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Approval voting and the Poisson-Myerson environment[1]

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Résumé: Dans ce papier, de nouveaux résultats sont fournis dans le modèle de Poisson-Myerson. Ces résultats se révèlent utiles pour l'étude du vote par assentiment. En effet, le Théorème d'Equivalence des Magnitudes (MET) réduit fortement la complexité du calcul des magnitudes des pivots. Un exemple est fourni qui contraste avec les résultats de Laslier (2004) sur le vote par assentiment. Dans une situation de vote avec trois candidats, le gagnant de l'élection ne coïncide pas avec le gagnant de Condorcet du profil à l'équilibre. Une discussion sur la stabilité de l'équilibre est fournie.

Abstract: In this paper, new results are provided in the Poisson-Myerson framework. These results are shown to be helpful in the study of approval voting. Indeed, the Magnitude Equivalence Theorem (MET) substantially reduces the complexity of computing the magnitudes of pivotal events. An example is provided that contrasts with Laslier (2004) results concerning approval voting. In a voting context with three candidates, the winner of the election does not coincide with the profile Condorcet winner in a three candidates contest. A discussion on the stability of the equilibrium is provided.

Mots clés : Vote par assentiment, Jeux de Poisson, Electeurs instrumentaux, Calcul de magnitudes, Gagnant de Condorcet

Key Words : Approval voting, Poisson games, Instrumental voters, Magnitude computation, Condorcet winner.

Classification JEL: D72

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1 Introduction

Most models in game theory consider the number of players as a constant known by all the players. This assumption, though realistic in bargaining contexts or small-group voting schemes, could be a problem in large games. In this kind of games, we can assume that players face some uncertainty about the total number of players. That is, we assume that agents have some beliefs about the total number of players. For example, in an electoral context, this uncertainty is quite realistic. Poisson games, introduced by [5], allow us to introduce such uncertainty. In such a framework, the number of players in the game is supposed to be drawn from a Poisson random variable of parameter n, the expected population.

[5] shows that these games are uniquely characterized by two fundamental features: the *environmental equivalence* and the *independent-actions* properties. The environmental equivalence can be defined as the fact that "any single player in the election should assess the same probability distribution for the vote profile that will be generated by all the other voters in the election counting everybody's ballots except his own". The property of independent actions states that the number of players who choose a given action (who cast a ballot in voting contexts) is independent of the number of players who choose all actions except this one. These two characteristics are an advantage for mathematical computations in this kind of games. Indeed, the existence of equilibria has been proved for Poisson games with compact type sets and finite action sets thanks to these properties.

However, even if Poisson uncertainty allows to work in a model with nice mathematical features, there exists a certain amount of complexity as we work in situations where the number of agents tends towards infinity. Indeed, we deal with the computations of events with extremely low probabilities. Therefore, instead of working with these probabilities, it seems more reasonable to work with a measure of the speed at which these probabilities tend towards zero: the magnitude.

Two seminal papers ([6], [7]) give the three results that could be considered as the state-of-the-art techniques in the characterization of limits of probabilities in this type of games. The *magnitude* theorem states a method to compute such a limit as the solution of a maximization problem with a concave and smooth objective function. The *offset* theorem characterizes the ratios between probabilities of events that differ by a finite additive translation. Finally, the *dual magnitude* theorem (DMT) gives a method of computing events that have the geometrical structure of a cone, in a simpler way than the Magnitude theorem. These three theorems give good tools to work in the Poisson-Myerson environment.

Despite these useful tools and their mathematical features, this type of games has not yet been applied in many different contexts. Mainly, they have been used for the study of voting situations. [6] studies in detail the properties of different scoring rules. In a scoring rule, each voter's ballot must be a vector that specifies the number of points that the voter gives to each candidate. Myerson proves that, in simple bipolar elections, equilibria are always majoritarian and efficient under approval voting, but not under other scoring rules. A bipolar election can be defined as the one where there are two types of voters and each candidate is associated with one of them (for instance, left and right). We will focus on this particular type of voting: approval voting or AV.

[3] suggests a new model to study the properties of AV in a large electorate. Under some sort of normally distributed uncertainty, his work shows that AV has two main features: it implies sincere behavior of the voters and the winner of the election is a Condorcet winner. In the present work, we show that this conclusion does not hold anymore in the Poisson-Myerson framework. Indeed, a simple voting situation is provided where the Condorcet winner does not coincide with the winner of the election. In this example, the presence of a small group of voters makes the majority of the population behave in a non-conventional way. The consequence of this deviation is the fact that at equilibrium the Condorcet winner does not win the election.

In order to clarify this simple voting situation, new mathematical tools are developed. Even if Myerson's seminal papers solved some of the main problems in order to work in this new kind of environments, some issues still needed to be elucidated. To start with, a general formula is provided for the computation of the magnitude of equalities between the scores of *candidates*. It should be noted that a previous result was given by [7] where the magnitude of getting equal number of votes in two disjoint sets of ballots (for instance, two different ballots) was explicitly expressed. Our result is therefore a generalization of the latter.

Besides, a formal definition of a pivotal event will be provided. Pivotal events and their probabilities are key to determine the strategic behavior of the voters. Voters are assumed to be instrumentally motivated[1] and consequently, their utility depends only on the candidate who wins the election. This implies that they only care about the influence their ballot can have in pivotally changing the result of the election. It should be noted that the definition of a pivotal event changes depending on the voting method used in the election. Therefore, the definition that we provide is only verified in voting contexts where voters are allowed to give at most one point to a candidate.

This new definition allows us to introduce a new theorem in this framework: the Magnitude Equivalence Theorem (MET). This theorem substantially reduces the computations of the magnitude of a pivotal event. Indeed, it allows us to compute the magnitude of a pivotal event by solving a single optimization problem. The offset-ratio concept, introduced by [3], and its interpretation will be key to understand the underlying intuition of this theorem. Indeed, a classification of the events depending on its offset ratios is given that clarifies in some way the previous results of the literature.

The usefulness of this theorem will be shown by the mean of an example that shows that [3] results do not apply in the Poisson-Myerson framework. A simple voting situation is analyzed where the expected winner of the election does not coincide with the profile Condorcet winner. A discussion on the stability of the given equilibrium and on its welfare implications is provided.

This paper is structured as follows. In Section 2 we introduce the basic model and

[1] In the literature, some authors have argued that since each single ballot has a negligible weight on the outcome of the election when turnout is large, assuming that voters are instrumental is misleading. However, [1] shows that, in opposition to the usual perceptions, assuming instrumental voting can reproduce several stylized facts. His work concludes that instrumental voting is not the main cause of the failure of "standard" models as far as explaining empirical regularities in elections is concerned.

Section 3 provides the results concerning the computations of magnitudes in the Poisson games environment. In Section 4, the Magnitude Equivalence Theorem *(MET)* is presented. Section 5 discusses in detail the situation where the Profile Condorcet Winner does not coincide with the Expected Winner of the election. Finally, Section 7 concludes.

2 The model

We let K denote the set of candidates or outcomes in the election. The aim of the election is to choose one candidate. The voters are asked to express their preferences among candidates through ballots. We suppose that the total number of voters is a random variable drawn from a Poisson distribution of parameter n. Each voter has a type t that determines her preferences over the candidates. We denote by T the set of types. A player's payoff (or utility) only depends on her type and on the candidate who is elected. The preferences of a voter with a type t will be denoted by $u_t = (u_t(k))_{k\in K}$. Thus, if for a given t we have that $u_t(k) > u_t(k')$, it implies that t-type voters strictly prefer candidate k to candidate k'.

Each player's type is independently drawn from T according to the *expected distribution of types* denoted by $r = (r(t))_{t\in T}$ [2]. By this, we mean that $r(t)$ represents the probability that a player randomly drawn from the population will have type t. The profile Condorcet winner ($P.C.W.$) of the election is defined as:

Definition 2.1 *A candidate k is called the Profile Condorcet Winner of the election (P.C.W.) of the election if*

$$\sum_{t\in T_{k,j}} r(t) > 1/2 \ \forall\, j \in K, j \neq k$$

where $T_{k,j}$ is the set of preference types where the candidate k is preferred to candidate j.

Each player i must choose a ballot c from a finite set of possible ballots (also called actions), denoted by C. There exists two main ways of considering the set of available ballots in the election. On one side, we can consider the ballot as a vector $c = (c_1, c_2, \dots, c_C)$ where c_i represents the number of votes given to candidate i (following Myerson's notations). This notation is quite general as it allows us to work with other voting systems. However, it should be noted that as we focus on AV, every component c_i is either one or nil. Therefore, this notation, used by [7] could seem unnecessarily complicated. Secondly, we can think of a ballot simply as a subset of the set of candidates (following Laslier's notation). For easiness of notation, we will mostly use the second one. The players are allowed to use mixed strategies. These strategies are determined by the *strategy function*[3] $\sigma(c|t)$ which is a function from T into the set of probability distributions over C ([7]). That is, $\sigma(c|t)$ determines the probability that a given player of type t will choose action c. Therefore, all the players with the same type choose the same strategy. Then,

[2] The expected distribution of types satisfies $r(t) > 0 \forall\, t \in T$ and $\sum_{t\in T} r(t) = 1$.

[3] The strategy function satisfies $\sigma(c|t) \geq 0 \ \forall\, c \in C$ and $\sum_{d\in C} \sigma(c|t) = 1$.

taking into account the expected distribution of voters and the strategy function, we can determine the *expected vote distribution* $\tau = (\tau(c))_{c\in C}$. Formally,

$$\tau(c) = \sum_{t\in T} r(t)\sigma(c|t)$$

As a consequence of the Poisson-Myerson environment, the number of players in the game who choose action c is drawn from a Poisson random variable with parameter $n\tau(c)$. The *action profile*, $x = (x(c))_{c\in C}$, is the vector that gives the number of players who are choosing action c, for all $c \in C$. We denote by $\mathcal{C}_k$ the set of available ballots where candidate k is approved by the voter.

We may define the random variable $s(k)$ that describes the score of candidate k by

$$s(k) = \sum_{c\in\mathcal{C}_k} x(c) \sim \mathcal{P}(n \sum_{c\in\mathcal{C}_k} \tau(c))$$

Definition 2.2 *A candidate k is called the Winner of the election if she has the highest expected score. Formally,*

$$\textit{The candidate } k \textit{ is the Winner of the election} \iff \sum_{c\in\mathcal{C}_k} \tau(c) > \sum_{c\in\mathcal{C}_j} \tau(c) \ \forall j \in K, j \neq k$$

The elected candidate is the one with the highest score. Ties are resolved by a fair lottery.

The parameters (T, n, r, C, U) define a Poisson game.

Taking into account the *independent-actions* property, we can write that, for any vote profile x, the probability that x is the players' action profile in the game is

$$P[x|\, n\tau] = \prod_{c\,\in\,C} \left(\frac{e^{-n\tau(c)}(n\tau(c))^{x(c)}}{x(c)!} \right)$$

We refer to $\{\sigma, \tau\}$ as an *equilibrium* of the finite voting game (T, n, r, C, U). However, as this paper deals with elections with a large number of voters, we shall look at the limits of equilibria as the expected number of voters n goes to infinity. Thus, we may refer to $\{\sigma, \tau\}$ as the equilibria sequence of the finite voting games, i.e. the limit of $\{\sigma_n, \tau_n\}$ when n goes to infinity.

Strategic behaviour of the voters.

Defining a pivotal event. We assume that each voter determines which ballot she will cast by maximizing her expected utility. As we have assumed that voters are instrumentally motivated, they care only about the influence of their own vote in determining the winner's identity. Thus, a voter needs to estimate the probability that any given set of candidates will be in a close race for first place where one ballot could pivotally change the result of the election: a *pivot*. We now introduce a formal definition of a pivot.

Definition 2.3 *For each non-empty subset Y of candidates, $Y = \{i_1, i_2, \ldots, i_Y\}$ we denote a pivot between all candidates in Y by:*

$$pivot(Y) = \{ \bigcup_{j_2=-1}^{1} \bigcup_{j_3=-1}^{1} \bigcup_{j_4=-1}^{1} \cdots \bigcup_{j_K=-1}^{1} (s(i_1) = s(i_2) + j_2 = s(i_3) + j_3 = \ldots$$
$$= s(i_l) + j_l = \ldots = s(i_Y) + j_Y > s(x)) |\, i_1, i_2, \ldots, i_K \in Y \, and \; \forall x \in K \setminus Y\}$$

The probability of such a sequence of pivotal events is small if n is large, but some of these sequences of events are nevertheless much less likely than others. The probability of any pivotal event will generally tend to zero as the expected population n becomes large. But we can identify which races are serious by comparing the rates at which their probabilities go to zero. These rates can be measured by a concept of magnitude, defined as follows: given an expected distributional strategy $\tau = (\tau(c))_{c \in C}$, the magnitude $\mu(M)$ of a sequence of events $(M_n)_{n \in \mathbb{N}}$ is

$$\mu(M) = \lim_{n \to \infty} \frac{1}{n} \log \mathrm{P}[M_n | n\tau]$$

Notice that the magnitude of a sequence of events must be inferior or equal to zero, since the logarithm of a probability is never positive. If we can show that a pivot between one pair of candidates has a magnitude that is strictly greater than the magnitude of a pivot between another pair of candidates, then the latter is not serious.

The Decision Process. Following [3], the decision process of the voters can be described as follows. Let x be a candidate. Let $\mathcal{B}$ and $\mathcal{B}'$ be two ballots such that $\mathcal{B}' = \mathcal{B} \cup \{x\}$. In order to evaluate which of the ballots the type-t voter is going to choose, she computes the sign of the following expression

$$\Delta = \sum_{x \in Y} P[pivot(Y) \mid n\tau] E[u_t(B' \mid Y) - u_t(B \mid Y)]$$

where $u_t(B \mid Y)$ represents the utility of a type-t voter when she casts ballot $\mathcal{B}$ knowing the event pivot(Y). The set Δ simply represents the expected utility of the effect of voting $\mathcal{B}'$ instead of $\mathcal{B}$ given all the events where switching from one ballot to the other can have an impact in the result of the election (the pivotal events where candidate x is involved). Then, as we are interested in elections with a large number of individuals, we can factor out by the pivotal(s) event(s) which has (or have) a higher magnitude than the others (if any). If we call A the pivotal event with the highest magnitude (if it exists), we can write

$$\mathrm{sign}(\Delta) = \mathrm{sign}\, E[u_t(B' \mid A) - u_t(B \mid A)]$$

This formula represents the expected utility of the effect of voting $\mathcal{B}'$ instead of $\mathcal{B}$ given all the events within a the pivotal event knowing the event pivot(A).

The following section introduces new mathematical results dealing with the subtleties of working with magnitudes. Particular attention is given to the interpretation of the offset-ratio concept.

3 Probabilities, offset ratios and magnitudes

3.1 A Tie between candidates

To begin with, an explicit formula for the magnitude of a tie between two given candidates is provided.

Theorem 3.1 *Let (T, n, r, C, U) be a Poisson game. Let $a, b \in K$. The magnitude of the tie between the scores of candidates a and b, i.e. the event $M = \{s(a) = s(b)\}$, is*

$$\mu(M) = -\left(\sqrt{\mathcal{C}_{a\backslash b}} - \sqrt{\mathcal{C}_{b\backslash a}}\right)^2$$

with

$$\mathcal{C}_{a\backslash b} = \sum\{\tau(c) \ : \ c \in \mathcal{C}_a \ , \ c \notin \mathcal{C}_b\}$$
$$\mathcal{C}_{b\backslash a} = \sum\{\tau(c) \ : \ c \in \mathcal{C}_b \ , \ c \notin \mathcal{C}_a\}$$

The intuition behind this result is the following one. When comparing the scores of two candidates, we can always express both of them as the sum of two Poisson random variables, one which is common to both while the other is independent. The closer to each other the averages of the independent parts of both scores ($\mathcal{C}_{a\backslash b}$ and $\mathcal{C}_{b\backslash a}$), the closer to zero the magnitude of a tie between these two candidates.

Proof: The score of candidate a could be written as:

$$\begin{aligned} s(a) &= \sum\{x(c) \ : \ c \in \mathcal{C}_a\} \\ &= \sum\{x(c) \ : \ c \in \mathcal{C}_{a\backslash b}\} + \sum\{x(c) \ : \ c \in (\mathcal{C}_a \cap \ \mathcal{C}_b)\} \\ &= s_a + s_{a,b} \end{aligned}$$

Similarly, the score of candidate b is

$$s(b) = s_b + s_{a,b}$$

So, each score is written down as the sum of two Poisson random variables: a common and an independent one. Then, as we are interested in the probability of having an equality between both scores, we are only interested on the equality of the independent parts of the sum.

$$P[\{s(a) = s(b)\} \mid n\tau] = P[\{s_a = s_b\} \mid n\tau]$$

where s_a and s_b are independent random variables that follow respectively $\mathcal{P}(n\,\mathcal{C}_{a\backslash b})$ and $\mathcal{P}(n\,\mathcal{C}_{b\backslash a})$. As a consequence of Myerson (2000) result, we can conclude that

$$\mu(A) = -\left(\sqrt{\mathcal{C}_{a\backslash b}} - \sqrt{\mathcal{C}_{b\backslash a}}\right)^2 \qquad \square$$

It should be emphasized that the new contribution here is to give an explicit formula for the magnitude of a tie between two sums of Poisson random variables that are *correlated* (that represent the scores of candidates) and not *independent* as in Myerson (2002) result.

Myerson (2000) argues that "the magnitude theorem is not useful for comparing the probabilities of the events that differ by adding or subtracting a fixed vector, because the difference between such events may seem small in large Poisson games and so they usually have the same magnitude. So relative probabilities of events that differ by a simple additive translation must be compared using the offset theorem instead". The following lemma extends the class of events described by Myerson that have the same magnitude.

Lemma 3.1 *Let $\{A_n\}_{n=1}^{\infty}$, $\{B_n\}_{n=1}^{\infty}$ be two sequences of events with a finite magnitude. Then, if*

$$\lim_{n\to\infty}\frac{P(A_n \mid n\tau)}{P(B_n \mid n\tau)} = \varepsilon_n$$

for some $\{\varepsilon_n\}_{n=1}^{+\infty}$, such that

$$\lim_{n\to\infty}\frac{\log[\varepsilon_n]}{n} = 0 \quad \textit{and } \varepsilon_n > 0 \ \forall\, n \ \in \mathbb{N}.$$

Then, we can write the following equality

$$\mu(A_n) = \mu(B_n)$$

Proof: This result is a direct consequence of the definition of magnitude □

Indeed, this lemma will allow us to extend the conclusions of the Theorem 3.1. However, some definitions are needed in order to enlarge its conclusion. Among these definitions, the concept of offset ratio will be of some importance.

3.2 The concept of Offset ratio: a measure in the Poisson-Myerson environment

In this section, a discussion about the difference between the probability and the magnitude of an event is presented. The offset ratio is shown to be a key concept in the Poisson-Myerson environment. Two examples are analysed in detail in order to explain the main features and subtleties of this concept. Indeed, this section introduces the reader in the spirit of the Magnitude Equivalence Theorem, which will be presented afterwards.

As stated by Myerson (2000), the concept of offset ratio can be defined as follows.

Definition 3.1 *Let (T,n,r,C,U) be a Poisson game. For an event M, we call $\alpha_M = (\alpha_M(c))_{c\in C}$ the offset-ratio vector of the vote profile x relative to the expected vote distribution τ, if for every action c in C, $\alpha_M(c) = \frac{x(c)}{n\tau(c)}$, provided that $x(c)$ occurs with a strictly positive probability. Each of the components $\alpha_M(c)$ of the vector α_M are called the c-offset ratios.*

So, the offset-ratio α_M is a vector which describes the number of players $x(c)$ who choose ballot c in the event M as a fraction of the expected number of players who were supposed to cast it. Obviously, this ratio is not constant in a game: it depends on the event under consideration. The concept of offset-ratio was shown to be important in this environment by the offset theorem. As [6] argues, "relative probabilities of events that differ by a simple additive translation must be compared using the offset theorem." Applied to an event M, the offset theorem could be expressed as follows,

Theorem 3.2 (Offset Theorem, Myerson (2000)) *Let w be any vector in $\mathbb{Z}^C$. For each action c such that $w(c) \neq 0$, suppose that $\tau(c) > 0$, and suppose that some number $\alpha_M(c)$ is the limit of the major c-offsets in the event M. Then*

$$\lim_{n \to \infty} \frac{P[M - w \mid n\tau]}{P[M \mid n\tau]} = \prod_{c \in C} \alpha_M(c)^{w(c)}$$

Thus, the offset ratio vector $\alpha_M = (\alpha_M(c))_{c \in C}$ has two main advantages.

First, it provides us a way of measuring the behavior of the electorate at a given event M with respect to the expected behavior.

Secondly, it is useful to compute the difference between the probabilities of two events that are "too close" (in the sense that they have the same magnitude).

We can divide the events to study in two categories: the *normal* and the *irregular* ones.

Definition 3.2 *A normal event can be defined as the event where all the offset ratios are different from zero.*

Definition 3.3 *An irregular event can be defined as the event where at least one offset ratio is equal to zero.*

This division comes from the fact that at a normal event there is no jump in the probability of the event whereas, in an irregular one, there is a jump in the probability of the event when adding or subtracting a fixed vector (when n tends towards infinity). Two simple voting examples are provided now that illustrate and clarify this classification.

Example 1: Where the offset ratios are different from zero In this first example, we focus on "well-behaved" situations where the offset ratios of each ballot are different from zero. Indeed, the assumption that every offset ratio is non negative implies that we can always express the ratio between the probabilities two events that differ by a linear transformation by some finite coefficient.

Let us consider a simple voting situation where there are two candidates (that we denote by a and b) and the voting rule is majority voting[4]. Therefore, the set of candidates is $K = \{a, b\}$ and the set of actions is equal to $C = \{1, 2\}$ (where 1 represents (resp. 2) the action of voting for candidate a (resp. candidate b)). We assume there is no abstention to keep things simple. Thus, the expected vote distribution τ is equal to $\tau = (\tau(1), \tau(2))$ with $\tau(1) + \tau(2) = 1$ and $\tau(1) > \tau(2) > 0$ w.l.o.g. We are interested in the event of a tie M between candidates a and b. Then, applying the DMT, we know that the magnitude $\mu(M)$ of such an event is equal to

$$\mu(M) = -\left(\sqrt{\tau(1)} - \sqrt{\tau(2)}\right)^2$$

[4] This example was studied by Myerson (2000) in a computational way. Here, the main interest is the interpretation of the results and not the mathematical results themselves.

and that the offset ratio vector $\alpha_M = (\alpha_M(1), \alpha_M(2))$ is such that

$$\alpha_M(1) = \sqrt{\frac{\tau(2)}{\tau(1)}} < 1 \quad \alpha_M(2) = \sqrt{\frac{\tau(1)}{\tau(2)}} > 1$$

There exists two interpretations to these formulas. First, this result implies that in order to get a tie (to be at the event M), a fraction $\alpha_M(1) = \sqrt{\frac{\tau(2)}{\tau(1)}}$ of the voters who were expected to cast ballot 1 (i.e. vote for candidate a), cast it. As $\alpha_M(1)$ is inferior to one, the number of voters who vote for candidate a is inferior to the average number of electors who were supposed to do it (that is $n\tau(1)$). Conversely, in the case of the event M, there is a fraction $\sqrt{\frac{\tau(1)}{\tau(2)}}$ of the voters who were supposed to vote for candidate b who vote for it.

Secondly, this result can also be interpreted as follows. It should be remarked that subtracting one ballot $\{a\}$ to the event M that could be written as $M - w$ where $w = (w(1) = 1, w(2) = 0) = (1, 0)$. So, applying the offset theorem, we can write

$$\lim_{n\to\infty} \frac{P(M - \{a\} \mid n\tau)}{P(M \mid n\tau)} = \lim_{n\to\infty} \frac{1}{n} \frac{P(M - w \mid n\tau)}{P(M \mid n\tau)} = \prod_{c\in C} \alpha_M(c)^{w(c)}$$

Thus, as the vector w is equal to $w = (1, 0) \in \mathbb{Z}^2$

$$\lim_{n\to\infty} \frac{P[M - \{a\} \mid n\tau]}{P[M \mid n\tau]} = \alpha_M(1) = \sqrt{\frac{\tau(2)}{\tau(1)}}$$

$$\lim_{n\to\infty} \frac{P[M + \{a\} \mid n\tau]}{P[M \mid n\tau]} = \alpha_M(1)^{-1} = \sqrt{\frac{\tau(1)}{\tau(2)}}$$

It should be noted that, as a consequence of Lemma 3.1 we can write

$$\mu(M - \{a\}) = \mu(M + \{a\}) = \mu(M)$$

meaning that the events M, $M - \{a\}$ and $M + \{a\}$ have the same magnitude. In some way, this method (using magnitudes) is "myopic" with respect to the linear transformations of an event. Both events do not have the same probability when n goes to infinity and have the same magnitude. Using the magnitudes, we can not state which of the events is more probable. However, using the offset theorem we can prevent this effect and state the ratio between the probabilities of these events when n goes to infinity. This shows the usefulness of the offset theorem.

Example 2: Where an offset ratio is equal to zero In this second example, we focus on what may be called an irregular event or a "corner solution". By this, we mean situations where there exists at least an offset ratio of one ballot which is equal to zero. In this case, we know there is ballot for which no voter has voted.

Let us consider a simple voting situation where there are two candidates (that we denote by a and b) and the voting rule is approval voting[5]. Therefore, the set of candidates

[5]This example was kindly suggested by J-F. Laslier.

is $K = \{a, b\}$ and the set of actions is equal to $C = \{1, 2, 3\}$ (assuming there is no abstention to keep things simple). Action 1 represents voting for candidate $\{a\}$, action 2 represents voting for candidate b and action 3 represents voting for both candidates. Suppose that, given the expected type distribution, voters cast only ballots 1 and 3. Thus, the expected vote distribution τ is equal to $\tau = (\tau(1), \tau(2) = 0, \tau(3))$ with $\tau(1) + \tau(3) = 1$ and $\tau(1) > \tau(3) > 0$ w.l.o.g. We denote by $x(1)$ (resp. $x(2)$) the random variable which describes the number of players who choose to cast ballot 1 (resp. 2). Thus, $x(i)$ is drawn from a Poisson distribution of parameter $n\tau(i)$ for i=1,2. We focus in the event of a tie M between candidates a and b. Then, applying the DMT, we know that the magnitude $\mu(N)$ of such an event is equal to

$$\mu(N) = -\tau(1)$$

and that the offset ratio vector is such that $\alpha_N = \alpha_N(1) = 0$, as provided that $x(1) = 0$ there is always a tie between candidates a and b.

Then, we know that in order to get a tie between the scores of the candidates we need that none of the voters ($\alpha_N(1) = 0$) who were supposed to vote for candidate $\{a\}$, vote for him. That is, in order to get a tie between both candidates in this situation, no one has to cast ballot 1.

The event of subtracting one vote for candidate $\{a\}$ to the tie N has no longer a sense (there is already no voter who votes only for him). However, denoting $w = (1, 0, 0)$, we can write the event $N + \{a\}$ as the event $N + w = N - (-w)$. Indeed, applying the offset theorem, we can write

$$\begin{aligned}
\lim_{n\to\infty} \frac{P[N + \{a\} \mid n\tau]}{P[N \mid n\tau]} &= \lim_{n\to\infty} \frac{P[N + w \mid n\tau]}{P[N \mid n\tau]} \\
&= \lim_{n\to\infty} \frac{P[N - (-w) \mid n\tau]}{P[N \mid n\tau]} \\
&= \alpha(1)^{-1} \\
&= +\infty
\end{aligned}$$

so that, the probability of getting a tie N becomes infinitesimal with respect to the probability of getting the event $N + \{a\}$. This is the "jump" we described before when establishing the division between normal and irregular events. This could be explained as follows,

$$\begin{aligned}
P[N \mid n\tau] &= P[x(1) = 0] \\
P[N + \{a\} \mid n\tau] &= P[x(1) = 1] = n\tau(1)P[x(1) = 0] = n\tau(1)P[N \mid n\tau]
\end{aligned}$$

Thus, the limit equality is verified. However, even if one of the probabilities becomes infinitesimal with respect to the other, they still have the same magnitude by Lemma 3.1. Indeed, we wan write that

$$\mu(N + \{a\}) = \mu(N)$$

meaning that the events (N) and $(N + \{a\})$ have the same magnitude even if the probabilities are infinitesimally different.

The conclusion we can get from these examples is that within a Poisson game there could be some events that infinitely more probable than others and that still have the same magnitude. The purpose of studying these examples is to show the existence some zones of absorption when working with magnitudes. That is, on both examples the tie has the same magnitude as the linear transformations of it. This "absorption" property will be used on the next section to prove the Magnitude Equivalence Theorem, a useful tool in this environment.

Prior to focusing on the MET, the extension of Theorem 3.1 is presented. The underlying logic of the proof is based on the "absorption" property.

Corollary 3.1 *Let (T, n, r, C, U) be a Poisson game. Let $a, b \in K$ and $j \in \mathbb{Z}$. The magnitude of the event $M_j = \{s(a) = s(b) - j\}$ is such that*

$$\mu(M_j) = \mu(M)$$

whenever M_j exists.

Proof: Let $w \in \mathbb{Z}^C$ such that $w(j) = 1$ and $w(c) = 0$ if $c \neq j$. In set-theoretical terms, we can write that

$$M_j = M - w$$

Therefore, applying the offset theorem, we can state that

$$\begin{aligned}
\lim_{n\to\infty} \frac{P[M_j \mid n\tau]}{P[M \mid n\tau]} &= \lim_{n\to\infty} \frac{P[M - w \mid n\tau]}{P[M \mid n\tau]} \\
&= \prod_{c\in C} \alpha_M(c)^{w(c)} \\
&= \alpha_M(j)
\end{aligned}$$

Thus, the magnitude of the event M_j is equal to

$$\begin{aligned}
\mu(M_j) &= \lim_{n\to\infty} \frac{1}{n} \log P[M_j \mid n\tau] \\
&= \lim_{n\to\infty} \frac{1}{n} \log\left[\alpha_M(j) P[M \mid n\tau]\right] \\
&= \lim_{n\to\infty} \frac{1}{n} \log P[M \mid n\tau] + \lim_{n\to\infty} \frac{1}{n} \log\left[\alpha_M(j)\right] \\
&= \mu(M) + \lim_{n\to\infty} \frac{1}{n} \log\left[\alpha_M(j)\right]
\end{aligned}$$

Then, we have two different cases: 1 $\alpha_M(j)$ finite for $c \in C$, 2 There exists at least one $\alpha_M(j) = 0$

Case 1: $\alpha_M(j)$ finite

In this case,

$$\lim_{n\to\infty} \frac{1}{n} \log\left[\alpha_M(j)\right] = 0.$$

The equality is verified.

Case 2: There exists at least one $\alpha_M(j) = 0$

Following similar reasonings to the ones explained in the Example 2, we can show that the equality is verified.

4 Magnitude equivalence theorem

The following result allows us to escape from the complexity of the structure of pivotal events. Using the offset theorem, we find a simpler expression of the magnitude of a pivotal event. The main advantage of this technique is that it allows us to use the *DMT* to compute magnitudes of pivotal events directly.

Theorem 4.1 (Magnitude Equivalence Theorem) *Let (T,n,r,C,U) be a Poisson game. Let $Y = \{i_1, i_2, \ldots, i_Y\}$ be a subset of K. If every $\alpha(c) \neq 0$ for every $c \in C$, we can state the following equality:*

$$\mu[pivot(Y)] = \mu[s(i_1) = s(i_2) = \ldots = s(i_Y) \geq s(x)\ \forall x \in K \setminus Y]$$

Proof: By definition, we can write the probability of a pivotal event as

$$\begin{aligned} P[pivot(Y) \mid n\tau] = P[\{ \bigcup_{j_2=-1}^{1} \bigcup_{j_3=-1}^{1} \bigcup_{j_4=-1}^{1} \cdots \bigcup_{j_K=-1}^{1} (s(i_1) = s(i_2) + j_2 = s(i_3) + j_3 = \ldots \\ \ldots = s(i_l) + j_l = \ldots = s(i_Y) + j_Y > s(x)) \mid i_1, i_2, \ldots, i_Y \in Y \text{and } \forall x \in K \setminus Y\}] \\ = P[\bigcup_{m \in M} A_m] \end{aligned}$$

For some arbitrary finite set M, the sets A_m represent the different cases in which we can have a pivotal event. However, it should be noted that this a disjoint union as $A_m \cap A_j = \emptyset$ if $m \neq j$. That is, the probability of the union is equal to the sum of the probabilities of every event, i.e.

$$P[pivot(Y) \mid n\tau] = \sum_{m \in M} P[A_m \mid n\tau]$$

Besides, we can express the relationship between every A_m and the simple cone event $A_0 = [s(i_1) = s(i_2) = \ldots = s(i_Y) \geq s(x)]$ as a single positive translation. We denote by $w_m = (w_m(c))_{c \in C} \in \mathbb{Z}^C$ the vector of translation. In set theoretical terms, we can write this as

$$A_m = A_0 - w_m$$

Then, by the offset theorem,

$$\begin{aligned} \lim_{n \to \infty} \frac{P[A_m \mid n\tau]}{P[A_0 \mid n\tau]} &= \lim_{n \to \infty} \frac{P[A_0 - w_m \mid n\tau]}{P[A_0 \mid n\tau]} \\ &= \prod_{c \in C} \alpha(c)^{w_m(c)} \end{aligned}$$

where $\alpha(c)$ represents the offset ratio of action c at the event A_0.

The proof proceeds in several steps. In Step 1, we prove the result when every offset ratio is different from zero and finite. Step 2 shows the result when there exists one offset ratio that is different from zero. Finally, Step 3 extends these results for the case where there exists two or more offset ratios that are equal to zero.

Step 1: Let us suppose first that every $\alpha(c)$ is different from zero and finite, where c represents a ballot which gives a point to at least one of the candidates in the set Y. In this case,

$$\lim_{n\to\infty} \frac{\log[\prod_{c\in C} \alpha(c)^{w_m(c)}]}{n} = 0$$

Therefore, by Lemma 3.1 both events have the same magnitude. Denoting for every $m \in M$, $\rho_m = \prod_{c\in C} \alpha(c)^{w_m(c)}$, we can write the magnitude of the pivotal event as

$$\begin{aligned}
\mu[pivot(Y)] &= \lim_{n\to\infty} \frac{1}{n} \log P[\text{pivot(Y)}|\, n\tau] \\
&= \lim_{n\to\infty} \frac{1}{n} \log \left[\sum_{m\in M} \rho_m P[A_0|\, n\tau] \right] \\
&= \lim_{n\to\infty} \frac{1}{n} \log \left[P[A_0|\, n\tau] \sum_{m\in M} \rho_m \right] \\
&= \lim_{n\to\infty} \frac{1}{n} \log P[A_0|\, n\tau] + \lim_{n\to\infty} \frac{1}{n} \log \sum_{m\in M} \rho_m \\
&= \lim_{n\to\infty} \frac{1}{n} \log P[A_0|\, n\tau] \\
&= \mu[s(i_1) = s(i_2) = \ldots = s(i_Y) \geq s(x)]
\end{aligned}$$

Step 2: Now, let us suppose that there exists an $\alpha(c)$ which is equal to zero, where c represents a ballot which gives a point to at least of the candidates in the set Y. Then, the event $A_0 - \{c\}$ (the event of subtracting a ballot $\{c\}$ to the event A_0) is infinitesimal with respect to the event A_0 when n tends to infinity. Conversely, the event $A_0 + \{c\}$ (the event of adding a ballot $\{c\}$ to the event A_0) is infinitely more probable than the event A_0 tends to infinity.

However, the event $A_{-c} = (A_0 - \{c\})$ does not occur with a positive probability. This could be explained as follows. Given that we are at event A_0, no voter who was supposed to vote for ballot $\{c\}$ has done it (as $\alpha(c) = 0$): $x(c) = 0$. The event $(A_0 - \{c\})$ would imply $x(c) = -1$. This is impossible given that $x(c)$ is a Poisson random variable defined only for the positive integers. Therefore, this event is not taken into account in the probability of the pivotal event.

The fact that the probabilities of A_0 and $A_c = (A_0 + \{c\})$ diverge when n tends to infinity, can be explained as follows. At event A_0, we know that $\alpha(c) = 0$ and thus that $x(c) = 0$. So, the event A_c implies that $(x(c) = 1)$. And,

$$\begin{aligned}
\lim_{n\to\infty} \frac{P[A_c|\, n\tau]}{P[A_0|\, n\tau]} &= \lim_{n\to\infty} \frac{P[x(c) = 1|\, n\tau]}{P[x(c) = 0|\, n\tau]} \\
&= \lim_{n\to\infty} \frac{n\tau(c) P[x(c) = 0|\, n\tau]}{P[x(c) = 0|\, n\tau]} \\
&= \lim_{n\to\infty} n\tau(c) = +\infty
\end{aligned}$$

Even if the ratio between these probabilities diverges, we can write that

$$\lim_{n\to\infty} \frac{\log[n\tau(c)]}{n} = 0$$

Thus, by Lemma 3.1, the events A_0 and A_c have the same magnitude.

By assumption, there is only one offset ratio which is equal to zero. Therefore, for every event A_m, $m \in M \setminus \{-c, c\}$, the following limit condition is still true,

$$\lim_{n\to\infty} \frac{P[A_m|\, n\tau]}{P[A_0|\, n\tau]} = \rho_m$$

for some ρ_m which is constant with respect to n.

Thus, for every event A_m, $m \in M \setminus \{-c, c\}$, we can write

$$\begin{aligned}
\lim_{n\to\infty} \frac{P[A_m|\, n\tau]}{P[A_c|\, n\tau]} &= \lim_{n\to\infty} \frac{\rho_m P[A_0|\, n\tau]}{P[A_c|\, n\tau]} \\
&= \lim_{n\to\infty} \frac{\rho_m P[A_0|\, n\tau]}{P[A_c|\, n\tau]} \\
&= \rho_m \lim_{n\to\infty} \frac{P[A_0|\, n\tau]}{P[A_c|\, n\tau]} = 0
\end{aligned}$$

Therefore, we can write that

$$\begin{aligned}
\mu[pivot(Y)] &= \lim_{n\to\infty} \frac{1}{n} \log P[\text{pivot(Y)} \mid n\tau] \\
&= \lim_{n\to\infty} \frac{1}{n} \log \sum_{m\in M} \rho_m P[A_c \mid n\tau]
\end{aligned}$$

with

$$\rho_m = \begin{cases} 1 & \text{if } m = c \\ 0 & \text{if not} \end{cases}$$

So,

$$\begin{aligned}
\mu[pivot(Y)] =& \mu[A_c] \\
=& \mu[A_0] \\
=& \mu[s(i_1) = s(i_2) = \ldots = s(i_Y) \geq s(x)]
\end{aligned}$$

Step 3: Finally, let us assume that there exists two ballots $\{c\}$ and $\{c^{'}\}$ such that $\alpha(c) = \alpha(c^{'}) = 0$, where $\{c\}$ and $\{c^{'}\}$ are both ballots which give a point to at least one of the candidates in the set Y. Given this hypothesis, we have that

$$\lim_{n\to\infty} \frac{P[A_0 + \{c\}|\, n\tau]}{P[A_0|\, n\tau]} = \lim_{n\to\infty} \frac{P[A_0 + \{c^{'}\}|\, n\tau]}{P[A_0|\, n\tau]} = +\infty$$

Unfortunately, we cannot repeat the same process as before as we have two different events that are infinitely more likely. However, as we can write the following limit condition

$$\lim_{n\to\infty} \frac{P[A_0 + \{c\}|\, n\tau]}{P[A_0 + \{c^{'}\}|\, n\tau]} = \lim_{n\to\infty} \frac{P[x(c) = 1|\, n\tau]}{P[x(c^{'}) = 1|\, n\tau]} = \frac{\tau(c)}{\tau(c^{'})}$$

and due to the fact that

$$\mu(A_0) = \mu(A_0 + \{c\}) = \mu(A_0 + \{c^{'}\})$$

the following equality still holds.

$$\mu[pivot(Y)] = \mu[A_0]$$

Similarly, we can prove the same result even if there exists more than two $\alpha(c)$ that are either equal or infinite.[6] □

Thus, the magnitude of a pivotal event is equivalent to the magnitude of a cone event which is part of it. This result allows us to directly use the DMT to compute the pivotal magnitudes.

[6] It should be noted that if there exists an offset ratio $\alpha(c) = 0$, then there always exists another $\alpha(c')$ which is infinite. If $\alpha(c)$ represents the effect of the translation $-\{c\}$ to the event A_0, it suffices to take for instance, the translation $\{c'\} = +\{c\}$.

5 An example where the Condorcet winner is not elected

In this section, an example is provided where, at equilibrium, the Winner of the election does not coincide with the profile Condorcet Winner. An interesting feature of this example is its simplicity. There are only three types of voters with ordinal preferences and the type distribution is quite simple. Furthermore, the equilibrium is shown to be quite stable.

Let us consider a voting context where there are three candidates $K = \{a, b, c\}$ and three types of voters, i.e. $T = \{t_1, t_2, t_3\}$. We describe the preference profiles of the different types as follows

$$u_{t_1}(a) > u_{t_1}(b) > u_{t_1}(c)$$
$$u_{t_2}(b) > u_{t_2}(a) > u_{t_2}(c)$$
$$u_{t_3}(c) > u_{t_3}(a) > u_{t_3}(b)$$

where $u_t(k)$ denotes the utility of type-t voters when candidate k wins the election. The expected type distribution is

$$r(t_1) = \frac{3}{32} \quad r(t_2) = \frac{18}{32} \quad r(t_3) = \frac{11}{32}$$

Under approval voting, we can find an equilibrium in which the Winner of the election does not coincide with the $P.C.W.$
Let us suppose that the strategy functions satisfy

$$\sigma(\{a\}|\, t_1) = \sigma(\{a, b\}|\, t_2) = \sigma(\{c\}|\, t_3) = 1$$

Note that in this example all the voters with the same type choose the same action. Therefore,

$$\tau(\{a\}) = r(t_1), \quad \tau(\{a, b\}) = r(t_2), \quad \tau(\{c\}) = r(t_3)$$

Once we have described the setting of the voting situation, we proceed to the computation of the magnitudes of the pivotal events. The solved minimization problems are included in the appendix.

Magnitude of a pivot between candidates a and b Following the MET, we can write the following equality

$$\mu(pivot(a, b)) = \mu(\{s(a) = s(b) \geq s(c)\})$$

According to the DMT, we know that this magnitude is equal to the solution of the following optimisation problem.

$$\mu(\{s(a) = s(b) \geq s(c)\}) = \min_{\lambda} \; r(t_1)\exp[\lambda_1 - \lambda_2 + \lambda_3] + r(t_2)\exp[\lambda_3] + r(t_3)\exp[-\lambda_3] - 1$$

such that $\lambda_i \geq 0 \,\forall\; i$.

Thus, numerically solving this constrained minimization problem, the magnitude of this pivotal event is such that

$$\mu(pivot(a, b)) = -0.09375$$

The offset ratio vector associated to this event which we call α_1, is equal to

$$\alpha_1(\{a\}) = 0 \;\; \alpha_1(\{a, b\}) = 1 \;\; \alpha_1(\{c\}) = 1$$

Magnitude of a pivot between candidates a and c Combining the MET and the DMT, the magnitude of a pivot between candidates a and c is equal to

$$\mu(pivot(a,c)) = \mu(\{s(a) = s(c) \geq s(b)\}) = -0.0500822$$

The offset ratio vector associated to this event which we call α_2, is equal to

$$\alpha_2(\{a\}) = \frac{1.44749}{2} = 0,7237 \;\; \alpha_2(\{a,b\}) = 0,7237 \;\; \alpha_2(\{c\}) = \frac{2}{1.44749} = 1,3817$$

Magnitude of a pivot between candidates b and c Combining the MET and the DMT, the magnitude of a pivot between candidates b and c is equal to

$$\mu(pivot(b,c)) = \mu(\{s(b) = s(c) \geq s(a)\}) = -0.120547$$

The offset ratio vector associated to this event which we call α_3, is equal to

$$\alpha_3(\{a\}) = 0 \;\; \alpha_3(\{a,b\}) = \frac{2}{2.55841} = 0,7817 \;\; \alpha_3(\{c\}) = \frac{2.55841}{2} = 1,2792$$

Therefore, the magnitudes of the pivotal events are ordered in the following way:

$$\mu(pivot(a,c)) > \mu(pivot(a,b)) > \mu(pivot(b,c))$$

Taking into account the ordering of the magnitudes, and following the assumption that we have instrumental voters, we can determine the action that each agent of a given type will choose. To clarify how the voters choose the ballot, the decision process of t_1 voters is now described in detail. When deciding between casting $\{a\}$ or $\{a,b\}$, they take into account the magnitudes of the pivots where candidate b is involved as they evaluate the effect on their expected utility of adding candidate b to their ballot $\{a\}$. Mathematically, this decision process could be expressed as evaluating the sign of Δ,

$$\begin{aligned}\Delta = E[\{a\}] - E[\{a,b\}] &= \sum_{b \in Y} P[pivot(Y)] E[u_{t_1}(\{a\}|pivot(Y)) - u_{t_1}(\{a,b\}|pivot(Y))] \\ &= P[pivot(a,b)] E[u_{t_1}(\{a\}) - u_{t_1}(\{a,b\})|pivot(a,b)] \\ &\quad + P[pivot(b,c)] E[u_{t_1}(\{a\}) - u_{t_1}(\{a,b\})|pivot(b,c)]\end{aligned}$$

As we know that $pivot(a,b)$ is the pivotal event where b is involved with the highest magnitude, we can write that

$$\lim_{n\to\infty} \frac{P[pivot(b,c)]}{P[pivot(a,b)]} = 0$$

Then, following Laslier (2004)'s reasoning, one can factor out P[pivot(a,b)] in Δ when n tends to infinity and then just evaluate the sign of the following expression

$$E[u_{t_1}(\{a\}) - u_{t_1}(\{a,b\})|pivot(a,b)]$$

Defining $A_j = (s(a) = s(b) + j \geq s(c))$ for $j \in \{-1,0,1\}$, this expression is equivalent to

$$E[u_{t_1}(\{a\}) - u_{t_1}(\{a,b\})|pivot(a,b)] = \sum_{j=-1}^{1} E[u_{t_1}(\{a\}) - u_{t_1}(\{a,b\})|A_j]\; P[A_j]$$

Using the offset ratios, we know that

$$P[A_{-1}] = 0$$
$$\lim_{n\to\infty} \frac{P[A_0 \mid n\tau]}{P[A_1 \mid n\tau]} = 0$$

and, by computing the expected utility of the voters, the following inequalities can be stated,

$$E[u_{t_1}(\{a\}) - u_{t_1}(\{a,b\})|A_{-1}] > 0$$
$$E[u_{t_1}(\{a\}) - u_{t_1}(\{a,b\})|A_0] > 0$$
$$E[u_{t_1}(\{a\}) - u_{t_1}(\{a,b\})|A_1] = 0$$

Thus, Δ is positive and so t_1 voters prefer to cast $\{a\}$ rather than $\{a,b\}$. Repeating the same decision process, it can be deduced that t_1 voters prefer to cast ballot $\{a\}$ to $\{a,c\}$. Similarly, the expected utility of adding one single candidate to a given ballot can be computed, obtaining the optimal strategy for each type of voter. Thus, the voters' optimal strategies are such that,

$$\sigma(\{a\}|t_1) = \sigma(\{a,b\}|t_2) = \sigma(\{c\}|t_3) = 1$$

Then, this is an equilibrium of the voting game. It should be noted that,

$$r(t_1) + r(t_2) > r(t_2) > r(t_3)$$

By this, we mean that candidate a is the Winner of the election. Besides, we also have:

$$r(t_2) > r(t_1) + r(t_3)$$
$$r(t_1) + r(t_2) > r(t_3)$$

that is, candidate b is the Profile Condorcet Winner. $\square$

On the stability of the equilibrium. Concerning the equilibrium presented here, there are two main aspects that should be underlined. First of all, in this game we can find another equilibrium where the *P.C.W.* coincides with the winner. For instance, if we assume that the expected type distribution is the following one,

$$r(t_1) = \frac{12}{32} \quad r(t_2) = \frac{12}{32} \quad r(t_3) = \frac{8}{32}$$

with the same strategy functions then we have such an equilibrium.
Secondly, we try to test the stability of this equilibrium. In order to attain this objective, we focus in situations where there are some part of the individuals who do not behave rationally. This reason could be some kind of bias towards the expected winner. It could also be thought as some kind of trembling-hand action due to some external factor. We show here that for non-negligible fractions of the voters showing lack of rationality, we still get the equilibrium where the winner does not coincide with the *P.C.W.*

Let us suppose that there is a fraction of t_2-voters who economically misbehave and vote only for candidate a. We will denote this class of voters by i_2 where I denotes their irrationality. Then, the expected type distribution is

$$r(t_1) = \frac{3}{32} \quad r(t_2) = \frac{18}{32} - \epsilon \quad r(i_2) = \epsilon \quad r(t_3) = \frac{11}{32}$$

Then, according to the DMT and the MET, we should solve the following optimization problem to compute the magnitude of a pivot between candidates a and b.

$$\min_{\lambda} \; r(t_1)UV + r(t_2)V + r(t_3)V^{-1} + r(i_2)UV$$

such that $U > 0$ and $V \geq 1$. We do not give explicit formulas for the optimization problems corresponding to the computation of the magnitudes of pivots between a, c and b, c as they are similar to the latter.

Solving this problem numerically, we find that whenever $\epsilon \in (0, \frac{6}{32})$, the pivotal events follow the same order as in our example,

$$\mu(pivot(a,c)) > \mu(pivot(a,b)) > \mu(pivot(b,c))$$

Therefore, the rational agents still behave in the same way. Then this is an equilibrium if and only if $\epsilon \in (0, \frac{6}{32})$. The upper bound of this interval is roughly equal to 0.21 implying that even if 21% (on average) of the population do not behave rationally we can still get an equilibrium where both concepts do not coincide. This process could be repeated assuming the same kind of irrationality among t_1 and t_3 voters leading us to the same result: the equilibrium holds. As a conclusion, we can say that the stability of this equilibrium is quite strong.

6 Conclusion

The structure of the Poisson-Myerson environment and the use of magnitudes is deeply analysed in this work. This article could be divided in two main sections. Firstly, the mathematical section where the attention is focused purely on the mathematical structure of the Poisson games. Secondly, the economic section where we focus on the economic implications of the tools previously developed.

Concerning the first section, the importance of the offset ratios associated to an event has been underlined. Indeed, the Magnitude Equivalence Theorem depends intrinsically on the concept of offset ratio and on its properties. This theorem allows us to compute the magnitudes of pivotal events (which have a difficult geometrical structure) in a quite simple way. We have been able to differentiate the events between the normal ones (where the offset ratios are different from zero) and the irregular ones (where there is at least offset ratio equal to zero).

Different objectives have been attained in the second section. We have been able to prove that, because of the presence of the "correlated" ballots, the expected winner of the election does not always coincide with the profile Condorcet winner and that voters do not behave sincerely. By "correlated" ballots, we mean ballots where voters are allowed to vote for more than one candidate. In the Poisson-Myerson framework, the number of voters who choose a ballot is independent of the number of voters who choose all other ones. However, as we allow voters to vote for more than one candidate, the scores of the candidates are correlated. This is the main difference with Laslier (2004) framework in which the scores of the candidates are assumed to be independent. Mathematically, the "correlated" ballots break the partial ordering stated by Laslier, which is a key feature of his model.

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7 Technical Appendix

Magnitude of a pivot between candidates a and b

$$\mu(\{s(a) = s(b) \geq s(c)\}) = \min_{\lambda} \; r(t_1)\exp[\lambda_1 - \lambda_2 + \lambda_3] + r(t_2)\exp[\lambda_3] + r(t_3)\exp[-\lambda_3] - 1$$

such that $\lambda_i \geq 0 \;\forall\; i$.

Solving the optimal problem we can state that,

$$\mu(\{s(a) = s(b) \geq s(c)\}) = -r(t_1) \text{ as } r(t_2) > r(t_3)$$

Similarly, the following equalities can be stated.

Magnitude of a pivot between candidates a and c

$$\mu(\{s(a) = s(c) \geq s(b)\}) = \min_{\lambda} \; r(t_1)\exp[\lambda_1 - \lambda_2 + \lambda_3] + r(t_2)\exp[\lambda_1 - \lambda_2] + r(t_3)\exp[-\lambda_1 + \lambda_2] - 1$$

such that $\lambda_i \geq 0 \;\forall\; i$.

$$\mu(pivot(a,c)) = \mu(\{s(a) = s(c) \geq s(b)\} = -(\sqrt{r(t_1) + r(t_2)} - \sqrt{r(t_3)})^2 = \mu(tie(a,c))$$

Magnitude of a pivot between candidates b and c

$$\mu(\{s(b) = s(c) \geq s(a)\}) = \min_{\lambda} \; r(t_1)\exp[-\lambda_3] + r(t_2)\exp[\lambda_1 - \lambda_2] + r(t_3)\exp[-\lambda_1 + \lambda_2] - 1$$

such that $\lambda_i \geq 0 \;\forall\; i$.

$$\begin{aligned}\mu(pivot(b,c)) = \mu(\{s(b) = s(c) \geq s(a)\}) &= -r(t_1) - (\sqrt{r(t_2)} - \sqrt{r(t_3)})^2 \\ &= -r(t_1) + \mu(tie(b,c))\end{aligned}$$