GREQAM

Groupement de Recherche en Economie
Quantitative d'Aix-Marseille - UMR-CNRS 6579
Ecole des Hautes Etudes en Sciences Sociales
Universités d'Aix-Marseille II et III

Document de Travail
n°2008-30

ASYMPTOTIC REFINEMENTS OF BOOTSTRAP TESTS IN A LINEAR REGRESSION MODEL; A CHM BOOTSTRAP USING THE FIRST FOUR MOMENTS OF THE RESIDUALS

Pierre-Eric TREYENS

Juin 2008

DT-GREQAM

Asymptotic refinements of bootstrap tests in a linear regression model; A CHM parametric bootstrap using the first four moments of the residuals. [1]

Pierre-Éric TREYENS
GREQAM
Université de la Méditerranée

Abstract

We consider linear regression models and we suppose that disturbances are either Gaussian or non Gaussian. Then, by using Edgeworth expansions, we compute the exact errors in the rejection probability (ERPs) for all one-restriction tests (asymptotic and bootstrap) which can occur in these linear models. More precisely, we show that the ERP is the same for the asymptotic test as for the classical parametric bootstrap test it is based on as soon as the third cumulant is nonnul. On the other side, the non parametric bootstrap performs almost always better than the parametric bootstrap. There are two exceptions. The first occurs when the third and fourth cumulants are null, in this case parametric and non parametric bootstrap provide exactly the same ERPs, the second occurs when we perform a t-test or its associated bootstrap (parametric or not) in the models $y = \mu + \sigma u_t$ and $y = \alpha_0 x_t + \sigma u_t$ where the disturbances have nonnull kurtosis coefficient and a skewness coefficient equal to zero. In that case, the ERPs of any test (asymptotic or bootstrap) we perform are of the same order.

Finally, we provide a new parametric bootstrap using the first four moments of the distribution of the residuals which is as accurate as a non parametric bootstrap which uses these first four moments implicitly. We will introduce it as the parametric bootstrap considering higher moments (CHM), and thus, we will speak about the CHM parametric bootstrap.

J.E.L classification : C10, C12, C13, C15.

Keywords : Non parametric bootstrap, Parametric Bootstrap, Cumulants, Skewness, Kurtosis.

[1] I thank the GREQAM and the Université de la Méditerranée for its financial and research support. Helpful comments, suggestions and corrections from Russell Davidson, John Galbraith and Jean-François Beaulnes.

0 Introduction

Beran (1988) asserts that when a test statistic is an asymptotic pivot, bootstrapping this statistic leads to an asymptotic refinement. A test statistic is said to be asymptotically pivotal if its asymptotic distribution is the same for all data generating processes (DGPs) under the null, and by asymptotic refinement we mean that the error in the rejection probability (ERP) of the bootstrap test, or the size distortion of the bootstrap test, is a smaller power of the sample size than the ERP of the asymptotic test that we bootstrap. As MacKinnon (2007) notes, although there is a very large literature on bootstrapping, only a small proportion of it is devoted to bootstrap testing, which is the purpose of the present paper. Instead, the focus is usually on estimating bootstrap standard errors and constructing bootstrap confidence intervals.

Recall that a single bootstrap test may be based on a statistic τ in an asymptotic p-value form. Rejection by an asymptotic test at level α is then the event $\tau < \alpha$. Rejection by the bootstrap test is the event $\tau < Q(\alpha, \mu^*)$, where μ^* is the bootstrap data-generating process and $Q(\alpha, \mu^*)$ is the (random) $\alpha-$quantile of the distribution of the statistic τ as generated by μ^*. Now, let us consider a bootstrap test computed from a t-statistic which tests $H_0 : \mu = \mu_0$ against $H_1 : \mu < \mu_0$ in a linear regression model where disturbances may be non-Gaussian but i.i.d., which is the framework of this paper. Moreover, we suppose that all regressors are exogenous. Under these assumptions, a t-statistic is an asymptotic pivot. In order to decrease ERP, it is obvious that the bootstrap test must use extra information compared with the asymptotic test. So the question becomes: where does this additional information come from?

By definition, we can write a t-statistic T as $T = \sqrt{n}(\hat{\mu} - \mu_0)\hat{\sigma}_{\hat{\mu}}^{-1}$ where n is the sample size and μ a parameter connected with any regressor of the linear regression model. Therefore computation of a t-statistic requires only $\hat{\mu}$, the estimator of μ, and the estimator of its variance $\hat{\sigma}_{\hat{\mu}}$. Moreover, as the limit in distribution of a t-statistic is $N(0, 1)$, we can find an approximation of the CDF of T by using an Edgeworth expansion. In fact, the main part of the asymptotic theory of the bootstrap is based on Edgeworth expansions of statistics which follow asymptotically standard normal distributions; see Hall (1992). Hence we can provide an approximation to the $\alpha-$quantile of T. Next we can obtain the $\alpha-$quantile of the bootstrap distribution by replacing the true values of the higher moments of the disturbances by their random estimates as generated by the bootstrap DGP. The extra information in the bootstrap test comes from these higher moments which the bootstrap DGP should estimate correctly.

A number of authors have stressed this point, including Parr (1983) who shows the effect of having a skewness coefficient equal to zero or not in the context of first order asymptotic theory for the jackknife and the bootstrap. See also Hall (1992) or Andrews (2005). For instance, quoting Buhlmann (1998), *"the key to the second order accuracy is the correct skewness of the bootstrap distribution"*. Intuitively, we can explain this remark in the framework of a t-test. If we first consider a parametric bootstrap test, the bootstrap DGP uses only the estimated variance of residuals $\hat{\sigma}$, which is directly connected to the estimated variance of the parameter tested. Indeed, bootstrap error terms are generated following a Gaussian distribution. So it is clear that we use the same information as when computing the t-statistic. More precisely, the $\alpha-$quantile of the parametric bootstrap distribution depends only on the higher moments of a centered Gaussian random variable, which is completely defined by its first two moments. So, the $\alpha-$quantile of the parametric bootstrap distribution is not random anymore. If we now consider a non parametric bootstrap, we implicitly use extra information which comes from higher moments of the distribution of the residuals. Indeed, when we resample the estimated residuals we provide random consistent estimators of these moments and therefore the $\alpha-$quantile of the parametric bootstrap distribution is random.

To clarify this example, consider a new parametric bootstrap using estimated higher moments; in this way we can provide a bootstrap framework which provides as much information as a non parametric bootstrap framework. We will refer to this new bootstrap as the CHM parametric bootstrap, from the acronym for *"Considering Higher Moments"*. Hall (1992) provides the ERP of a non parametric bootstrap which tests the sample mean and only the magnitude of the ERP for the test of a slope parameter, but he does not derive ERPs for other tests in a linear regression model (asymptotic and parametric bootstrap tests, or tests for a slope parameter with or without an intercept in the model). In the present paper we go further and provide the exact ERPs in the following models; $y = \mu_0 + \sigma u$ where we will test $H_0 : \mu = \mu_0$ against $H_1 : \mu < \mu_0$, $y = \mu_0 + \beta_0 x + \sigma u$ where we will test $H_0 : \beta = \beta_0$ against $H_1 : \beta < \beta_0$, and $y = \beta_0 x + \sigma u$ where we will test $\beta = \beta_0$ against $H_1 : \beta < \beta_0$. These three cases all include one-restriction tests which can occur in a linear regression model. For each of these three models, we will use a classical parametric bootstrap, a non parametric bootstrap (or residual bootstrap) and finally, a CHM parametric bootstrap which will use the estimators of the first four moments of the residuals. Our method is slightly different from Hall's, so we also use it in the case of a simple mean, although Hall did establish the exact ERP of the non parametric bootstrap test in the model $y_t = \mu + \sigma u_t$.

Moreover, we stress the role of the fourth cumulant for the bootstrap refinements in a linear regression model. When disturbances are Gaussian, we show that ERPs of the different tests are of lower order compared with non-Gaussian disturbances. In fact, disturbances do not need to be Gaussian, but rather to have third and fourth cumulants equal to zero, as is the case for Gaussian distributions. Finally, in the models $y = \mu_0 + \sigma u$ and $y = \beta_0 x + \sigma u$, we introduce a new family of standard distributions defined by its third and fourth cumulants that theoretically provides the same ERPs for the non parametric bootstrap and CHM parametric bootstrap as the Gaussian distribution; this family admits the Gaussian distribution as a special case.

These results are given in the second and third sections of this paper; the first section is devoted to preliminaries. In the fourth section, we proceed to simulations and explain how to apply the CHM parametric bootstrap. The fifth section concludes.

1 Preliminaries

In this paper, we show how higher moments of the disturbances in a linear regression model influence either asymptotic and bootstrap inferences. In this way, we have to consider non Gaussian distributions whose skewness and/or kurtosis coefficients are not zero. For any centered random variable X, if we define its characteristic function f_c by $f_c(u) = E\left(e^{iux}\right)$ we can obtain, by using a MacLaurin expansion,

$$ln\left(f_c(u)\right) = \kappa_1(iu) + \frac{\kappa_2(iu)^2}{2!} + \frac{\kappa_3(iu)^3}{3!} + \cdots + \frac{\kappa_k(iu)^k}{k!} + \ldots \tag{1.1}$$

In this equation, the κ_k are order k cumulants of the distribution of X. Moreover, for a centered standardised random variable the four first cumulants are

$$\kappa_1 = 0,\ \kappa_2 = 1,\ \kappa_3 = E(X^3),\ \text{and}\ \kappa_4 = E(X^4) - 3 \tag{1.2}$$

In particular, κ_3 and κ_4 are the skewness and kurtosis coefficients. One of the main problems when we deal with higher moments is how we can generate centered standardised random variable fitting these coefficients.

Treyens (2006) provides two methods to generate random variables in this way and we use them because they are the existing fastest methods. Let us consider three independent random variables p, N_1 and N_2 where N_1 and N_2 are two Gaussian variables of expectations μ_1 and μ_2 and of standard error σ_1 and σ_2 and define $X = pN_1 + (1-p)N_2$. If p is a uniform distribution $U(0,1)$, the set of admissible couples (κ_3, κ_4) this method can provide is Γ as showed on the figure 1.1 and it will be called the unimodal method. If p is a binary variable and if $\frac{1}{2}$ is the probability that $p = 1$, the set of admissible couples is Γ_1 and this method will be called the bimodal method. On figure 1.1, the parabola and the straight line are just structural constraints which connect κ_4 to κ_3. Now, if a centered standardised random variable X has κ_3 and κ_4 as skewness and kurtosis coefficients, we will write $X \rightarrow \Delta(0, 1, \kappa_3, \kappa_4)$. In this paper, all disturbances will be distributed as $u_t \rightarrow ii\Delta(0, 1, \kappa_3, \kappa_4)$.

In order to estimate the error in the rejection probability, we are going to use Edgeworth expansions. With this theory, we can express the error in the rejection probability as a quantity of the order of a negative power of n, where n is the size of the sample from which we compute the test statistic. Let t be a test statistic which asymptotically follows a standard normal distribution, and $F(.)$ be the CDF of the test statistic. Almost with a classic Taylor expansion, we can develop the function $F(.)$ as the CDF $\Phi(.)$ of the standard normal distribution plus an infinite sum of its successive derivatives that we always can write as a polynomial in t multiplied by the PDF $\phi(.)$ of the standard normal distribution. Precisely, we have

$$F(t) = \Phi(t) - n^{-1/2}\phi(t)\sum_{i=1}^{\infty}\lambda_i He_{i-1}(t) \tag{1.3}$$

In this equation, $He_i(.)$ is the Hermite polynomial of degree i and the λ_i are coefficients which are at most of the order of unity. Hermite polynomials are implicitly defined by the relation $\phi^{(i)}(x) = (-1)^i He_i(x)\phi(x)$, as a function of the derivatives of $\phi(.)$

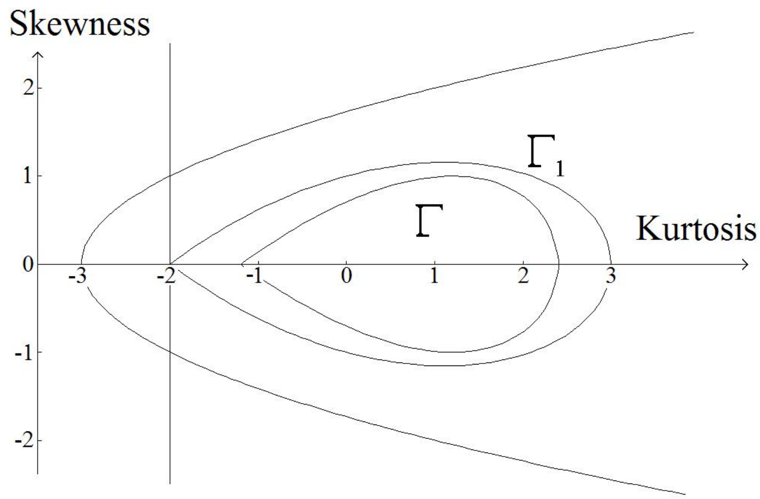

Figure 1.1: Sets of admissible couples Γ and Γ_1

which gives the recurrence relation $He_0(x) = 1$ and $He_{i+1}(x) = xHe_i(x) - He_i'(x)$, and the coefficients λ_i are defined as the following function of uncentered moments of the test statistic t, $\lambda_i = \frac{n^{1/2}}{i!} E\left(He_i(t)\right)$. Moreover, we will use in computations several random (or not) variables which are functions of the disturbances and of parameters of the models, all these variables (w, q, s, k, X, Q, m_1, m_3 and m_4) are described in the Appendix.

2 Testing a simple mean

Let us consider the model $y_t = \mu_0 + \sigma u_t$ with $u_t \rightarrow ii\Delta(0, 1, \kappa_3, \kappa_4)$. In order to test $H_0 : \mu = \mu_0$ against $H_1 : \mu < \mu_0$, we use a Student statistic and we bootstrap it. The t-statistic is obviously $T = \sqrt{n}\left(\frac{\hat{\mu}-\mu_0}{\hat{\sigma}_\mu}\right)$, where $\hat{\mu}$ is the estimator of the mean of the sample and $\hat{\sigma}$ is the unbiased estimator of the standard error of the OLS regression. So, we can give an approximate value at order $o_p\left(n^{-1}\right)$ of the test statistic T :

$$T = w\left(1 - n^{-1/2}\frac{q}{2} + n^{-1}\left(\frac{w^2}{2} - \frac{1}{2} + \frac{3q^2}{8}\right)\right) + o_p\left(n^{-1}\right) \qquad (2.1)$$

In order to give the Edgeworth expansion $F_{1,T}(.)$ of the CDF of this test statistic, we have to check two points. First, the asymptotic distribution of the test statistic must be a standard normal distribution. Secondly, all expectations of its successive power must exist. The first point is easy to check, indeed, by applying the central limit theorem on w, we see that the asymptotic distribution of w is a standard normal distribution. Moreover, the limit in probability of the right hand of the equation 2.1 divided by w is deterministic and equal to 1. In order to check the second, we will just compute successive expectations of powers of T. This will allow us to deduce easily expectations of Hermite polynomials of T and in that way we obtain an estimate of $F_{1,T}(.)$ at order n^{-1}. Now, we can compute an approximation q_α of the $\alpha-$quantile

$Q(\alpha, \Theta)$ of the test statistic at order n^{-1}, where Θ is the DGP which generated the original data y. To find this approximation, we introduce $q_\alpha = z_\alpha + n^{-1/2} q_{1\alpha} + n^{-1} q_{2\alpha}$, the Cornish-Fisher expansion of $Q(\alpha, \Theta)$, where z_α is the α-quantile of a standard normal distribution, with actually, $Q(\alpha, \Theta) = z_\alpha + n^{-1/2} q_{1\alpha} + n^{-1} q_{2\alpha} + o(n^{-1})$. If we now evaluate $F_{1,T}(.)$ in q_α, then we can find the expression of $q_{1\alpha}$ and $q_{2\alpha}$.

$$q_{1\alpha} = -\frac{\kappa_3 (1 + 2z_\alpha^2)}{6} \tag{2.2}$$

$$q_{2\alpha} = -\frac{z_\alpha^3 (6\kappa_4 - 20\kappa_3^2 - 18) + z_\alpha (-18\kappa_4 + 5\kappa_3^2 - 18)}{72} \tag{2.3}$$

And so, the ERP of the asymptotic test is obviously at order $n^{-1/2}$. Indeed, estimating $F_{1,T}(.)$ in z_α, we can provide the ERP of the asymptotic test

$$\begin{aligned} ERP_{as}^1 &= n^{-1/2} \phi(z_\alpha) \left[\frac{\kappa_3 (1 + 2z_\alpha^2)}{6} \right] \\ &\quad + n^{-1} \phi(z_\alpha) \left[\frac{\kappa_3^2 (3z_\alpha - 2z_\alpha^3 - z_\alpha^5)}{18} + \frac{\kappa_4 (z_\alpha^3 - 3z_\alpha)}{12} - \frac{z_\alpha (1 + z_\alpha^2)}{4} \right] \end{aligned} \tag{2.4}$$

Now, to compute the ERP of a bootstrap test, we have first to find q_α^* the $\alpha-$quantile of the bootstrap statistic's distribution. So, we replace κ_3 by κ_3^* which is its estimate as generated by the bootstrap DGP. For κ_4, the bootstrap DGP just needs to provide any consistant estimator because it does not appear in $q_{1\alpha}$ but only in $q_{2\alpha}$. The rejection condition of the bootstrap test is $T < q_\alpha^*$ at the order we consider and this condition is equivalent to $T - q_\alpha^* + q_\alpha < q_\alpha$. Now, we just have to compute the Edgeworth expansion $F^*(.)$ of $T - q_\alpha^* + q_\alpha$ to provide an estimate of its CDF and to eveluate it at q_α to find the ERP of the bootstrap test. If we consider a non parametric bootstrap or a CHM parametric bootstrap, we obtain exactly the same estimator $\hat{\kappa}_3$ of κ_3. Indeed, the first uses the empirical distribution of the residuals and the second the estimate of κ_3 provided by these residuals. Both these methods lead us to $\kappa_3^* = \kappa_3 + n^{-1/2} \left(s - 3w - \frac{3}{2}\kappa_3 q \right) + o_p\left(n^{-1/2}\right)$ which is random because s, w and q are. Then, we compute the Edgeworth expansion as described earlier and we obtain an ERP equal to zero at order $n^{-1/2}$ and nonnull at order n^{-1}. More precisely, we obtain

$$ERP_{BT_{nonpar}}^1 = n^{-1} z_\alpha \phi(z_\alpha) \frac{(1 + 2z_\alpha^2)(3\kappa_3^2 - 2\kappa_4)}{12} \tag{2.5}$$

Actually, Hall (1992) already obtained this result with a method quite different. Then, the parametric bootstrap DGP just uses a centered normal distribution to generate bootstrap samples. So, its third and fourth cumulants are zero and thus, they are not random. By using exactly the same method as previously, we obtain an ERP at order $n^{-1/2}$ as for an asymptotic test. Actually, this ERP is

$$\begin{aligned} ERP_{BT_{par}}^1 &= -n^{-1/2} \phi(z_\alpha) \left[\frac{\kappa_3 (1 + 2z_\alpha^2)}{6} \right] \\ &\quad + n^{-1} \phi(z_\alpha) \left[\frac{\kappa_4}{4} - \frac{7\kappa_3^2}{36} - \frac{\kappa_4 z_\alpha^2}{12} - \frac{\kappa_3^2 z_\alpha^4}{18} \right] \end{aligned} \tag{2.6}$$

So, for any κ_3 and κ_4, the ERP of the non parametric or of the CHM parametric bootstrap test is at order n^{-1}. If the disturbances are Gaussian, $\kappa_3 = \kappa_4 = 0$ then the ERP of the non parametric bootstrap test or, in an equivalent way, of the CHM parametric bootstrap is now at order $n^{-3/2}$. On the other hand, by considering 2.4 and 2.6, we see that if $\kappa_3 \neq 0$ then the dominant term of both ERP is the same and it is at order $n^{-1/2}$. Thus, if disturbances are asymmetrical, the parametric bootstrap fails to decrease ERP. However, if both κ_3 and κ_4 are null, i.e. if the first four moments of their distribution are the same as for a standard normal distribution, whichever bootstrap we use, then we obtain the same accuracy at order $n^{-3/2}$. Now if $\kappa_3 = 0$ and $\kappa_4 \neq 0$, the three tests have the same accuracy. This result is quite surprising, indeed it contradicts the Beran's assertion recalled in the introduction for asymptotic pivotal statistics. Actually, it just occurs because the ERP of the non parametric bootstrap test at order n^{-1} depends on the kurtosis coefficient κ_4 and not only on the skewness coefficient. Another special case appears in equation 2.5 , when we have $3\kappa_3^2 = 2\kappa_4$, the ERP of the non parametric bootstrap test is now at order $n^{-3/2}$. In the next part we find this condition again in the model $y = \beta_0 x + \sigma u$. We will test this special case in the simulation part. In the next part, we will use a Student not on the intercept but on other variables. We will consider two cases, with or without an intercept in addition to this variable.

3 A linear model

3.1 With intercept

Now, we consider linear models $y_t^{\cdot} = \mu_0^{\cdot} + \alpha_0 x_t^{\cdot} + Z_t\gamma + \sigma u_t^{\cdot}$ with $u_t^{\cdot} \rightarrow ii\Delta(0, 1, \kappa_3, \kappa_4)$ and where Z is a $n \times k$ matrix. By projecting both the left and the right hand side of the defining equation of the model on $M_{\iota Z}$[2] and by using Frisch-Waugh-Lovell Theorem (FWL Theorem), we obtain the model $M_{\iota Z} y_t^{\cdot} = \alpha M_{\iota Z} x_t^{\cdot} + residuals$ with $\sum_{t=1}^{n} M_{\iota Z} y_t = \sum_{t=1}^{n} M_{\iota Z} x_t = 0$. Obviously, in this last model, if we want to test the null $H_0 : \alpha = \alpha_0$ the test statistic is a Student with $k+2$ degrees of freedom. In this part, we just consider the model $y_t = \mu_0 + \alpha x_t + \sigma u_t$ with $u_t \rightarrow ii\Delta(0, 1, \kappa_3, \kappa_4)$. Or in an equivalent way, the model $y_t = \alpha x_t + \sigma \left(u_t - n^{-1/2} w\right)$ with $\sum_{t=1}^{n} y_t = \sum_{t=1}^{n} x_t = 0$ and two degrees of liberty. Moreover, we suppose that $Var(x) = 1$ without loss of generality. We obtain the asymptotic test

$$T = X \left(1 - n^{-1/2}\frac{q}{2} + n^{-1}\left(\frac{X^2}{2} + \frac{w^2}{2} - 1 + \frac{3q^2}{8}\right)\right) \tag{3.1}$$

The limit in probability of T is a standard normal distribution and we use Edgeworth expansions to provide an approximation of the CDF $F(.)$ of T at order n^{-1}. Following the same framework as in the previous part, we compute an approximation $q_\alpha =$

[2] $M_{\iota Z}$ is a projection matrix defined by $M_{\iota Z} = I - [\iota Z]\left([\iota Z]^{\perp}[\iota Z]\right)^{-1}[\iota Z]^{\perp}$ with ι a $n \times 1$ vector of 1, Z a matrix $n \times k$ of explicatives and I the matrix identity $(n+1) \times (n+1)$.

$z_\alpha + n^{-1/2}q_{\alpha 1} + n^{-1}q_{\alpha 2}$ of the $\alpha-$quantile of T.

$$q_{\alpha 1} = \frac{\kappa_3 m_3 \left(z_\alpha^2 - 1\right)}{6} \tag{3.2}$$

$$q_{\alpha 2} = z_\alpha^3 \left(\frac{(3\kappa_4 + 9)m_4 - 4\kappa_3^2 m_3^2 - 9\kappa_4 + 18}{72}\right) + z_\alpha \left(\frac{(-9\kappa_4 - 27)m_4 + 10\kappa_3^2 m_3^2 + 27\kappa_4 + 18}{72}\right) \tag{3.3}$$

And so, we obtain the ERP of the asymptotic test

$$ERP_{as}^2 = n^{-1/2}\kappa_3 m_3 \phi(z_\alpha)\left[\frac{1 + z_\alpha^2}{6}\right] + o\left(n^{-\frac{1}{2}}\right) \tag{3.4}$$

We recall that the rejection condition at order n^{-1} of the bootstrap test is $T < q_\alpha^*$ where q_α^* is the approximation of the $\alpha-$quantile of the bootstrap distribution and now, we can write this condition as $T < q_\alpha^* - q_\alpha + q_\alpha$. In order to obtain q_α^*, we just replace κ_3 by its estimate as generated by the bootstrap DGP in $q_{\alpha 1}$ and $q_{\alpha 2}$. If we deal with the non parametric or CHM parametric bootstrap, we obtain the same estimator as in the last part. We recall that $\kappa_3^* = \kappa_3 + n^{-1/2}\left(s - 3w - \frac{3}{2}\kappa_3 q\right) + o_p\left(n^{-1/2}\right)$. Now, we just use exactly the same framework as in the previous part and in this case, the CDF $F^*(.)$ of $T - q_\alpha^* + q_\alpha$ is the same as $F(.)$ the CDF of T at order n^{-1}. So when we evaluate $F^*(.)$ in q_α we obviously find an ERP at order $n^{-3/2}$.

Intuitively, we thought we would find an ERP of the bootstrap test at order n^{-1} as in the previous part. But according to Davidson and MacKinnon (2000), independence between the test statistic and the bootstrap DGP improves bootstrap inferences by an $n^{-1/2}$ factor, this is the reason why we have $F(.) = F^*(.)$ up to order n^{-1} and why we find an ERP of the bootstrap test at order $n^{-3/2}$. Actually, we do not have independence but a weaker condition. Let B be the bootstrap DGP, it comes from the random part of κ_3^*, and so the random part of B is the same as the random part of κ_3^*. Here, we just have $E\left(T^k B\right) = o\left(n^{-1/2}\right)$ for all $k \in \aleph$. But this is enough for $F^*(.)$ to be equal to $F(.)$ at order n^{-1}. In fact, we obtain this result just because we have $m_1 = 0$ by introducing the intercept in linear model. Then, for the parametric bootstrap the estimator of κ_3 is still zero. We proceed in the same way as for the non parametric bootstrap or CHM parametric bootstrap and we obtain an ERP at order $n^{-1/2}$ which is exactly the same than ERP_{as}^2 as defined in equation 3.4. This result is natural, at least when we consider the order of the ERP, indeed we use as much information to perform asymptotic and parametric bootstrap tests, the estimators of the parameter α and of the variance; and so, no more information, no more accuracy.

Now, let us consider $\kappa_3 = 0$ and $\kappa_4 \neq 0$, i.e. symmetrical distributions for the disturbances. The ERP of both the non parametric bootstrap and CHM parametric bootstrap are still at order $n^{-3/2}$. This is logical, indeed whether $\kappa_3 = 0$ or not, the κ_3^* we use in the bootstrap DGP is a random variable with the true κ_3 as expectation. So, we do not use more information coming from the true DGP which generates the original data. However, the ERP of asymptotic and parametric bootstrap tests are now at order n^{-1} but they are no longer equal. Indeed, when $\kappa_3 = 0$ the ERP of the parametric bootstrap test is

$$ERP^2_{BT_{par}} = n^{-1}\phi\left(z_\alpha\right)\kappa_4 \frac{\left(z_\alpha^3 - z_\alpha\right)\left(3 - m_4\right)}{24} + o\left(n^{-1}\right) \tag{3.5}$$

So the distribution of these two statistics are not the same, which we could have thought by only considering the case $\kappa_3 \neq 0$ and so, the parametric bootstrap still fails to improve the accuracy of inferences. The last case we have to consider is $\kappa_3 = \kappa_4 = 0$. Here, the parametric bootstrap estimates κ_3 and κ_4 perfectly because for a centered Gaussian distribution they are both equal to zero and it decreases its ERP at order $n^{-3/2}$ exactly as the non parametric bootstrap and CHM parametric bootstrap. Such a result just occurs because we use extra information by chance, using a bootstrap DGP very close to the original DGP. So, the parametric bootstrap test is better than the asymptotic test only when disturbances have skewness and kurtosis coefficients equal to zero whereas the non parametric bootstrap and CHM parametric bootstrap always improve the quality of the asymptotic t test. In particular, when disturbances are Gaussian, the parametric bootstrap has the same accuracy as the non parametric bootstrap and the CHM parametric bootstrap.

3.2 Without intercept

Let us consider linear models $y_t^{,} = \lambda_0 x_t^{,} + Z_t\beta + \sigma^{,} v_t$ with $v_t \rightarrow ii\Delta(0, 1, \kappa_3^{,}, \kappa_4^{,})$ and Z a matrix $n \times k$. In order to test $H_0 : \lambda = \lambda_0$, we can use the FWL theorem to test this hypothesis in the model $M_Z y_t = \lambda M_Z x_t + residuals$ in an equivalent way. So, any Student test can be seen as a particular case of the Student test connected to α_0 in the model $y_t = \alpha_0 x_t + \sigma u_t$ with $u_t \rightarrow ii\Delta(0, 1, \kappa_3, \kappa_4)$ and with $k+1$ degrees of freedom rather than only one. Now, we just consider this last model with only one degree of freedomand where we impose $n^{-1}\sum_{t=1}^{n} x_t^2 = 1$ without loss of generality. The t statistic we compute is given by

$$T = X\left(1 - n^{-1/2}\frac{q}{2} + n^{-1}\left(\frac{X^2}{2} - \frac{1}{2} + \frac{3q^2}{8}\right)\right) \tag{3.6}$$

As the limit in probability of T is still a standard normal distribution, we can follow exactly the same procedure as in the previous part in order to obtain the approximation of the CDF $F(.)$ of T at order n^{-1} by using Edgeworth expansions and then an approximation $q_\alpha = z_\alpha + n^{-1/2}q_{\alpha 1} + n^{-1}q_{\alpha 2}$ at order n^{-1} of the $\alpha-$quantile of T. Computations provide

$$q_{\alpha 1} = \frac{\left(\kappa_3 m_3 - 3\kappa_3 m_1\right) z_\alpha^2 - \kappa_3 m_3}{6} \tag{3.7}$$

$$\begin{aligned} q_{\alpha 2} = {} & z_\alpha^3\left(\frac{\left(3\kappa_4 + 9\right) m_4 - 4\kappa_3^2 m_3^2 + 6\kappa_3^2 m_1 m_3 + 18\kappa_3^2 m_1^2 - 9\kappa_4 + 18}{72}\right) \\ & + z_\alpha\left(\frac{\left(-9\kappa_4 - 27\right) m_4 + 10\kappa_3^2 m_3^2 - 6\kappa_3^2 m_1 m_3 - 9\kappa_3^2 m_1^2 + 27\kappa_4 - 18}{72}\right) \end{aligned} \tag{3.8}$$

As previously, the ERP of the asymptotic test T is at order $n^{-1/2}$ because $q_{\alpha 1} \neq 0$. Moreover, estimating $F(.)$ in z_α and if $\phi(.)$ is the PDF of a standard normal distribution, then we find that

$$ERP_{as}^{3} = n^{-1/2}\kappa_3\phi(z_\alpha)\left[\frac{m_3 + z_\alpha^2(m_3 - 3m_1)}{6}\right] + o\left(n^{-\frac{1}{2}}\right) \tag{3.9}$$

Considering this last equation, we see that if $m_1 = 0$, we have $ERP_{as}^3 = ERP_{as}^2$ Now, whatever the bootstrap we consider, we have to center the residuals to provide a valid bootstrap DGP because the intercept does not belong to the model. We recall again that the rejection condition at order n^{-1} of the bootstrap test is $T < q_\alpha^*$ where q_α^* is the approximation of the $\alpha-$quantile of the bootstrap distribution and now, we can write this condition as $T < q_\alpha^* - q_\alpha + q_\alpha$. In order to obtain q_α^*, we just replace κ_3 by its estimate as generated by the bootstrap DGP in $q_{\alpha 1}$ and $q_{\alpha 2}$.

If we deal with the non parametric or CHM parametric bootstrap, we obtain the same estimator as in the last part. We recall that $\kappa_3^* = \kappa_3 + n^{-1/2}\left(s - 3w - \frac{3}{2}\kappa_3 q\right) + o_p\left(n^{-1/2}\right)$. Now,we just use exactly the same framework as in the previous part. In this part, we do not have anymore $m_1 = 0$, so we do not obtain $E\left(T^k B\right) = o\left(n^{-1/2}\right)$ for all $k \in \aleph$ and in this case, the CDF $F^*(.)$ of $T - q^* + q$ is not equal to $F(.)$ the CDF of T. Now, by estimating $F^*(.)$ in q_α, we find the ERP of both non parametric and CHM parametric bootstrap

$$ERP_{BT_{nonpar}}^{3} = n^{-1}\phi\left(z_\alpha\right)\frac{m_1 z_\alpha\left(2\kappa_4 - 3\kappa_3^2\right)\left(m_3\left(z_\alpha^2 - 1\right) - 3m_1 z_\alpha^2\right)}{12} + o\left(n^{-1}\right) \tag{3.10}$$

Considering last result, we see this ERP is at order $n^{-3/2}$ if $\kappa_3 = \kappa_4 = 0$. However, there is an other way to obtain this order for the ERP, this is the special case we already obtained by testing a simple mean in the previous chapter, i.e. when $2\kappa_4 - 3\kappa_3^2 = 0$. We will consider this case in the simulation part in order to know if we can find the same accuracy as for Gaussian distributions or if it is just a theoritical result. Now, let us consider a parametric bootstrap test, the estimator of κ_3 used by bootstrap DGP is still zero, we proceed exactly in the same way as previously and we obtain an ERP which is the same than ERP_{as}^3 as defined in equation 3.9 but when, distribution of the disturbances is symmetrical, i.e. when $\kappa_3 = 0$, we now have

$$ERP_{BT_{par}}^{3} = n^{-1}\kappa_4 z_\alpha\frac{\left(m_4 - 3\right)\left(z_\alpha^2 - 3\right)}{24} + o\left(n^{-1}\right) \tag{3.11}$$

And now, explanations are the same as at the end of part 3.1

4 Simulation evidence

In the different figures provided in appendix, we seek to estimate the power of these four tests when the level of significance is $\alpha = 0.05$. For the asymptotic test, there are 100000 repetitions and for the bootstrap tests we limit the number of repetitions to 20000 and bootstrap repetitions to 999. We want to examine the convergence rate

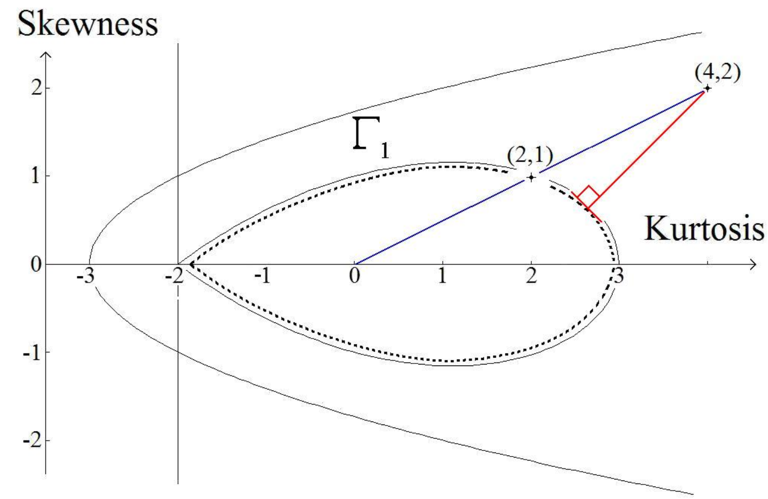

Figure 4.1: Methods of projection inside of Γ_1

when skewness and/or kurtosis coefficients of the distribution of the disturbances vary in the set Γ_1. So we fit the kurtosis or the skewness with a specific value and we allow the other one to vary in Γ_1. Asymptotic tests, and parametric and non parametric bootstrap tests, are performed in the usual way. By usual way we mean that we estimate the different models under the null and then the bootstrap residuals are Gaussian for the parametric bootstrap and resampled from the empirical distribution of the estimated residuals for the non parametric bootstrap, obviously, they are centered if the intercept does not belong to the model under H_0.

Then, in order to estimate the level of CHM parametric bootstrap tests, a new problem arises in generating bootstrap samples. Indeed, even if we generate disturbances following a standard distribution belonging to the set Γ_1, then the estimated standardised residuals do not always provide an estimate $(\hat{\kappa}_4, \hat{\kappa}_3)$ which belongs to Γ_1. So, we cannot directly use the bimodal method to generate bootstrap samples. This problem happens because estimates of higher moments are not very reliable for small sample size. We correct it by multiplying $(\hat{\kappa}_4, \hat{\kappa}_3)$ by a constant $k \in [0, 1]$. In our algorithm, we choose $k = \frac{10-i}{10}$ with i the first integer in $[0, 10]$ which satisfies $(k\hat{\kappa}_4, k\hat{\kappa}_3) \in \Gamma_1 \setminus Fr(\Gamma_1)$. Actually, this homothetic transformation respects the signs of both estimated cumulants $\hat{\kappa}_3$ and $\hat{\kappa}_4$ and never provides a couple on the frontier of Γ_1. Indeed, on this frontier, the distributions connected with the couple (κ_4, κ_3) which defines it are not continuous. We provide an example in figure 4.1 with $(\hat{\kappa}_4, \hat{\kappa}_3) = (4, 2)$, here we have $k = 5$ and we obtain the couple $(2, 1)$. Actually, we prefer this method rather than a method projecting directly on to a subset very close to the frontier of Γ_1, as described in figure 4.1, because it projects in the direction of the cumulants of a standard normal distribution, i.e. $(\kappa_4, \kappa_3) = (0, 0)$.

A last problem can occur when κ_3 is very close to 0. It is not a theoretical problem because solutions always exist in the set Γ_1; it is just a computational problem. So, if $\hat{\kappa}_3 < 2.10^{-2}$, then we fit κ_3 to zero, in order to suppress all the algorithmic problems which can occur.

4.1 $y_t = \mu_0 + u_t$

Here, we suppose that $\mu_0 = 0$ and $u_t \sim iid(0,1)$. Then, we test $H_0 : \mu = 0$ against $H_1 : \mu < 0$ because we deal with unilateral tests. We consider the following couples κ_3 and κ_4.

Couples $(\kappa_3; \kappa_4)$	$(0,8;0)$	$(0,4;0)$	$(0;0)$	$(-0,4;0)$	$(-0,8;0)$
Couples $(\kappa_3; \kappa_4)$	$(0,8;1)$	$(0,4;1)$	$(0;1)$	$(-0,4;1)$	$(-0,8;1)$

By considering figures 7.1 to 7.4, we check that parametric bootstrap tests and asymptotic ones provide the same rejection probabilities, in agreement with the theoretical results. In fact, even the signs of the ERP are the ones predicted by our computations. Then, as soon as $\kappa_3 \neq 0$, we check that asymptotic and parametric bootstrap tests have the same accuracy. Thus, we check that the parametric bootstrap test does not use more information than the asymptotic test when $\kappa_3 \neq 0$. Now, if we consider the next four figures, we first observe that the nonparametric bootstrap and CHM parametric bootstrap have the same convergence rates and they are better than parametric bootstrap or asymptotic tests. Thus, at the order we consider, we do not use more information than what is contained in the first four moments. Moreover, we can observe under-rejection and over-rejection phenomena; these are in agreement with theory. Actually, when $\kappa_3 > 0$, the tails of distributions are thicker on the left, so we have more chance to find a realization in the rejection area and to obtain over-rejection. Then, when $\kappa_3 < 0$, it is exactly the reverse.

4.2 $y_t = \mu_0 + \alpha_0 x_t + u_t$

Here, we suppose that $\mu_0 = 2$ and $\alpha_0 = 0$ with $u_t \sim iid(0,1)$ and $Var(x) = 1$. Then, we still have an unilateral test and we test $H_0 : \alpha = 0$ against $H_1 : \alpha < 0$. We consider the following couples κ_3 and κ_4.

Couples $(\kappa_3; \kappa_4)$	$(0,8;0)$	$(0,8;1)$	$(0;0)$	$(0;1)$

In this subsection, we observe quite the same results. When $\kappa_3 \neq 0$, convergence rates are less fast for both asymptotic and parametric bootstrap tests and they are the same in the both cases. On the other hand, non parametric bootstrap and CHM parametric bootstrap provide exactly the same results and these two methods provide better convergence rates especially when κ_3 is very different from zero.

4.3 $y_t = \alpha_0 x_t + u_t$

Here, we suppose that $\alpha_0 = 0$ with $u_t \sim iid(0,1)$ and $Var(x) = 1$. Then, we still have an unilateral test and we test $H_0 : \alpha = 0$ against $H_1 : \alpha < 0$. We consider the following couples κ_3 and κ_4.

Couples $(\kappa_3; \kappa_4)$	$(0,8;0)$	$(0,8;1)$	$(0;0)$	$(0;1)$

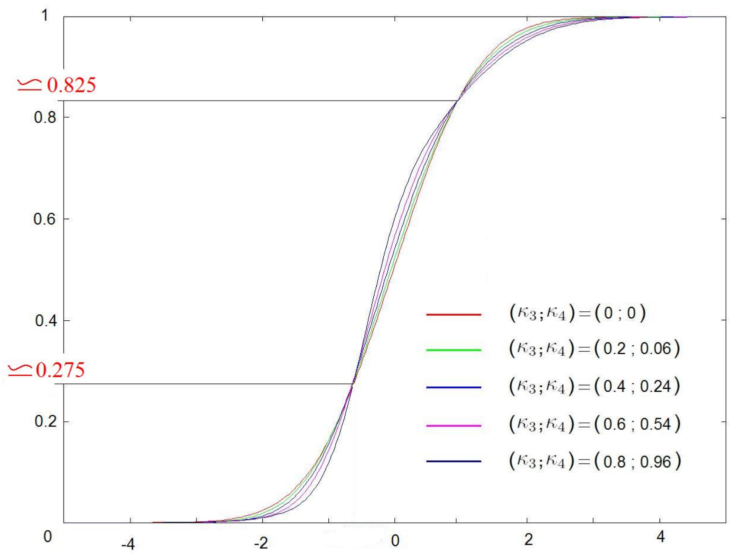

Figure 4.2: Examples of distributions such as $2\kappa_4 - 3\kappa_3^2 = 0$

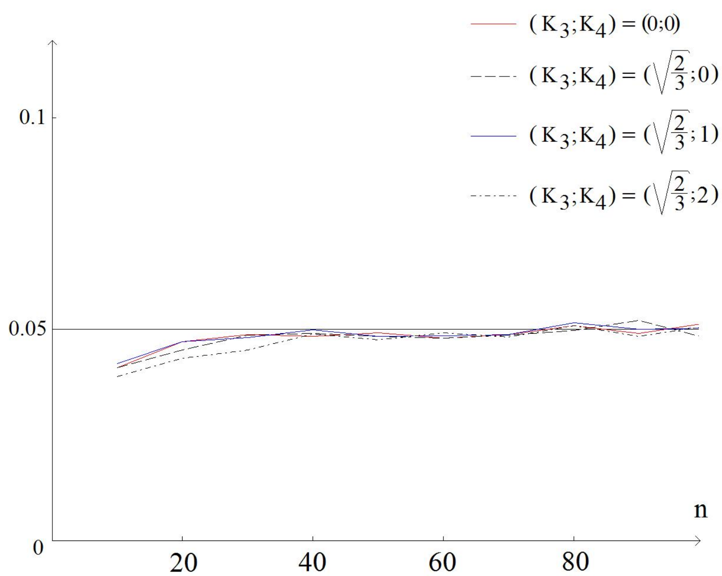

Figure 4.3: Rejection probabilities when $2\kappa_4 - 3\kappa_3^2 = 0$

Finally, in this last subsection, we still obtain the same results with convergence rates faster when $\kappa_3 = 0$ for asymptotic and parametric bootstrap tests than when $\kappa_3 = 1$. Moreover, convergence rates are the same for both methods. Then, for non parametric bootstrap and CHM parametric bootstrap, convergence rates are the same and we still observe over-rejection when $\kappa_3 > 0$.

4.4 The case $2\kappa_4 - 3\kappa_3^2 = 0$

In the parts 2 and 3.2, we show we theoretically can obtain an extra refinement for the non parametric bootstrap and the CHM parametric bootstrap if the third and fourth cumulants of the disturbances check $2\kappa_4 - 3\kappa_3^2 = 0$. On the figure 4.2, we see that the distributions of this family have the same quantiles 0.275 and 0.825. In order to check if these special cases provide a refinment, we just simulate ERPs for the non-parametric bootstrap framework and for the model $y_t = \mu + u_t$. In this model, we use four different disturbances; the gaussian distribution as a special case satisfying the equality $2\kappa_4 - 3\kappa_3^2 = 0$, an other distribution which satisfies it, $(\kappa_3; \kappa_4) = (\sqrt{2/3}; 1)$ and two other distributions defined by $(\kappa_3; \kappa_4) = (\sqrt{2/3}; 0)$ and $(\kappa_3; \kappa_4) = (\sqrt{2/3}; 2)$. Then, we observe on the figure 4.3 that we obtain a very slight refinment compared to the case where $\kappa_4 = 2$. When $\kappa_4 = 0$, the ERP is almost the same and it is difficult to conclude against an improvement other than theoritical.

5 Conclusion

In this paper, we provide the ERPs for all one-restriction tests which can occur in a linear regression model for asymptotic tests and different bootstrap tests by using Edgeworth expansions. These results clarify how the bootstrap DGP needs to estimate correctly not only the third cumulant but also the fourth, at least at order of the unity, to provide first and second order refinements. So, we introduce a new parametric bootstrap method which uses the four first moments of the estimated residuals. Asymptotically, this method has the same convergence rates as the non parametric bootstrap and they are better than asymptotic and parametric bootstrap when $\kappa_3 \neq 0$. Actually, the accuracy of a test is directly linked to the information it uses. As both asymptotic and parametric tests use the same information coming from the first two moments (except when $\kappa_3 = \kappa_4 = 0$), they provide the same convergence rates. On the other hand, non parametric bootstrap and CHM parametric bootstrap use extra information from third and fourth moments and they provide better convergence rates. We resume the different results of this paper in the tables A and B.

Test	$\kappa_3 \neq 0 \neq \kappa_4$	$\kappa_3 = 0 \neq \kappa_4$	$\kappa_3 = \kappa_4 = 0$
Asymptotic	$O\left(n^{-\frac{1}{2}}\right)$	$O\left(n^{-1}\right)$	$O\left(n^{-1}\right)$
Parametric bootstrap	$O\left(n^{-\frac{1}{2}}\right)$	$O\left(n^{-1}\right)$	$O\left(n^{-\frac{3}{2}}\right)$
Non parametric and CHM bootstrap	$O\left(n^{-1}\right)$	$O\left(n^{-1}\right)$	$O\left(n^{-\frac{3}{2}}\right)$

Table A. "Models $y_t = \mu_0 + \sigma_0 u_t$ and $y_t = \alpha_0 x_t + \sigma_0 u_t$"

Actually, even if we did not do the simulations, it seems logical to think that the results would be the same for any test in a linear regression model with exogenous explicatives and disturbances which are iid. Obviously, it could be very different for other models. Now, let us imagine another model with rejection probabilities such as those in the figure 5.1.

Test	$\kappa_3 \neq 0 \neq \kappa_4$	$\kappa_3 = 0 \neq \kappa_4$	$\kappa_3 = \kappa_4 = 0$
Asymptotic	$O\left(n^{-\frac{1}{2}}\right)$	$O\left(n^{-1}\right)$	$O\left(n^{-1}\right)$
Parametric bootstrap	$O\left(n^{-\frac{1}{2}}\right)$	$O\left(n^{-1}\right)$	$O\left(n^{-\frac{3}{2}}\right)$
Non parametric and CHM bootstrap	$O\left(n^{-\frac{3}{2}}\right)$	$O\left(n^{-\frac{3}{2}}\right)$	$O\left(n^{-\frac{3}{2}}\right)$

Table B. "Model $y_t = \mu_0 + \alpha_0 x_t + \sigma_0 u_t$"

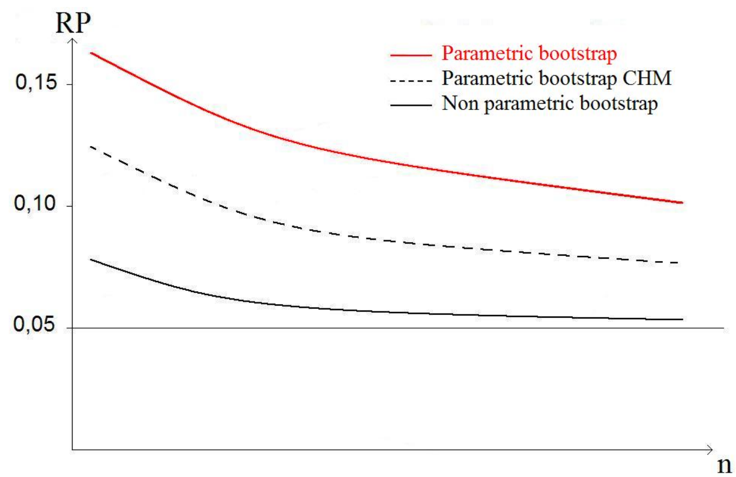

Figure 5.1: Hypothetical rejection probabilities

In this example, it would be obvious that other cumulants than the first four ones appear in the dominant term of the rejection probability. Actually, if we could develop other methods to control more than the first four cumulants of a distribution, it would be possible to know the information a bootstrap test uses because non parametric bootstrap always uses all the estimated moments of the residuals. Now, the obvious question is : "Does a non parametric bootstrap test always use the information contained in the first four moments?". CHM parametric bootstrap could help to answer this question.

Then, our simulations show that if disturbances are normal, parametric bootstrap can provide better results than non parametric bootstrap or CHM parametric bootstrap, however, it is quite impossible in small samples to know if the distribution is normal or not. Moreover, even if CHM parametric bootstrap and non parametric bootstrap have almost the same rejection probabilities, the first can reject H_0 when

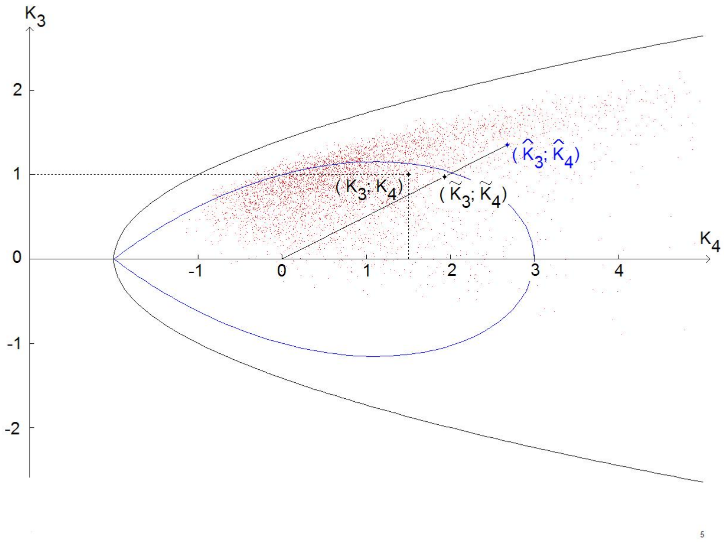

Figure 5.2: Distribution of couples $(\kappa_3; \kappa_4)$ estimated.

the second does not and conversely. We show with the help of the figure 5.2 why the CHM parametric bootstrap can be more accurate than the non parametric bootstrap. In this figure, there are 5000 points which are estimated couples $(\hat{\kappa}_3, \hat{\kappa}_4)$ from a distribution for which $(\kappa_3, \kappa_4) = (1, 1.5)$. By considering this figure, we immediately see that a lot of couples $(\hat{\kappa}_3, \hat{\kappa}_4)$ are outside the set Γ_1 as defined in the figure 4.1. In these cases, CHM parametric bootstrap, by projecting towards normality uses a DGP closer to the true distribution than the non parametric bootstrap. And so, CHM parametric bootstrap will provide better inferences than the non parametric bootstrap.

So, we think that we must use a principle of precaution using bootstrap and compute the three bootstrap tests. Actually, this procedure (using the three tests) could be seen as a very restricted maximized Monte-Carlo test (MMC). Instead of using a grid of values for the couple $(\kappa_3; \kappa_4)$ which are the nuisance parameters of the model, we only use these three bootstrap tests and so three couples (κ_3, κ_4). If one of the three tests does not reject the null, we do not reject it. Thus, this method would decrease the computing times compared with a MMC test, a current research seek to know if the rejection probabilities are the same in both cases.

Actually, we try to go further and to connect more closely the MMC and the bootstrap by using the framework of the CHM parametric bootstrap. Let us suppose we can provide a $1 - \alpha_1$ confidence region for the couple $(\kappa_3; \kappa_4)$ where α_1 must be selected correctly. The main difference with the classical MMC is that the confidence region depends on the estimates of κ_3 and κ_4 as obtained by estimating the initial model and which belong to the confidence region. Then, we build a grid on this confidence region and for each point of this grid we use the framework introduced for the CHM parametric bootstrap and we compute a p-value. Finally, we maximize the p-value on this grid and we do not reject the null if the result of this maximization is larger than the selected level. This procedure could be a good alternative to the bootstrap, however we will have to check if it is not too conservative.

6 Bibliography

Andrews, D. K. W. (2005). "Higher-order Improvements of the Parametric Bootstrap for Markov Processes, *Identification and Inference for Econometric Models: A Festschrift in Honor of Thomas J. Rothenberg*, ed. by D.W.K. Andrews and J.H. Stock. Cambridge, UK: Cambridge University Press.

Beran, R. (1988). "Prepivoting test statistics: A bootstrap view of asymptotic refinements", *Journal of the American Statistical Association*, 83, 687-697.

Buhlman, P. (1998). "Sieve Bootstrap for Smoothing in Nonstationary Time Series", *Annals of Statistics*, 26, 1, 48-83.

Davidson, R. and MacKinnon J. G. (1993). "Estimation and inference in econometrics", *New-York, Oxford University Press.*

Davidson, R. and MacKinnon J. G. (1999*b*). "The size distortion of bootstrap tests", *Econometric Theory*, **15**, 361-76.

Davidson, R. and MacKinnon J. G. (2000). "Improving the reliability of bootstrap tests with the fast double bootstrap", *Computational Statistics and Data Analysis*, **51**, 3259–3281.

Davidson, R. and MacKinnon J. G. (2001). "The power of asymptotic and bootstrap tests", *Journal of Econometrics*, **133**, 421-441.

Davidson, R. and MacKinnon J. G. (2004). "Econometric theory and methods", *New York, Oxford University Press.*

Dufour, J. M.(2006). "Monte Carlo tests with nuisance parameters: A general approach to finite-sample inference and nonstandard asymptotics in econometrics", *Journal of Econometrics*, **133**, 443-477.

Hall P. (1992). "The bootstrap and the Edgeworth expansion", *New York, Springer Velag.*

Hill G. W. et Davis, A. W. (1968). "Generalized asymptotic expansions of Cornish-Fisher type", *Ann. Math. Statist. 39: 1264-1273.*

MacKinnon J. G. (2007). "Bootstrap hypothesis testing", *working paper n1127*, Queen's Economic Department.

Parr W. C. (1983). "A Note on the Jackknife, the Bootstrap and the Delta Method Estimators of Bias and Variance", *Biometrika 70, 3, 719-722.*

Treyens, P.-E. (2005). "Two methods to generate centered distributions controlling skewness and kurtosis coefficients", *working paper n2007-12*, GREQAM, Université de la Méditerranée.

7 Appendix

7.1 $y_t = \mu_0 + u_t$

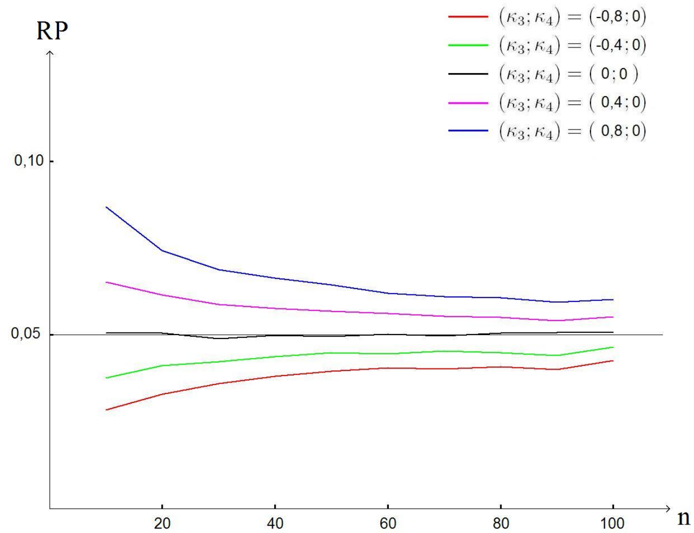

Figure 7.1: RP of asymptotic tests with $\kappa_4 = 0$ and κ_3 varying.

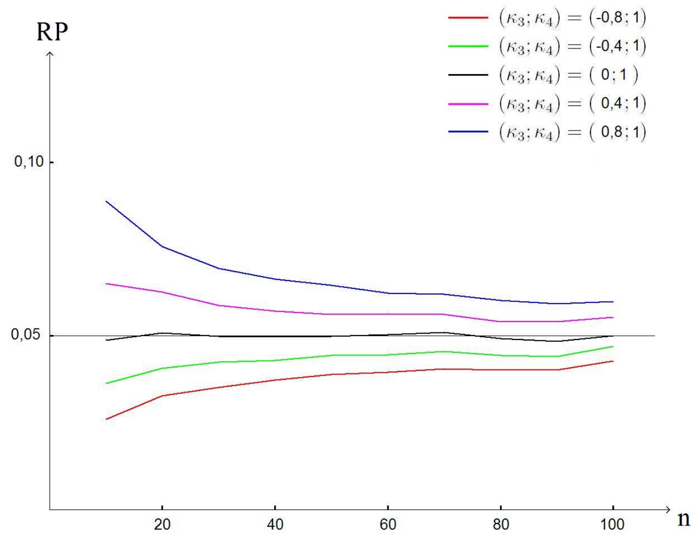

Figure 7.2: RP of asymptotic tests with $\kappa_4 = 1$ and κ_3 varying.

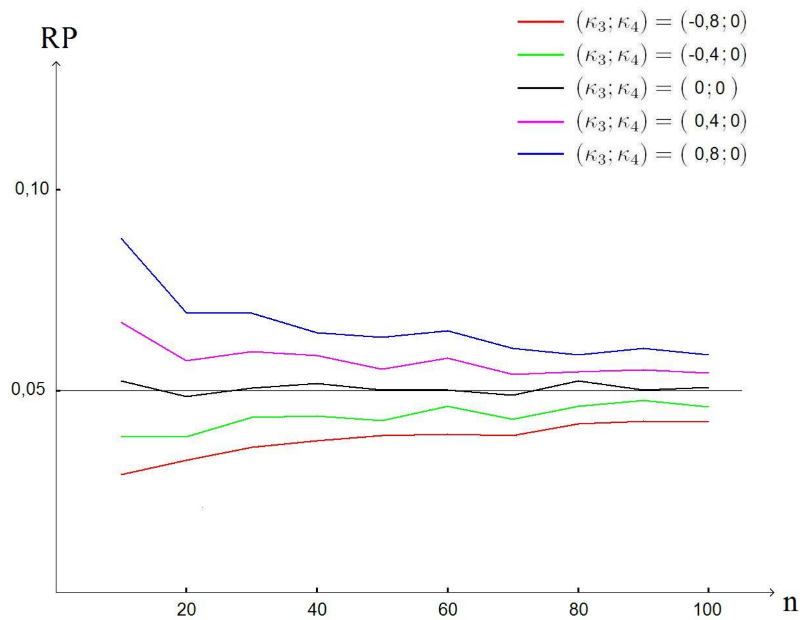

Figure 7.3: RP of parametric bootstrap tests with $\kappa_4 = 0$ and κ_3 varying.

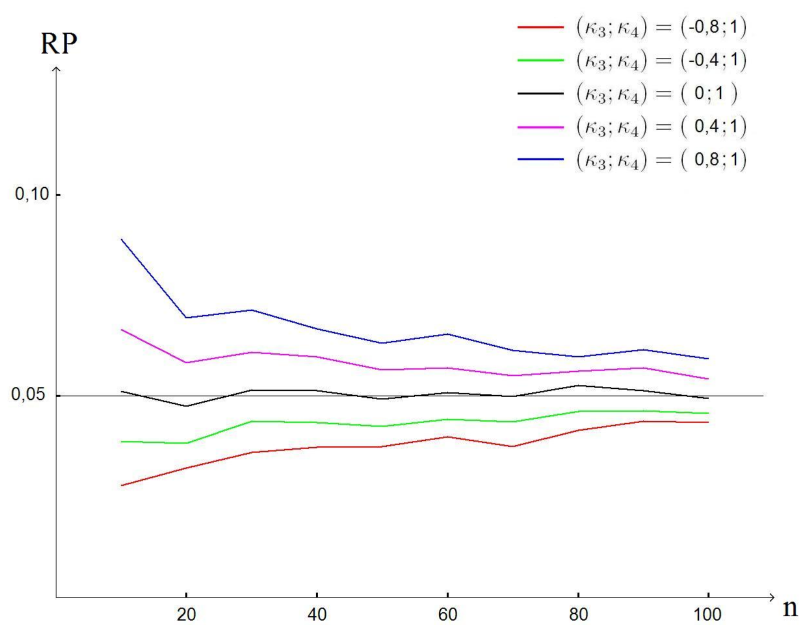

Figure 7.4: RP of parametric bootstrap tests with $\kappa_4 = 1$ and κ_3 varying.

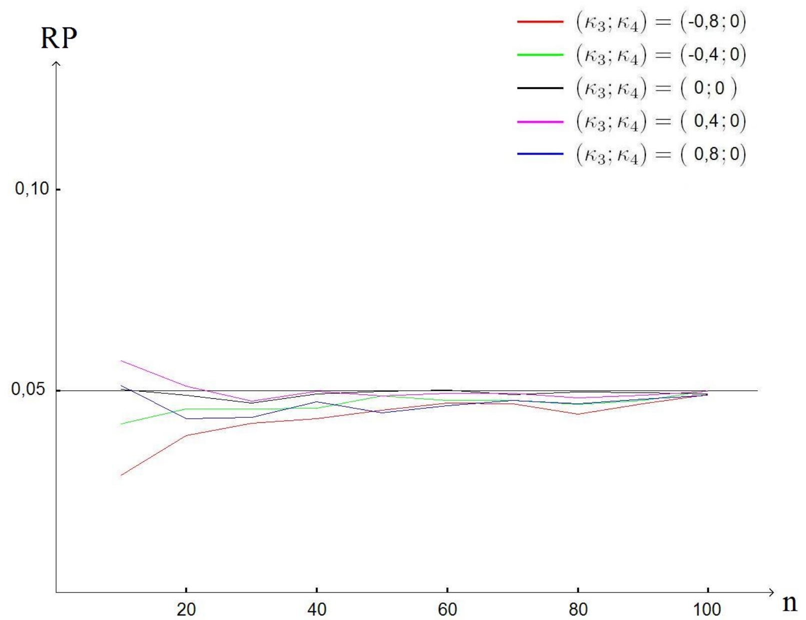

Figure 7.5: RP of non-parametric bootstrap tests with $\kappa_4 = 0$ and κ_3 varying.

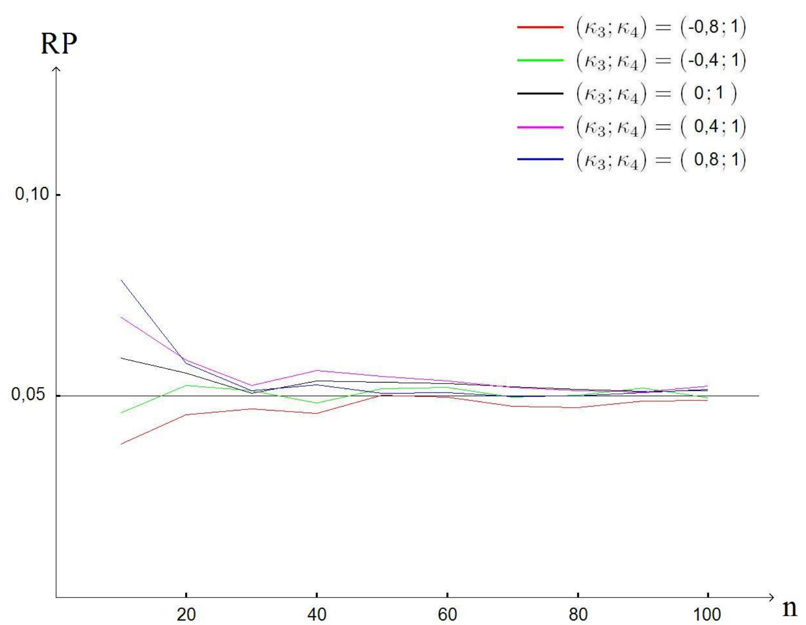

Figure 7.6: RP of non-parametric bootstrap tests with $\kappa_4 = 1$ and κ_3 varying.

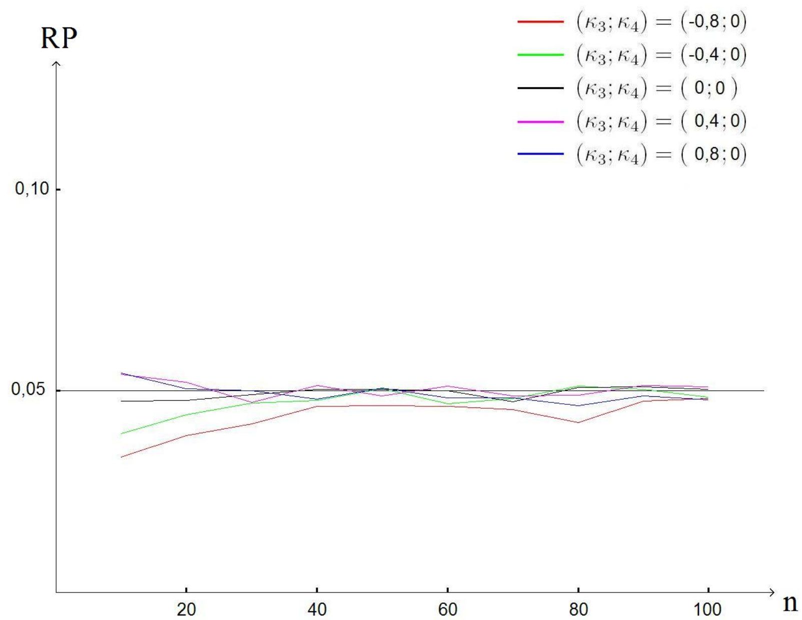

Figure 7.7: RP of CHM parametric bootstrap tests with $\kappa_4 = 0$ and κ_3 varying.

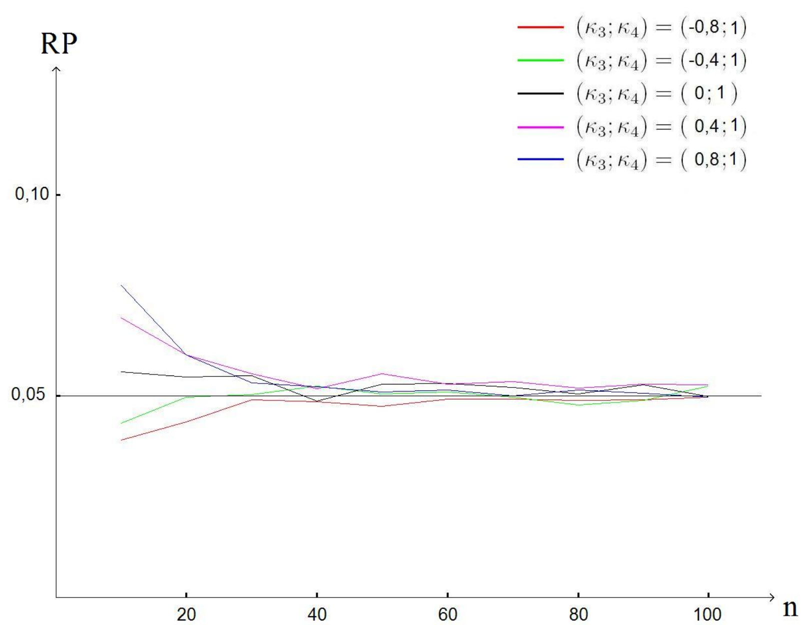

Figure 7.8: RP of CHM parametric bootstrap tests with $\kappa_4 = 1$ and κ_3 varying.

7.2 $y_t = \mu + \alpha_0 x_t + u_t$

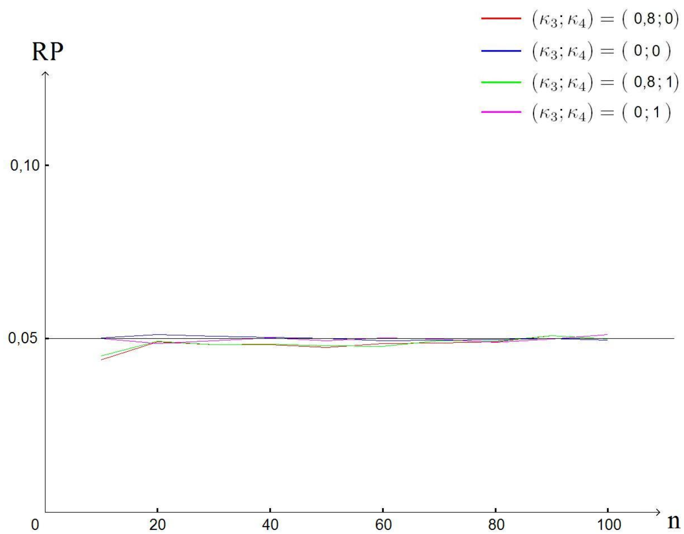

Figure 7.9: RP of the asymptotic tests.

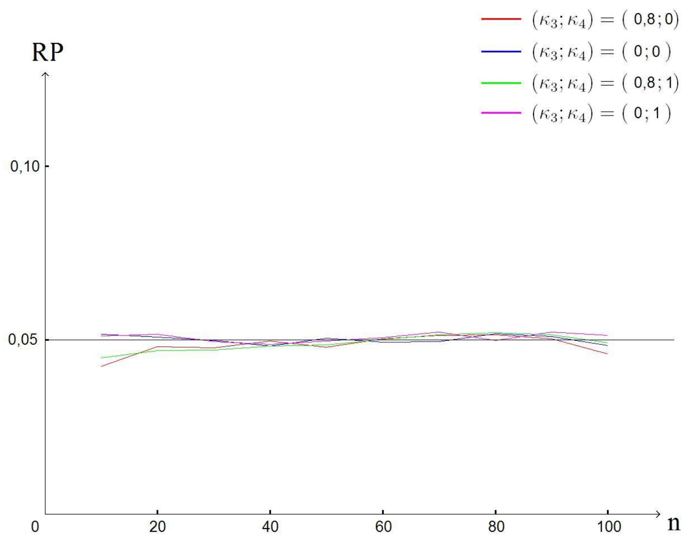

Figure 7.10: RP of the parametric bootstrap tests.

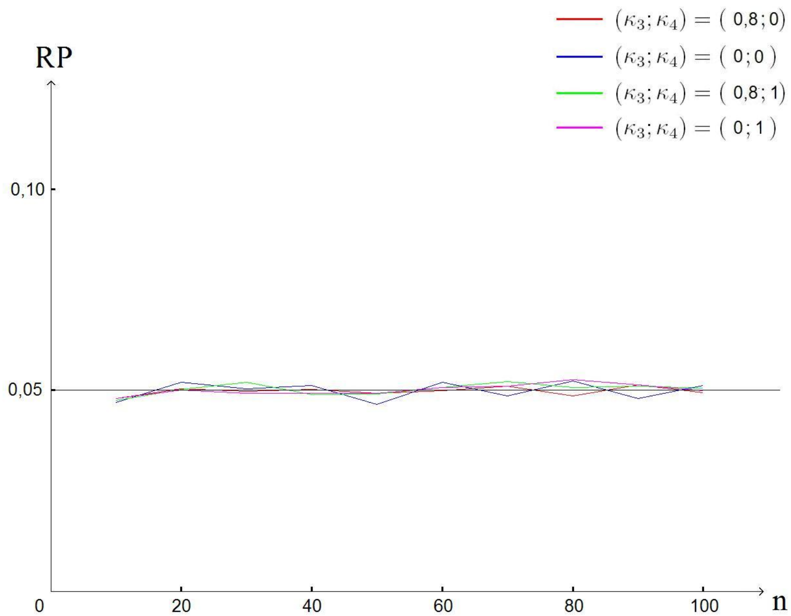

Figure 7.11: RP of the non-parametric bootstrap tests.

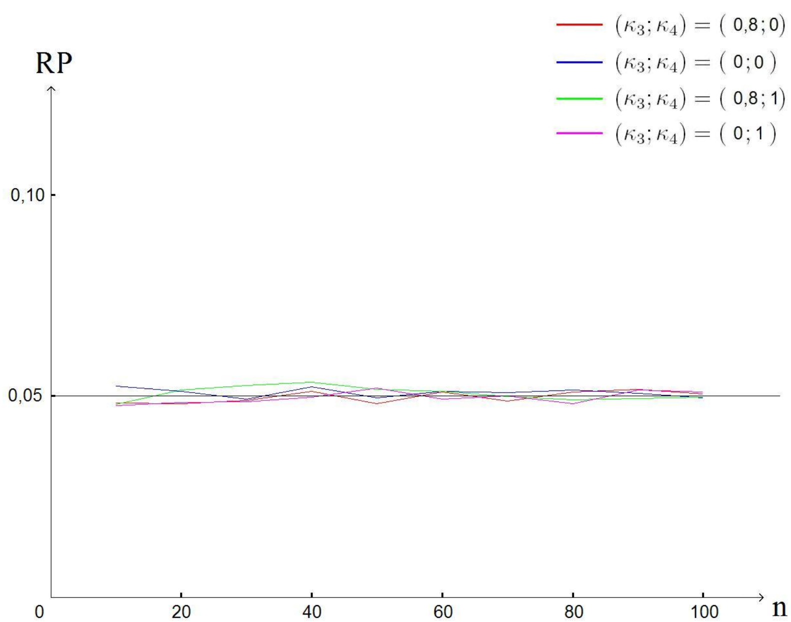

Figure 7.12: RP of the CHM parametric bootstrap tests.

7.3 $y_t = \alpha_0 x_t + u_t$

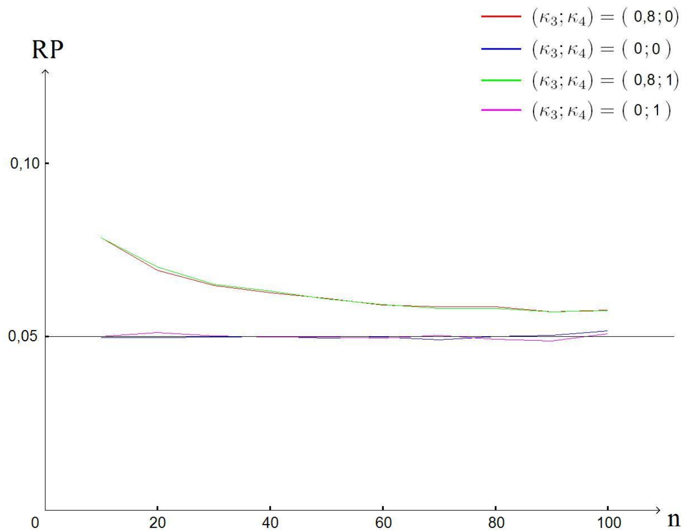

Figure 7.13: RP of the asymptotic tests.

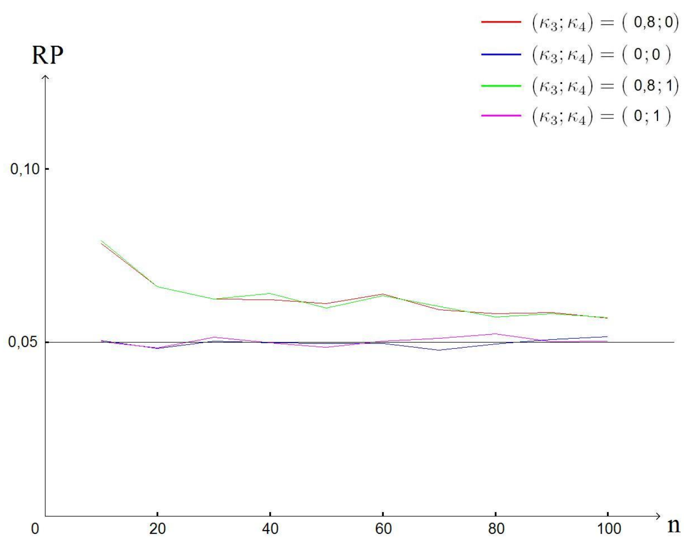

Figure 7.14: RP of the parametric bootstrap tests.

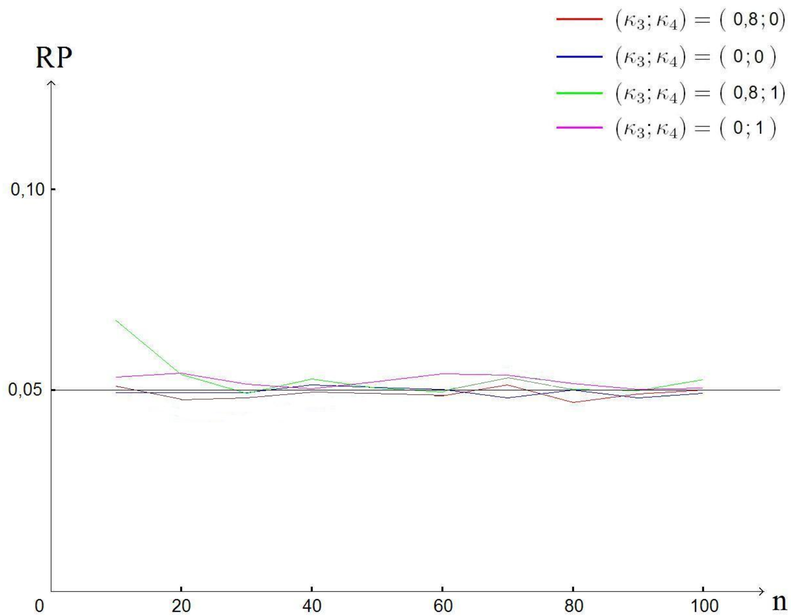

Figure 7.15: RP of the non-parametric bootstrap tests.

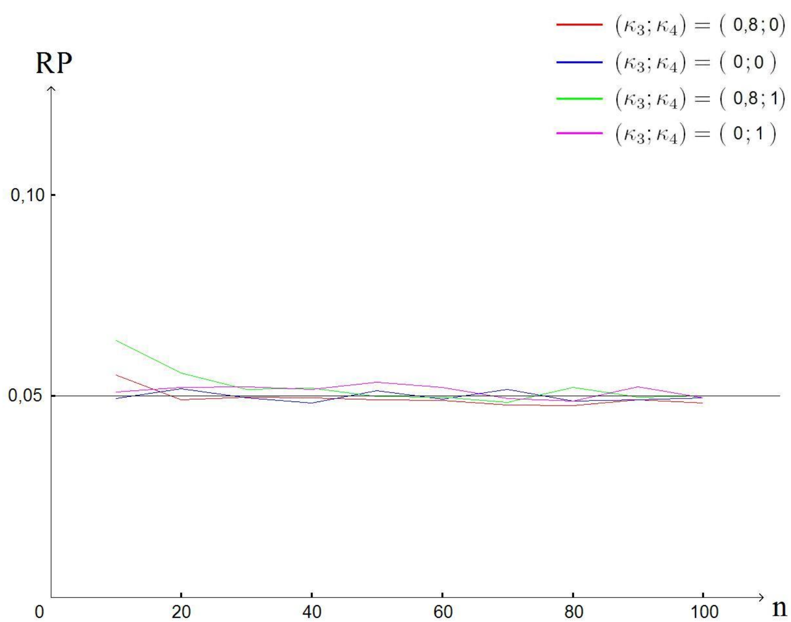

Figure 7.16: RP of the CHM parametric bootstrap tests.

7.4 Different variables.

We give all variables we use to compute the ERPs of this paper. Here, μ_i denotes the uncentered moment of the disturbances distribution at order i.

$$m_1 \equiv n^{-1} \sum_{t=1}^{n} x_t \quad with \quad \lim_{n \to \infty} m_1 = O(1) \tag{7.1}$$

$$m_2 \equiv n^{-1} \sum_{t=1}^{n} x_t^2 \quad with \quad \lim_{n \to \infty} m_2 = O(1) \tag{7.2}$$

$$m_3 \equiv n^{-1} \sum_{t=1}^{n} x_t^3 \quad with \quad \lim_{n \to \infty} m_3 = O(1) \tag{7.3}$$

$$m_4 \equiv n^{-1} \sum_{t=1}^{n} x_t^4 \quad with \quad \lim_{n \to \infty} m_4 = O(1) \tag{7.4}$$

$$w \equiv n^{-1/2} \sum_{t=1}^{n} u_t \quad with \quad p \lim_{n \to \infty} w = N(0, 1) \tag{7.5}$$

$$q \equiv n^{-1/2} \sum_{t=1}^{n} \left(u_t^2 - 1\right) \quad with \quad p \lim_{n \to \infty} q = N(0, 2 + \kappa_4) \tag{7.6}$$

$$s \equiv n^{-1/2} \sum_{t=1}^{n} \left(u_t^3 - \kappa_3\right) \quad with \quad p \lim_{n \to \infty} s = N(0, \mu_6 - \kappa_3^2) \tag{7.7}$$

$$k \equiv n^{-1/2} \sum_{t=1}^{n} \left(u_t^4 - 3 - \kappa_4\right) \quad with \quad p \lim_{n \to \infty} q = N\left(0, \mu_8 - (3 + \kappa_4)^2\right)\right) \tag{7.8}$$

$$X \equiv n^{-1/2} \sum_{t=1}^{n} (u_t x_t) \quad with \quad p \lim_{n \to \infty} X = N(0, m_2) \tag{7.9}$$

$$Q \equiv n^{-1/2} \sum_{t=1}^{n} \left((u_t^2 - 1) x_t\right) \quad with \quad p \lim_{n \to \infty} Q = N(0, (2 + \kappa_4) m_2) \tag{7.10}$$