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A REFINEMENT OF PRUDENT CHOICES

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Abstract: We charaterize the choice correspondences that can be rationalized by a procedure that is a refinement of the prudent choices exposed in [Houy, 2008]. Our characterization is made by means of the usual expansion axiom and by a weakening of the usual contraction axiom α.

Key Words : Prudent choices, multi-criteria decision making.

Classification JEL: D0

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1 Introduction

The litterature dealing with multi-rationales decision making has known new interesting developments in the last few years. We can divide these developments into three main categories.

The first series of papers considers that two binary relations should be composed by constructing a composite binary relation as follows.[1] For any two alternatives, x and y, x is strictly prefered to y if it is so according to the first rationale or if y is not strictly prefered to x according to the first rationale and x is strictly prefered to y according to the second rationale. Then, the choice is made by maximizing the composite binary relation. This procedure of decision making has the advantage to show nice properties of rationality. On the contrary, it is often empty and in these cases it does not help make a decision.

The second series of papers deals with the problem of multi-rationale decision making as follows.[2] In order to choose in a set, first, a decision is made by maximizing the first binary relation. Second, the choice is made by maximizing the second binary relation in the set of elements that have been preselected in the first step. Contrary to the first procedure, this one always makes non-empty choices under mild assumptions. However, it fails at satisfying the usual rationality properties.[3]

The third procedure that has been imagined in order to compose two rationales works has follows. In a given set, a prudent composition of the rationales is constructed. It is a binary relation that contains the first rationale and as many elements of the second rationale as possible such that the prudent composition remains acyclic. Obviously, a prudent composition is not necessarily unique. Then, an element is chosen if there exists a prudent composition such that the considered element is maximal according

[1] See [Tadenuma, 2002], [Tadenuma, 2005].
[2] See [Manzini and Mariotti, 2007a], [Manzini and Mariotti, 2007b].
[3] See [Houy, 2007] and [Houy and Tadenuma, 2007].

to this prudent composition. This procedure has been introduced and axiomatized by [Houy, 2008]. This procedure always makes non-empty choices under mildest assumptions than the ones required for the second procedure. However, as the second procedure, it fails at satisfying the usual rationality properties.

This paper is an attempt to add a new procedure in the line of what has been done in the latter stream of litterature. We propose to construct the set of prudent compositions as it has been defined above. However, we will consider that an element is chosen if for any prudent composition, the considered element is maximal according to this prudent composition. Hence, what we define is a refinement of the prudent choice procedure. The purpose of this article is to give an axiomatization of this refinement and show some of its properties.

In Section 2, we intraoduce the notation. In Section 3, we give the axioms and our characterization result.

2 Notation

Let X be a finite set of alternatives. Let $\mathcal{X} = 2^X \setminus \emptyset$ be the set of all non-empty subsets of X. A choice correspondence on X is a function $C : \mathcal{X} \to \mathcal{X}$ such that $\forall S \in \mathcal{X}, C(S) \subseteq S$.[4] Let $\mathcal{C}(X)$ be the set of all choice correspondences on X.

Let $C \in \mathcal{C}(X)$. For any subset $S \in \mathcal{X}$, we define the two-tier domination relation $(TD(S,C))$ as follows. Let $R, T \in \mathcal{X}$ be such that $R, T \subseteq S$. $(R,T) \in TD(S,C)$ if and only if $R \cup T = S$, $R \cap T = \emptyset$ and $\forall r \in R, \forall t \in T, \exists S' \in \mathcal{X}, a \in C(S')$ and $b \in S'$. Hence, R two-tier dominates T in S for C if and only if $\{R,T\}$ is a partitioning of S such that there exists no element in T that menu-indenpendently dominates any element of R.

Let $P \subseteq X \times X$ be a binary relation on X. For any subset S of X, $P\mid_S$ is

[4]Notice that by definition, $\forall S \in \mathcal{X}, C(S) \neq \emptyset$.

the restriction of P to S, *i.e.* $P\mid_S = \{(a,b) \in P, a, b \in S\}$. P^t is the transitive closure of P *i.e.* $\forall a, b \in X$, $(a,b) \in P^t$ if and only if $\exists n \in \mathbb{N}$, $\exists a_1, ..., a_n \in X$ such that $\forall i \in \{1, ..., n-1\}$, $(a_i, a_{i+1}) \in P$, $a_1 = a$ and $a_n = b$. We say that P is irreflexive if and only if $\forall a \in X, (a,a) \notin P$. We say that P is asymmetric if and only if $\forall a, b \in X$, $(a,b) \in P$ implies $(b,a) \notin P$. We say that P is acyclic if and only if $\forall a \in X, (a,a) \notin P^t$.[5] An asymmetric binary relation will be called a preference relation.

Let P_1 and P_2 be two preference relations on X. We define $Q(P_1, P_2)$ by $\forall a, b \in X$, $(a,b) \in Q(P_1, P_2)$ if and only if $(a,b) \in P_1$ or $[(b,a) \notin P_1$ and $(a,b) \in P_2]$.

Let (P_1, P_2) be an ordered pair of preference relations on X such that P_1 is acyclic. Let $S \in \mathcal{X}$. We say that $P \subseteq X \times X$ is a prudent composition of P_1 and P_2 on S if

- $P = P_1\mid_S \bigcup Q$ with $Q \subseteq P_2\mid_S$,
- P is acyclic and,
- $\forall Q'$ such that $Q \subset Q' \subseteq P_2\mid_S$, $P_1\mid_S \bigcup Q'$ is cyclic on S.

Then, a prudent composition of P_1 and P_2 on S is a binary relation containing P_1 and as many elements of P_2 as possible with the constraint that the prudent composition is not cyclic. We denote by $(\widehat{P_1, P_2})(S)$ the set of all prudent compositions of P_1 and P_2 on S. Notice that by definition, $(\widehat{P_1, P_2})(S)$ is non-empty if and only if it is well defined or, said differently, if $P_1\mid_S$ is acyclic.[6]

Let $P \subseteq X \times X$ be a preference relations on X. We define $C_P : \mathcal{X} \to 2^X$ by

$$\forall S \in \mathcal{X}, C_P(S) = \{a \in S, \forall b \in S, (b,a) \notin P\}.$$

[5] Notice that, by definition, acyclicity implies asymmetry and asymmetry implies irreflexivity.

[6] Obviously, if $(\widehat{P_1, P_2})(S) = \{\emptyset\}$, we still have $(\widehat{P_1, P_2})(S) \neq \emptyset$.

If C_P is choice correspondence, *i.e.* $\forall S \in \mathcal{X}, C_P(S) \neq \emptyset$, we say that P rationalizes the choice correspondence C_P. Let $P_1, P_2 \subseteq X \times X$ be two preference relations on X with P_1 acyclic. We define $C^{\cup}_{(P_1,P_2)}$ by

$$\forall S \in \mathcal{X}, a \in C^{\cup}_{(P_1,P_2)}(S) \Leftrightarrow a \in S \text{ and } \exists P \in (\widehat{P_1, P_2})(S) \text{ such that}$$

$$\forall b \in S, (b,a) \notin P.$$

If $C^{\cup}_{(P_1,P_2)}$ is a choice correspondence, we say that (P_1, P_2) $\cup$-prudently rationalizes C. We define $C^{\cap}_{(P_1,P_2)}$ by

$$\forall S \in \mathcal{X}, a \in C^{\cap}_{(P_1,P_2)}(S) \Leftrightarrow a \in S \text{ and } \forall P \in (\widehat{P_1, P_2})(S),$$

$$\forall b \in S, (b,a) \notin P.$$

If $C^{\cap}_{(P_1,P_2)}$ is a choice correspondence, we say that (P_1, P_2) $\cap$-prudently rationalizes C.

3 Characterization

The first three lemmas characterize the choices made by $C_{Q(P_1,P_2)}$, $C^{\cup}_{(P_1,P_2)}$ and $C^{\cap}_{(P_1,P_2)}$. The first lemma is straightforward and left to the reader as a simple exercise. The second lemma has been proved in [Houy, 2008].

Lemma 1
Let P_1, P_2 be two preference relations on X. Let $S \in \mathcal{X}$ and $a \in S$. $a \in C_{Q(P_1,P_2)}(S)$ if and only if:

- $\forall b \in S, (b,a) \notin P_1$ *and,*
- $\forall b \in S$ *such that* $(b,a) \in P_2$, $(a,b) \in P_1$.

Lemma 2
Let P_1, P_2 be two preference relations on X with P_1 acyclic. Let $S \in \mathcal{X}$ and $a \in S$. $a \in C^{\cup}_{(P_1,P_2)}(S)$ if and only if:

- $\forall b \in S, (b,a) \notin P_1$ *and,*

- *$\forall b \in S$ such that $(b,a) \in P_2$, $(a,b) \in (Q(P_1,P_2)\mid_S)^t$.*

Lemma 3

Let P_1, P_2 be two preference relations on X with P_1 acyclic. Let $S \in \mathcal{X}$ and $a \in S$. $a \in C^{\cap}_{(P_1,P_2)}(S)$ if and only if:

- *$\forall b \in S, (b,a) \notin P_1$ and,*

- *$\forall b \in S$ such that $(b,a) \in P_2$, $(a,b) \in (P_1\mid_S)^t$.*

Proof. If: Let $S \in \mathcal{X}$ and let $a \in S$. Assume that $\forall b \in S, (b,a) \notin P_1$ and, $\forall b \in S$ such that $(b,a) \in P_2$, $(a,b) \in (P_1\mid_S)^t$. Assume that $a \notin C^{\cap}_{(P_1,P_2)}(S)$. Then, by definition, $\exists P \in (\widehat{P_1,P_2})(S)$ such that $\exists b \in P, (b,a) \in P$. By definition, $P_1\mid_S \subseteq P \subseteq P_1\mid_S \cup P_2\mid_S$. Then, either $(b,a) \in P_1$ or $(b,a) \in P_2$. The first case can not occur by assumption. In the second case, by assumption, $(a,b) \in (P_1\mid_S)^t$ and then P is cyclic which is impossible by definition. Hence a contradiction.

Only if: Let $S \in \mathcal{X}$ and let $a \in S$. 1) Assume that $\exists b \in S, (b,a) \in P_1$. Obviously, by definition, $\forall P \in (\widehat{P_1,P_2})(S)$, $P_1\mid_S \subseteq P$. Then, $\forall P \in (\widehat{P_1,P_2})(S), \exists b \in S, (b,a) \in P$. Hence, by definition, $a \notin C^{\cap}_{(P_1,P_2)}(S)$. 2) Assume that $\exists b \in S, (b,a) \in P_2$ and $(a,b) \notin (P_1\mid_S)^t$. Then, $P_1 \cup \{(a,b)\}$ is acyclic on S. Hence, there exists $P \in (\widehat{P_1,P_2})(S)$ such that $P_1 \cup \{(a,b)\} \subseteq P$. Hence, by definition, $a \notin C^{\cap}_{(P_1,P_2)}(S)$. $\square$

As a simple corollary, we can show that $C^{\cap}_{(P_1,P_2)}$ is a refinement of $C^{\cup}_{(P_1,P_2)}$ and $C_{Q(P_1,P_2)}$ is a refinement of $C^{\cap}_{(P_1,P_2)}$. As we have stated in the introduction, $C^{\cup}_{(P_1,P_2)}$ never makes empty choices under very mild assumptions. On the contrary, $C_{Q(P_1,P_2)}$ often makes empty choices. Hence, the introduction of $C^{\cap}_{(P_1,P_2)}$ can be seen as an attempt to refine $C^{\cup}_{(P_1,P_2)}$ without showing all the weaknesses of $C_{Q(P_1,P_2)}$.

Corollary 1

Let P_1, P_2 be two binary relations on X such that P_1 is acyclic. For all $S \in \mathcal{X}, C_{Q(P_1,P_2)}(S) \subseteq C^{\cap}_{(P_1,P_2)}(S) \subseteq C^{\cup}_{(P_1,P_2)}(S)$.

The remainder of this article is a characterization of $C^{\cap}_{(P_1,P_2)}$ by its properties. The following axioms are needed.

The first axiom is an usual one in choice theory and is given in [Sen, 1993] for instance. It states that if an alternative is chosen from different sets, then, it is chosen from the union of these sets.

Axiom 1 (γ)
Let $C \in \mathcal{C}(X)$. The choice correspondence C satisfies γ if and only if $\forall n \in \mathbb{N}$ and $\forall S_1, ..., S_n \in \mathcal{X}$,
$a \in \bigcap_{i \in \{1,...,n\}} C(S_i)$ implies $a \in C(\bigcup_{i \in \{1,...,n\}} S_i)$.

The second axiom is a weak version of the usual Chernoff or α consistency condition. Let us interpret it. Assume that the set S can be divided into two distinctive subsets, T and R such that R two-tier dominates T, *i.e.* there is no element of T that menu-independently dominates any element of R. Now, consider that there is an element a in T that is not chosen when considered with an element b in R. Then, a is dominated by b, not only because a is not chosen from the set $\{a, b\}$, but also because a is an element of T, two-tier dominated by R, of which b is an element. Then, Axiom Weak α states that in this case of double domination, a should not be chosen from S.

Axiom 2 (Weak α)
Let $C \in \mathcal{C}(X)$. The choice correspondence C satisfies Weak α if and only if $\forall S \in \mathcal{X}$ and $\forall a \in S$,
if $\exists (R,T) \in TD(S,C)$ such that $a \in T$ and $\exists b \in R, a \notin C(\{a,b\})$, then $a \notin C(S)$.

Notice that Axiom Weak α is weaker than the usual Axiom α. In fact, it is also weaker than the Axiom $\alpha 2$ which is itself a weak version of Axiom α, see [Sen, 1977].[7] Indeed, if Axiom $\alpha 2$ is assumed, $a \notin C(S)$ is a consequence

[7] In our framework, the usual Axiom α can be written as follows: the choice correspondence C satisfies α if and only if $\forall S, T \in \mathcal{X}$ such that $T \subseteq S$ and $\forall a \in T$, if $a \notin C(T)$, then $a \notin C(S)$. Axiom $\alpha 2$ can be written as follows: the choice correspondence C satisfies $\alpha 2$ if and only if $\forall S \in \mathcal{X}$ and $\forall a \in S$, if $\exists b \in S, a \notin C(\{a,b\})$, then $a \notin C(S)$.

of $\exists b \in S, a \notin C(\{a,b\})$ only and the two-tier domination requirement is not necessary.

Let us show by means of an example that Axiom Weak α is strictly stronger than Axioms α and $\alpha 2$. Let us have $X = \{a,b,c\}$ and $C \in \mathcal{C}(X)$ be such that $C(\{a,b\}) = \{a\}$, $C(\{a,c\}) = \{c\}$, $C(\{b,c\}) = \{b\}$ and $C(X) = \{c\}$. Obviously, C does not satisfy $\alpha 2$ since $C(\{b,c\}) = \{b\}$ should imply $c \notin C(X)$. Since Axiom α is stronger than Axiom $\alpha 2$, it is obviously not satisfied by C. However, C satisfies Weak α since even though $C(\{b,c\}) = \{b\}$, we have $TD(X,C) = \{(\{c\},\{a,b\}),(\{c,a\},\{b\})\}$ and then Weak α is satisfied.

Notice also that Axiom Weak α is stronger than the axiom of consistency stated in [Houy, 2008] (we will call this axiom WWα). Indeed, the latter states that $\forall S \in \mathcal{X}$ and $\forall a \in S$ if [there exists a partitioning $\{R,T\}$ of S such that $\forall d \in R$, $\forall e \in T$, $d \in C(\{e,d\})$ and such that $a \in T$ and $\exists b \in R, a \notin C(\{a,b\})$], then $a \notin C(S)$. Let us show with the following example that Weak α is strictly stronger than WWα. Let us have $X = \{a,b,c\}$ and $C \in \mathcal{C}(X)$ be such that $C(\{a,b\}) = \{a\}$, $C(\{a,c\}) = \{c\}$, $C(\{b,c\}) = \{b\}$ and $C(X) = X$. Obviously, WWα is satisfied. On the contrary, Weak α is not. Indeed, $(\{c\},\{a,b\}) \in TD(X,C)$ and $a \notin C(\{a,c\})$. Then, if Weak α was satisfied, we would have $a \notin C(X)$.

Proposition 1 shows that a choice correspondence can be $\cap$-prudently rationalized if and only if it satisfies γ and Weak α.

Proposition 1

Let C in $\mathcal{C}(X)$. The choice correspondence C satisfies γ and Weak α if and only if there exist two preference relations P_1, P_2 with P_1 acyclic such that (P_1, P_2) $\cap$-prudently rationalizes C.

Proof. If: Let $C \in \mathcal{C}(X)$ be $\cap$-prudently rationalized by the ordered pair of binary relations (P_1, P_2) with P_1 acyclic.

Let us show that C satisfies γ. Let $n \in \mathbb{N}$ and let $S_1, ..., S_n \in \mathcal{X}$. Assume $a \in \bigcap_{i\in\{1,...,n\}} C(S_i)$. By Lemma 3, $\forall i \in \{1,...,n\}$, $\forall b \in S_i$,

$(b,a) \notin P_1$. Then, $\forall b \in \bigcup_{i\in\{1,...,n\}} S_i$, $(b,a) \notin P_1$. Moreover, by Lemma 3, $\forall i \in \{1,...,n\}$, $\forall b \in S_i$, $(b,a) \in P_2 \Rightarrow (a,b) \in (P_1 \mid_{S_i})^t$. Then, by definition, $(a,b) \in (P_1 \mid_{\bigcup_{i\in\{1,...,n\}} S_i})^t$. Then, $\forall b \in \bigcup_{i\in\{1,...,n\}} S_i$, $(b,a) \in P_2$ implies $(a,b) \in (P_1 \mid_{\bigcup_{i\in\{1,...,n\}} S_i})^t$. Then, by Lemma 3, $a \in C(\bigcup_{i\in\{1,...,n\}} S_i)$.

Let us show that C satisfies Weak α. Let $S \in \mathcal{X}$ and $a \in S$ be such that $\exists (R,T) \in TD(S,C)$ such that $a \in T$ and $\exists b \in R, a \notin C(\{a,b\})$. $a \notin C(\{a,b\})$ implies $(b,a) \in P_1$ or $((b,a) \in P_2$, $(a,b) \notin P_1$ and $(b,a) \notin P_1)$. If $(b,a) \in P_1$, then $a \notin C(S)$ follows by Lemma 3. Assume $(b,a) \in P_2$, $(a,b) \notin P_1$ and $(b,a) \notin P_1$. By definition, $(R,T) \in TD(S,C)$ implies $\forall r \in R, \forall t \in T$, $(t,r) \notin P_1$. Hence, $(a,b) \notin (P_1 \mid_S)^t$. Then, by Lemma 3, $a \notin C(S)$.

Only if: Let $C \in \mathcal{C}(X)$ satisfy γ and Weak α. Let us show that there exist two preference relations P_1, P_2 with P_1 acyclic such that (P_1, P_2) $\cap$-prudently rationalizes C.

Let us define $P_1 = \{(a,b) \in X \times X, \forall S \in \mathcal{X}, a \in S \Rightarrow b \notin C(S)\}$ and $P_2 = \{(a,b) \in X \times X, C(\{a,b\}) = \{a\} \text{ and } \exists S \in \mathcal{X}, a \in S, b \in C(S)\}$.

1) P_1 is acyclic. Assume on the contrary that P_1 is cyclic. Then $\exists n \in \mathbb{N}\backslash\{1\}$, $\exists a_1, ..., a_n$, $\forall i \in \{1, ..., n-1\}, (a_i, a_{i+1}) \in P_1$ and $(a_n, a_1) \in P_1$. Then, by definition, $C(\{a_1, ..., a_n\}) = \emptyset$ which contradicts the fact that C is a choice correspondence.

Let us show that (P_1, P_2) $\cap$-prudently rationalizes C. Let $S \in \mathcal{X}$ and let $a \in S$.

2) Assume $\exists b \in S, (b,a) \in P_1$. Then, by definition, $a \notin C(S)$.

3) Assume $\exists b \in S, (b,a) \in P_2$ and $(a,b) \notin (P_1 \mid_S)^t$. Let us define $a(P_1 \mid_S)^t = \{c \in S, (a,c) \in (P_1 \mid_S)^t\} \cup \{a\}$. By definition, $a(P_1 \mid_S)^t \neq \emptyset$ since $a \in a(P_1 \mid_S)^t$. Moreover, $S \setminus a(P_1 \mid_S)^t \neq \emptyset$ since $b \in S \setminus a(P_1 \mid_S)^t$. Obviously, $\forall d \in a(P_1 \mid_S)^t$, $\forall e \in S\backslash a(P_1 \mid_S)^t$, $(d,e) \notin P_1$ (else $e \in a(P_1 \mid_S)^t$ by definition), *i.e.* $\exists S' \in \mathcal{X}, d \in S', e \in C(S')$. Hence, $(a(P_1 \mid_S)^t, S\backslash a(P_1 \mid_S)^t)) \in TD(S,C)$. Then, by Weak α, $a \notin C(S)$.

4) Assume $\forall b \in S, (b,a) \notin P_1$ and $[(b,a) \in P_2$ implies $(a,b) \in (P_1 \mid_S)^t]$. Let us define $B = \{b \in S, (b,a) \in P_2\}$. Obviously, $\forall d \in S \setminus B$, $(d,a) \notin P_1 \cup P_2$, then, by definition, $a \in C(\{a,d\})$. Then, by γ, $a \in C(S \setminus B)$. Let $b \in B$. By assumption, $\exists n \in \mathbb{N} \setminus \{1\}, \exists a_1, ..., a_n \in S$ such that $\forall i \in \{1, ..., n-1\}, (a_i, a_{n+1}) \in P_1$ with $a_1 = a$ and $a_n = b$. Let us define $S_b = \{a_1, ..., a_n\}$. Then, by definition, $\forall i \in \{2, ..., n\}, a_i \notin C(S_b)$. Since C is a choice correspondence, we necessarily have $\{a\} = C(S_b)$. This reasoning holds for all elements of B and then, $a \in \bigcap_{b \in B} C(S_b)$. Hence, by γ, $a \in C(\bigcup_{b \in B} S_b)$. With $a \in C(S \setminus B)$ and still by γ, we have $a \in C(S)$. $\square$

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