

# A class of models satisfying a dynamical version of the CAPM\*

Elyès Jouini<sup>†</sup>

Université Paris IX-Dauphine and CREST

Clotilde Napp

Université Paris IX-Dauphine and CREST

June 26, 2007

## Abstract

Under a comonotonicity assumption between aggregate dividends and the market portfolio, the CCAPM formula becomes more tractable and more easily testable. In this paper, we provide theoretical justifications for such an assumption.

## 1. Introduction

In this paper we provide a dynamical version of the CAPM, i.e., an equilibrium model, analogous to the CCAPM, replacing in the resulting formula aggregate consumption by the market portfolio. Indeed, we show in a continuous time framework that the instantaneous rate of return of any security in excess of the riskless interest rate is a multiple common to all securities of the “instantaneous covariance” of the

---

\*Keywords : CAPM, CCAPM, market beta, equilibrium, financial markets. JEL numbers D50, G12.

<sup>†</sup>Corresponding author: Elyès Jouini, CEREMADE, Université de Paris IX Dauphine, Place du Maréchal de Lattre de Tassigny, 75116 Paris. e-mail : [jouini@ceremade.dauphine.fr](mailto:jouini@ceremade.dauphine.fr)

return with the market portfolio changes. In order to derive such a dynamical version of the CAPM, we need an important assumption: the stocks prices are nondecreasing functions of the dividends. The main contribution of this paper is to prove that for almost all the classical utility functions, there exist equilibria satisfying this monotonicity assumption.

The main advantage of our approach is that in order to estimate the risk premium of any asset we do not need to observe the aggregate consumption process, which, as underlined by Breeden, Gibbons and Litzenberger (1989), is a delicate issue. Furthermore, as underlined by Mankiw and Shapiro (1986) “a stock’s market beta contains much more information on its return than does its consumption beta”.

## 2. The model

We fix a finite-time horizon  $T > 0$  and we set  $\mathbf{T} \equiv [0, T]$ . As usual, we let  $(\Omega, \mathcal{F}, \mathbf{P})$  denote a fixed probability space on which is defined a  $k$ -dimensional Brownian motion  $W = (W_t^1, \dots, W_t^k)_{t \in \mathbf{T}}^*$  and a  $\mathbf{P}$ -augmentation of the natural filtration generated by  $W$  denoted by  $(\mathcal{F}_t)_{t \in \mathbf{T}}$ .

We consider a complete financial market with  $(k + 1)$  securities whose prices are denoted by  $S^0, S^1, \dots, S^k$ . Each security  $S^i$  is in total net supply equal to  $\alpha_i$  with  $\alpha_0 = 0$  and we denote by  $D^i$  the dividends process associated to asset  $S^i$  for  $i = 1, \dots, k$ , and by  $D = \sum_{i=1}^k \alpha_i D^i$  the aggregate dividends process. We assume that

$$dS_t^0 = S_t^0 r_t dt, S_0^0 = 1 \tag{2.1}$$

$$dS_t^i = S_t^i [(\mu_t^i - \delta_t^i) dt + \sigma_t^i dW_t] \text{ for } i = 1, \dots, k, \tag{2.2}$$

where the real valued processes  $r, \delta^i$  (the dividend yield process),  $\mu^i$  as well as the  $(1 \times k)$ -matrix valued process  $\sigma^i$  are progressively measurable and uniformly bounded. We assume that for all  $t$  in  $\mathbf{T}$ ,  $(\sigma_t)^{-1} =$

$(\sigma_j^i(t))_{1 \leq i, j \leq k}^{-1}$  exists and is uniformly bounded. The  $k$ -dimensional process  $\theta$  can then be defined by:

$$\theta_t \equiv (\sigma_t)^{-1} [(\mu_t - r_t \mathbf{1}_k)]$$

where  $\mu = (\mu^1, \dots, \mu^k)^*$  and where  $\mathbf{1}_k$  denotes the  $k$ -dimensional vector whose every component is one.

We let for all  $t \in \mathbf{T}$ ,  $M_t \equiv \mathcal{E}_t(-\theta) \equiv \exp \left\{ \int_0^t (-\theta_s)^* dW_s - 1/2 \int_0^t \|\theta_s\|^2 ds \right\}$ .

The market is complete, so we can apply the results of Huang (1987) and our economy can be supported by a representative agent, endowed with one unit of the market portfolio  $N \equiv \sum_{i=1}^k \alpha_i S^i$  and whose preferences for consumption and terminal wealth are characterized by

$$U(c, X) = E \left[ \int_0^T u(t, c_t) dt + V(X) \right]$$

where the utility functions  $u$  and  $V$  satisfy the following assumption.

**Assumption (U)** *The function  $u : \mathbf{T} \times R_+ \rightarrow R$  is continuous. The function  $V : R_+ \rightarrow R$  as well as for all  $t \in \mathbf{T}$ ,  $u(t, \cdot)$  are  $C^3$  on  $(0, \infty)$ , strictly increasing, strictly concave and satisfy  $\inf_x u_c(t, x) = \inf_x V'(x) = 0$ . The function  $I_u : \mathbf{T} \times R_+^* \rightarrow R$  is of class  $C^{1,2}$ , where  $I_u(t, \cdot)$  denotes the generalized inverse of  $u_c(t, \cdot)$ .*

Recall that an equilibrium consists of a collection  $\{S^0, \dots, S^k; (\hat{\pi}; \hat{c})\}$  such that  $(\hat{\pi}; \hat{c})$  is an optimal investment-consumption process for the representative agent as defined in Merton (1971) and such that markets clear, i.e.,  $\hat{c}_t = D_t$  and  $\hat{\pi}_t^i = \alpha_i S_t^i$  for all  $t \in \mathbf{T}$ . At the equilibrium the optimal consumption process of the representative agent must equal the aggregate dividends process, and we deduce from e.g. Karatzas (1989) that a necessary and sufficient condition for an equilibrium to be reached is that there exists  $\gamma > 0$  such that

$$\beta_t M_t = \gamma u_c(t, D_t), \quad t \in \mathbf{T} \quad (2.3)$$

$$\beta_T M_T = \gamma V'(N_T) \quad (2.4)$$

where  $\beta_t \equiv 1/S_t^0$  and with no additional condition because in this case, the budget constraint, i.e.  $E \left[ \int_0^T \beta_t M_t D_t dt + \beta_T M_T N_T \right] = N_0$ , is automatically binding.

This leads to the following classical CCAPM formula, where the market clearing conditions permit to replace aggregate consumption by aggregate dividends

$$\mu^i(t) - r_t = \Gamma_t \sigma^i(t) (\sigma_D(t))^* \quad \text{for all } t \in \mathbf{T} \text{ and for all } i = 1, \dots, k \quad (\text{CCAPM})$$

where  $\Gamma_t = -u_{cc}(t, D_t) / u_c(t, D_t)$  and  $dD_t = \mu_D(t)dt + \sigma_D(t)dW_t$  for all  $t \in \mathbf{T}$ .

The market portfolio value satisfies a stochastic differential equation of the form

$$dN_t = N_t ((\mu_N(t) - \delta_N(t)) dt + \sigma_N(t)dW_t).$$

When the process  $N$  can be written in the form  $N_t = \{n(t, D_t); t \in \mathbf{T}\}$  where  $n : \mathbf{T} \times R_+ \rightarrow R_+$  is of class  $C^{1,2}$  with  $n_x(t, x) > 0$  (**Assumption (N)**), then Itô's Lemma gives us the following Dynamical CAPM formula

$$\mu^i(t) - r_t = \bar{\Gamma}_t \sigma^i(t) (\sigma_N(t))^* \quad \text{for all } t \in \mathbf{T} \text{ and for all } i = 1, \dots, k. \quad (\text{Dynamical CAPM})$$

This means that it suffices to observe e.g. an index (seen as a *proxy* of the market portfolio) in order to estimate the risk premium of any asset.

In the next, we prove the existence of equilibria satisfying Assumption (N) for a large class of utility

functions.

### 3. Existence of equilibria satisfying Assumption (N)

More precisely, we shall establish that for a given choice of utility functions and of risk level, there exists an equilibrium satisfying Assumption (N).

**Proposition 1.** *Let  $\sigma$  be given satisfying the regularity assumptions of Section 2. For the following choices of  $(u, V)$ , there exists an equilibrium satisfying Assumption (N) :*

1. *Logarithmic utility function:  $u(t, c) = \exp(-\alpha t) \alpha \log(c + \alpha)$ ,  $V(x) = \exp(-\alpha T) \log(x + 1)$ ,  $n(t, x) = (1/\alpha)x$  for some positive constant  $\alpha$ .*
2. *Power utility function:  $u(t, c) = \frac{c^\delta}{\delta}$  for  $\delta \in ]0, 1[$ ,  $V(x) = \frac{a^{(\delta-1)}}{\delta} x^\delta$ ,  $n(t, x) = (1/a)x$  for some positive constant  $a$ .*
3. *Exponential utility function:  $u(t, c) = 1 - \exp(-c)$ ,  $V(x) = \frac{(1+x)^\delta}{\delta}$ ,  $n(t, x) = \exp\{x/(1-\delta)\} - 1$  for some  $\delta \in ]0, 1[$ .*

**Proof** An equilibrium is characterized by (2.3) and (2.4). We proceed as in Karatzas-Lehoczký-Shreve (1990) and by Itô's Lemma, we obtain that (2.3) and (2.4) as well as Assumption (N) are satisfied if there exists a function  $d$  of class  $C^{1,2}$  such that for all  $t$ ,  $d_x(t, \cdot) > 0$  and

$$\left\{ \begin{array}{l} \mu_N(t) \left[ 1 + \frac{f_x(t, N_t)}{f(t, N_t)} N_t \right] = -\frac{f_t(t, N_t)}{f(t, N_t)} + \frac{f_x(t, N_t)}{f(t, N_t)} \left[ d(t, N_t) - \|\sigma_N(t)\|^2 N_t \right] - \frac{1}{2} \frac{f_{xx}(t, N_t)}{f(t, N_t)} \|\sigma_N(t)\|^2 N_t^2 \\ \mu^i(t) - r_t = -\frac{f_x(t, N_t)}{f(t, N_t)} N_t (\sigma^i(t)) (\sigma_N(t))^* \\ d(T, N_T) \equiv I_u[T, V'(N_T)] \end{array} \right.$$

where  $f(t, x) \equiv u_c(t, d(t, x)) / u_c(0, d(0, N_0))$ .

In our examples, solutions are given by

1.  $\mu_t^i = \alpha + \frac{N_t}{N_t+1} \sigma^i(t) (\sigma_N(t))^*$ ,  $r_t = \alpha$ ,  $d(t, x) = \alpha x$
2.  $\mu_t^i = r_t + (1 - \delta) \sigma^i(t) (\sigma_N(t))^*$ ,  $r_t = \frac{1}{2} (\delta - 1) \|\sigma_N(t)\|^2 + \frac{\alpha(\delta-1)}{\delta}$ ,  $d(t, x) = \alpha x$
3. 
$$\begin{cases} \mu_N(t) = -(\delta - 1)^2 \frac{\log(1+N_t)}{1+\delta N_t} - (\delta - 1) \frac{N_t}{1+\delta N_t} \|\sigma_N(t)\|^2 \left[ 1 + \frac{1}{2} (\delta - 2) \frac{N_t}{(1+N_t)} \right] \\ r_t = \frac{1}{2} \delta (\delta - 1) \frac{(N_t)^2}{(1+N_t)(1+\delta N_t)} \|\sigma_N(t)\|^2 - (\delta - 1)^2 \frac{\log(1+N_t)}{(1+\delta N_t)}, \quad d(t, x) = (1 - \delta) \log(1 + x) \end{cases}$$

which completes the proof. ■

**Proposition 2.** *If  $\sigma$  is given satisfying the regularity assumptions of Section 2 and if  $x \frac{V''(x)}{V'(x)}$  and  $x^2 \frac{V'''(x)}{V'(x)}$  are bounded, if  $\frac{V'(x)}{u_c(t,x)}$  is bounded in a neighborhood of 0 and if  $\inf_x x \frac{V''(x)}{V'(x)} > -1$ , then there is an equilibrium satisfying Assumption (N).*

**Proof** Immediate using the proof of Proposition 1 and taking  $d(t, x) = I_u[t, V'(x)]$  and

$$\begin{aligned} r_t &= \frac{\beta_t^V}{\left[ 1 + \frac{V''(N_t)}{V'(N_t)} N_t \right]} - \frac{V'(N_t)}{V(N_t)} N_t \|\sigma_N(t)\|^2 \\ \mu_t^i &= r_t + \frac{V''(N_t)}{V'(N_t)} N_t \sigma_t^i (\sigma_N(t))^* \end{aligned}$$

where  $\beta_t^V = -\frac{V''(N_t)}{V'(N_t)} N_t \|\sigma_N(t)\|^2 + \frac{V''(N_t)}{V'(N_t)} I_u[t, V'(N_t)] - \frac{1}{2} \frac{V'''(N_t)}{V'(N_t)} \|\sigma_N(t)\|^2 (N_t)^2$ . ■

We have so far obtained existence of an equilibrium satisfying Assumption (N) when  $\sigma$  is given. Another way of modelling the introduction of uncertainty is to replace uncertainty on the securities price process by uncertainty on the dividends process.

**Proposition 3.** *Let  $\sigma_D, u$  and  $V$  be given.*

1. *If  $\sigma_D = \bar{\sigma} D$  for  $\bar{\sigma} = (\bar{\sigma}^1, \dots, \bar{\sigma}^k)$  in  $R^k$ , if  $V(\cdot) = u(T, \cdot)$  and if  $\tilde{u} : x \mapsto x u_c(t, x)$  is increasing, then for all dividend process  $D = \{D_t; t \in \mathbf{T}\}$  of the form  $dD_t = \bar{\mu} D_t dt + \sigma_D(t) dW_t$ , for  $\bar{\mu} \in \mathbb{R}$ , there exist price processes  $S^0, \dots, S^k$  such that the equilibrium conditions (2.3) and (2.4) are satisfied as well as Assumption (N).*

2. In the general case, for all utility functions  $u$  and  $V$ , there exists  $\mu_D$  such that  $N_t = n(t, D_t)$  for  $n(t, \cdot) = I_V [u_c(t, \cdot)]$  and the equilibrium conditions (2.3) and (2.4) are satisfied.

**Proof** We must have  $n(T, x) = I_V [u_c(T, x)]$ . Adopting the same approach as in the proof of Proposition 1, it is easy to obtain that there is an equilibrium with  $N_t = n(t, D_t)$  if  $n$  satisfies the following partial differential equation

$$\begin{aligned} & n_t(t, D_t) + n_x(t, D_t) \mu_D(t) + \frac{1}{2} n_{xx}(t, D_t) \|\sigma_D(t)\|^2 + D_t - A_t n(t, D_t) \\ &= -\frac{u_{cc}(t, D_t)}{u_c(t, D_t)} n_x(t, D_t) \|\sigma_D(t)\|^2, \end{aligned} \quad (3.1)$$

for  $A_t \equiv -\frac{1}{u_c(t, D_t)} [u_{ct}(t, D_t) + u_{cc}(t, D_t) \mu_D(t) + \frac{1}{2} u_{ccc}(t, D_t) \|\sigma_D(t)\|^2]$  and with terminal condition  $n(T, x) = I_V [u_c(T, x)]$ .

This is in turn equivalent to the following equation

$$-Z_t(t, x) = Z_{xx}(t, x) + F(t, x)$$

with terminal condition  $Z(0, x) = H(x) \equiv (u_c I_V \circ u_c) \left( T, \exp \frac{1}{\sqrt{2}} \|\bar{\sigma}\| x \right)$  where  $Z(t, x) \equiv (n u_c) [T - t, \exp(\alpha x + \beta t)]$ ,  $F(t, x) \equiv \exp(\alpha x + \beta t) u_c(T - t, \exp(\alpha x + \beta t))$ ,  $\alpha = \frac{1}{\sqrt{2}} \|\bar{\sigma}\|$  and  $\beta = \alpha^2 - \bar{\mu}$

The solution of this partial differential equation (see Cannon [1984]) is given by

$$Z(t, x) = \int_{-\infty}^{+\infty} K(t, x - y) H(y) dy + \int_0^t \int_{-\infty}^{+\infty} K(t - \tau, x - y) F(\tau, y) dy d\tau \quad (3.2)$$

where

$$K(t, x) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right).$$

By differentiation, we obtain

$$Z_x(t, x) = K(t, \cdot) * H'(x) + \int_0^t K(t - \tau, \cdot) * F_x(\tau, \cdot)(x) d\tau.$$

We check then that under our conditions we have  $H'(x) > 0$  and  $F_x(t, x) \geq 0$ . Hence  $Z_x(t, x) > 0$  which implies that  $n_x(t, \cdot) > 0$ . This ends the proof of 1.

If we do not impose a particular choice for  $\mu_D$ , it suffices to impose  $n(t, \cdot) = I_V(u_c(t, \cdot))$  in the previous partial differential equation and to choose  $\mu_D$  accordingly. ■

## 4. Conclusion

We proved that there exist equilibria satisfying our comonotonicity condition for a large class of utility functions. In such equilibria, the state price density  $M_t$  decreases with the market portfolio value and the consumption *beta* is replaced by the market *beta* in the CCAPM formula. These two features are consistent with empirical evidence as underlined by Aït-Sahalia and Lo (2000) for the first one and by Mankiw and Shapiro (1986) for the second one.

## References

- [1] Aït-Sahalia, Y., Lo, A., 2000. Nonparametric risk management and implied risk aversion. *Journal of Econometrics*, 94, 9-51.
- [2] Breeden, D.T., M.R. Gibbons and R.H. Litzenberger, 1989. Empirical tests of the consumption oriented CAPM. *Journal of Finance*, 44 (2), 231-262.
- [3] Cannon, 1984. The one-dimensional heat equation, in *Encycl. of Math.*, vol.23, ed. G-C. Rota, Cambridge University Press.



- [4] Huang, C.F., 1987. An Intertemporal General Equilibrium Asset Pricing Model: The Case of Diffusion Information. *Econometrica* , 55, 117-142.
- [5] Karatzas, I., 1989. Optimization problems in the theory of continuous trading, *SIAM J. Control Optimization*, 27, 1221-1259.
- [6] Karatzas, I., Lekoczy, J-P., Shreve, S.E., 1990. Existence and uniqueness of multi-agent equilibrium in a stochastic dynamic consumption/investment model. *Mathematics of Operations Research*, 15 (1), 80-128.
- [7] Mankiw, N.G., Shapiro M.D., 1986. Risk and return : Consumption beta versus market beta. *The Review of Economics and Statistics*, 68 (3), 452-459.
- [8] Merton, R., 1971, Optimum consumption and portfolio rules in a continuous time model, *Journal of Economic Theory*, 3, 373-413.