure (IS), we can write that

$$P[a_1, a_k] >> P[a_2, a_k] >> \ldots >> P[a_l, a_k] >> \ldots \text{ for any } l \in \{1, K-1\}$$

One can thus factor out $Pr[a_1, a_k]$ in Δ and write

$$\frac{\Delta}{P[a_1, a_k]} = a + E[u_t(c^{'}|[a_1, a_k]) - u_t(c|[a_1, a_k])] + \sum_{Y} o_Y$$

for

$$a = E[u_t(c^{'}|[a_1, a_k]) - u_t(c|[a_1, a_k])]$$

and

$$o_Y = \frac{P[pivot(Y)]}{P[a_1, a_k]} E[u_t(c^{'}|[pivot(Y)]) - u_t(c|[pivot(Y)])]$$

with a being strictly positive and with o_Y tending towards zero when n tends to infinity. Therefore, when n is big enough, the expression Δ is strictly positive. This proves the fact that ballot c is not a best response.

(ii) Let us suppose that there exists $k \in K$ such that $u_t(k) < u_t(a_1)$ and $k \in B$. Let then $c^{'} = c \setminus k$. By similar reasonings as in the previous case, one can show that c is not a best response ballot.

(iii) Let us suppose now that $u_t(a_2) > u_t(a_1)$, that is $t \in L$. Similarly, one can prove that t-type voter best response ballot is such that

$$\text{If } t \in L, \; B_t^* = \{a_i \in K : u_t(a_i) \geq u_t(a_2)\}$$

Every voter with a type in the set R votes for the front-runner a_1 and every voter with a type in the set L votes for the main challenger a_2 and no voter votes for both of them. Therefore, as by assumption, the front-runner is the *P.C.W.* (and thus there is a majority of the electorate who prefers a_1 rather than a_2), we can conclude that the Winner of the election coincides with the Profile Condorcet winner under Approval voting.

Step 4: Uniqueness of the equilibrium

Lemma 4.2 *If preferences are single-peaked, the Winner of the election coincides with the Profile Condorcet Winner whenever the latter exists is the only possible equilibrium.*

Proof: Let us assume that there exists an equilibrium where the candidate a_1 is the *P.C.W.* and is *not* the Front Runner of the election.

Let us first assume that there exists a candidate b which is the Front Runner and that candidate a is the *P.C.W.* of the election and the main challenger. In this case, one can, by similar reasonings that the above-mentioned ones, state that at equilibrium every voter votes for either candidate b or for the *P.C.W.* of the election. However, the fact candidate a is the *P.C.W.* implies that there is a majority of the electorate who prefers candidate a to candidate b. Therefore, candidate a must be the Front Runner at equilibrium.
Let us now assume that there exists a couple of candidates b and c which respectively are the Front Runner and the Main Challenger. Candidate a is assumed to represent the *P.C.W.* of the election. In this case, one can, by similar reasonings that the above-mentioned ones, state that at equilibrium every voter votes for either candidate b or for candidate c always following the same heuristics. Formally, one can write that

$$E[s(b)] + E[s(c)] = 1$$

As we have assumed that candidate b is the Front Runner, we have that $E[s(b)] > E[s(c)]$ and so $E[s(c)] < \frac{1}{2}$. However, the fact candidate a is the *P.C.W.* implies that there is a majority of the electorate who prefers candidate a to candidate b. Therefore, the expected score of candidate a is strictly superior to $\frac{1}{2}$, contradicting the fact that candidate c is the main challenger. $\square$

5 An example where the Condorcet Winner is not elected

In this section, an example is provided where, in equilibrium, the Winner of the election does not coincide with the Profile Condorcet Winner. An interesting feature of this example is its simplicity. Indeed, as will be shown, the majority of population (t_1-voters) votes against its own interest due to the available information. There are only three types of voters with ordinal preferences and the type distribution is quite simple. This fact is quite paradoxical and is due to the presence of a small group in the population (the t_3-voters) and to the correlation between the scores of the candidates that naturally arise in the Poisson-Myerson framework. Furthermore, the equilibrium is shown to be quite stable. It should be emphasized that in this simple voting context that two of the three pivotal events are irregular, i.e. have at least one nil offset ratio. As previously described, counter-intuitive situations are more suitable to arise in this kind of situations.

I consider a voting context where there are three candidates $K = \{a, b, c\}$ and three types of voters, i.e. $T = \{t_1, t_2, t_3\}$. The voters' preferences are as follows

$$u_{t_1}(a) > u_{t_1}(b) > u_{t_1}(c)$$
$$u_{t_2}(b) > u_{t_2}(a) > u_{t_2}(c)$$
$$u_{t_3}(c) > u_{t_3}(a) > u_{t_3}(b)$$

where $u_t(k)$ denotes the utility of type-t voters when candidate k wins the election. The expected type distribution is

$$r(t_1) = \frac{3}{32} \quad r(t_2) = \frac{18}{32} \quad r(t_3) = \frac{11}{32}$$

Under approval voting, there is an equilibrium in which the Winner of the election does not coincide with the *P.C.W.*
I suppose that the strategy functions satisfy

$$\sigma(\{a\}|\, t_1) = \sigma(\{a, b\}|\, t_2) = \sigma(\{c\}|\, t_3) = 1$$

Note that in this example all the voters with the same type choose the same ballot. Therefore,

$$\tau(\{a\}) = r(t_1), \quad \tau(\{a, b\}) = r(t_2), \quad \tau(\{c\}) = r(t_3)$$

Once the setting of the voting situation has been described, I now proceed to the computation of the magnitudes of the pivotal events. The solved minimization problems are included in the appendix.

Magnitude of a pivot between candidates a and b Following the *MET*, one can write the following equality

$$\mu(pivot(a, b)) = \mu(\{s(a) = s(b) \geq s(c)\})$$

According to the *DMT*, we know that this magnitude is equal to the solution of the following optimisation problem.

$$\mu(\{s(a) = s(b) \geq s(c)\}) = \min_{\lambda} \; r(t_1)\exp[\lambda_1 - \lambda_2 + \lambda_3] + r(t_2)\exp[\lambda_3] + r(t_3)\exp[-\lambda_3] - 1$$

such that $\lambda_i \geq 0 \, \forall \, i$.

Thus, numerically solving this constrained minimization problem, the magnitude of this pivotal event is such that

$$\mu(pivot(a,b)) = -0.09375$$

The offset-ratio vector associated with this event, called α_1, is equal to

$$\alpha_1(\{a\}) = 0 \;\; \alpha_1(\{a,b\}) = 1 \;\; \alpha_1(\{c\}) = 1$$

Magnitude of a pivot between candidates a and c Combining the MET and the DMT, the magnitude of a pivot between candidates a and c is equal to

$$\mu(pivot(a,c)) = \mu(\{s(a) = s(c) \geq s(b)\}) = -0.0500822$$

The offset ratio vector associated with this event called α_2, is equal to

$$\alpha_2(\{a\}) = \frac{1.44749}{2} = 0,7237 \;\; \alpha_2(\{a,b\}) = 0,7237 \;\; \alpha_2(\{c\}) = \frac{2}{1.44749} = 1,3817$$

Magnitude of a pivot between candidates b and c Combining the MET and the DMT, the magnitude of a pivot between candidates b and c is equal to

$$\mu(pivot(b,c)) = \mu(\{s(b) = s(c) \geq s(a)\}) = -0.120547$$

The offset ratio vector associated with this event called α_3, is equal to

$$\alpha_3(\{a\}) = 0 \;\; \alpha_3(\{a,b\}) = \frac{2}{2.55841} = 0,7817 \;\; \alpha_3(\{c\}) = \frac{2.55841}{2} = 1,2792$$

Therefore, the magnitudes of the pivotal events are ordered as follows:

$$\mu(pivot(a,c)) > \mu(pivot(a,b)) > \mu(pivot(b,c))$$

This ordering is not intuitive at all as candidates a and b both have a higher expected score than candidate c and therefore one would expect that the most probable pivotal event would take place between them. This a consequence of the correlations between the scores of the candidates that arise in the Poisson Games. For instance, this ordering is not possible under the Score uncertainty model à la Laslier.

Taking into account the ordering of the magnitudes, and assuming that voters are instrumental, one can determine the ballot that each voter of a given type chooses. To clarify how the voters choose the ballot, the decision process of t_1 voters is now described in detail. When deciding between casting $\{a\}$ or $\{a,b\}$, they take into account the magnitudes of the pivots where candidate b is involved as they evaluate the effect on their expected utility of adding candidate b to their ballot $\{a\}$. Mathematically, this decision process could be expressed as evaluating the sign of Δ,

$$\begin{aligned}\Delta = E[\{a\}] - E[\{a,b\}] &= \sum_{b \in Y} P[pivot(Y)] E[u_{t_1}(\{a\}|pivot(Y)) - u_{t_1}(\{a,b\}|pivot(Y))] \\ &= P[pivot(a,b)] E[u_{t_1}(\{a\}) - u_{t_1}(\{a,b\})|pivot(a,b)] \\ &\quad + P[pivot(b,c)] E[u_{t_1}(\{a\}) - u_{t_1}(\{a,b\})|pivot(b,c)]\end{aligned}$$

By assumption, $pivot(a,b)$ is the pivotal event where b is involved with the highest magnitude. Therefore,

$$\lim_{n\to\infty} \frac{P[pivot(b,c)]}{P[pivot(a,b)]} = 0$$

Then, following Laslier (2004)'s reasoning, one can factor out $P[pivot(a,b)]$ in Δ when n tends to infinity and then just evaluate the sign of the following expression

$$E[u_{t_1}(\{a\}) - u_{t_1}(\{a,b\})|pivot(a,b)]$$

Defining $A_j = (s(a) = s(b) + j \geq s(c))$ for $j \in \{-1,0,1\}$, this expression is equivalent to

$$E[u_{t_1}(\{a\}) - u_{t_1}(\{a,b\})|pivot(a,b)] = \sum_{j=-1}^{1} E[u_{t_1}(\{a\}) - u_{t_1}(\{a,b\})|A_j]\ P[A_j]$$

Using the offset ratios, one can write that

$$P[A_{-1}|\ n\tau] = 0$$
$$\lim_{n\to\infty} \frac{P[A_0 \mid n\tau]}{P[A_1 \mid n\tau]} = 0$$

and, by computing the expected utility of the voters, the following inequalities can be stated,

$$E[u_{t_1}(\{a\}) - u_{t_1}(\{a,b\})|A_{-1}] > 0$$
$$E[u_{t_1}(\{a\}) - u_{t_1}(\{a,b\})|A_0] > 0$$
$$E[u_{t_1}(\{a\}) - u_{t_1}(\{a,b\})|A_1] \geq 0$$

Thus, Δ is positive and so t_1 voters prefer to cast $\{a\}$ rather than $\{a,b\}$. Repeating the same decision process, it can be deduced that t_1 voters prefer to cast ballot $\{a\}$ to $\{a,c\}$. Similarly, the expected utility of adding one single candidate to a given ballot can be computed, obtaining the optimal strategy for each type of voter. Thus, the voters' optimal strategies are such that,

$$\sigma(\{a\}|t_1) = \sigma(\{a,b\}|t_2) = \sigma(\{c\}|t_3) = 1$$

Then, this is an equilibrium of the voting game. It should be noted that,

$$\tau(\{a\}) + \tau(\{a,b\}) > \tau(\{a,b\}) > \tau(\{c\})$$

The previous inequality implies that candidate a is the Winner of the election. Besides,

$$r(t_2) > r(t_1) + r(t_3)$$
$$r(t_1) + r(t_2) > r(t_3)$$

which implies that candidate b is the Profile Condorcet Winner. $\square$

On the stability of the equilibrium. Concerning the equilibrium presented here, there are two main aspects that should be underlined. First of all, in this game there is another equilibrium where the *P.C.W.* coincides with the Winner. For instance, assuming that the expected type distribution is the following one,

$$r(t_1) = \frac{12}{32} \quad r(t_2) = \frac{12}{32} \quad r(t_3) = \frac{8}{32}$$

with the same strategy functions leads us to such an equilibrium.
Secondly, I try to test the stability of this equilibrium. In order to attain this objective, the main focus on situations where some proportion of the individuals do not behave rationally. This could be due to some kind of bias towards the expected Winner. It could also be thought as some kind of trembling-hand ballot due to some external factor. I show here that for non-negligible fractions of the voters showing a lack of rationality, the model still reaches an equilibrium where the Winner does not coincide with the *P.C.W.*

Let us suppose that there is a fraction of t_2-voters who economically misbehave and vote only for candidate a. This class of voters is denoted by i_2 where i stands for their irrationality. Then, the expected type distribution is

$$r(t_1) = \frac{3}{32} \quad r(t_2) = \frac{18}{32} - \epsilon \quad r(i_2) = \epsilon \quad r(t_3) = \frac{11}{32}$$

Then, combining the DMT and the MET, the solution to the following optimization problem corresponds with the magnitude of a pivot between candidates a and b.

$$\min_{\lambda} r(t_1)UV + r(t_2)V + r(t_3)V^{-1} + r(i_2)UV$$

such that $U > 0$ and $V \geq 1$. Explicit formulas for the optimization problems corresponding to the computation of the magnitudes of pivots between a, c and b, c are not provided as they are similar to the latter.
Numerically solving this problem, one can find that whenever $\epsilon \in (0, \frac{6}{32})$, the pivotal events follow the same order as in our example,

$$\mu(pivot(a, c)) > \mu(pivot(a, b)) > \mu(pivot(b, c))$$

Therefore, the rational voters still behave in the same way. This is an equilibrium if and only if $\epsilon \in (0, \frac{6}{32})$. The upper bound of this interval is roughly equal to 0.21 implying that even if 21% (on average) of the population do not behave rationally one can still reach an equilibrium where neither concept coincides. This process could be repeated assuming the same kind of irrationality among t_1 and t_3 voters leading us to the same result: the equilibrium holds. One can therefore conclude that the stability of this equilibrium is quite strong.

6 Conclusion

Setting up a game-theoretical framework for the study of Large elections which is both trackable and realistic is not an easy matter. In this work, I have analysed one of the main models that have attempted to do so and tried to understand its strategic properties in this framework through the perspective of AV, a voting rule advocated by its simplicity and good properties in classical models.

Large Poisson Games assume that population is not constant and therefore introduce some voters' prior beliefs concerning the "seriousness" of the candidates in the election. This work shows that the introduction of such beliefs drastically improves the predictive power of Nash equilibrium in voting environments, in line with Myerson's results. Indeed, the main result of this work states that, under the assumption of single-peaked preferences, approval voting ensures both the sincerity of the voters and the Condorcet Consistency of the election. This a positive result that strengthens the positive results previously drawn over approval voting by other scholars.

However, this work also shows that the way this uncertainty is introduced induces the fact that paradoxical situations can arise. As a consequence of the independent-actions property, Large Poisson Games have a number of drawbacks which are worthy to be underlined. When the voting rule allows to vote for more than one candidate, the fact that the number of people who cast a given ballot is independent of the number of people who cast another one naturally implies that the scores of the candidates are correlated. The main problem is that, because of this correlation, the Winner of the election does not always coincide with the Profile Condorcet Winner and that voters can behave in a counter-intuitive manner at equilibrium. Indeed, the set of equilibria includes some situations where counter-intuitive behaviour could be best response.

Whether these differences still hold under other Population Uncertainty models such as the one proposed by Milchtaich (2004) is a matter of future agenda.

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7 Technical Appendix

Magnitude of a pivot between candidates a and b

$$\begin{aligned}\mu(\{s(a)=s(b)\geq s(c)\}) &= \min_{\lambda}\,(r(t_1))\exp[\lambda_1-\lambda_2+2\lambda_3]+r(t_2)\exp[\lambda_1-\lambda_2-\lambda_3]-1\\ &= \min_{U>0,V\geq 1}\; r(t_1)UV^2+r(t_2)UV^{-1}-1=-(\sqrt{\tau(1)}-\sqrt{\tau(2)})^2\end{aligned}$$

such that $\lambda_i \geq 0\;\forall\; i$.

Magnitude of a pivot between candidates a and c

$$\begin{aligned}\mu(\{s(a)=s(c)\geq s(b)\}) &= \min_{\lambda}\,(r(t_1))\exp[2\lambda_1-2\lambda_2+\lambda_3]+r(t_2)\exp[-\lambda_1+\lambda_2+\lambda_3]-1\\ &= \min_{U>0,V\geq 1}\; r(t_1)U^2V+r(t_2)U^{-1}V-1=-(\sqrt{\tau(1)}-\sqrt{\tau(2)})^2\end{aligned}$$

such that $\lambda_i \geq 0\;\forall\; i$.

Magnitude of a pivot between candidates b and c

$$\mu(\{s(b)=s(c)\geq s(a)\}) = \min_{U>0,V\geq 1}\; r(t_1)UV^{-1}+r(t_2)U^{-2}V^{-1}=-(\sqrt{\tau(1)}-\sqrt{\tau(2)})^2$$

such that $\lambda_i \geq 0\;\forall\; i$.