A prospective study of the k - factor Gegenbauer Processes with heteroscedastic errors and an application to inflation rates

Guégan Dominique. [1]

Abstract

In this paper we investigate some statistical properties of the k-factor Gegenbauer process with particular noises. We make an extensive study concerning the behavior of these models in the stationary and non stationary cases using Monte Carlo simulations. We apply some of the results on a data set of inflation rates.

Introduction

Since the sixties', a fairly large number of power spectra have been estimated using economic data and, as observed by Granger (1966) for instance, these spectra exhibit a spike at the very low frequencies. At this time, the spectral methods are based upon the idea of decomposition of a stochastic process into a (possible non finite) number of orthogonal components, each of which being associated with a "frequency". It is well known that such a decomposition is always possible for processes that are covariance stationary, i.e., have variances and covariances independent of real time, and that the power spectrum records the contribution of the components belonging to a given frequency band to the total variance of the process. Thus the existence of peaks at low frequencies suggests that there exists, for these economic data, long term fluctuations associated with the frequency components. Generally the spectrum does not consist of a series of peaks of decreasing size corresponding to different cycles, but is rather a smooth, decreasing curve with no (significant) peaks corresponding to periods longer than the one of the lowest frequency component (see for examples Granger and Hatanaka, 1964). It has, of course, long been realized that these cycles are not strictly periodic fluctuations and do not correspond to a specific seasonal component.

The idea of persistence and long memory behavior inside these kinds of data become more precise in the eighties' and at this time Granger (1980), Granger and Joyeux (1980) and Hosking (1981) propose a parsimonious representation of the long memory effect with the FARMA process. A possible trend in means within the data can be also the subject of certain confusion: the definition of such a notion is not obvious. It is not sufficient to think that a trend in mean can be approximated by a simple polynomial or exponential expression, and a trend in a short series would not be so considered if the series were longer. Indeed, in order to understand better the problem of existence of trend in mean in data, which is close to detect the existence of non stationarity, a

[1] **Professor, E.N.S. Cachan, Department of Economy and Managment, 61 Avenue du President Wilson, 94235, Cachan Cedex, France, e-mail : guegan@ecogest.ens-cachan.fr**

lot of unit-root tests has been developed by Dickey-Fuller (1981), Tiao and Tsay (1983), Phillips-Perron (1988) and Kwiakowski, Phillips, Schmidt and Shin (1992). Thus, when some non stationarity or long memory behaviors are observed within real data, it is classical for practicians to use both approaches to try to detect effectively the existence of persistence within the data. It is important to think that a non stationary behavior is often associated with a linear approach, whereas the research of a long memory behavior is associated with a nonlinear point of view.

In that way, in economic policy, the development of the inflation rate plays a prominent role, and from the Maastricht treaty the convergence of inflation rates of prospective members of the European Monetary Union is a very important criterion. The inflation rates have been extensively studied in the literature and most of the authors detect non stationarity using unit root tests, for particular periods, see for instance Barsky (1987), Rose (1988), McDonald and Murphy (1989), Kirchgasser and Wolters (1993). These results are very sensitive with respect to the different sample periods. Because of the contracdictory results, using these different unit root tests, the choice of modelling long memory behavior upon these kinds of data seem a more appropriate approach. Concerning inflation rates, this has been done in particular by Chung and Baillie (1993), Baillie, Chung and Tieslau (1996a), Hassler and Wolters (1995) and Franses and Ooms (1997). Other financial data sets have been also studied with this approach, see for instance, for stock markets, Smith and Yadov (1994), Bollerslev and Mikkelsen (1996), Breidt and Davis (1998) and Willinger, Taqqu and Teverovski (1999), for stock market trading volume, Bollerslev and Jubinski (1999) and Lobato and Velasco (2000) and for exchanges rates, Beran and Ocker (1999) and Ferrara and Guégan (2000, 2001b, 2002).

Among all the studies, some suggest that the long memory coefficient, called here d, can take some values greater than or equal to $1/2$. This magnitude of long memory has been reported in some of the previous papers including Bollerslev and Mikkelsen (1996), Hassler and Wolters (1995), Ferrara and Guégan (2000) and Lobato and Velasco (2000).

On the other side, the study of most of these real data prevails on the existence of more complex behavior including both persistence and heteroscedasticity. Thus, it appears pertinent to associate the possibility of modeling long term persistence with quasi periodic behavior and heteroscedasticity. These different phenomena have been already detected in a lot of financial data, see for instance, Avouyi-Dovi, Guégan and Ladoucette (2002), Campbell and Shiller (2001), Ferrara and Guégan (2001a), Bollerslev, Cai and Song (2000), Bollerslev and Wright (2000), Lobato and Velasco (2000), Chambers (1998), Hassler and Wolters (1995) and Ding, Granger and Engle (1993).

Guégan (2000) in a recent article proposed a new model which permits to model long memory behavior with quasi periodic behavior in the conditional variance of the observed data. This model is an extension of the FIGARCH, FIEGARCH and the FARMA-GARCH models introduced respectively by Baillie, Bollerslev and Mikkelsen (1996b), Bollerslev and Mikkelsen (1996) and Ling and Li (1997). A general representation of the model proposed by Guégan (2000) is given by the following expression:

$$\phi(B)\prod_{i=1}^{k}(I-2u_iB+B^2)^{d_i}X_t = m+(1-\theta(B))\eta_t \qquad (1)$$

with $L(\eta_t \mid I_t) \sim N(0,\sigma^2)$ and $\sigma_{t+1}=\varphi(\eta_t,\sigma_t)$, where B denotes the backshift operator: $BX_t = X_{t-1}$, and the function $\varphi: R^2 \to R_{\geq 0}$ will be some measurable function, which will have to be specified in the following. $\phi(z)=1-\phi_1 z-\cdots-\phi_p z^p$ and $\theta(z)=1-\theta_1 z-\cdots-\theta_q z^q$ are p-th and q-th degree polynomials, $(I-2uB+B^2)^{d_i}$, i=1, ..., k is a difference operator defined by means of Gegenbauer polynomials, that we present with more details below. This operator has a finite number of zeros or singularities of order $d_1,\cdots,d_k$, ($|d_i|$<1/2, i =1, ...,k) on the unit circle which allows the modelling of long-short memory data containing seasonal periodicities, m is some constant. As for the random shocks, or innovations, $(\eta_t)_t$, we will make the usual hypothesis that they are independent and have a conditional Gaussian law, with mean 0 and variance 1. The set I_t is the σ-field generated by the past values $(\eta_{t-1},\eta_{t-2},\cdots)$. This model can provide a useful way of analysing the relationship between the conditional mean and variance of a process exhibiting long memory and show decay in its level, yet with time varying volatility.

In this paper we are interested by the statistical properties of the model (1), in order to use it on real data. This long memory process whose covariance function is not absolutely summable is characterized by an explosive behavior of its spectral density in frequency λ_j, $0<\lambda_j<\pi$, $1\leq j\leq k$. We will present these properties for a specific class of function φ in (1), which corresponds to the GARCH models, introduced by Bollerslev (1986). All the results can be extended in a lot of cases that we precise latter. Apart from a theoretical discussion we present the models fitted on consumer price index and inflation rates. We apply fractionally integrated models to describe the behaviors of these series. We find that the difference parameters are significantly different from 1 as well as from 0, and we do not reject a priori the hypothesis of non stationarity. The same kind of assumption has been done for instance by Hassler and Wolters (1995) and Baillie, Bollerslev and Mikkelsen (1996b).

As we have specified in the title, this work is a preliminary study on the k-factor Gegenbauer processes. We prove certain results, but a lot are still opened problems.

In a first step (Section 1) we present the model on which we work and we give some properties (existence of a stationary solution, asymptotic behavior of the autocovariance function and spectral density, ergodicity, mixing, asymptotic distribution for the k-factor GIGARCH process). Section 2 reports the results of several simulations experiments for different classes of GIGARCH processes defined in (1), showing trajectories, autocorrelation function, spectral densities and empirical distributions. Until now it does not exist an identification theory which permits to decide between the different classes of non linear models, thus the knowledge of the second order properties empirically and graphically of such a model can be fundamental to go through the choice of a model when we observe real data. We discuss, in Section 3, empirical results which explain how non stationary fractional integration may arise through specific series. Indeed this work is important because a lot of data sets can produce long memory behavior and non stationarity. Section 4 applies the GIGARCH model to different data sets on consumer price index and inflation rates. The Section 5 gives a summary of the approach developped in this paper and some possible extensions. We postponed the proofs of main results of section 1 in an appendix based in Section 6.

I - Properties of the stationary k -factor GIGARCH model.

In this section we precise the function φ in (1) and using the same notations as above, we consider the k-factor GIGARCH model introduced first in Guégan (2000) and defined by the following expression (here $p = q = 0$ in (1)):

$$\prod_{i=1}^{k} (I - 2u_i B + B^2)^{d_i} X_t = \eta_t \qquad (2)$$

where $L(\eta_t \mid I_t) \sim N(0, \sigma_t^2)$ and where:

$$\sigma_t^2 = a_0 + \sum_{i=1}^{r} a_i \eta_{t-i}^2 + \sum_{j=1}^{s} b_j \sigma_{t-j}^2 .$$

We assume that $a_0 > 0$, $a_i \in R$, i = 1, ...r, $b_j \in R$, j = 1, ..., s, $(d_i, u_i) \in R^2$, i = 1, ..., k. In this paper, the conditional law of the process $(\eta_t)_t$ is Gaussian, nevertheless this assumption can be relaxed (the theoretical results given above can be extended, but the computations are not straightforward and we do not give them in this paper). In the next section we examine some simulations in this extended context. We give now some statistical properties of the process (2) under smooth assumptions.

The k -factor Gegenbauer process defined by:

$$\prod_{i=1}^{k} (1 - 2u_i B + B^2)^{d_i} X_t = \eta_t, \qquad (3)$$

where $(\eta_t)_t$ is a Gaussian noise has been extensively studied in the literature, see for instance Giraitis and Leipus (1995), Wayne, Woodward, Cheng and Gray (1998) Ferrara (2000), Collet and Guégan (2001) and Collet, Guégan and Valdès (2002). This model exhibits both long term persistence and quasi periodic behaviour, Hosking (1981) was the first to mention this phenomena. Here, we consider an extension of this model when the noise $(\eta_t)_t$ is dependent and follows a GARCH(r,s) model. In order to give the properties of the process (2), we introduce the Gegenbauer polynomials $(\psi_j^{(d,u)})_{j \in Z}$ defined by:

$$(1 - 2uz + z^2)^{-d} = \sum_{j \geq 0} \psi_j^{(d,u)} z^j, \qquad (4)$$

where $|z| \leq 1$ and $|u| \leq 1$. They can be easily computed by the following recursion formula:

$$\begin{cases} \psi_0^{(d,u)} = 1 \\ \psi_1^{(d,u)} = 2du \\ \psi_j^{(d,u)} = 2u\left(\frac{d-1}{j} + 1\right)\psi_{j-1}^{(d,u)} - \left(2\frac{d-1}{j} + 1\right)\psi_{j-2}^{(d,u)} \end{cases} \qquad (5).$$

An explicit expression of $\psi_j^{(d,u)}$ is the following:

$$\psi_j^{(d,u)} = \sum_{k=0}^{[j/2]} \frac{(-1)^k (2u)^{j-2k} \Gamma(d-k+j)}{k!(j-2k)!\Gamma(k)}, \quad (6)$$

with $\Gamma(.)$ the Gamma function. See for instance Magnus, Oberhettinger and Soni (1966) or Rainville (1960) for further details on Gegenbauer polynomials.

Now, we recall some well known properties of the process $(\eta_t)_t$ which appears in (1). Because $(\eta_t)_t$ is conditionally Gaussian, we can write:

$$\eta_t^2 = a_0 Z_t^2 + \sum_{i=1}^{r} a_i Z_t^2 \eta_{t-i}^2 + \sum_{j=1}^{s} b_j Z_t^2 \sigma_{t-j}^2, \quad (7)$$

where Z_t is a standard normal random variable independent of I_{t-1}. In a vector form, the expression (7) becomes:

$$\xi_t = A_t \xi_{t-1} + \zeta_t, \quad (8)$$

where

$$\xi_t = (\eta_t^2, \cdots, \eta_{t-r+1}^2, \sigma_t, \cdots, \sigma_{t-s+1})',$$

and

$$\zeta_t = (a_0 Z_t^2, 0, \cdots, 0, a_0, 0, \cdots, 0)',$$

(the first component is $a_0 Z_t^2$ and the $(r+1)$th component is a_0 and Σ' denotes the transpose of the matrix Σ and A_t is given by:

$$\begin{bmatrix} a_1 Z_t^2 & \cdots & a_r Z_t^2 & b_1 Z_t^2 & \cdots & b_s Z_t^2 \\ I_{(r-1)(r-1)} & \cdots & 0_{(r-1)x1} & \cdots & 0_{(r-1)x1} & \cdots \\ a_1 & \cdots & a_r & b_1 & \cdots & b_s \\ \cdots & 0_{(r-1)x1} & \cdots & I_{(r-1)(s-1)} & \cdots & 0_{(s-1)x1} \end{bmatrix}$$

where $I_{(r \times r)}$ denotes the $r \times r$ identity matrix.

In (8), we got a relationship which is a stochastic differential equation. The existence of a solution for such an equation has been studied by Kesten (1973) for instance. Mikosch and Starica (2000) apply his result to get a solution for a GARCH(1,1) model and Starica (1999) for the GARCH(r,s) model. Then, under the previous assumptions, a solution for (7) can easily be derived and it has the following expression:

$$\eta_t = Z_t [a_0 + \sum_{j=1}^{\infty} \delta' (\Pi_{i=1}^{j} A_{t-1}) \zeta_{t-j}]^{1/2}, \quad (9)$$

where $\delta = (a_1, \cdots, a_r, b_1, \cdots, b_s)'$. A solution for the process $(X_t)_t$ defined by (2) is given in the following proposition:

Proposition 1: Assume that $a_0 > 0$, $a_i > 0$, i = 1, ..., r and $b_j > 0$, j = 1, ..., s, $\sum_{i=1}^{r} a_i + \sum_{j=1}^{s} b_j < 1$, $d_i < 1/2$, i = 1, ..., k and $u_i < 1$, i = 1, ..., k, then the process defined by (2) has a unique second order stationary solution (X_t, η_t). The solution has a causal representation given by

$$\begin{cases} X_t = \sum_{n=0}^{\infty} \psi_n^{(d,u)} \eta_{t-n} \\ \eta_t = Z_t (a_0 + \sum_{j=1}^{\infty} \delta^{'} (\prod_{i=1}^{j} A_{i-1}) \zeta_{t-j})^{1/2} \end{cases} \quad (10)$$

where

$$\psi_j^{(d,u)} = \sum_{0 \le l_1,..,l_k \le j, l_1+...+l_k=j} \psi_{l_1}^{(d_1,u_1)} \dots \psi_{l_k}^{(d_k,u_k)}, \quad (11)$$

with $(\psi_{l_i}^{(d,u)})_{l_i \in Z}$ the Gegenbauer polynomials defined in (6).

Proof: See the Appendix.

Now we precise the asymptotic behavior of the autocorrelation function and of the spectral density of the model (2). Their autocorrelation function resembles a superposition of hyperbolically damped sin waves, as also mentioned and illustrated in Guégan (2000).

Proposition 2: Assume that the hypotheses of proposition 1 are verified, then the autocorrelation function $\gamma_X(h) = E[X_h X_0]$ of the process defined by (2), in the case $\max_{j=1,,k} d_j > 0$ has the asymptotics:

$$\gamma_X(h) = \sum_{j=1,,k, d_j>0} \psi_j h^{2d_j^*-1} (\cosh \lambda_j + o(1)), \quad (12)$$

as $h \to \infty$, with :
- $\psi_j = \psi'_j$ if $\lambda_j = 0$ or π;
- $\psi_j = 2\psi'_j$, if $0 < \lambda_j < \pi$,

and

$$\psi'_j = f_\eta(\lambda) \Gamma(1-2d_j^*) \sin(d_j^* \pi) D^2(j),$$

where $f_\eta(\lambda)$ is the spectral density of the process $(\eta_t)_t$. Moreover,
- $d_j^* = d_j$ if $0 < \lambda_j < \pi$;
- $d_j^* = 2d_j$ if $\lambda_j = 0$ or π,

and finally:

$$D(j) = |2 \sin \lambda_j|^{-d_j} \prod_{j \ne l} |2(\cos\lambda_j - \cos\lambda_l)|^{-d_j}, \text{ if } 0 \prec \lambda_j \prec \pi \quad (13)$$
$$= \prod_{j \ne l} |2(\cos\lambda_j - \cos\lambda_l)|^{-d_j}, \text{ if } \lambda_j = 0 \text{ or } \pi.$$

The spectral density $f_X(\lambda)$ of the process $(X_t)_t$ is equal to:

$$f_X(\lambda) = f_\eta(\lambda) \prod_{j=1}^{k} |2(\cos\lambda - \cos\lambda_j)|^{-2d_j}, \quad (14)$$

and has the asymptotics

$$f_X(\lambda) \sim f_\eta(\lambda) D(j)^2 |\lambda - \lambda_j|^{-2d_j}, \text{ as } \lambda \to \lambda_j, \text{ j = 1, ..., k.} \quad (15)$$

Proof: See the Appendix.

We remark that, as $\lambda \to \lambda_i$, for $i = 1, \ldots, k$, the spectral density $f_X(\lambda)$ defined in (14) becomes unbounded when 0 < d < 1/2. Thus, the spectral density of a k-factor Gegenbauer process with heteroscedastic noise clearly possesses k peaks on the interval [0,π].

We denote now $A_t^{\otimes n}$ the Kronecker product of n matrices A_t and we use it to study the moments of the process (2).

Proposotion 3: Assume that the hypotheses of proposition 1 are verified and that the process $(X_t)_t$ is defined by (2), then,

- If $\rho(E(A_t^{\otimes 2})) < 1$, then the fourth-order moments of $(X_t)_t$ and $(\eta_t)_t$ are finite,
- If $\rho(E(A_t^{\otimes 4})) < 1$, then the eighth-order moments of $(X_t)_t$ and $(\eta_t)_t$ are finite,

with A_t defined previously.

Proof: See the Appendix.

One of the goal of this paper is to give tools which permit to use this model to explain the behavior of certain data sets in finance or macroeconomics. If the notion of stationarity is important, in order to apply some results concerning the extreme value theory, it is important to know if this class of models is ergodic and mixing and on which conditions. The ergodicity and mixing properties of different classes of GARCH models is now established, for instance for the GARCH(1,1) model, we refer to Lu (1996), for the β-ARCH model to Guégan and Diebolt (1994) and for the general GARCH(p,q) model to Carrasco and Chen (1999) and references therein. If $|\sigma_t| < \infty$, $E|\eta_t^2| < \infty$ and (η_0, σ_0) are initialized from the invariant measure, then the process (X_t, η_t) is ergodic and mixing with exponential decay. These conditions are clearly verified for the process $(\eta_t)_t$ which appears in the model (2). On the other hand the mixing properties of the long memory process are not so documented. In a recent article Guégan and Ladoucette (2001) establish the non mixing property of a Gegenbauer process defined by (3). We can establish the same property for the process (2) using the same way.

First of all, we recall some definitions concerning the notion of dependence for stationary processes. Let $(X_t)_t$ be a second order stationary process in $\Re$, and F_n^m the σ-algebra generated by $X_n, X_{n+1}, \ldots, X_m$ and let L_n^m denote the closure in L^2 of the vector space spanned by $X_n, X_{n+1}, \ldots, X_m$. We say that the process $(X_t)_t$ satisfies :

- the strong mixing condition if $\lim_{k \to +\infty} \alpha(k) = 0$, where :
$$\alpha(k) = \sup_{A \in F_{-\infty}^n, B \in F_{n+k}^\infty} |P(A \cap B) - P(A)P(B)|$$
- the completely regular condition if $\lim_{k \to +\infty} r(k) = 0$, where :

$$r(k) = \sup_{\xi_1,\xi_2} |corr(\xi_1,\xi_2)|,$$

when the random variables ξ_1 and ξ_2 are respectively measurable with respect to the σ-algebras $F_{-\infty}^{n}$ and F_{n+k}^{∞}.

- the completely linearly regular condition if $\lim_{k\to+\infty} \rho(k) = 0$, where :

$$\rho(k) = \sup_{\xi_1 \in L_{-\infty}^{n}, \xi_2 \in L_{n+k}^{\infty}} |corr(\xi_1,\xi_2)|.$$

We have the following relationships between these coefficients : α(k) ≤ r(k) and ρ(k) ≤ r(k); and for the Gaussian processes, ρ(k) = r(k) and α(k) ≤ r(k) ≤ 2πα(k), see for instance Ibrajimov and Rozanov (1974). Here we get the following result:

Proposition 4: Assume that the hypotheses of proposition 1 are verified, then the process defined by (2), is not completely linearly regular.

Proof: See the Appendix.

All the previous results are still true for the general process (2) with any finite p and q degrees in the polynomials ϕ(B) and θ(B). Using the same notations as before, we consider the following model:

$$\begin{cases} \phi(B)\prod_{i=1}^{k}(I-2u_iB+B^2)^{d_i} X_t = m + (I-\theta(B))\eta_t \\ L(\eta_t \mid I_t) \sim N(0,\sigma_t^2) \\ \sigma_t^2 = a_0 + \sum_{i=1}^{r} a_i\eta_{t-i}^2 + \sum_{j=1}^{s} b_j\sigma_{t-j}^2 \end{cases} \qquad (16)$$

The next lemma summarizes some properties of this general model (16):

Lemma 1: Let the process $(X_t)_t$ defined by the equations (16). We assume that all roots of ϕ(B) and θ(B) lie outside the unit circle, $a_0 > 0$, $a_i > 0$, i = 1, ..., r and $b_j > 0$, j= 1, ..., s and $\sum_{i=1}^{r} a_i + \sum_{j=1}^{s} b_j < 1$. Now, if we assume that k=1 (for simplicity), then,

- If d < ½ and |u| < 1,the process defined by (16) has a unique second order stationary ergodic solution (X_t,η_t). The solution has a causal representation given by

$$\begin{cases} X_t = \phi(B)^{-1}\theta(B)\sum_{n=0}^{\infty}\psi_n^{(d,u)}\eta_{t-n} \\ \eta_t = Z_t(a_0 + \sum_{j=1}^{\infty}\delta'(\prod_{i=1}^{j}A_{i-1})\zeta_{t-j})^{1/2} \end{cases} \qquad (17)$$

where $\psi_n^{(d,u)}$ is given by (6).

- If d > - ½ and |u| < 1, the process defined by (16) is invertible: that is $(\eta_t)_t$

can be written as a function of $(X_t)_t$ in the following way:

$$X_t = \phi(B)^{-1}\theta(B)\sum_{n=0}^{\infty}\pi_n^{(d,u)}X_{t-n} \qquad (18)$$

where

$$\pi_j^{(d,u)} = \sum_{0\le l_1,\ldots,l_k \le j, l_1+\ldots+l_k=j} \psi_{l_1}^{(-d_1,u_1)}\ldots\psi_{l_k}^{(-d_k,u_k)}, \quad (19)$$

where $\psi_n^{(d,u)}$ are the Gegenbauer polynomials defined in (11).

- The autocovarance function of (16) has the same asymptotics as in Proposition 2: the weights ψ_j are multiplied by $|\frac{\theta(e^{-i\lambda})}{\phi(e^{-i\lambda})}|^2$ and the spectral density is equal to

$$f(\lambda) = |\frac{\theta(e^{-i\lambda})}{\phi(e^{-i\lambda})}|^2 f_X(\lambda) \quad (20)$$

where $f_X(\lambda)$ is given by (14).

- If $\rho(E(A_t^{\otimes m})) < 1$, then the 2mth-order moments of the processes $(X_t)_t$ and $(\eta_t)_t$ are finite.
- The process (16) is not completely linearly regular.

All the properties given in this lemma are straightforward generalizations of the previous propositions.

Another kind of property which is important concerns the asymptotic distribution of the process $(X_t)_t$ or of the maxima of the process defined by $M_n = \max(X_1,\ldots,X_n)$, n ≥ 2. This property is fundamental, in particular, in order to use these models in a context of risk management. Indeed, the computations of measures of risks are more and more important and these measures are based on the knowledge of the limiting distributions associated to a given model.

Here the process $(\eta_t)_t$ which appears in (2) is a conditionally Gaussian GARCH model. If $(\eta_t)_t$ follows a GARCH(1,1) process, (this means, r = s = 1), then Mikosch and Starica (2000) proves that :

$$P(\eta \succ x) \sim cx^{-\kappa} \text{ where } \kappa \text{ satifies } E(a_1 Z^2 + b_1)^{\kappa} = 1. \quad (21)$$

This indicates that the GARCH(1,1) model have extremely heavy tails. Now, we have seen, in proposition 1 that the process $(X_t)_t$ defined by (2) can be expressed like an infinite moving average of the process $(\eta_t)_t$ which is regularly varying at ∞ and - ∞ with index κ/2. Then we can say that $(\eta_t)_t$ in (21) is an extremal process generated by an extreme value distribution that we denote G and MDA(G) the corresponding domain of attraction. Then the extremal distribution function F of the process $(X_t)_t$ belongs to MDA(G). This can derive for instance from Resnick (1987, chapter 4).

On the other side, the extreme value theory for Gaussian processes cannot be used here directly. Previously we have proven that the process (2) is not linearly completely regular, thus it does not verify the classical dependence conditions (see for instance Leadbetter, Lindgren and Rootzen (1983)). This condition is very important to establish the limit distribution of the maxima (M_n) of stationary process $(X_t)_t$. On the other side, it is easy to verify that for the process $(X_t)_t$ defined in (2) the Berman's (1964) condition, $\gamma_X(n)\ln n \to 0$ when n $\to \infty$, holds. But the process (2) is not Gaussian and we cannot derive directly the limiting distribution of the maxima (M_n) of the process using this result. Thus, the knowledge of the asymptotic distribution and the norming constants of the maxima (M_n) of the process $(X_t)_t$ is an opened problem. Nevertheless, empirically it appears that this limiting distribution belongs to Gumbel domain, see the simulations below. It is also important to remark that, if the Berman's condition is theoretically verified for most of the classical time series models, nevertheless it can be difficult to observe its convergence towards zero for finite samples,. This has been illustrated for instance in Diop and Guégan (2001) for SVM models with GED distributions.

II - Simulations

Using the package S-PLUS, see Ferrara and Guégan (2002), we have made some simulations to illustrate the properties of the k--factor GIGARCH model investigated in the previous section. To simplify, we have simulated, in this Section, the GIGARCH model defined in (1) with k=1, p = q= 0 and different noises $(\eta_t)_t$ that we precise later. All along this section, the values of the long memory parameters are d = 0.4 and u = 0.45. On each figure, we present a realization of length n = 1000, the autocorrelation function, the spectral density and the histogram of the process $(X_t)_t$. The simulated realizations shown in this section are based on m = 0. Thus, the expression of the Gegenbauer process which has been used in the following is:

$$(I-2(0.45)B+B^2)^{0.4}X_t=\eta_t \ . \quad (22)$$

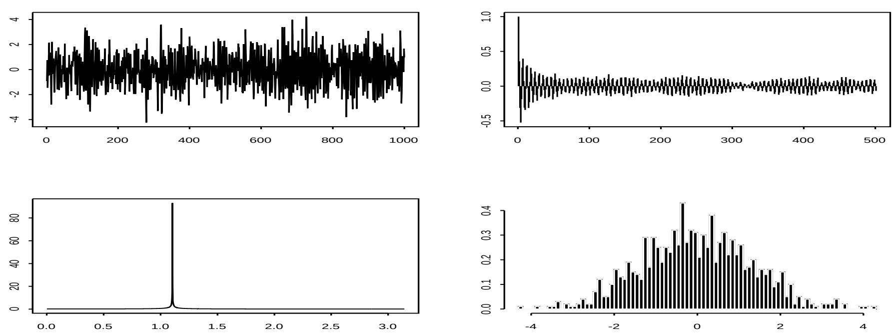

Figure 1: Realization, Autocorrelation function, spectral density, histogram of $(I-0.90B+B^2)^{0.4}X_t=\eta_t$, with $(\eta_t)_t$ a Gaussian noise.

The figure (1) shows a realization from the Gegenbauer process (22) simulated with a sequence of i.i.d. Gaussian noise $(\eta_t)_t$, mean 0 and variance 1.

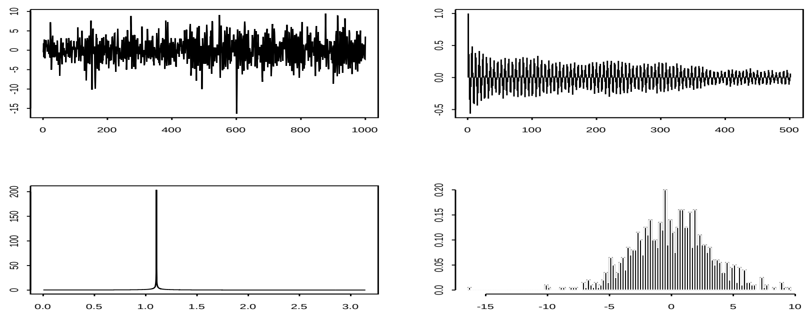

Figure 2: Realization, Autocorrelation function, spectral density, histogram of $(I-0.90B+B^2)^{0.4}X_t=\eta_t$, with $(\eta_t)_t$ a Student (4) noise.

The figure (2) is obtained using a sequence of i.i.d. Student(4) noise $(\eta_t)_t$ (the distribution of the process $(\eta_t)_t$ is a Student distribution with 4 degrees of freedom).

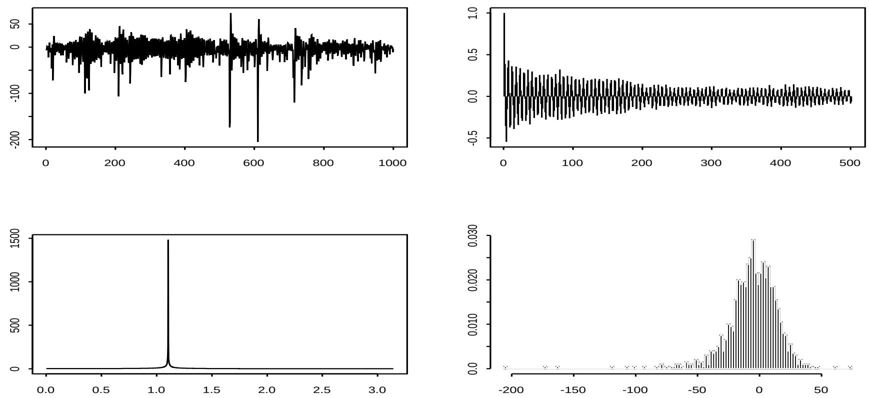

Figure 3: Realization, Autocorrelation function, spectral density, histogram of $(I-0.90B+B^2)^{0.4}X_t=\eta_t$, with $(\eta_t)_t$ a Weibull noise.

The figure (3) is obtained using a sequence of i.i.d. Weibull noise $(\eta_t)_t$ (the distribution of the process $(\eta_t)_t$ is a Weibull distribution with parameters (1, 0.5)).

We observe on these three trajectories and on the corresponding autocorrelation functions, the existence of periodic behaviors. As we expected the spectral density explodes in the G-frequency $\cos^{-1}(0.90)$, and the autocorrelation function decreases very slowly. The influence of the distribution law of the noise $(\eta_t)_t$ on the one of the process $(X_t)_t$ appears mainly in figure (3) as soon as we use a Weibull distribution for $(\eta_t)_t$. In that last case the distribution law of the process $(X_t)_t$ is far from a Gaussian law. This will be very important for estimation theory as soon as actually most of the results concerning the properties of the estimated long memory parameter are based on Gaussian Gegenbauer processes. For the process whose illustration is given in figure (3), we can also observe that the trajectory displays a behavior which presents intermittencies or explosions. On the other side, the use of Student distribution for the noise (figure (2)), even with a small degree of freedom, here 4, does not provide behaviors far from the one observed with Gaussian noise, figure (1).

The presence of long memory behavior inside real data driven by non Gaussian law arises very often. This is well known in hydrology, see for instance Collet and Guégan (2001) and references therein, and in finance, see Avouyi – Dovi, Guégan and Ladoucette (2002). The estimation of the long memory parameter in that case is more difficult. In Avouyi – Dovi, Guégan and Ladoucette (2002), we investigate empirically the importance of the law of the noise in the estimation procedures for the parameters. If we do not use Gaussian law, the variability of the estimates is great with respect to the method used.

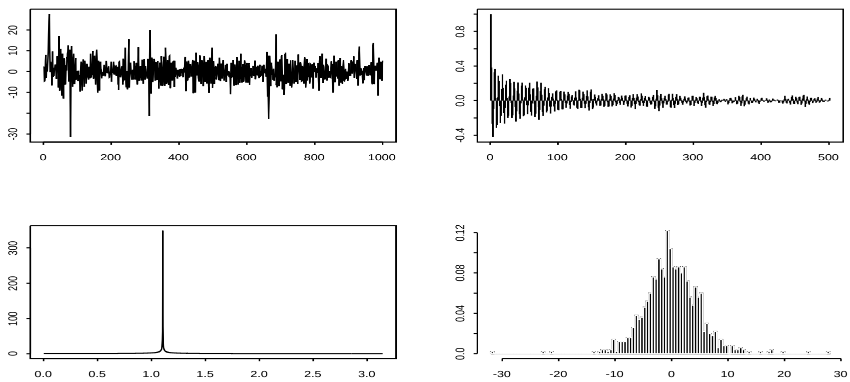

Figure 4: Realization, Autocorrelation function, spectral density, histogram of $(I-0.90B+B^2)^{0.4}X_t=\eta_t$, with $(\eta_t)_t$ an ARCH(1) noise (23) with $a_0=1$ and $a_1=0.8$.

We present now some trajectories of the Gegenbauer processes (22) simulated from a sequence of conditionally Gaussian noises $(\eta_t)_t$, mean zero and conditional variance σ_t^2, defined by:

$$\sigma_t^2=a_0+a_1\eta_{t-1}^2 \quad (23)$$

for the ARCH(1) noise (figure (4)), and

$$\sigma_t^2=a_0+a_1\eta_{t-1}^2+b_1\sigma_{t-1}^2 \quad (24)$$

for the GARCH(1,1) noise (figure (5)).

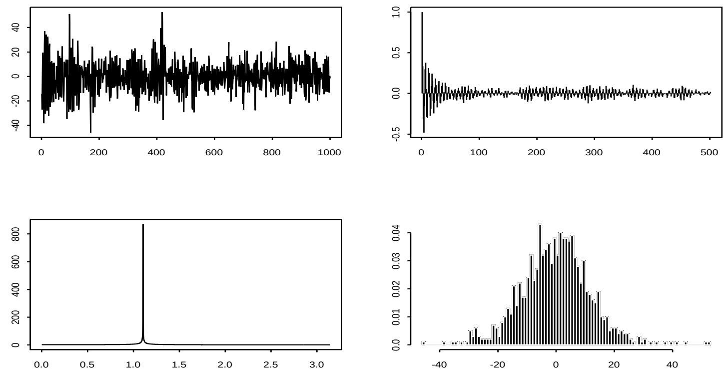

Figure 5: Realization, Autocorrelation function, spectral density, histogram of $(I-0.90B+B^2)^{0.4}X_t=\eta_t$, with $(\eta_t)_t$ an GARCH(1,1) noise (24) with $a_0=1$ and $a_1=0.2$ and $b_1=0.7$.

It is interesting to note - on these two figures (4) and (5) - the presence of a strong volatility on the trajectories. The spectral densities present an explosion always in the frequency $\cos^{-1}(0.90)$ and the autocorrelation functions decrease slowly but no so slowly than on the previous graphs. The histograms of the process $(X_t)_t$ are similar to those of a Gaussian law, in both cases, and in that cases, it seems possible to use the classical estimation theory to estimate the long memory parameters.

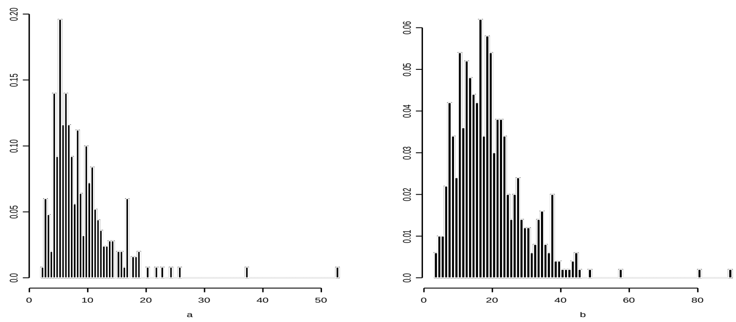

Figure 6: Distribution of 1000 maxima of the process $(I-0.90B+B^2)^{0.4}X_t=\eta_t$, with $(\eta_t)_t$ driven by an ARCH(1) noise (23) with $a_0=1$ and $a_1=0.8$ (a), and with $(\eta_t)_t$ driven by an GARCH(1,1) noise (24) with $a_0=1$ and $a_1=0.2$ and $b_1=0.7$ (b).

In figure (6) we present the distribution of the maxima of 1000 points generated with the recursive scheme (2), respectively with r = 1, s= 0 and r = 1, s = 1. It appears that the distributions of these maxima belong to domain of Extreme value distributions, using QQ-PLOT method. Precisely, we get a Fréchet distribution for the maxima of 1000 points generated with the recursive scheme (23), with an ARCH(1). The estimation that we got for the parameter ξ is, ξ = 0.2234, (0.0385). The estimated values of the location parameter is μ = 6.3257, (0.0385) and of the scale parameter σ

is, σ = 2.9021 (0.1197). We get a Gumbel distribution for the maxima of 1000 points generated with the recursive scheme (24), with an GARCH(1,1). Here the parameter ξ = 0.0651, (0.0237) is not significant. The values of the location parameter is μ = 14.4786, (0.2499) and the scale parameter σ = 6.9994 (0.1864). A complete study on the distributions of the maxima for more general classes of heteroscedatic models can be found in Avouyi - Dovi and Guégan (2000).

III - The non stationary Gegenbauer Processes

Very often, when we investigate real data and when we study the long memory behavior of these data, the long memory parameter d is greater than ½ . This means that we are in presence of a non stationary long memory process. This class of models has not yet been really studied, nevertheless the importance of this kind of behavior requires that we look at it in more details. A natural approach when d > ½ is to make a transformation on the real data set to make them stationary. This approach has been a lot studied using unit roots tests. A competitive way consists to keep this "non stationarity " and to use it to adjust a model on the data including this "kind of non stationarity". This means that we need to develop new classes of non stationary processes. Here we are going to investigate, with an empirical approach, the second order properties of the non stationary long memory processes with independent noises and dependent noises. We are going to consider two cases, ½ < d < 1 and d > 1.

To make our study complete, we begin to present some simulations concerning the stationary FARMA processes (which are particular cases of non stationary Gegenbauer processes) defined by

$$(I - B)^d X_t = \eta_t \qquad (25)$$

where $(\eta_t)_t$ can be either a sequence of independant identically random variables or an heteroscedastic noise driven by (23) or by (24). Secondly we present some non stationary FARMA and Gegenbauer processes defined by (2) with the same kind of noises used before.

- 1. **Stationary FARMA processes** defined by (25).

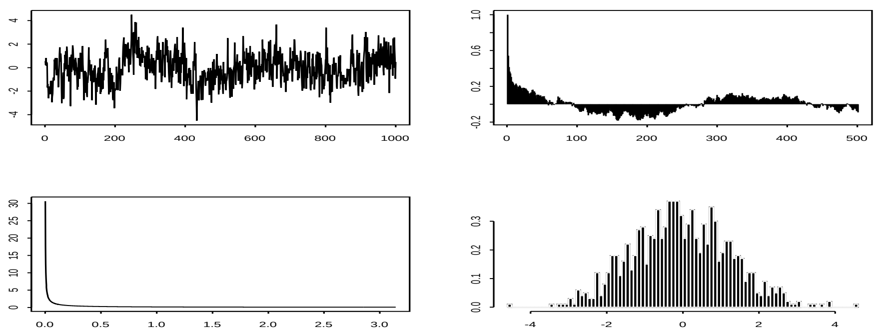

Figure 7 : Realization, Autocorrelation function, spectral density, histogram of $(I-B)^{0.4}X_t=\eta_t$, with $(\eta_t)_t$ a Gaussian noise.

On figures (7), (8), (9), respectively we present a Gaussian FARMA(0, 0.4, 0) process, a Weibull FARMA(0, 0.4, 0) and a FARMA(0, 0.4, 0) process with an ARCH noise.

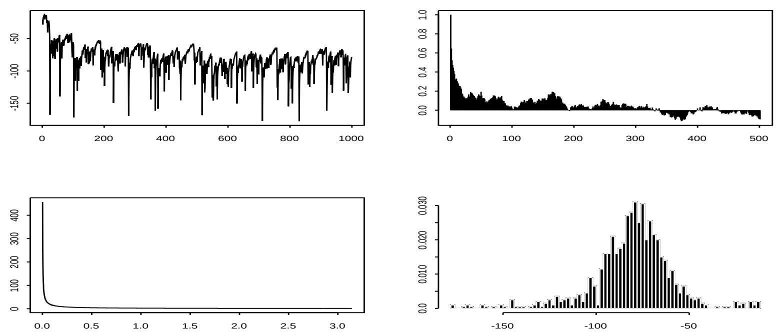

Figure 8 : Realization, Autocorrelation function, spectral density, histogram of $(I-B)^{0.4}X_t=\eta_t$, with $(\eta_t)_t$ a Weibull noise.

The trajectories are very different from each others. These processes are by definition stationary and simulated with specific noises.

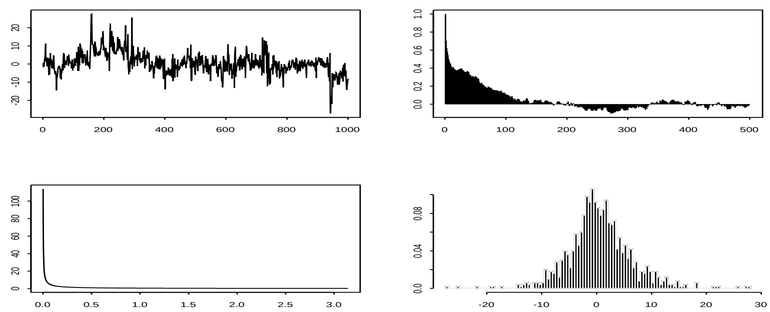

Figure 9 : Realization, Autocorrelation function, spectral density, histogram of $(I-B)^{0.4}X_t=\eta_t$, with $(\eta_t)_t$ an ARCH(1) noise with $a_1=0.8$.

The autocorrelation function decreases slowly and the spectral density explodes in zero in all cases. We do not observe a great difference in the behavior of the empirical distribution in figures (7) and (9). It is not so clear that we are close to the Normal distribution in figure (8).

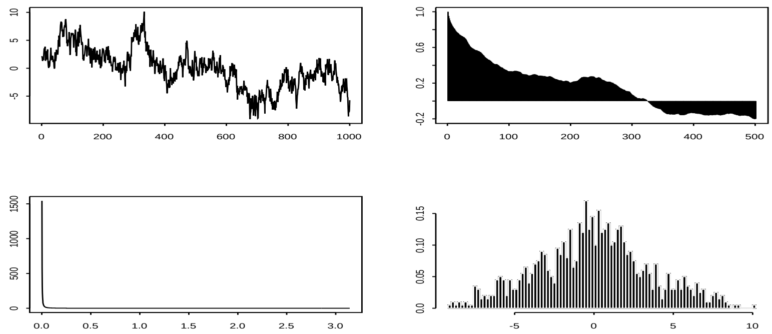

Figure 10 : Realization, Autocorrelation function, spectral density, histogram of $(I-B)^{0.7}X_t=\eta_t$, with $(\eta_t)_t$ a Gaussian noise.

- 2. Non stationary FARMA processes: ½ < d < 1. We use d = 0.7 in all simulations.

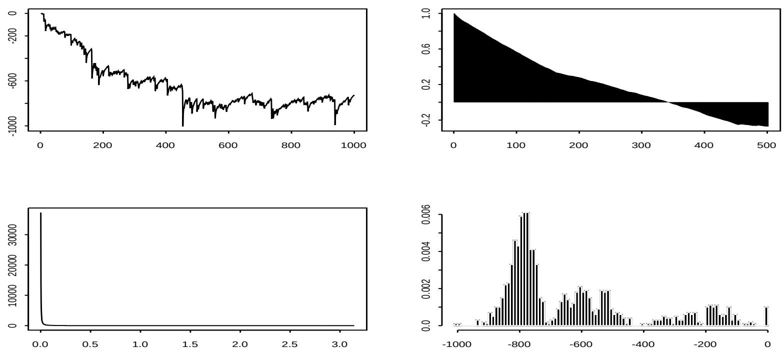

Figure 11: Realization, Autocorrelation function, spectral density, histogram of $(I-B)^{0.7}X_t=\eta_t$, with $(\eta_t)_t$ a Weibull noise.

In figures (10), (11) and (12), the slow decreasing rate of the autocorrelation function is amplified when we compare these figures with the three previous ones.

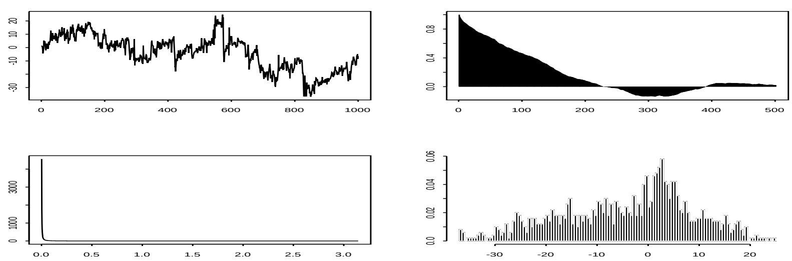

Figure 12 : : Realization, Autocorrelation function, spectral density, histogram of $(I-B)^{0.7}X_t=\eta_t$, with $(\eta_t)_t$ an ARCH(1) noise with $a_1=0.8$**.**

In figure (11), the empirical distribution is far from the Gaussian one.

- 3. Non stationary Gegenbauer processes: ½ < d < 1. We use d= 0.7 and u = 0.45 in all simulations. First of all, we use independent noises. We get the figures (13), (Gaussian noise) and the figure (14), (Weibull noise).

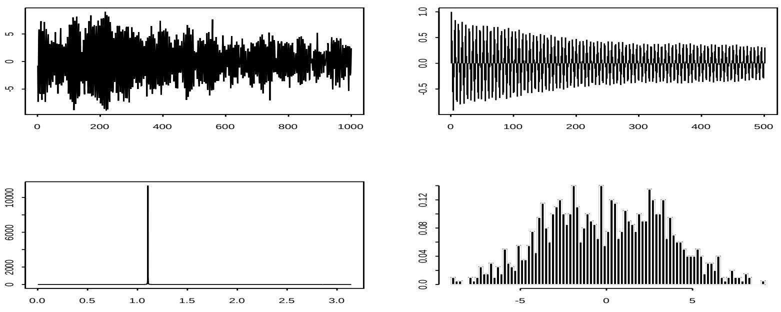

Figure 13 : : Realization, Autocorrelation function, spectral density, histogram of $(I-0.90B+B^2)^{0.7}X_t=\eta_t$, with $(\eta_t)_t$ a Gaussian noise.

The trajectories appear very noisy and present a great instability. The autocorrelation functions do not decrease towards zero and show sin waves.

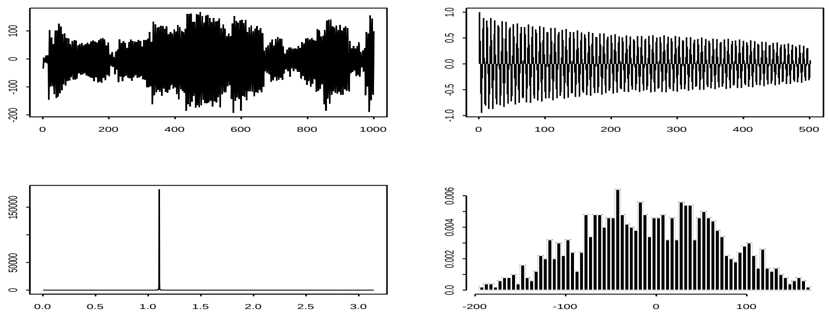

Figure 14 : : Realization, Autocorrelation function, spectral density, histogram of $(I-0.90B+B^2)^{0.7}X_t=\eta_t$ with $(\eta_t)_t$ a Weibull noise.

In figure (15), we use a conditionally Gaussian ARCH noise and in figure (16), a conditionally Gaussian GARCH noise.

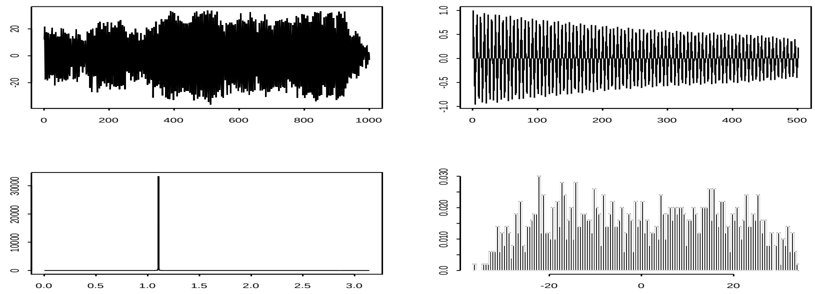

Figure 15: : Realization, Autocorrelation function, spectral density, histogram of $(I - 0.90B + B^2)^{0.7} X_t = \eta_t$ with $(\eta_t)_t$ an ARCH(1) noise with $a_1 = 0.8$.

The variability of the trajectories seem amplified. We have the same behavior as before for the autocorrelation function.

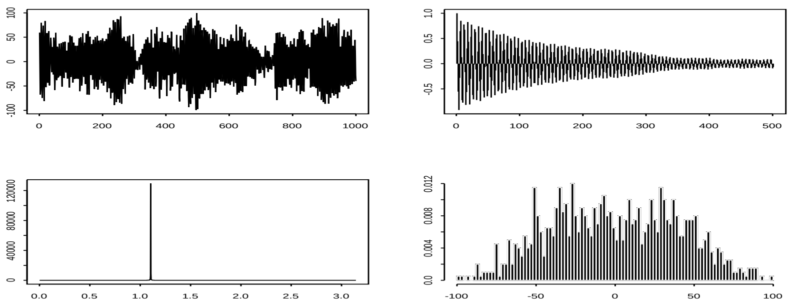

Figure 16: : Realization, Autocorrelation function, spectral density, histogram of $(I - 0.90B + B^2)^{0.7} X_t = \eta_t$ with $(\eta_t)_t$ an GARCH(1,1) noise with $a_1 = 0.2$, $b_1 = 0.7$.

For all the graphs the spectral density explodes in a frequency far from zero and the histogram is generally difficult to interpret.

- 4. Non stationary FARMA and Gegenbauer processes: d > 1. We use d = 1.2 in all simulations.

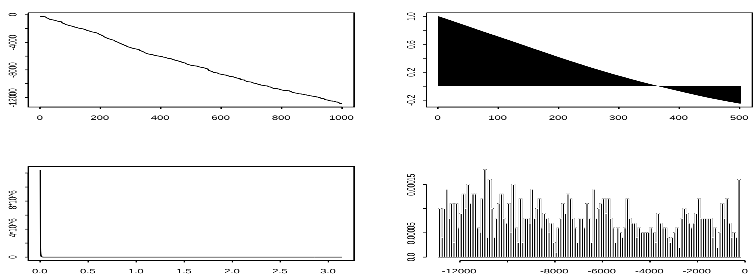

Figure 17: Realization, Autocorrelation function, spectral density, histogram of $(I - B)^{1.2} X_t = \eta_t$, with $(\eta_t)_t$ a Weibull noise.

In figures (17) and (18) we present non stationary FARMA processes with different noises and in figures (19) and (20) non stationary Gegenbauer processes (u = 0.45) with the same noises as in the previous items. (We do not present here all the graphs, because they are very close to each other and not very informative).

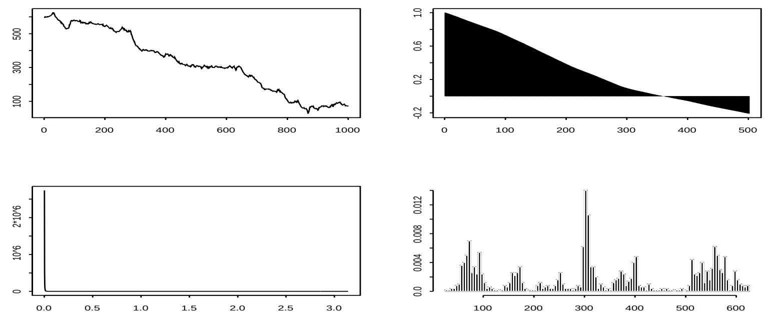

Figure 18 : Realization, Autocorrelation function, spectral density, histogram of $(I-B)^{1.2} X_t = \eta_t$, with $(\eta_t)_t$ an ARCH(1) noise with $a_1 = 0.8$.

If in the case of the non stationary FARMA processes, it seems possible to identify directly a model on the series, this appears impossible with the non stationary Gegenbauer processes. The series are too noisy.

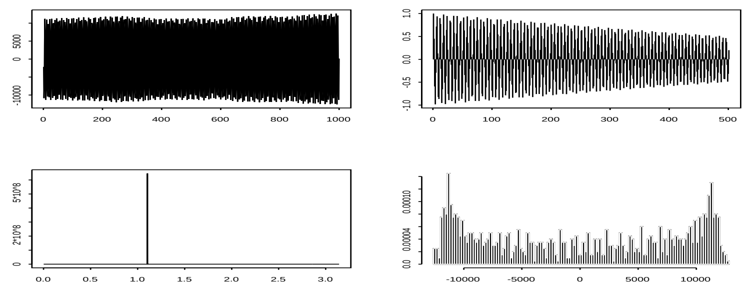

Figure 19 : Realization, Autocorrelation function, spectral density, histogram of $(I-0.90B+B^2)^{1.2} X_t = \eta_t$, with $(\eta_t)_t$ a Weibull noise (1, 0.5).

In that latter case, a preliminary transformation on the series will be necessary before investigate them.

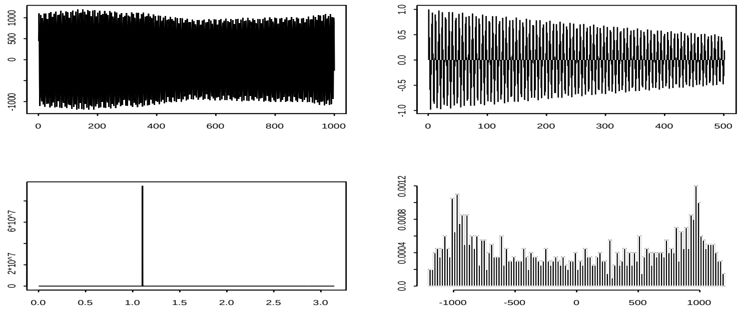

Figure 20 : Realization, Autocorrelation function, spectral density, histogram of $(I-0.90B+B^2)^{1.2}X_t=\eta_t$, with $(\eta_t)_t$ an ARCH(1) noise with $a_1=0.8$.

These different graphs show that, when the long memory parameter is greater than 1, it will be necessary to transform the series first, before considering any identification procedure. It could be interesting, in the other cases, ½ < d < 1, to detect directly the values of the parameter d without transformation on the data. In that last case, even if the series are non stationary, keeping the real data set can be strongly informative in order to rebuild the series or to make predictions. Thus, a theory in that context seems really important. Let us discuss now this point.

While there are many methods in the time series literature that deal with certain type of trends and seasonnalities, the large variety of possible models is often confusing to the applied data analysis. Finding an appropriate model, and in particular, defining realistic forecasts, is therefore a challenging task in practice. A possibility to solve this problem is to set up a unified framework in which flexible modelling of deterministic and stochastic components is possible, and objective data driven inference can be made to decide which of the components (deterministic trends or seasonnalities, stochastic trends or seasonnalities, spurious effects, stationary and nonstationary components) may be present, see Guégan (2001, 2003). The model described here combine parametric modelling of stochastic components using stationary and non stationary fractional Gegenbauer factors with in addition non stationary conditional variance. Thus the nonstationary GIGARCH models can be a challenge for the problem discussed previously. The estimation theory needs to be done and it is in preparation.

Nevertheless because many time series appear to be non stationary in different degrees in the reality, the interest to extend the concept of long memory to non stationary processes has been already addressed in the literature from the Gaussian FARMA process (25). Most of the works concern results on the functional convergence of the partial sums of such model, see Robinson (1995), Marinucci and Robinson (1998), Lu (1998) and Velasco (1999). These results are very important in estimation theory and for aplications, see for instance Ling and li (1997), Beran and Ocker (1999) and Beran and Ghosh (2000).

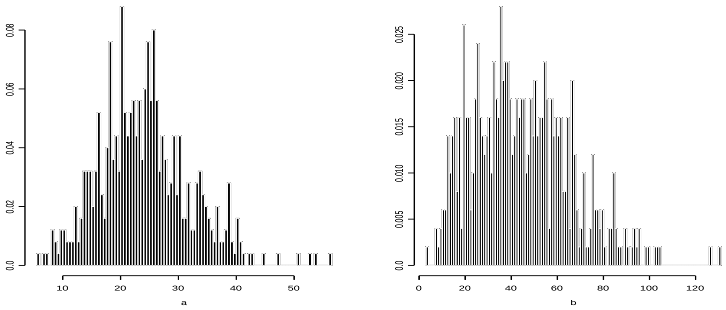

Figure 21 : Distribution of 1000 maxima of the process $(I-0.90B+B^2)^{0.7}X_t=\eta_t$ with $(\eta_t)_t$ driven by an ARCH(1) noise (23) with $a_0=1$ and $a_1=0.8$ (a), and with $(\eta_t)_t$ driven by an GARCH(1,1) noise (24) with $a_0=1$ and $a_1=0.2$ and $b_1=0.7$ (b).

In figure (21) we give the distribution of the 1000 maxima obtained from a non stationary Gegenbauer process with d = 0.7 and u = 0.45 and driven respectively with an ARCH(1) process and a GARCH(1,1) process. In these two cases it has not been possible to find precisely the distribution function of the maxima. The non stationarity of the process can affect the possibility of estimation of the limiting distribution of the maxima.

IV - An illustrative example: are inflation rates non stationary and long memory?

In this section we are interested by the evolution of the Consumer price index (CPI) and inflation rate. Let denote the CPI by $(V_t)_t$ and the inflation rate by $(W_t)_t$, with $W_t=\frac{V_t-V_{t-1}}{V_{t-1}}$. For all the models that we propose in the following, we first compute the empirical mean and we work with the centered process.

Thus, we consider the series of CPI monthly data in France during the period 01/01/70 to 01/12/98. This represents 348 points. The mean of the series in m = 72.544, the variance σ^2 = 1092.793, the skewness κ_3 = 0.225 and the kurtosis κ_4 = 1.505. We present in figure (22) the trajectory, the autocorrelation function, the spectral density and the histogram of $(V_t)_t$.

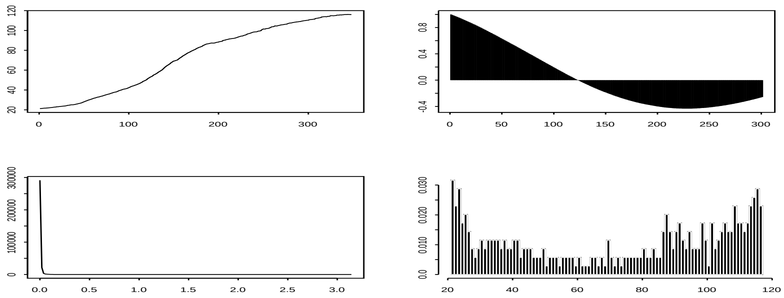

Figure 22: : Realization, Autocorrelation function, spectral density, histogram of the CPI $(V_t)_t$ from 01/01/70 to 01/12/98.

We observe that the series seem not stationary. Now if we compare the figure (22) with some previous figures, we can try first to adjust a FARMA model (probably with d > 1, when we observe the autocorrelation function), on $(V_t)_t$ before any transformation. We get different models:

- a. First we adjust a **FARMA process:**

$$(I-B)^{1.002}V_t = \varepsilon_t \qquad (26)$$

with $\hat{\sigma}_d = 0.024$ and $\hat{\sigma}_\varepsilon = 0.197$. The value obtained for $\hat{d}$ confirms that the CPI is non stationary. Thus, we do not go on with this approach.

- b. We decide to adjust a **Gegenbauer process** on the centered process $(V_t)_t$ using the frequency λ = 0.05 (we choose a frequency which is close to zero). We get the following model:

$$(I-2\cos(0.05)B+B^2)^{0.531}V_t = \varepsilon_t \qquad (27)$$

with $\hat{\sigma}_d = 0.022$ and $\hat{\sigma}_\varepsilon = 2.237$. The value of the estimate $\hat{d}$ is significantly different of 0.5, thus this adjustment seems interesting. It confirms that if we do not consider that the explosion of the spectral density is not exactly in zero, then the Gegenbauer estimation can be considered. Here we get a non stationary Gegenbauer process. The variance of the residual seems important. Thus, we estimate the series of the residual using the recursive scheme (27) and we adjust on these residuals $(\varepsilon_t)_t$, an ARCH(1) process (23). The empirical mean is $\hat{m}_\varepsilon = 0.144$, and the estimated parameters of the ARCH(1) model are $\hat{a}_0 = 0.014$ and $\hat{a}_1 = 1.055$ with $\hat{\sigma}_{a_0} = 0.005$ and $\hat{\sigma}_{a_1} = 0.221$. To get these values we work forcing the variance of the process $(\eta_t)_t$, which appears in the expression of the ARCH(1) model (23) to be equal to one (this corresponds to the construction of the ARCH(1) process). Now in order to justify this adjustment, we need to study the distribution function of the residuals (by construction they have to be Gaussian). We give it in figure (23) using a Q-Qplot approach.

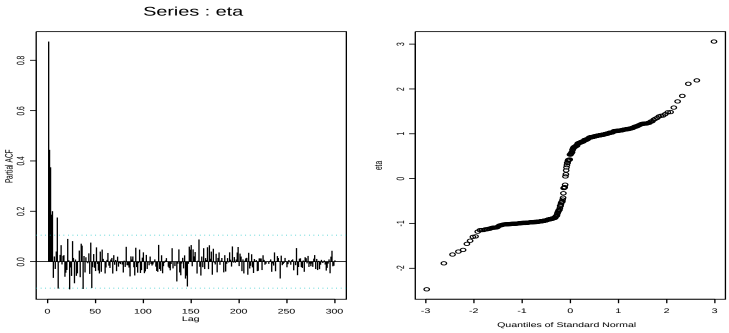

Figure 23: Behavior of the ACF and of the QQ-plot of the distribution function of the residuals against Gaussian law, when we use a Gegenbauer (0.45; 1.02) with an ARCH(1) noise for the CPI $(V_t)_t$ **.**

Clearly the distribution of the residual is far from the Gaussian distribution. Moreover, we observe that the estimation of the parameter a_1 is greater than 1. This value is not possible when we use ARCH(1) model (in that case the process in non mixing for instance, see Guégan and Diebolt, 1994), thus we decide to differentiate the series $(V_t)_t$.

- c. Now we consider the **differentiated process** $((I-B)V_t)_t$, its mean is m = 0.273. The representation of this process is given in figure (24). The trajectory does not present any feature of non stationarity. Nevertheless the autocorrelation function does not decrease quickly towards zero and the spectral density does not appear to be bounded.

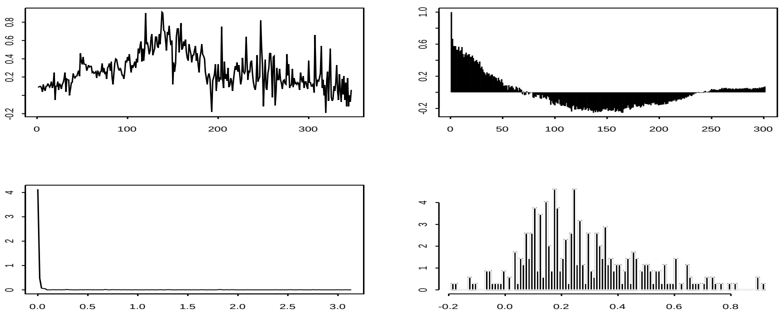

Figure 24: Realization, Autocorrelation function, spectral density, histogram of the differentiated CPI $((I-B)V_t)_t$ from 01/01/70 to 30/11/98.

We first adjust a **FARMA process**. We get:

$$(I-B)^{0.415}(I-B)V_t = \varepsilon_t \quad (28)$$

with $\hat{\sigma}_d = 0.047$ and $\hat{\sigma}_\varepsilon = 0.136$. The estimation for d is significant. We observe that the variance of the residuals is smaller than the one obtained with the previous approach. We adjust on the residual $(\varepsilon_t)_t$, using the same way as previously, an

ARCH(1) process. We get $\hat{m}_\varepsilon = -0.004$ and $\hat{a}_0 = 0.016$ and $\hat{a}_1 = 0.118$ with $\hat{\sigma}_{a_0} = 0.00009$ and $\hat{\sigma}_{a_1} = 0.071$. We give in figure (25) the ACF of the residuals and the QQ-PLOT of the distribution of the residuals with respect of a Gaussian law. This last representation is completely different on the representation observed in figure (23).

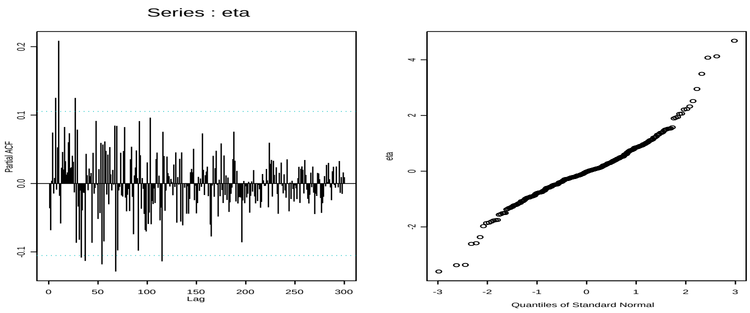

Figure 25: Behavior of the ACF and of the QQ-plot of the distribution function of the residuals against Gaussian law, when we use a FARMA(0,0.415,0) with an ARCH(1) noise for the differentiated CPI $((I-B)V_t)_t$ **.**

.

Here, the adjustment with a Gaussian law can be accepted.

Now, to try to take into account some "cycle" observed on the autocorrelation function represented in figure (24), we adjust **a Gegenbauer process** in the frequency λ = 0.05 (close to zero). We get the following model:

$$(I - 2\cos(0.05)B + B^2)^{0.224}(I-B)V_t = \varepsilon_t \quad (29)$$

with $\hat{\sigma}_d = 0.025$ and $\hat{\sigma}_\varepsilon = 0.138$. The estimate of $\hat{d}$ is significant. We can keep it. The variance of the residual is greater than the one obtained for the previous model. Thus, we adjust on the residual $(\varepsilon_t)_t$, an ARCH(1) process. We get $\hat{m}_\varepsilon = 3.22 \times 10^{-4}$, $\hat{a}_0 = 0.017$ and $\hat{a}_1 = 0.118$, with $\hat{\sigma}_{a_0} = 0.0009$ and $\hat{\sigma}_{a_1} = 0.070$. The QQ-PLOT of the distribution function of the residuals with respect to a Gaussian law is given in figure (26).

The values obtained for the different parameters are very close to those obtained with the model (26). Nevertheless comparing the figures (25) and (26), whether we have to decide between the models (28) and (29), we prefer the adjustment obtained for the model (29).

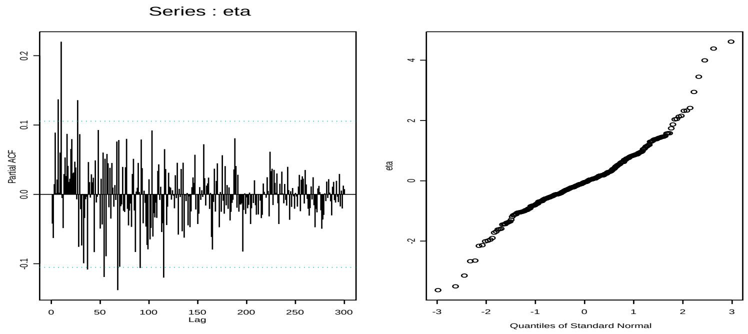

Figure 26 : Behavior of the ACF and of the QQ-plot of the distribution function of the residuals against Gaussian law, when we use a GARMA(0.05,0.415) with an ARCH(1) noise for the differentiated CPI $((I-B)V_t)_t$ **.**

Now we start the investigation for the **inflation rates** $(W_t)_t$. The mean of this process is $m = 4.9\times10^{-3}$. Looking at the figure (27), we observe that the trajectory does not present any trend, but the autocorrelation function does not decrease and the spectral density is not bounded into a frequency close to zero. We are going to try the same kinds of models as before to take this kind of non stationarity into account.

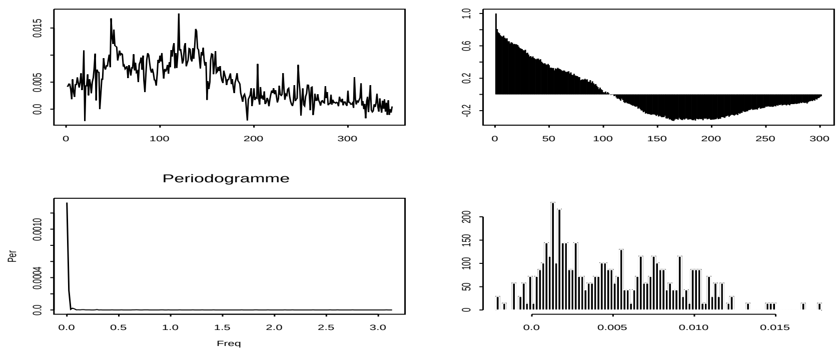

Figure 27 : Realization, Autocorrelation function, spectral density, histogram of the inflation rates $(W_t)_t$ from 01/01/70 to 30/11/98.

- d. We first adjust **a FARMA process**. We get:

$$(I-B)^{0.499}W_t = \varepsilon_t \quad (30)$$

with $\hat{\sigma}_d = 0.052$ and $\hat{\sigma}_\varepsilon = 1.99\times10^{-3}$. The estimation obtained for the long memory parameter seems close to 0.5. Thus we prefer to stop the investigation of these data by this way and to try a Gegenbauer process.

- e. With the **Gegenbauer approach**, we get the following model, using the frequency λ = 0.05. :

$$(I - 2\cos(0.05)B + B^2)^{0.275} W_t = \varepsilon_t \quad (31)$$

with $\hat{\sigma}_d = 0.027$ and $\hat{\sigma}_\varepsilon = 2.057 \times 10^{-3}$. The value obtained for the long memory parameter d is significant. We decide to try to adjust on the residuals $(\varepsilon_t)_t$ an ARCH(1) process although the variance appears small . We get $\hat{m}_\varepsilon = 1.368 \times 10^{-6}$, then on the centered process, we get: $\hat{a}_0 = 3.45 \times 10^{-6}$ and $\hat{a}_1 = 0.188$ with $\hat{\sigma}_{a_0} = 10^{-7}$ and $\hat{\sigma}_{a_1} = 0.049$. The value of $\hat{a}_1$ gives interesting information on the volatility of the residuals. We compare the distribution of the residuals with the one of a Gaussian law using the QQ-plot method. This is given in figure (28).

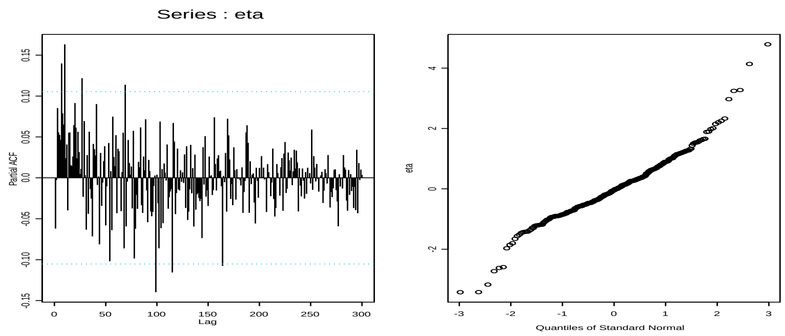

Figure 28 : : Behavior of the ACF and of the QQ-plot of the distribution function of the residuals against Gaussian law, when we use a GARMA(0.05,0.275) with an ARCH(1) noise for the inflation rates $(W_t)_t$ **.**

It seems that this modelization can be useful to make predictions for instance. These problems will be investigated in another paper.

In the introduction, we have recalled different ways to investigate some kinds of non stationarity observed on the inflation rates. Here we apply Gegenbauer models to describe the fluctuations of the inflation rates. In this sense our approach is quite different of Hassler and Wolters (1995) for instance. Indeed, we do not need to make any transformation on the process of inflation rates $(W_t)_t$, and the model (31) permits to take into account both the long memory behavior (often assimilated to non stationarity) and the existence of cycle (also often assimilated to non stationarity). We think that this approach can be promising and can permit to propose new set of predictions taking into account the evolution of the different cycles displayed by the model (31).

V - Summary and Conclusion

In this paper, we have investigated some probabilistic and statistical properties of a k-factor Gegenbauer process driven with dependent noises. We have focused on an empirical investigation of these different properties in order to detect specific features which can permit to build an identification theory for these processes in the non stationary context. We have tried to adjust some of these models on inflation rates and our approach here remains prospective.

In this paper we have not studied the properties of the empirical moments of the process $(X_t)_t$ and their asymptotic behavior, nor the estimation of the parameters. Some interesting problems concern the estimation of the parameters of the process (16) because to get nice properties of the estimates, we need to prove that the estimates obtained from the ARMA part on one side, from the long memory part on the other side and finally from the heteroscedastic part, are robust, altough the models are driven by dependent noises. This seems not do be done actually.

On the other side, we have not investigated the effect of aggregated time in the data, nor the aggregation with macro economic data, which is also an important point of view, to interpret any aggregated price levels. One important issue concerns how to aggregate prices when they respond to shocks. We know that in case of heteroscedastic models, this has been already studied, see for instance Drost and Nijman (1993), Guégan (1996) and Duan (1997). Recently this problem has been studied for high frequency data sets which present long memory behaviour, see for instance Willinger, Taqqu and Teverovsky (1999) and Bollerslev and Wright (2000) and for financial prices Chambers (1998) and Avouyi-Dovi, Guégan and Ladoucette (2002 a). In the context of the k-factor GIGARCH model it is an open way of research.

All the properties that we have shown here for the k-factor GIGARCH process defined in (2) can be extended for more general expression for the volatility process σ_t: for instance the EGARCH model, see Nelson (1990), the TARCH model, see Zakoian (1994) or the Asymmetric power ARCH model, see Ding, Granger and Engle(1993).

VI - Appendix

A - Sketch of the proof of Proposition 1: For k = 1 in (2), the existence of the causal solution (10) is obtained as soon as

$$X_{n,t} = \sum_{k=0}^{n} \psi_k \eta_{t-k}$$

where the coefficients ψ_j given by (11) satisfy $\sum \psi_j^2 \prec \infty$. In the following the notation ψ_i is used in place of $\psi_j^{(d,u)}$ for simplicity. The uniqueness of the solution follows from the Theorem 12.4.1, in Brockwell and Davis (1987). The stationarity of the solution is verified thanks to the condition imposed on the parameters, see Bollerslev (1986) and Wayne, Woodward, Cheng and Gray (1998). Moreover, as soon as there exist finite number of zeroes or peaks in the spectrum (k > 1) the proof requires a different approach in particular to get the asymptotics for (10) (we do not give them here), but this can be deduced directly from theorem 1 in Giraitis and Leipus (1995) using Darboux method, see Szegö (1959).

B. Sketch of the proof of Proposition 2: As soon as the spectral density of the process $(\eta_t)_t$ has bounded derivative and is slowly varying in any point $0 \le \lambda \le \pi$ such that $f_\eta(\lambda) \neq 0$, then the proof is similar to the lemma 3 in Giraitis and Leipus

(1995). The behavior of the autocovariance function follows from the Corollary 1 in the same article.

C. Sketch of the proof of Proposition 3: For simplicity, we assume that k = 1 in (2). It is known that the 2mth-order moments of the process (8) are finite, as soon as $\rho(E(A_t^{\otimes m})) \prec 1$, see for instance Bougerol and Picard (1992). Under the assumptions of the proposition 2, we know that $E(\eta_t^4) \leq \infty$, therefore,

$$EX_t^4 = \sum_{k_1,k_2}^{\infty} \psi_{k_1}^2 \psi_{k_2}^2 E(\eta_{t-k_1}^2 \eta_{t-k_2}^2) \quad (33)$$

$$\leq \sum_{k_1,k_2}^{\infty} \psi_{k_1}^2 \psi_{k_2}^2 \sqrt{E(\eta_{t-k_1}^4 \eta_{t-k_2}^4)}$$

$$= (\sum_{k}^{\infty} \psi_k^2)^2 E(\eta_t^4),$$

where ψ_k is given by (6). The proof for the eighth-order moment is the same as soon as $E(\eta_t^4) \leq \infty$.

D. Sketch of the proof of Proposition 4: It has been established that if a stationary process (with a spectral density f) is completely linearly regular, then the function f^r is summable for every positive constant r, see for instance Ibrajimov and Rozanov (1974). Thus a stationary process, with spectral density f, is not completely linearly regular if there exists a positive constant p such that the function f^p is not summable. Now, from proposition 3, we know the asymptotic behavior of the spectral density of the process $(X_t)_t$ defined by (2): it is given by (15). Then,

$$\int_{-\pi}^{\pi} f_X^p(\lambda) d\lambda = \infty \Leftrightarrow \exists i \in \{1,\dots k\} : \int_{\lambda_i}^{\pi} f_X^p(\lambda) d\lambda = \infty$$

$$\Leftrightarrow \exists i \in \{1,\dots k\} : \int_{0}^{\pi-\lambda_i} \lambda^{2d_i^{'} p} d\lambda = \infty$$

$$\Leftrightarrow \exists i \in \{1,\dots k\} : 2d_i^{'} p \geq 1 .$$

For the process (2), the vector $d = (d_1, \cdots, d_k)$, and thus the vector $d = (d_1^{'}, \cdots, d_k^{'})$ with 0 < $d_i^{'}$ < 1/2, i = 1, ..., k is fixed. Hence, there exists an i for which we can find a positive constant p such that $2d_i^{'} p \geq 1$, so that $|\lambda - \lambda_i|^{-2d_i^{'} p}$, and thus $f_X{}^p$ is not summable.

References

Avouyi-Dovi S., Guégan D. (2000) " Valeurs extrêmes et séries temporelles: Applications à la finance", *Preprint University of Evry*, France.

Avouyi-Dovi S., Guégan D., Ladoucette S. (2002), "Indices boursiers et longue mémoire", *Note de Recherche Banque de France, NR*- **95**, Paris, France.

Avouyi-Dovi S., Guégan D., Ladoucette S. (2002 a), « What is the best approach to measure the interdependence between different markets », *Note de Recherche Banque de France, NR*- **94**, Paris, France.

Baillie R.T (1996) "Long memory processes and fractional integration in econometrics", *Journal of econometrics*, , 5 - 59.

Baillie R.T, Chung C.F, Tieslau M.A. (1996a) "Analysis inflation by the fractionally integrated ARFIMA-GARCH model", *Journal of Applied Econometrics*, , 23 - 40.

Baillie R.T., Bollerslev T., Mikkelsen H.O. (1996b), "Fractionally integrated generalized autoregressive conditional heteroscedasticity", *Journal of Econometrics*, , 3-30.

Barsky R.B. (1987) "The Fisher hypothesis and the forecastability and persistence of inflation", *Journal of Monetary Economics*, , 3-24.

Beran J., Ockers D. (1999) "SEMIFAR forecasts with application to foreign exchange rates", *Journal of Statistical Planning and Inference*, , 137 - 153.

Beran J., Ghosh S. (2000) Estimation of the dominating frequency for stationary and non stationary fractional autoregressive models, *Journal of Time series analysis*, , 513 - 533.

Berman S.M. (1964), "Limit theorems for the maximum term in stationary sequences", *Ann. Math. Statist.*, , 502–516.

Bollerslev T. (1986), "Generalized autoregressive conditional heteroscedasticity", *Journal of Econometrics*, , 307-327.

Bollerslev T., Cai J., Song F.M. (2000), "Intraday periodicity, long memory volatility and macroeconomic announcement effect in the U.S. Treasury bond market", *Journal of Empirical Finance*, , 37-55.

Bollerslev T., Wright J.H. (2000), "Semiparametric estimation of long memory dependencies : the role of high-frequency data", *Journal of Econometrics*, , 81-106.

Bollerslev T., Jubinski D. (1999) Equity trading volumeand volatlity: latent information arrival and common long run dependencies, *Journal of Business and economic statistics*, , 9 - 21.

Bollerslev T., Mikkelsen H.O. (1996), "Modeling and pricing long memory in stock market volatility", *Journal of Econometrics*, , 1-29.

Bougerol P., Picard N. (1992) "Stationarity of GARCH processes and of some non negative time series", *Journal of Econometrics*, , 115 - 127.

Breidt F.J., Davis R.A. (1998) "Extremes of stochastic volatility models", *The annals of Applied probability*, , 664 - 675.

Brockwell P.J., Davis R.A. (1987), Time series: Theory and Methods, Wiley, New York.

Chambers M. (1998), "Long memory and aggregation in macroeconomic time series", *International Economic Review*, , 1053-1072.

Campbell J., Shiller B. (2001), "Valuation ratios and the long run Stock market outlook: an update", NBER Working Paper, .

Carrasco M., Chen X. (1999) "[Trial mode]-mixing and moment properties of various GARCH, Stochastic volatility and ACD models", *Preprint INSEE*, , Paris.

Collet J., Guégan D. (2001) "Forecasting with non Gaussian long memory processes", NR02-02, GRID, Preprint ENS-Cachan, France.

Collet J., Guégan D., Valdès P. (2002) "How shall we determine the number and the location of the Gegenbauer frequencies, Preprint ENS-Cachan, France.

Chung C.F., Baillie R.T. (1993) "Small sample biais in conditional sum of squares estimators of fractionally integrated ARMA models", *Empirical Economics*, 791 - 806.

Dickey D.A., Fuller W.A. (1981) "Likelihood ratio statistics for autoregressive time series with a unit root", *Econometrica*, , 1057 - 1072.

Ding Z., Granger C.W.J., Engle R.F. (1993), "A long memory property of stock markets return and a new model", *Journal of Empirical Finance*, , 83-106.

Diop A., Guégan D. (2001) "Extreme distribution for generalzed stocastic volatility model", *Preprint GRID-ENS Cachan*, **- 11**, France.

Drost F.C., Nijman T.E. (1993), "Temporal agregation of GARCH processes", *Econometrica*, , 909-927.

Drost F.C., Werker B.J.M. (1996), "Closing the GARCH gap: continuous time GARCH modeling", *Journal of Econometrics*, , 31-57.

Duan J.C. (1997), "Augmented GARCH([Trial mode]) process and its diffusion limit", *Journal of Econometrics*, , 97-127.

Ferrara L. (2000) "Processus longue mémoire généralisé: estimation, prévision et applications", PhD Thesis, Paris XIII, France.

Ferrara L., Guégan D. (2000) "Forecasting financial time series with generalized long memory processes", in "Advanced in Quantitative Asset Managment", C. Dunis ed., Bluver Academic Press, Chapter 14, 319 - 342.

Ferrara L., Guégan D. (2001a), "Comparison of parameter estimation methods in cyclical long memory time series", in *Developments in Forecast Combination and Portfolio Choice*, Dunis C., Timmermann A. and Moody J. eds., J. Wiley, New York, chapter 8, 179 - 195.

Ferrara L., Guégan D. (2001b) "Forecasting with k-factor Gegenbauer processes: theory and applications, *Journal of forecasting*, 20, 1 - 21.

Ferrara L., Guégan D. (2002) "Analyser les séries chronologiques avec S-Plus : une approche paramétrique », *Presses Universitaires de Rennes*, France.

Franses P.H., Ooms M. (1997) "A periodic long memory model for quartely UK inflation", *International Journal of forecasting*, , 117 - 126.

Giraitis L., Leipus R. (1995) "A generalized fractioanlly differencing approach in long memory modelling", *Lithuanian Mathematical Journal*, , 65 - 81.

Granger C.W.J. (1966) "The typical spectral shape of an economic variable", *Econometrica*, , 150 - 161.

Granger C.W.J. (1980) "Long memory relationship and the aggregation of dynamic models", *Journal of Econometrics*, , 227 - 338.

Granger C.W.J., Hatanaka M. (1964) Spectral analysis of times series, Princeton.

Granger C.W.J., Joyeux R. (1980) "An introduction to long memory time series models and fractional differencing", *Journal of Time Series Analysis*, , 15-29.

Guégan D. (1994) “Séries chronologiques non linéaires à temps discret », Economica, Paris.

Guégan D. (1996) "Some recent developments in nonlinear time series", del convegno in onore di Oliviero Lessi, Universita degli studi di Padova, Italy, 16-38.

Guégan D. (2000) "A new model : the k-factor GIGARCH process", *Journal of Signal Processing*, , 265-271.

Guégan D. (2001) “Long memory behavior for simulated chaotic time series, IEICE Trans Fundamentals, E84-A, 2145 – 2154.

Guégan D. (2003) “Le chaos en finance: Approche statistique », Economica, Paris.

Guégan D. and Diebolt J. (1994) "Probabilistic properties of the β-ARCH model, *Statistica Sinica*, , 71 -87.

Guégan D., Ladoucette S. (2001) "Non mixing properties of long memory processes", *C.R.A.S.*, **t. 333**, Série I, 373 - 376, Paris.

Hassler V., Wolters J. (1995), "Long memory in inflation rates : international evidence", *Journal of Business and Economic Statistics*, , 37-46.

Hosking J.R.M. (1981) "Fractional Differencing", *Biometrika*, , 165 - 176.

Ibragimov I.A., Rozanov Y.A. (1974) Processus aléatoires Gaussiens, Mir Ed.

Kesten H. (1973) "Random Difference Equations and Renewal Theory for Products of Random Matrices", *Acta Math.*, , 207-248.

Kirchgasser G., Wolters (1993) "Are real interest rates stables? An international comparison", in Studies in applied econometrics, eds. Schneeweiss H. and Zimmermann K.F., Heidelberg, Physica-Verlag, 214 -238.

Kwiatowski D., Phillips P.C.B., Schmidt P., Shin Y. (1992) "Testing the null hypothesis stationarity against the alternative of a unit root", *Journal of Econometrics*, , 159 - 178.

Leadbetter M.R., Lindgren G., Rootzen H. (1983) Extremes and Related Properties of Random Sequences and Processes, Springer-Verlag, Berlin.

Ling S., li W.K. (1997), "On fractionally integrated autoregressive moving average time series models with conditional heteroscedasticity", *Journal of American Statistical Association*, , 1184 - 1187.

Lobato I., Velasco C. (2000), "Long memory in stock-market trading volume", *Journal of American Statistical Association*, , 410-427.

Lu Z. (1996) "A note on geometric ergodicity of autoregressive conditional heteroscedasticity model", *Statistic and Probability Letters*, , 305 - 311.

Lu M., (1998) Asymptotics of non stationary fractional integrated series, *Econometric theory*, , 641 - 662.

Magnus, W., Oberhettinger, F. and Soni, R.P. (1966) Fomulas and Theorems for the Special Functions of Mathematical Physics, Springer, Berlin.

Marinucci D., Robinson P. (1998) Weak convergence of multivariate fractional processes, Preprint L.S.E., G.B.

McDonald R., Murphy P.D. (1989) "testing for the long run relationship between nominal interest rates and inflation using cointegration techniques", *Applied Statistics*, , 439 - 447.

Mikosch T., Starica C. (2000) "Limit Theory for the Sample Autocorrelations and Extremes of a GARCH(1,1) Process", *Annals of Statistics*, , 1427 - 1451.

Nelson D.B. (1990) Conditional heteroscedastic in asset returns: a new approach", *Econometrica*, , 347 - 370.

Phillips P.C.P., Perron P. (1988) "testing for a unit root in time series regression", *Biometrika*, , 335 - 346.

Rainville E.D., Special functions, Mac Millan, New York, 1960.

Resnick S.I. (1987) Extreme values, Regular variation and Point processes, Springer Verlag.

P. Robinson (1995) Gaussian semiparametric estimation of non stationary time series, *annals of Statistics*, , 1630 - 16661.

Rose A.K. (1988) " Is the real interest rate stable?", *Journal of finance*, , 1095 - 1112.

Smith J., Yadov S. (1994) "Forecasting cost incurred from unit differencing fractionally integrated processes, *International Journal of Forecasting*, , 507 - 514.

Starica C. (1999) "Estimation of the Extreme Value Distribution for Constant Conditional Correlation Models", *Dpt of Statistics, The Wharton School, University of Pennsylvan*

ia USA and Chalmers University of Technology, Gothenburg, Sweden, Preprint.

Szegö G. (1959) Orthogonal polynomials, A.M.S., New York.

Tiao G. C., Tsay R.S. (1983) "Consistency properties of least squares estimates of autoregressive parameters in ARMA models", *Annals of Statistics*, , 856 - 871.

Velasco C. (1999) Gaussian semi parametric estimation of non stationary time series, *Journal of Time Series Analysis*, , 87 - 127.

Wayne A., Woodward W.A., Cheng A.C. and Gray H.L. (1998)," A k-factor GARMA long memory model", *Journal of Time Series Analysis*, , 485-504.

Willinger W., Taqqu M.S., Teverovsky (1999) "Stock market prices and long range dependence", *Finance and Stochastics*, , 1 - 13.

Zakoian M. (1994) "Threshold heteroskedastic models", *Journal of Economic Dynamic and Control*, , 931 - 955.