GREQAM
Groupement de Recherche en Economie
Quantitative d'Aix-Marseille - UMR-CNRS 6579
Ecole des Hautes Etudes en Sciences Sociales
Universités d'Aix-Marseille II et III

Document de Travail
n°2007-12

TWO METHODS TO GENERATE CENTERED DISTRIBUTIONS CONTROLLING SKEWNESS AND KURTOSIS COEFFICIENTS

Pierre-Eric TREYENS

Juin 2007

Two methods to generate centered distributions controlling skewness and kurtosis coefficients

Pierre-Eric TREYENS

Université de la Méditerranée
GREQAM

Université de Montréal
CIREQ

June 7, 2007

Abstract

Whatever the econometric model which we study; any simulation requires a perfectly definite DGP. Thus, even if all software can generate standard normal distributions, we need methods not programmed to control higher moments. For all these methods, we need to estimate the parameters connected to the desired values of the higher moments. Within the framework of Monte Carlo experiments, the computing times of this estimate are then not very important. Indeed, once these parameters estimated, we can re-use them and the computing time of simulations does not suffer from it. On the other hand, for a parametric bootstrap which would consider the first four moments, the computing time is then multiplied by the number of desired simulations. So we understand the importance to provide a method which makes it possible to find the parameters attached to the first four estimated moments as quickly as possible. So, we must trade off between speed and possibilities of the method. The goal of this paper is to provide two new methods which control these first four moments and to compare their speed with that of the already existing methods.

0 Introduction.

In econometrics, we must check most theoretical researches by using simulations. In this framework, the first thing to do is to specify suitably the model we want to study or in an equivalent way defining a suitable data generating process (DGP). Of course, disturbances are the most important part of the model; indeed, they are the only random part of the DGP. So, their distribution will condition all the statistical properties of the model, not only the properties of the estimators, but the ones of the tests too. Obviously, we are always able to centre and to standardise the distribution of residuals, but it is harder to fit exact values ex-ante to third and fourth moments of this distribution. Nevertheless, it is very interesting to control these moments. For example, according to Pearson and Please [1975], we can find many variables which do not admit Gaussian distributions. So, we can suppose that disturbances in any model are not always Gaussian and so, we must provide methods to consider non-normal distributions for the disturbances. We recall that the cumulant of order three, which we usually call the skewness coefficient, calculates the asymmetry of the distribution. When the distribution is Gaussian, this coefficient is equal to zero. If it is positive, the probability density function (PDF) is leaning to the left; if it is negative it is leaning to the right. The cumulant of order four, which we call the kurtosis coefficient from the Greek word meaning shoulder, calculates as far as it is concerned the concentration of the distribution. Like the skewness coefficient, it is equal to zero when the distribution is Gaussian. If it is positive the PDF data are concentrated around the mean; if it is negative data are distributed more uniformly.

Fleishman [1978] provided a method which made it possible to control the first four moments of a variable Y by defining it by

$$Y = a + bZ + cZ^2 + dZ^3 \tag{0.1}$$

Where Z is a standard normal distribution. Later, Tadikamalla [1980] referred to several methods, among which the Fleishman method, which also made it possible to control these moments. For each one of these methods, it was then necessary to solve a nonlinear system of four equations in four unknowns (the first four moments). Tadikamalla then calculated the computing times necessary to the resolution of these systems. According to him, the Fleishman method is fastest, the computing times of the other methods being at least higher by almost 40%. We do not provide these computing times, the introduction of new processors having made them obsolete, but the orders of magnitude are the same.

In the preliminary part, we will define cumulants of a distribution exactly and we will see how they are linked to the moments of this distribution. In the second part, we will present a method to generate data from unimodal distributions of which we will control exactly second, third and fourth moments. Even if we can control these three moments, we will study standard distributions more precisely and we will compare the theoretical set of admissible couples (κ_3, κ_4) with the set we can obtain with our method. The third part will present another method which generates possibly bimodal distributions of which we will control second, third and fourth moments. As in the last part, we will particularly study standardised distributions we can obtain using this method and we will compare the set of couples (κ_3, κ_4) we can obtain with the last sets. In the fourth part, we will provide examples of PDFs we can obtain with these methods. Then, we will compare the PDFs of different distributions with the same first four moments in order to know how moments higher than four modify graphs of the PDF. This part will also show how we can provide a better approximation of the distribution of the residuals than the estimation used by a paametric bootstrap, i.e. a Gaussian distribution. In the fifth and last part, we will compare the computing times necessary for the two methods and the Fleishman one by using the freeware Maxima[1].

1 Preliminaries

Let X be a real random variable. Its first characteristic function $f_X(u)$ is the Fourier transform of its probability density $f_X(u) = E\left(e^{iuX}\right)$. So if we suppose that for all $k \in \aleph$, moments of order k exist, we can compute them by using Taylor expansion of this exponential to obtain $E\left(X^k\right) = \frac{1}{i^k}\left[\frac{\partial f_X(u)}{\partial u^k}\right]_{/u=0}$. The second characteristic function $f_X^{(2)}(u)$ of X is the logarithm of its first function. By using a Taylor-McLaurin development of $f_X^{(2)}(u)$, some constants κ_k appear. They are the cumulants of order k of the distribution of X and we can compute them by using the following formula $\kappa_k(X) = \frac{1}{i^k}\left[\frac{\partial^k f_X^{(2)}(u)}{\partial u^k}\right]_{/u=0}$. So, for any centered random variable and defining μ_k as the moment of order k of its distribution we obtain

$$\kappa_1 = 0, \quad \kappa_2 = \mu_2, \quad \kappa_3 = \mu_3, \quad \kappa_4 = \mu_4 - 3\mu_2^2 \quad and \quad \kappa_5 = \mu_5 - 5 - 10\mu_2\mu_3 \tag{1.1}$$

By definition, cumulants of order three and four are the skewness and kurtosis coefficients. We are going to be interested especially in these two coefficients; indeed, according to the values we are going to assign them, we could obtain an asymmetric distribution by varying the skewness coefficient and/or concentrate the data by varying the kurtosis. Now, we give two new definitions to simplify notations subsequent

Definition 1 *Let X be a random variable of which the first four cumulants exist and are respectively equal to κ_1, κ_2, κ_3 and κ_4. So, we will write*

$$X \sim \Delta(\kappa_1, \kappa_2, \kappa_3, \kappa_4) \tag{1.2}$$

Thus, for a standard normal distribution, we directly obtain

$$Z \sim \Delta(0, 1, 0, 0)$$

Definition 2 *Let $Y = (y_1, \ldots, y_n)$ be a vector of which each element y_t is a draw independently and identically distributed from $X \sim \Delta(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$. So we will write*

$$y_t \sim ii\Delta(\kappa_1, \kappa_2, \kappa_3, \kappa_4) \tag{1.3}$$

[1] This freeware can be downloaded on http://www.ma.utexas.edu/maxima.html

2 A method to generate unimodal distributions.

We developed this method to generate easily data for simulations with non Gaussian disturbances of which the skewness and/or kurtosis coefficients are not equal to zero. As the fundamental property of disturbances in an econometric model is that their expectation is equal to zero, we can just consider the system of equations (1.1).
Let p be a uniform random variable defined on the interval $[0, 2a]$, with $a > 0$. Let N_1 and N_2 be two normal random variables of respective expectations μ_1 and μ_2 and of respective variances σ_1 and σ_2. We suppose these three random variables are independent and we define X in the following way

$$X = pN_1(\mu_1, \sigma_1) + (2a - p)N_2(\mu_2, \sigma_2) \tag{2.1}$$

Remembering we seek centered distributions, we can easily compute the first four moments of X. Obviously, these four equations just consider moments of the random variable X. Well, it is clearly more natural to consider its cumulants rather than its moments. Indeed, thanks to the cumulants we can have ex-ante a precise idea of the distribution of X. As we already saw, skewness and kurtosis coefficients are respectively cumulants of order three and four. So, if we choose a positive (respectively negative) value for skewness coefficient of random variable X, the graph of its PDF will lean toward the left (respectively right) and/or if we choose a positive (respectively negative) value for kurtosis coefficient, it will be less (respectively more) uniformly distributed compared to the graph of a normal random variable. So, if we consider the cumulants rather than the moments, we obtain the following system that we will have to solve under the constraints $\sigma_1 \geq 0$ and $\sigma_2 \geq 0$.

$$\begin{aligned}
\kappa_2 &= \frac{4a^2\left(\sigma_1 + \sigma_2 + \mu_1^2\right)}{3} \\
\kappa_3 &= 4a^3\left(\mu_1\sigma_1 - \mu_1\sigma_2\right) \\
\kappa_4 &= \frac{16a^4\left(3\sigma_1^2 + 3\sigma_2^2 + \sigma_1\sigma_2 + 4\mu_1^2\sigma_2 + 4\mu_1^2\sigma_1 + \mu_1^4\right)}{5} - 3\kappa_2^2
\end{aligned} \tag{2.2}$$

Theorem 1 *Let Z_1 and Z_2 be two independent normal random variables of respective expextations and variances μ_1 and μ_2 and σ_1 and σ_2.*
Let p be a uniform random variable on $[0, 2a]$ independent of Z_1 and Z_2.
Now, let X be the random variable defined by $X = pZ_1 + (1-p)Z_2$ of which the four first cumulants are κ_1, κ_2, κ_3 and κ_4.
If $\kappa_1 = 0$, then the set of couples (κ_3, κ_4) that X is able to admit is invariant to the uniform distribution p selected.

Proof 1 *See Appendix*

Thanks to this theorem, we can choose any value for a. In order to simplify computations, we will suppose that $a = \frac{1}{2}$. Even if it is possible to provide exact solutions when the second cumulant $\kappa_2 = 1$, these last ones are too complex to use. So we prefer work with the approximated solutions provided by freeware *Maxima* fitting κ_3 and κ_4. Now, we only consider standard distributions. If we want the variance of

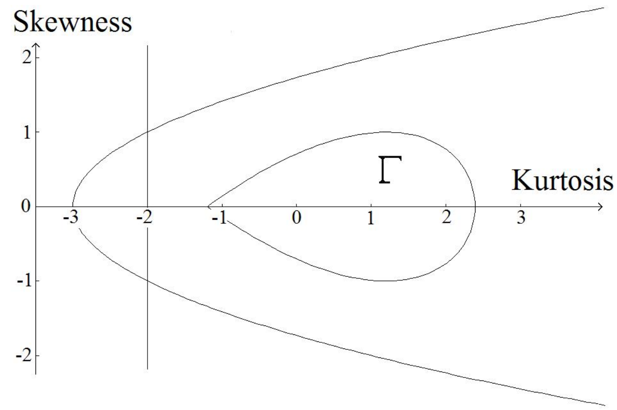


Figure 2.1: Set of admissible couples Γ

the disturbances to vary, we just have to multiply them by the appropriate standard error. By doing this, we modify in a non linear way the cumulants of the distribution, where by non linear, we just mean that for all random variable $Y \sim \Delta(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$, we have $\sigma Y \sim \Delta(\sigma\kappa_1, \sigma^2\kappa_2, \sigma^3\kappa_3, \sigma^4\kappa_4)$. Obviously, doing this does not change the general graph of the PDF, indeed, standard error is positive and multiplying by a positive integer does not change the sign of cumulants.

To begin with, we will give the theoretical couples (κ_3, κ_4) which can exist. After, we will give the couples this method can provide. Let X be a continuous random variable of which each moment exists and of which the PDF is $f_X(x)$. Let V be the vector space of the functions $g : \Re \to \Re$ which satisfy $\int_{-\infty}^{\infty} f_X(x)g^2(x)dx < \infty$. For all $(g, h) \in V^2$, we define $\langle .,. \rangle : V^2 \to \Re$ by

$$\langle .,. \rangle : (g, h) \to \langle g, h \rangle = \int_{-\infty}^{\infty} f_X(x)g(x)h(x)dx \tag{2.3}$$

As $\langle .,. \rangle : V^2 \to \Re$ is a scalar product, by applying the Cauchy-Schwartz inequality, we obtain $\langle g, h \rangle^2 \leq \langle g, g \rangle \langle h, h \rangle \quad \forall (g, h) \in V^2$. Now, suppose that $g : x \to g(x) = 1$ and $h : x \to h(x) = x^2$. By using previous results, we obtain $\kappa_4 \geq -2$. Suppose then that $g : x \to g(x) = x$ and $h : x \to h(x) = x^2$ we obtain $\kappa_3^2 \leq 3 + \kappa_4$. We emphasize the fact that even if many other constraints connect κ_3 and κ_4 with other cumulants, this is the only one which connect only κ_3 and κ_4 together. By using previous results, we can obtain the following system of equations where Θ_1 and Θ_2 are two continuous functions in σ_1 and σ_2.

$$\kappa_3 = \pm\Theta_1(\sigma_1, \sigma_2) = \pm\frac{(\sigma_1 - \sigma_2)\sqrt{3 - \sigma_1 - \sigma_2}}{2}$$
$$\kappa_4 = \Theta_2(\sigma_1, \sigma_2) = \frac{6\sigma_1 + 6\sigma_2 - 5\sigma_1\sigma_2 - 6}{5} \tag{2.4}$$

The natural constraint of this system is just that σ_1 and σ_2 are positive or equal to zero. The set of the acceptable couples (κ_3, κ_4) for which σ_1 and/or σ_2 are equal to zero is the frontier of the convex set Γ of the figure (2.1). We obtain this frontier by setting one variance to zero and by solving the system (2.4). Under these conditions, we have

$$\kappa_3 = \pm \frac{(5\kappa_4 + 6)\sqrt{\frac{12-5\kappa_4}{6}}}{12} \qquad \text{with} \qquad \kappa_4 \in [-1,2;2,4] \tag{2.5}$$

As the functions Θ_1 and Θ_2 are continuous in σ_1 and, σ_2 and as on the frontier of Γ at least one of the variances σ_1 or σ_2 is equal to zero, the couples $(\kappa_3, \kappa_4) \in \Gamma$ are associated with positive variance and so all these couples are acceptable. On the other side, the couples $(\kappa_3, \kappa_4) \notin \Gamma$ are related with negative or complex variances, and so they can not be accepted.

3 A method to generate bimodal distributions.

As in the previous part, let N_1 and N_2 be two normal random variables. Now, let p be a binary variable and a be the probability that $p = 1$. We suppose these three variables are independent and we define

$$B = pN_1(\mu_1, \sigma_1) + (1-p)N_2(\mu_2, \sigma_2) \tag{3.1}$$

By using the same method than in the previous part, we obtain the following equations, still under the constraints $\sigma_1 \geq 0$ and $\sigma_2 \geq 0$.

$$\begin{aligned} \kappa_2 &= \mu_1^2 + a\sigma_1 + (1-a)\sigma_2 \\ \kappa_3 &= \mu_1^3(2a-1) + 3(a-1)\mu_1\sigma_2 + 3a\mu_1\sigma_1 \\ \kappa_4 &= -2\mu_1^4 + 3a(1-a)(\sigma_1-\sigma_2)^2 \end{aligned} \tag{3.2}$$

We find the greatest set of admissible couples (κ_3, κ_4) by setting a to $\frac{1}{2}$ and as in the previous part we just consider standard distributions. Now, by using equations (3.2) we can obtain the following system of equations where Φ_1 and Φ_2 are two continuous functions in σ_1 and σ_2.

$$\begin{aligned} \kappa_3 &= \pm\Phi_1(\sigma_1, \sigma_2) = \pm\frac{3(\sigma_1-\sigma_2)\sqrt{2-\sigma_1-\sigma_2}}{2\sqrt{2}} \\ \kappa_4 &= \Phi_2(\sigma_1, \sigma_2) = \frac{(\sigma_1-\sigma_2)^2 + 8\sigma_1 + 8\sigma_2 - 8\sigma_1\sigma_2 - 8}{4} \end{aligned} \tag{3.3}$$

Now, by using the same arguments as previously, we can find the equations which connect κ_3 and κ_4 and which provide the frontier of the set of admissible couples (κ_3, κ_4) this method provides.

$$\kappa_3 = \pm 3\sqrt{3 - \sqrt{6+\kappa_4}}\left(\sqrt{6+\kappa_4} - 2\right) \tag{3.4}$$

On the figure (3.1), we see immediatly that Γ is included in the set Γ_1 of admissible couples for the bimodal case. Moreover, by considering figures $D.a$, $D.b$ and $E.a$ we see this method does not just generate bimodal distributions. Obviously we can also find unimodal distributions which are particular cases of this method as a special case of this method.

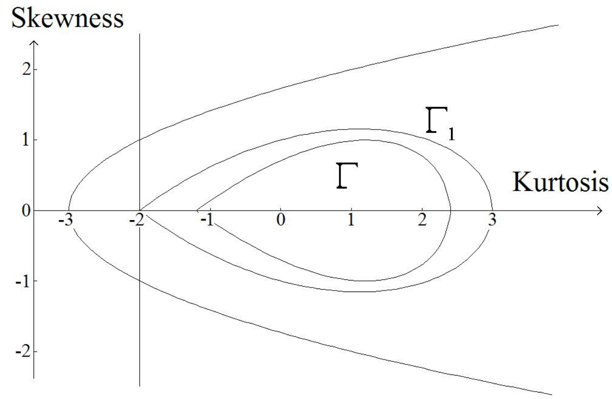


Figure 3.1: Sets of admissible couples Γ and Γ_1

4 Simulation evidence

4.1 Examples of PDF

Considering the set Γ, we see six special cases. Firstly, when $\kappa_3 = 0$ we see on figures $(A.a)$ and $(B.b)$ that the PDFs oscillate between a graph where data are concentrated all around zero and a uniform distribution. Secondly, when $\kappa_4 = 0$, in figures $(C.a)$ and $(C.b)$ we see that PDF are nearly discontinuous. Figures $(C.a)$ and $(C.b)$ just represent the highest and the lowest values that κ_3 can take. Now, by continuous distortion we can have a good idea of all PDFs this method can provide. Now, when we consider the graphs of the PDFs provided by our second method, it is not as clear. Indeed, for couples (κ_3, κ_4) on the frontier, we always obtain discontinuous distributions because σ_1 or σ_2 is zero. So figures $(G.a)$, $(G.b)$ and H give us more information. For example, with figure H we can have an more accurate idea of what a continuous distortion between couples $(\kappa_3, \kappa_4) = (0, -2)$ and $(\kappa_3, \kappa_4) = (0, 3)$ can provide. Obviously, to go from figure $(E.a)$ to figure $(E.b)$ we could find some graphs of unimodal distributions as a special case of the bimodal method.

Now, by using both these methods and the Fleishman one we can fit κ_3 and κ_4 with specific values and compare the PDF we obtain. This will enable us to know how the fifth and the sixth moment of a standard distribution influence the graph of its PDF.

4.2 Example of estimation

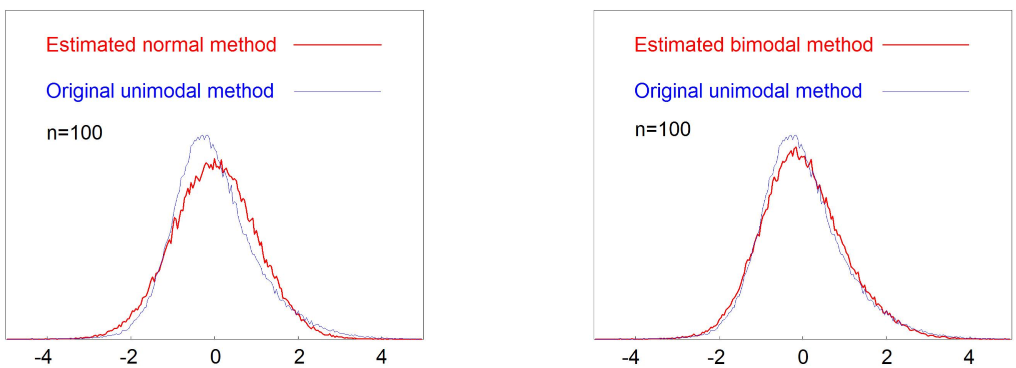
In this subsection, we will show we can provide a better estimation of the disturbances in a linear model of regression by considering the first four moments than by considering only the first two; I.e. what we use for parametric bootstrap. Let consider for example the following DGP Θ

$$y_t = 3x_t + 2 + u_t \quad \text{with} \quad u_t \sim ii\Delta\,(0, 1, 0.6, 1) \quad \text{and} \quad t \in [0; n], \quad n = 100, 1000 \tag{4.1}$$

Now, by using the first method we can provide the random variable u which satisfies $u \sim \Delta(0, 1, 0.6, 1)$ with $p \sim U[0, 1]$. Then, this random variable generates the sample y and we estimate it by using OLS. So, we obtain the estimated centered moments μ_i^n.

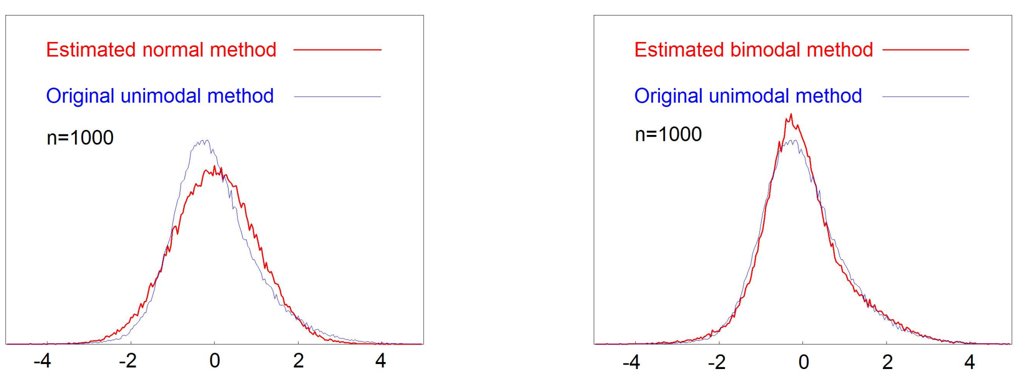
n	$\hat{\mu}_2^n$	$\hat{\mu}_3^n$	$\hat{\mu}_4^n$
100	0.998002	0.321370	0.299812
1000	1.018832	0.552135	1.283436

$$u = pN\,(\,0.68637431480031\,,\ 2.138603696098563\,) \\ +(1-p)N\,(\,-0.68637431480031\,,\ 0.39028658936775\,)$$



(4.1.a) $\Delta\,(0, 1, 0.6, 1)$ and $N\,(0, \hat{\kappa}_2)$ (4.1.b) $\Delta\,(0, 1, 0.6, 1)$ and $\Delta\,(0, \hat{\kappa}_2, \hat{\kappa}_3, \hat{\kappa}_4)$

Figure 4.1: Comparaison of two estimation methods with $n = 100$



(4.2.a) $\Delta\,(0, 1, 0.6, 1)$ and $N\,(0, \hat{\kappa}_2)$ (4.2.b) $\Delta\,(0, 1, 0.6, 1)$ and $\Delta\,(0, \hat{\kappa}_2, \hat{\kappa}_3, \hat{\kappa}_4)$

Figure 4.2: Comparaison of two estimation methods with $n = 1000$

Then, we just have to use the bimodal method to provide two random variable $u^n \sim \Delta(0, \hat{\mu}_2^n, \hat{\mu}_3^n, \hat{\mu}_4^n)$ for $n = 100, 1000$. We want to make it clear that we do not use the same method to generate and to estimate the model. Indeed, a parametric bootstrap using Gaussian disturbances will use also Gaussian terms for the bootstrap DGP and a bias could appear. These new random variables are defined by

$$u^{100} = pN\,(\,0.3214607754734\,,\,1.22790422701744\,) \\ +(1-p)N\,(\,-0.3214607754734\,,\,0.56142575373345\,)$$

$$u^{1000} = pN\,(\,0.29294677628732\,,\,1.561268209083119\,) \\ +(1-p)N\,(\,-0.29294677628732\,,\,0.30476009861559\,)\,,$$

where p is the bimodal random variable defined in the previous part. Then, by comparing figures $(4,1,a)$ and $(4,1,b)$ we immediately see that estimation obtained with our method is better than estimation which only uses the first two moments. Indeed, the two PDF are completely superposed on figure $(4.1.b)$ and clearly not on figure $(4.1.a)$ and we also obtain the same result when the sample size is $n = 1000$. Then, our first idea will obviously be to use this method in a parametric bootstrap framework using all the first four moments of the distribution of the residuals and not only the first two moments as we do by generating bootstrap disturbances following a centered Gaussian distribution.

5 Comparison of the computing times

In order to apply these methods in a parametric bootstrap framework considering the first four moments of the residuals, we now have to consider the computing times of these methods. Indeed, for a classical parametric bootstrap we just compute an estimate of the variance of the residuals and we use this estimation to generate bootstrap disturbances following a Gaussian distribution and so we should not solve a system of equations. Now, if we deal with a parametric bootstrap considering higher moments (CHM),we first have to estimate the first four moments of the residuals and then to solve a system of four equations with four unknowns. This is the computing times of this part of the method which will change the final computing times of the parametric bootstrap CHM considerably.

According to Tadikamalla [1980], the Fleishman method fails to generate all couples (κ_3, κ_4) which can exist theoretically, but on the other hand, it is much faster than all the other existing methods. Both methods developed in this paper are also unable to generate all these couples. In fact some couples can be generated with the bimodal method and not with the Fleishman one and conversely. But if we suppose that disturbances can not be distributed in a very different way from the economic variables they are connected with, these methods can generate the majority of the distributions which we can accept for disturbances, see Pearson and Please [1975]. Moreover, the frontier of admissible couples we can obtain with the Fleishman method is just provided approximately whereas both bimodal and unimodal distribution provide the set of admissible couples exactly. This will be important when we bootstrap a test. Indeed we could know ex-ante if we can provide a solution linked with the estimates of the first four moments. And so, there will be no risk for couples with no solution which would lengthen the computing time.

In order to estimate these different systems of equations, we will use the freeware Maxima [2]. We choose couples (κ_3, κ_4) such that $\kappa_3 \in [-0.5, 0.5]$ and $\kappa_4 \in [-0.4, 1.5]$, so we are sure that the three methods have solution for the couples they must estimate and we will just evaluate the different computing times. We choose 15 couples arbitrarily and we provide the computing time. Then, we provide the average computing time for each method. So, we obtain

(κ_3, κ_4)	Unimodal	Bimodal	Fleishman
$(0.5, -0.4)$	66.7 ms	66.7 ms	12.45 s
$(0.5, 0)$	66.7 ms	66.7 ms	10.60 s
$(0.5, 0.4)$	66.7 ms	66.7 ms	29.95 s
$(0.5, 0.8)$	66.7 ms	66.7 ms	61.21 s
$(0.5, 1.2)$	66.7 ms	50.0 ms	24.32 s
$(0.25, -0.4)$	66.7 ms	66.7 ms	12.88 s
$(0.25, 0)$	66.7 ms	50.0 ms	11.73 s
$(0.25, 0.4)$	66.7 ms	66.7 ms	13.90 s
$(0.25, 0.8)$	66.7 ms	66.7 ms	12.52 s
$(0.25, 1.2)$	50.0 ms	66.7 ms	12.13 s
$(0, -0.4)$	66.7 ms	116.7 ms	2.27 s
$(0, 0)$	133.3 ms	0.0 ms	2.30 s
$(0, 0.4)$	33.3 ms	116.7 ms	2.27 s
$(0, 0.8)$	150.0 ms	116.7 ms	2.25 s
$(0, 1.2)$	166.7 ms	116.7 ms	2.37 s
Average computing times	80.02 ms	73.36 ms	14.21 s

It appears clearly that the Fleishman method is very slow compared with both other methods which provide nearly the same computing times. So, we cannot consider a parametric bootstrap using the first four moments with the Fleishman method. Simulations would be really too long. Actually, if you just want to deal with Monte-Carlo experiments with extreme distributions you could generate what you wish by using the Fleishman method whereas both unimodal and bimodal methods fail. Now, if you just want classical distributions, these two methods are enough and they are the only ones which make it possible to bootstrap a test considering higher moments.

6 Concluding remarks

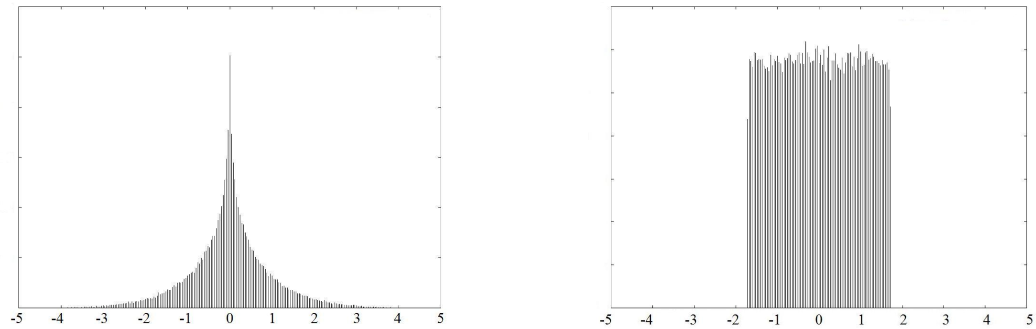
As we saw , these two methods can easily provide a very wide panel of distributions and almost all distributions which can occur in any economics framework, see Pearson

[2]All computations were made with a processor AMD 4400+ dualcore and a ram of 2GO

and Please [1975]. Indeed, we can choose exactly the sort of PDF we want either concentrating data by fitting κ_4 or providing an asymmetrical PDF by fitting κ_3. In the last part, we showed how we could use these methods in order to estimate residuals in an econometric model. The only problem was to estimate correctly both cumulants κ_3 and κ_4. Indeed, their estimates are very volatile and when the sample size is too small, it is very difficult to provide sharp ones. This is the reason why we consider $t = 1000$ in this part. Now, the idea will be to use this method in a parametric bootstrap framework to see if we can decrease the error in the rejection probability of parametric bootstrap tests. Then, as estimates of (κ_3, κ_4) are rarely sharp enough for small size samples, and as $\Gamma \in \Gamma_1$ the best way to simulate a bootstrap test will be to generate the error terms of our DGP with the unimodal method and to estimate (κ_3, κ_4) with the bimodal one. Moreover, this method will make it possible to disconnect the DGP which generates the disturbances from the bootstrap DGP. Indeed, with a classical parametric bootstrap, simulations often use Gaussian disturbances at the same time for the disturbances and the bootstrap disturbances, which is very questionable.

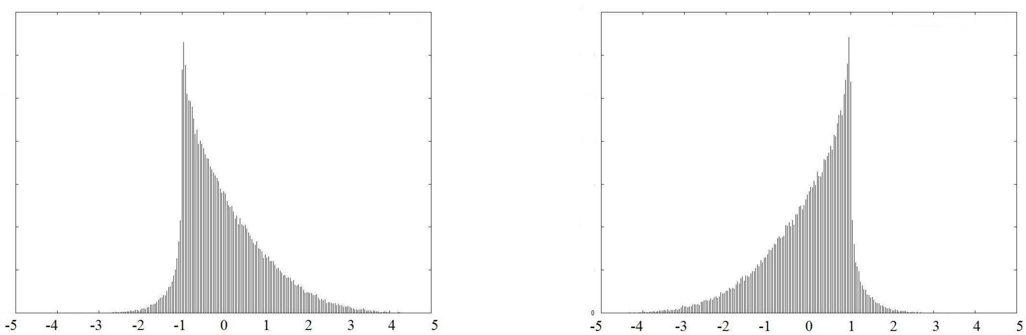
7 Figures

7.1 Unimodal method



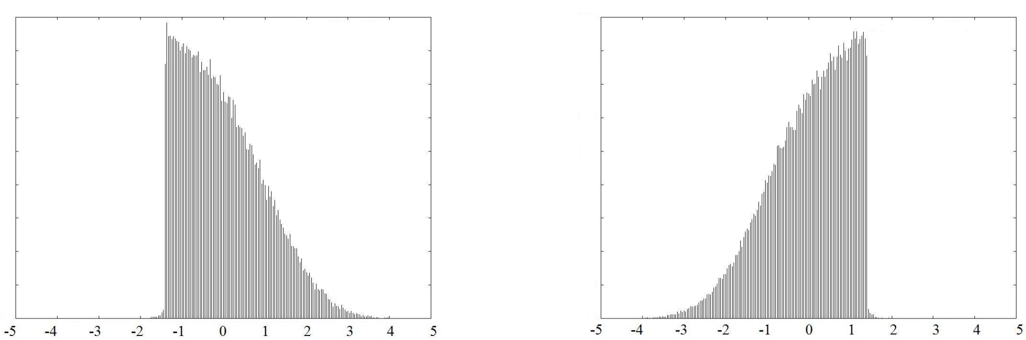
(A.a) $\Delta\,(0, 1, 0, 2.4)$ (A.b) $\Delta\,(0, 1, 0, -1.2)$

Figure A: Unimodal method



(B.a) $\Delta\,(0, 1, 1, 1.2)$ (B.b) $\Delta\,(0, 1, -1, 1.2)$

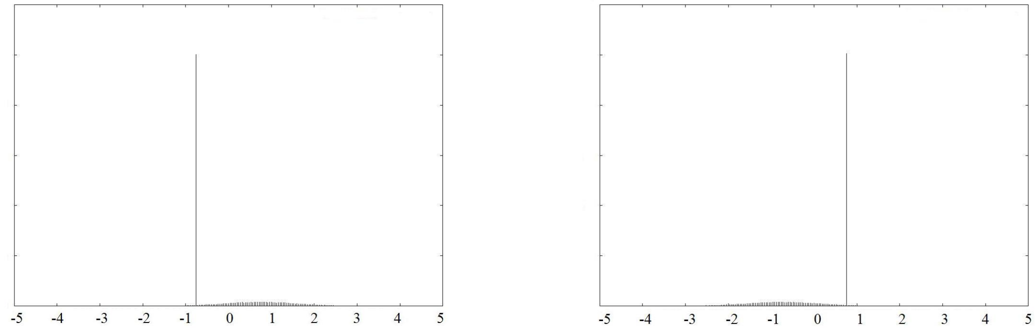
Figure B: Unimodal method



(C.a) $\Delta\left(0, 1, \frac{\sqrt{2}}{2}, 0\right)$ (C.b) $\Delta\left(0, 1, -\frac{\sqrt{2}}{2}, 0\right)$

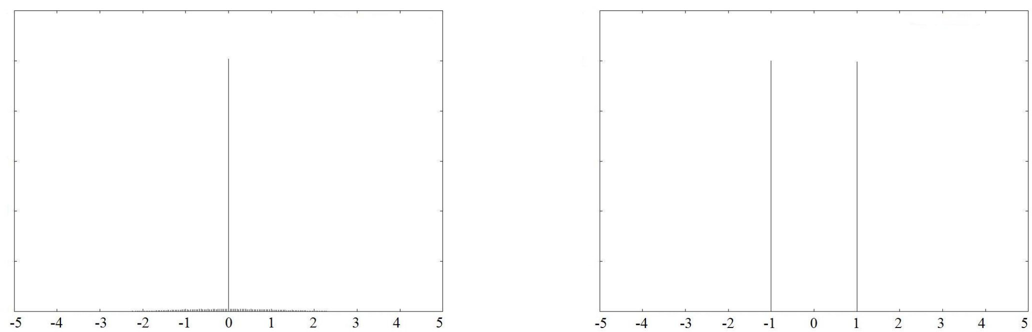
Figure C: Unimodal method

7.2 Bimodal method



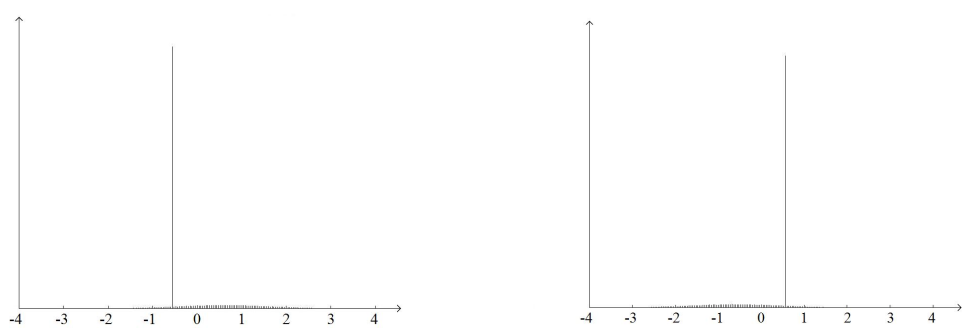
(D.a) $\Delta\left(0,1,3\left(\sqrt{6}-2\right)\sqrt{3-\sqrt{6}},0\right)$ (D.b) $\Delta\left(0,1,-3\left(\sqrt{6}-2\right)\sqrt{3-\sqrt{6}},0\right)$

Figure D: Bimodal method



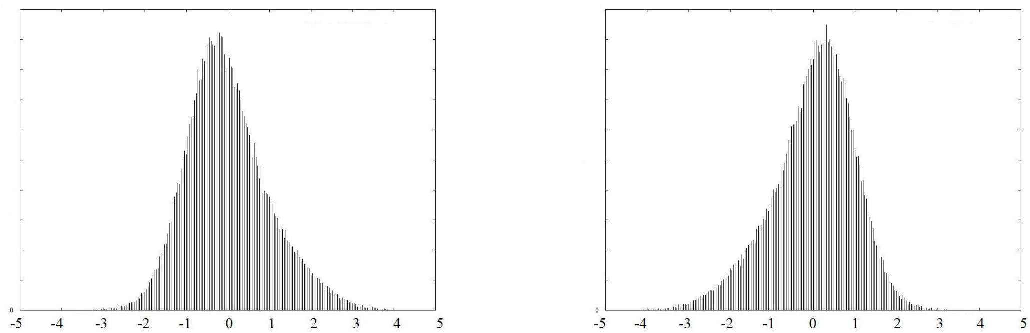
(E.a) $\Delta\left(0,1,0,3\right)$ (E.b) $\Delta\left(0,1,0,-2\right)$

Figure E: Bimodal method



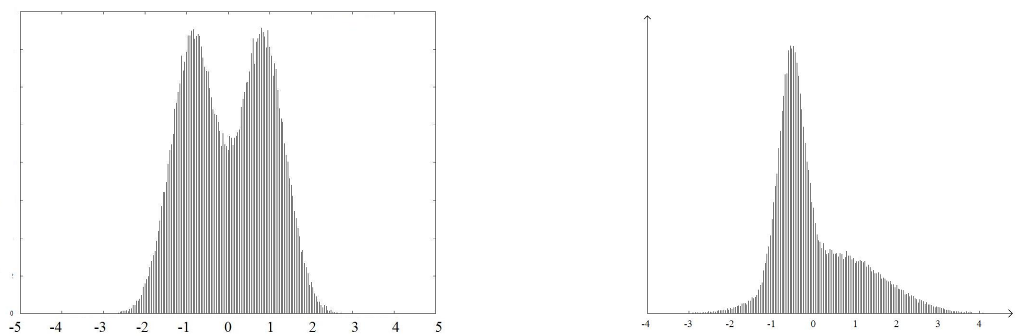
(F.a) $\Delta\left(0,1,\frac{2}{\sqrt{3}},\frac{10}{9}\right)$ (F.b) $\Delta\left(0,1,-\frac{2}{\sqrt{3}},\frac{10}{9}\right)$

Figure F: Bimodal method



(G.a) $\Delta(0, 1, 0.5, 0.5)$ (G.b) $\Delta(0, 1, -0.5, 0.5)$

Figure G: Bimodal method



(H.a) $\Delta(0, 1, 0.5, 0.5)$ (H.b) $\Delta(0, 1, -0.5, 0.5)$

Figure H: Bimodal method

6 Appendix

Theorem 1 *Let Z_1 and Z_2 be two independent normal random variables of respective means and variances μ_1 and μ_2 and σ_1 and σ_2.*
Let p be a uniform random variable on $[0, 2a]$ independent of Z_1 and Z_2.
Now, let X be the random variable defined by $X = pZ_1 + (1-p)Z_2$ of which the four first cumulants are κ_1, κ_2, κ_3 and κ_4.
If $\kappa_1 = 0$, so the set of couples (κ_3, κ_4) that X is able to admit is invariant to the uniform distribution p selected.

Proof 1 *We consider the following system of equations*

$$\kappa_2 = \frac{4a^2\left(\sigma_1 + \sigma_2 + \mu_1^2\right)}{3}$$

$$\kappa_3 = 4a^3\left(\mu_1\sigma_1 - \mu_1\sigma_2\right)$$

$$\kappa_4 = \frac{16a^4\left(3\sigma_1^2 + 3\sigma_2^2 + \sigma_1\sigma_2 + 4\mu_1^2\sigma_2 + 4\mu_1^2\sigma_1 + 16\mu_1^4\right)}{5} - 3\kappa_2^2$$

By considering the first equation, we can provide easily μ_1 in function of other variables and now, by replacing μ_1 by this value in both other equations, we obtain the new system

$$\kappa_3 = \pm 4a^3\left(\sigma_1 - \sigma_2\right)\sqrt{\frac{3\kappa_2}{4a^2} - \sigma_1 - \sigma_2}$$

$$\kappa_4 = \frac{24a^2\kappa_2\sigma_1 + 24a^2\kappa_2\sigma_2 - 6\kappa_2^2 - 80a^4\sigma_1\sigma_2}{5}$$

As on the frontier of the admissible set of couples (κ_3, κ_4), σ_1 or σ_2 is equal to zero and as these two variances are symmetrical, we choose to fit σ_2 to zero. At present, with the second equation, we can provide σ_1 as a function of a, κ_2 and κ_4. By replacing this value in κ_3, we obtain

$$\kappa_3 = \pm\frac{5\kappa_4 + 6\kappa_2^2}{6\kappa_2}\sqrt{\frac{18\kappa_2 - 5\kappa_4 - 6\kappa_2^2}{24\kappa_2}}$$

And we see immediately that a does not appear in the equation which connects the two cumulants. And so, the set of admissible couples' frontier does not depend of the parameter of uniformity a.