The Choquet integral for the aggregation of interval scales in multicriteria decision making[*]

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Abstract

This paper addresses the question of which models fit with information concerning the preferences of the decision maker over each attribute, and his preferences about aggregation of criteria (interacting criteria). We show that the conditions induced by these information plus some intuitive conditions lead to a unique possible aggregation operator: the Choquet integral.

Keywords: multicriteria decision making, Choquet integral, axiomatic approach.

1 Introduction

Let us consider a decision problem that depends on n points of views described by the attributes X_1, ..., X_n. The attributes can be any set. They can be of cardinal nature (for instance, the maximum speed of a car) or of ordinal nature (for instance, the color {red,blue,...}). We wish to model the preferences $\succeq$ of the decision maker (DM) over acts, that is to say over elements of $X = X_1 \times \cdots \times X_n$. For any $x \in X$, we use the following notation:

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$x = (x_1, \ldots, x_n)$ where $x_1 \in X_1$, ..., $x_n \in X_n$. A classical way is to model $\succeq$ with the help of an overall utility function $u : X \to \mathbb{R}$ [11]:

$$\forall x, y \in X \;\;,\;\; x \succeq y \;\Leftrightarrow\; u(x) \geq u(y) \;. \tag{1}$$

The overall utility function u is often taken in the following way [11]

$$u(x) = F(u_1(x_1), \ldots, u_n(x_n)) \;,$$

where the u_i's are the so called *utility functions*, and F is the aggregation function. The main question is then how to determine the u_i's and F. The utility functions and the aggregation function are generally constructed in two separate steps. Each criterion is considered separately in the first step. The utility function u_i represents the preferences of the DM over the attribute X_i. The Macbeth approach [1, 2] provides a methodology to construct the u_i's as scales of difference (see section 6.1). In the second step, all criteria are considered together. As example of aggregation functions, one can find the weighted sum, the Choquet integral or the Sugeno integral. When F is a Choquet integral, several methods for the determination of the fuzzy measure are available. For instance, linear methods [13], quadratic methods [4, 6] and heuristic-based methods [5] are available in the literature. For the Sugeno integral, heuristics [7] and methods based on fuzzy relations [16, 17] can be found. Unfortunately, these methods address only the problem of constructing the aggregation function and refer more to learning procedures than to true decision making approaches. The way the first step is dealt with is generally not explained, especially with elaborate aggregators such as fuzzy integrals.

Yet, the two steps are intimately related. For instance, the notion of utility function has no *absolute* meaning. One cannot construct the u_i's without any a priori knowledge of what kind of F will be considered. However, when constructing the u_i's, F is not already known. As a consequence, the construction of the utility functions is more complicated than what it seems. One method addresses both points in a way that is satisfactory in the measurement theory standpoint: this is the Macbeth approach. However, it is restricted to the weighted sum. In such model, there is no interaction between criteria. The aim of this paper is to extend the Macbeth approach in such a way that interaction between criteria is allowed.

The u_i's and F shall be determined from some information obtained from the DM about his preference over acts, e.g. elements of X. The information we will consider in this paper can be seen as a generalization of the information needed in the Macbeth technology [1]. As in Macbeth, the information is

based on the introduction of two absolute reference levels over each attribute, and the determination of scales of difference, which, put together, ensures commensurateness. We will consider more general information than in the Macbeth approach in such a way that interaction between criteria will be allowed. We address then the question of which models fit with this information. The information we have and the measurement conditions imply some conditions on the aggregation function F. As an example, one condition is that F shall enable the construction of the u_i's. Some conditions given on specific acts are naturally extended to wider sets of acts. These conditions rule out a wide range of families of aggregators. We see finally which aggregation functions comes up. With these new properties, we show that the only possible model is the Choquet integral. We obtain an axiomatic representation of the Choquet integral that is a weak version of a result shown by J.L. Marichal [15].

We already addressed the same kind of problem for ratio scales in [10]. For ratio scales, the Šipoš integral seems to be the right aggregation function. The other difference between this paper and [10] is that we restricted ourself only to the necessary information to construct the model (preferences over each attribute and the aggregation of criteria) and do not add any a priori assumption as in [10].

2   Available information

The set of all criteria is denoted by $N = \{1, \ldots, n\}$. Considering two acts $x, y \in X$ and $A \subset N$, we use the notation (x_A, y_{-A}) to denote the act $z \in X$ such that $z_i = x_i$ if $i \in A$ and $z_i = y_i$ otherwise.

It is well-known in MCDM that the preference relation $\succeq$ can be modeled by an overall utility function u (see (1)) if and only if all attributes are set commensurate in some way [11]. The notion of commensurateness hinges on the idea that we shall not try to aggregate directly attributes but rather aggregate values that represent the same kind of quantity. This is the commensurate scale.

The modeling of the preference relation of the DM depends entirely on what type of scale the DM can work on. Here, we assume that the DM can handle a scale of difference. This means that, on top of being able to rank two acts $x, y \in X$, the DM can also give an assessment of the difference $u(x) - u(y)$ between the overall utility of x and y. In other words, if $x \succeq y$, the DM can give the intensity with which x is preferred to y. We assume furthermore that the underlying scale is a bounded unipolar scale. It is bounded from above and below. In this case, the commensurate scale depicts

the satisfaction degree of the DM over attributes. The satisfaction degree is typically a number belonging to the interval $[0, 1]$. The two bounds 0 and 1 have a special meaning. The 0 satisfaction value is the value that is considered completely unacceptable by the DM. The 1 satisfaction value is the value that is considered perfectly satisfactory by the DM. We assume that these two values 0 and 1 of the satisfaction scale can be identified by two particular elements $\mathbf{0}_i$ et $\mathbf{1}_i$ for each attribute X_i. These two particular elements have an absolute meaning throughout the attributes. We assume that $\mathbf{0}_i$ is the worst element of X_i, that is to say

$$\forall x_i \in X_i \quad , \quad (x_i, \mathbf{0}_{-i}) \succeq (\mathbf{0}_i, \mathbf{0}_{-i}) \ . \tag{2}$$

Similarly, $\mathbf{1}_i$ is the best element of X_i :

$$\forall x_i \in X_i \quad , \quad (x_i, \mathbf{0}_{-i}) \preceq (\mathbf{1}_i, \mathbf{0}_{-i}) \ . \tag{3}$$

The introduction of $\mathbf{0}_i$ and $\mathbf{1}_i$ enables us to construct intra-criterion and inter-criteria information.

2.1   Intra-criterion information

For each criterion i, the mapping (denoted by u_i) from the attribute X_i to the satisfaction scale $[0, 1]$ must be explicited. This corresponds to the preferences of the DM over each attribute. Commensurateness implies that the elements of one attribute shall be compared to the elements of any other scale. Taking a simple example involving two criteria (for instance consumption and maximal speed), this amounts to know whether the DM prefers a consumption of 5 liters/100km to a maximum speed of 200 km/h. This does not generally make sense to the DM, so that he is not generally able to make this comparison directly. In order to solve this problem out, the Macbeth approach is based on the idea that a scale of difference is constructed separately on each attribute. A scale of difference is given up to two degrees of freedom. Fixing two points on the scale determines entirely the scale. These two points are chosen in order to enforce the overall commensurateness. As a consequence, these two points are the only elements of the attribute that have to be compared to the elements of the other attributes. These elements are actually the $\mathbf{0}_i$'s and $\mathbf{1}_i$'s. All the $\mathbf{0}_i$'s have the same meaning : $u_1(\mathbf{0}_1) = \cdots = u_n(\mathbf{0}_n)$. Similarly, $u_1(\mathbf{1}_1) = \cdots = u_n(\mathbf{1}_n)$.

It is natural for a DM to give his preferences over acts. On the other hand, the DM shall not be asked information directly on the parameters of the model. Therefore, it is not assumed that the DM can isolate attributes and give information regarding directly the u_i's. To this end, as in the

Macbeth methodology, we consider the subset $X\rfloor_i$ (for $i \in N$) of X defined by

$$X\rfloor_i := \{(x_i, \mathbf{0}_{-i}) \ , \ x_i \in X_i\} \ .$$

We ask the DM not only the ranking of the elements of $X\rfloor_i$ but also the difference of satisfaction degree between pairs of elements of $X\rfloor_i$. From this, u_i can be defined by :

(Intra$_a$) $\forall x_i, y_i \in X_i$, $u_i(x_i) \geq u_i(y_i) \Leftrightarrow (x_i, \mathbf{0}_{-i}) \succeq (y_i, \mathbf{0}_{-i})$.

(Intra$_b$) $\forall x_i, y_i, w_i, z_i \in X_i$ such that $u_i(x_i) > u_i(y_i)$ and $u_i(w_i) > u_i(z_i)$, we have

$$\frac{u_i(x_i) - u_i(y_i)}{u_i(w_i) - u_i(z_i)} = k(x_i, y_i, w_i, z_i) \ , \quad k(x_i, y_i, w_i, z_i) \in \mathbb{R}^+$$

if and only if the difference of satisfaction degree that the DM feels between $(x_i, \mathbf{0}_{-i})$ and $(y_i, \mathbf{0}_{-i})$ is $k(x_i, y_i, w_i, z_i)$ times as big as the difference of satisfaction between $(w_i, \mathbf{0}_{-i})$ and $(z_i, \mathbf{0}_{-i})$.

(Intra$_c$) $u_i(\mathbf{0}_i) = 0$ and $u_i(\mathbf{1}_i) = 1$.

Condition **(Intra$_b$)** means that $k(x_i, y_i, w_i, z_i) = \frac{u(x_i, \mathbf{0}_{-i}) - u(y_i, \mathbf{0}_{-i})}{u(w_i, \mathbf{0}_{-i}) - u(z_i, \mathbf{0}_{-i})}$. The u_i correspond to a scale of difference. Such a scale is always given up to a shift and a dilation. Condition **(Intra$_c$)** fixes these two degrees of freedom.

In order to be able to construct a unique scale from **(Intra$_a$)**, **(Intra$_b$)** and **(Intra$_c$)**, some consistency assumptions shall be made :

(Intra$_d$) $\forall x_i, y_i, w_i, z_i, r_i, s_i \in X_i$ such that $u_i(x_i) > u_i(y_i)$, $u_i(w_i) > u_i(z_i)$ and $u_i(r_i) > u_i(s_i)$,

$$k(x_i, y_i, w_i, z_i) \times k(w_i, z_i, r_i, s_i) = k(x_i, y_i, r_i, s_i) \ .$$

2.2 Inter-criteria information

We are concerned here in the determination of the aggregation function F. When F is taken as a weighted sum, it is enough to obtain from the DM some information regarding the importance of each criterion. This is in particular the case of the Macbeth methodology [1]. In the Macbeth approach, the following set of acts is introduced

$$\tilde{X} := \{(\mathbf{0}_N)\} \bigcup \{(\mathbf{1}_i, \mathbf{0}_{-i}) \ , \ i \in N\} \ ,$$

where $(\mathbf{0}_N)$ denotes the alternatives that is unacceptable on every criteria. On top of giving his preferences on the elements of $\tilde{X}$, the DM provides some information on the difference of satisfaction between any two elements of $\tilde{X}$. Thanks to that information, an interval scale that represents the satisfaction degree of the elements of $\tilde{X}$ can be constructed, provided the information is consistent. Then, the importance of criterion i is defined as being proportional to the difference of the satisfaction degrees between $(\mathbf{1}_i, \mathbf{0}_{-i})$ and $(\mathbf{0}_N)$. The constant of proportionality is fixed from the property that all the importances shall sum up to one.

In order to generalize the Macbeth approach, we introduce the following set $X]_{\{0,1\}}$ of acts :

$$X]_{\{0,1\}} := \{(\mathbf{1}_A, \mathbf{0}_{-A}) \; , \; A \subset N\} \; .$$

As before, we ask information from which one can obtain a satisfaction scale defined on $X]_{\{0,1\}}$. Define $\mathcal{P}(N)$ as the set of all subsets of N. Let $\mu : \mathcal{P}(N) \to [0,1]$ defined by

(Inter$_a$) $\forall A, B \subset N, \mu(A) \geq \mu(B) \Leftrightarrow (\mathbf{1}_A, \mathbf{0}_{-A}) \succeq (\mathbf{1}_B, \mathbf{0}_{-B})$.

(Inter$_b$) $\forall A, B, C, D \subset N$ such that $\mu(A) > \mu(B)$ and $\mu(C) > \mu(D)$, we have

$$\frac{\mu(A) - \mu(B)}{\mu(C) - \mu(D)} = k(A, B, C, D) \; , \quad k(A, B, C, D) \in \mathbb{R}^+$$

if and only if the difference of satisfaction degree that the DM feels between $(\mathbf{1}_A, \mathbf{0}_{-A})$ and $(\mathbf{1}_B, \mathbf{0}_{-B})$ is $k(A, B, C, D)$ times as big as the difference of satisfaction between $(\mathbf{1}_C, \mathbf{0}_{-C})$ and $(\mathbf{1}_D, \mathbf{0}_{-D})$.

(Inter$_c$) $\mu(\emptyset) = 0$ and $\mu(N) = 1$.

The last condition is rather natural since $\mu(\emptyset) = 0$ means that the act which is completely unacceptable on all attributes is also completely unacceptable as a whole, and $\mu(N) = 1$ means that the act which is perfectly satisfactory on all attributes is also perfectly satisfactory as a whole. This depicts the idea of commensurateness.

In order to be able to construct a unique scale from **(Inter$_a$)**, **(Inter$_b$)** and **(Inter$_c$)**, some consistency assumptions shall be made :

(Inter$_d$) $\forall A, B, C, D, E, F \subset N$ such that $\mu(A) > \mu(B), \mu(C) > \mu(D)$ and $\mu(E) > \mu(F)$

$$k(A, B, C, D) \times k(C, D, E, F) = k(A, B, E, F) \; .$$

2.3 Measurement conditions

The u_i's and μ correspond to scales of difference. Considering for instance u_i, this means that the only type of quantity that makes sense is ratios of the form $\frac{u_i(x_i)-u_i(y_i)}{u_i(w_i)-u_i(z_i)}$. Henceforth, it would have been possible to replace condition $(\mathbf{Intra}_c)$ by $u_i(\mathbf{0}_i) = \beta$ and $u_i(\mathbf{1}_i) = \alpha + \beta$, with $\alpha > 0$ and $\beta \in \mathbb{R}$. In other words, we could replace u_i given by conditions $(\mathbf{Intra}_a)$, $(\mathbf{Intra}_b)$ and $(\mathbf{Intra}_c)$ by $\alpha u_i + \beta$. Since the u_i's map the attributes onto a single commensurate scale, one must change all u_i's in $\alpha u_i + \beta$ with the same α and β (one cannot change only one scale). The measurement conditions for scales of difference imply that the preference relation $\succeq$ and the ratios $\frac{u(x)-u(y)}{u(z)-u(t)}$ for all $x, y, z, t \in X$ shall not be changed if all the u_i's are changed into $\alpha u_i + \beta$ with $\alpha > 0$ and $\beta \in \mathbb{R}$. We assume in this paper that this property holds only for the acts that we are considering in our construction, that is to say for $x, y, z, t \in X\rfloor_i$ and $x, y, z, t \in X\rfloor_{\{0,1\}}$. This leads to the following requirement :

$(\mathbf{Intra}_e)$ The preference relation $\succeq$ and the $\frac{u(x)-u(y)}{u(z)-u(t)}$ for $x, y, z, t \in X\rfloor_i$ and for $x, y, z, t \in X\rfloor_{\{0,1\}}$ shall not be changed if all the u_i's are changed into $\alpha u_i + \beta$ with $\alpha > 0$ and $\beta \in \mathbb{R}$.

This condition is no more than the requirement that the u_i's are commensurate scales of differences.

Looking at $X\rfloor_{\{0,1\}}$ as a generalization of the set $\tilde{X}$ used in the Macbeth approach, $\mu(\{i\})$ represents some kind of *importance of criterion i*. More precisely, the term $\mu(\{i\})$ corresponds to the difference of the satisfaction degrees between $(\mathbf{1}_i, \mathbf{0}_{-i})$ and $(\mathbf{0}_N)$. Generalizing this to coalitions of any cardinality, $\mu(A)$ corresponds in fact to the difference of the satisfaction degrees between the alternatives $(\mathbf{1}_A, \mathbf{0}_{-A})$ and $(\mathbf{0}_N)$. Applying this to $A = \emptyset$, the value $\mu(\emptyset)$ shall always be equal to zero, whatever the interval scale attached to $\tilde{X}$ may be. If the interval scale attached to $\tilde{X}$ is changed to another interval scale, the values of $\mu(A)$ could vary, except $\mu(\emptyset)$ that will always vanish. More precisely, μ will be replaced by $\gamma\mu$, with $\gamma \in \mathbb{R}$. Henceforth, μ corresponds to a ratio scale. As a consequence, it would be possible to replace condition $(\mathbf{Inter}_c)$ by $\mu(\emptyset) = 0$ and $\mu(N) = \gamma$, with $\gamma \in \mathbb{R}$. As previously, the following requirement is imposed :

$(\mathbf{Inter}_e)$ The preference relation $\succeq$ and the $\frac{u(x)-u(y)}{u(z)-u(t)}$ for $x, y, z, t \in X\rfloor_i$ and for $x, y, z, t \in X\rfloor_{\{0,1\}}$ shall not be changed if μ is changed into $\gamma\mu$ with $\gamma \in \mathbb{R}$.

3   Conditions on the model

In measurement theory, it is classical to split the overall evaluation model u into two parts [11]: the utility functions (that map the attributes onto a single satisfaction degree scale), and the aggregation function (that aggregates commensurate scales). The function u_i corresponds to satisfaction degrees over criterion i. Thanks to assumptions $(\mathbf{Intra}_a)$ and $(\mathbf{Intra}_b)$, u_i are scales of difference. These scales are commensurate by condition $(\mathbf{Intra}_c)$. The term $\mu(A)$ represents the importance that the DM gives to the coalition A in the DM process. Consequently, it is natural to write u as follows:

$$u\left(x\right) = F_\mu\left(u_1(x_1), \ldots, u_n(x_n)\right) \;, \tag{4}$$

where F_μ is the aggregation operator. F_μ depends on μ in a way that is not known for the moment. From now on, we assume that the preferences of the DM can be modeled by u given by (4).

As example of aggregation operators, F_μ could be a weighted sum

$$F_\mu\left(u_1, \ldots, u_n\right) = \sum_{i=1}^{n} \alpha_i\, u_i \;.$$

For a normalized weighted sum (i.e. when the weights α_i sum up to one), one clearly has $F_\mu(\beta, \ldots, \beta) = \beta$ for any β. This property is explained by the fact that F_μ aggregates commensurate scales. It can be naturally generalized to any F_μ. The condition that the weighted sum is normalized becomes that μ is normalized, that is to say $\mu(N) = 1$. Henceforth, it is natural to assume that (whenever μ satisfies $\mu(N) = 1$)

$$F_\mu(\beta, \ldots, \beta) = \beta \;\;, \;\forall \beta \in [0, 1] \;.$$

Due to condition $(\mathbf{Intra}_c)$ and relations (2) and (3), $u_i(x_i) \in [0, 1]$ for any $x_i \in X_i$. This is why $\beta \in [0, 1]$ in previous relation. However, condition $(\mathbf{Intra}_e)$ implies that $u_i(x_i)$ could take virtually any real value. Hence, F_μ shall be defined on $\mathbb{R}^n$. This shows that previous relation on F_μ shall hold for any $\beta \in \mathbb{R}$:

$$F_\mu(\beta, \ldots, \beta) = \beta \;\;, \;\forall \beta \in \mathbb{R} \;.$$

Consider now the case when $\mu(N) \neq 1$. Since μ corresponds to a ratio scale, it is reasonable to assume that μ acts as a dilation of the overall evaluation scale of u. In other words, when $\mu(N) \neq 1$, we should have

$$F_\mu(\beta, \ldots, \beta) = \beta\mu(N) \;\;, \;\forall \beta \in \mathbb{R} \;. \tag{5}$$

3.1 Intra-criterion information

Lemma 1 *If u satisfies (4), and if conditions $(\mathbf{Intra}_a)$, $(\mathbf{Intra}_b)$, $(\mathbf{Intra}_c)$, $(\mathbf{Intra}_d)$, $(\mathbf{Intra}_e)$, $(\mathbf{Inter}_a)$, $(\mathbf{Inter}_b)$, $(\mathbf{Inter}_c)$, $(\mathbf{Inter}_d)$ and $(\mathbf{Inter}_e)$ are fulfilled, then for all $a_i, b_i, c_i, d_i \in [0,1]$, and for all $\alpha > 0$, $\gamma, \beta \in \mathbb{R}$,*

$$\frac{F_{\gamma\mu}\left(\alpha a_i + \beta, \beta_{-i}\right) - F_{\gamma\mu}\left(\alpha b_i + \beta, \beta_{-i}\right)}{F_{\gamma\mu}\left(\alpha c_i + \beta, \beta_{-i}\right) - F_{\gamma\mu}\left(\alpha d_i + \beta, \beta_{-i}\right)} = \frac{a_i - b_i}{c_i - d_i} \ . \tag{6}$$

Proof : $u\left(x_i, \mathbf{0}_{-i}\right)$ and $u_i(x_i)$ correspond to two possible scales of difference related to the same act $(x_i, \mathbf{0}_{-i}) \in X\rfloor_i$. Henceforth

$$\frac{u\left(x_i, \mathbf{0}_{-i}\right) - u\left(y_i, \mathbf{0}_{-i}\right)}{u\left(w_i, \mathbf{0}_{-i}\right) - u\left(z_i, \mathbf{0}_{-i}\right)} = \frac{u_i(x_i) - u_i(y_i)}{u_i(w_i) - u_i(z_i)} \ ,$$

which gives

$$\frac{F_\mu\left(u_i(x_i), u_{-i}(\mathbf{0}_{-i})\right) - F_\mu\left(u_i(y_i), u_{-i}(\mathbf{0}_{-i})\right)}{F_\mu\left(u_i(w_i), u_{-i}(\mathbf{0}_{-i})\right) - F_\mu\left(u_i(z_i), u_{-i}(\mathbf{0}_{-i})\right)} = \frac{u_i(x_i) - u_i(y_i)}{u_i(w_i) - u_i(z_i)} \ . \tag{7}$$

From $(\mathbf{Intra}_e)$, one might change all u_j in $\alpha u_j + \beta$ at the same time without any change in (7). In this case, the utility functions will take values in the interval $[\beta, \alpha + \beta]$. From $(\mathbf{Inter}_e)$, one might change μ into $\gamma\mu$ $(\gamma \in \mathbb{R})$ without any change in (7). Consequently, one must have

$$\frac{F_{\gamma\mu}\left(\alpha u_i(x_i) + \beta, \alpha u_{-i}(\mathbf{0}_{-i}) + \beta\right) - F_{\gamma\mu}\left(\alpha u_i(y_i) + \beta, \alpha u_{-i}(\mathbf{0}_{-i}) + \beta\right)}{F_{\gamma\mu}\left(\alpha u_i(w_i) + \beta, \alpha u_{-i}(\mathbf{0}_{-i}) + \beta\right) - F_{\gamma\mu}\left(\alpha u_i(z_i) + \beta, \alpha u_{-i}(\mathbf{0}_{-i}) + \beta\right)}$$
$$= \frac{(\alpha u_i(x_i) + \beta) - (\alpha u_i(y_i) + \beta)}{(\alpha u_i(w_i) + \beta) - (\alpha u_i(z_i) + \beta)} \ .$$

Now, setting $a_i = u_i(x_i)$, $b_i = u_i(y_i)$, $c_i = u_i(w_i)$ and $d_i = u_i(z_i)$, we have $a_i, b_i, c_i, d_i \in [0,1]$ by (2), (3) and $(\mathbf{Intra}_c)$. Hence, the lemma is proved. ∎

3.2 Inter-criteria information

Lemma 2 *If u satisfies (4), and if conditions $(\mathbf{Intra}_a)$, $(\mathbf{Intra}_b)$, $(\mathbf{Intra}_c)$, $(\mathbf{Intra}_d)$, $(\mathbf{Intra}_e)$, $(\mathbf{Inter}_a)$, $(\mathbf{Inter}_b)$, $(\mathbf{Inter}_c)$, $(\mathbf{Inter}_d)$ and $(\mathbf{Inter}_e)$ are fulfilled, then for any $\alpha > 0$ and $\gamma, \beta \in \mathbb{R}$, it holds that*

$$\frac{F_{\gamma\mu}\left((\alpha + \beta)_A, \beta_{-A}\right) - F_{\gamma\mu}\left((\alpha + \beta)_B, \beta_{-B}\right)}{F_{\gamma\mu}\left((\alpha + \beta)_C, \beta_{-C}\right) - F_{\gamma\mu}\left((\alpha + \beta)_D, \beta_{-D}\right)} = \frac{\mu(A) - \mu(B)}{\mu(C) - \mu(D)} \ . \tag{8}$$

Proof : By conditions $(\mathbf{Inter}_a)$ and $(\mathbf{Inter}_b)$, $u\left(\mathbf{1}_A, \mathbf{0}_{-A}\right)$ and $\mu(A)$ correspond to two possible interval scales related to the act $\left(\mathbf{1}_A, \mathbf{0}_{-A}\right) \in X\rceil_{\{0,1\}}$. Hence

$$\frac{u\left(\mathbf{1}_A, \mathbf{0}_{-A}\right) - u\left(\mathbf{1}_B, \mathbf{0}_{-B}\right)}{u\left(\mathbf{1}_C, \mathbf{0}_{-C}\right) - u\left(\mathbf{1}_D, \mathbf{0}_{-D}\right)} = \frac{\mu(A) - \mu(B)}{\mu(C) - \mu(D)} \ ,$$

which gives

$$\frac{F_\mu\left(u_A(\mathbf{1}_A), u_{-A}(\mathbf{0}_{-A})\right) - F_\mu\left(u_B(\mathbf{1}_B), u_{-B}(\mathbf{0}_{-B})\right)}{F_\mu\left(u_C(\mathbf{1}_C), u_{-C}(\mathbf{0}_{-C})\right) - F_\mu\left(u_D(\mathbf{1}_D), u_{-D}(\mathbf{0}_{-D})\right)} = \frac{\mu(A) - \mu(B)}{\mu(C) - \mu(D)} \ . \qquad (9)$$

As previously, from $(\mathbf{Intra}_e)$ and $(\mathbf{Inter}_e)$, one can change u_j in $\alpha u_j + \beta$, and μ in $\gamma\mu$ without any change in (9) :

$$\frac{F_{\gamma\mu}\left(\alpha u_A(\mathbf{1}_A) + \beta, \alpha u_{-A}(\mathbf{0}_{-A}) + \beta\right) - F_{\gamma\mu}\left(\alpha u_B(\mathbf{1}_B) + \beta, \alpha u_{-B}(\mathbf{0}_{-B}) + \beta\right)}{F_{\gamma\mu}\left(\alpha u_C(\mathbf{1}_C) + \beta, \alpha u_{-C}(\mathbf{0}_{-C}) + \beta\right) - F_{\gamma\mu}\left(\alpha u_D(\mathbf{1}_D) + \beta, \alpha u_{-D}(\mathbf{0}_{-D}) + \beta\right)}$$
$$= \frac{\gamma\mu(A) - \gamma\mu(B)}{\gamma\mu(C) - \gamma\mu(D)} \ .$$

Hence the lemma is proved. ∎

4   Generalized conditions on the model

Lemma 3 *If u satisfies (4), and if conditions (5), $(\mathbf{Intra}_a)$, $(\mathbf{Intra}_b)$, $(\mathbf{Intra}_c)$, $(\mathbf{Intra}_d)$, $(\mathbf{Inter}_a)$, $(\mathbf{Inter}_b)$, $(\mathbf{Inter}_c)$ and $(\mathbf{Inter}_d)$ are fulfilled, then we have for any $\eta, \beta \in \mathbb{R}$*

$$F_{\gamma\mu + \delta\mu'}\left(\eta_A, \beta_{-A}\right) = \gamma F_\mu\left(\eta_A, \beta_{-A}\right) + \delta F_{\mu'}\left(\eta_A, \beta_{-A}\right) \ . \qquad (10)$$

Proof : Taking (8) with $\gamma = 1$, $B = D = \emptyset$ and $C = N$, it holds that

$$F_\mu\left((\alpha + \beta)_A, \beta_{-A}\right) = \alpha\mu(A) + \beta\mu(N) \ . \qquad (11)$$

This relation holds for any μ satisfying $\mu(\emptyset) = 0$. Consequently, replacing μ by $\gamma\mu + \delta\mu'$, we obtain for any $\beta \in \mathbb{R}$, $\alpha > 0$

$$F_{\gamma\mu + \delta\mu'}\left((\alpha + \beta)_A, \beta_{-A}\right) = \gamma F_\mu\left((\alpha + \beta)_A, \beta_{-A}\right) + \delta F_{\mu'}\left((\alpha + \beta)_A, \beta_{-A}\right) \ .$$

Hence, setting $\eta = \alpha + \beta$, we obtain for any $\beta \in \mathbb{R}$ and $\eta > \beta$,

$$F_{\gamma\mu + \delta\mu'}\left(\eta_A, \beta_{-A}\right) = \gamma F_\mu\left(\eta_A, \beta_{-A}\right) + \delta F_{\mu'}\left(\eta_A, \beta_{-A}\right) \ .$$

Replacing A by $N \setminus A$ in previous relation, we obtain for any $\beta \in \mathbb{R}$ and $\eta > \beta$,
$$F_{\gamma\mu+\delta\mu'} \left(\beta_A, \eta_{-A} \right) = \gamma F_\mu \left(\beta_A, \eta_{-A} \right) + \delta F_{\mu'} \left(\beta_A, \eta_{-A} \right) .$$

Putting together last two results, we get for any $\eta, \beta \in \mathbb{R}$ with $\beta \neq \eta$

$$F_{\gamma\mu+\delta\mu'} \left(\eta_A, \beta_{-A} \right) = \gamma F_\mu \left(\eta_A, \beta_{-A} \right) + \delta F_{\mu'} \left(\eta_A, \beta_{-A} \right) .$$

Previous relation holds also when $\beta = \eta$ by (5). Hence, we have for any $\eta, \beta \in \mathbb{R}$

$$F_{\gamma\mu+\delta\mu'} \left(\eta_A, \beta_{-A} \right) = \gamma F_\mu \left(\eta_A, \beta_{-A} \right) + \delta F_{\mu'} \left(\eta_A, \beta_{-A} \right) .$$

■

We wish to generalize (10) in the following way

$$F_{\gamma\mu+\delta\mu'}(x) = \gamma F_\mu(x) + \delta F_{\mu'}(x) \quad , \ \forall x \in \mathbb{R}^n, \ \forall \gamma, \delta \in \mathbb{R} .$$

We should have $F_{\gamma\mu}(x) = \gamma F_\mu(x)$ ($\gamma \in \mathbb{R}$) since μ corresponds to a ratio scale. One should have $F_{\frac{\mu+\mu'}{2}}(x) = \frac{1}{2} \left(F_\mu(x) + F_{\mu'}(x) \right)$ for the following reason. If two decision makers give the importances of the coalitions (μ and μ' respectively), the consensus of these two decision makers can be performed by taking the mean value of these information (that is to say $\frac{\mu+\mu'}{2}$). Then it is reasonable that the overall aggregation functions equals the mean value of the aggregation for the two DM. Based on these considerations, we propose the following axiom :

Linearity wrt the Measure (LM): For all $x \in \mathbb{R}^n$ and $\gamma, \delta \in \mathbb{R}$,
$$F_{\gamma\mu+\delta\mu'}(x) = \gamma F_\mu(x) + \delta F_{\mu'}(x) . \tag{12}$$

This axiom cannot be completely deduced from the construction of the u_i's and μ. However, Lemma 3 proves that formula (12) is obtained from the construction of the u_i's and μ for particular alternatives x.

Since F_μ aggregates satisfaction scales, it is natural to assume that $x \mapsto F_\mu(x)$ is increasing.

Increasingness (In): $\forall x, x' \in \mathbb{R}^n$,
$$x_i \leq x'_i \ \forall i \in N \ \Rightarrow F_\mu(x) \leq F_\mu(x')$$

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This axiom is not deduced from the construction of the u_i's and μ. It is a necessary requirement for F_μ to be an aggregation function [3].

Let us give now the last two axioms.

Properly Weighted (PW): If μ satisfies condition **(Inter$_c$)**, then $F_\mu\left(1_A, 0_{-A}\right) = \mu(A)$, $\forall A \subset N$.

Stability for the admissible Positive Linear transformations (weak SPL): If μ satisfies condition **(Inter$_c$)**, then for all $A \subset N$, $\alpha > 0$, and $\beta \in \mathbb{R}$,

$$F_\mu\left((\alpha + \beta)_A, \beta_{-A}\right) = \alpha F_\mu\left(1_A, 0_{-A}\right) + \beta$$

This axiom is called **(weak SPL)** since it is a weak version of the axiom **(SPL)** introduced by J.L. Marichal [14, 15] :

(SPL): For all $x \in \mathbb{R}^n$, $\alpha > 0$, and $\beta \in \mathbb{R}$,

$$F_\mu\left(\alpha x + \beta\right) = \alpha F_\mu\left(x\right) + \beta$$

Lemma 4 *Axioms* **(PW)** *and* **(weak SPL)** *can be deduced from the construction of the u_i's and μ.*

Proof : Axiom **(PW)** is obtained from (11) with $\beta = 0$ and $\alpha = 1$. Finally, using (11) and **(PW)**, **(weak SPL)** holds. ∎

Next lemma shows that axioms **(LM)**, **(PW)** and **(weak SPL)** contain the two relations (6) and (8).

Lemma 5 *Conditions* **(LM)***,* **(PW)** *and* **(weak SPL)** *imply relations (6) and (8).*

Proof : Consider F_μ satisfying **(LM)**, **(PW)** and **(weak SPL)**. By **(LM)** and **(weak SPL)**, we have for any $\alpha > 0$ and $a_i \in \mathbb{R}^+$,

$$F_{\gamma\mu}\left(\alpha a_i + \beta, \beta_{-i}\right) = \gamma F_\mu\left(\alpha a_i + \beta, \beta_{-i}\right) = \gamma \alpha a_i F_\mu\left(1_i, 0_{-i}\right) + \gamma\beta \ .$$

Consequently, (6) is fulfilled. Condition (8) is clearly satisfied from **(LM)**, **(PW)** and **(weak SPL)**. ∎

To end up this section, let us recall that the construction of the u_i's (conditions **(Intra$_a$)**, **(Intra$_b$)**, **(Intra$_c$)**, **(Intra$_d$)**), **(Intra$_e$)**) and μ (conditions **(Inter$_a$)**, **(Inter$_b$)**, **(Inter$_c$)**, **(Inter$_d$)**) and **(Inter$_e$)**, plus assumptions (4) and (5) lead to axioms **(PW)** and **(weak SPL)**, and partially to axiom **(LM)** (see Lemma 3). Moreover, we saw that axiom **(In)** is a very natural requirement.

5 Expression of the aggregation function

In this section, we wish to find which aggregation functions F_μ are characterized by conditions (**LM**), (**In**), (**PW**) and (**weak SPL**). It can be noticed that the Choquet integral satisfies these conditions. Our main result shows that this is the only aggregator satisfying these conditions.

5.1 Background on the Choquet integral

Thanks to condition (**Inter$_c$**) and axiom (**In**), the set function μ satisfies

(i) $\mu(\emptyset) = 0$, $\mu(N) = 1$.

(ii) $A \subset B \subset N$ implies $\mu(A) \leq \mu(B)$.

These two properties define the so-called *fuzzy measures*. As a consequence, F_μ could be a Choquet integral.

The Choquet integral [3, 8] is a generalization of the commonly used weighted sum. Whereas the weighted sum is the discrete form of the Lebesgue integral with an additive measure, we consider as an aggregator the discrete form of the generalization of Lebesgue's integral to the case of non-additive measures. This generalization is precisely the Choquet integral.

Definition 1 *Let μ be a fuzzy measure on N. The* discrete Choquet integral *of an element $x \in \mathbb{R}^n$ with respect to μ is defined by*

$$C_\mu(x) = \sum_{i=1}^{n} x_{\tau(i)} [\mu(\{\tau(i), \ldots, \tau(n)\})$$
$$-\mu(\{\tau(i+1), \ldots, \tau(n)\})]$$

where τ is a permutation satisfying $x_{\tau(1)} \leq \cdots \leq x_{\tau(n)}$.

Besides, the Choquet integral can model typical human behavior such as the veto. This operator is also able to model the importance of criteria and the interaction between criteria. Conversely, the Choquet integral can be interpreted in term of the importance of criteria, the interaction between criteria, and veto [6, 8, 3].

5.2 Expression of F_μ

The result shown here is close to a result proved in [15]. Before giving the final result, let us give an intermediate lemma.

Lemma 6 *If $F : \mathbb{R}^n \to \mathbb{R}$ satisfies **(In)** and **(weak SPL)**, $F(0_N) = 0$, $F(1_N) = 1$ and that $F(1_A, 0_{-A}) \in \{0, 1\}$ for all $A \subset N$, then $F \equiv C_\mu$, where the fuzzy measure μ is given by $\mu(A) = F(1_A, 0_{-A})$ for all $A \subset N$.*

Proof : Let σ be a permutation of N. For $i \in N$, let

$$\theta_\sigma(i) := F\left(1_{\{\sigma(i),...,\sigma(n)\}}, 0_{\{\sigma(1),...,\sigma(i-1)\}}\right) .$$

We have $\theta_\sigma(i) \in \{0, 1\}$. From **(In)**, there exists $k_\sigma \in N$ such that

$$\theta_\sigma(i) = 0 \text{ for } i \in \{k_\sigma + 1, \dots, n\} \quad \text{and} \quad \theta_\sigma(i) = 1 \text{ for } i \in \{1, \dots, k_\sigma\} .$$

We have $k_\sigma \in \{1, \dots, n-1\}$ since $F(0_N) = 0$ and $F(1_N) = 1$. For $x \in \mathbb{R}^n$, the following notation $[x_1, \dots, x_n]_{\sigma^{-1}}$ denotes the element $y \in \mathbb{R}^n$ such that $y_i = x_{\sigma^{-1}(i)}$. With this notation, $x \in \mathbb{R}^n$ reads $x = [x_{\sigma(1)}, \dots, x_{\sigma(n)}]_{\sigma^{-1}}$.

Let $\mathcal{B}_\sigma := \{x \in \mathbb{R}^n , \ x_{\sigma(1)} \leq \dots \leq x_{\sigma(n)}\}$ and $x \in \mathcal{B}_\sigma$. From **(In)** and **(weak SPL)**, and since $x_{\sigma(k_\sigma)} \leq x_{\sigma(n)}$, we have

$$
\begin{aligned}
F(x) &= F\left([x_{\sigma(1)}, \dots, x_{\sigma(n)}]_{\sigma^{-1}}\right) \\
&\leq F([\underbrace{x_{\sigma(k_\sigma)}, \dots, x_{\sigma(k_\sigma)}}_{k_\sigma \text{ times}}, \underbrace{x_{\sigma(n)}, \dots, x_{\sigma(n)}}_{(n-k_\sigma) \text{ times}}]_{\sigma^{-1}}) \\
&= x_{\sigma(k_\sigma)} + \left(x_{\sigma(n)} - x_{\sigma(k_\sigma)}\right) F([\underbrace{0, \dots, 0}_{k_\sigma \text{ times}}, \underbrace{1, \dots, 1}_{(n-k_\sigma) \text{ times}}]_{\sigma^{-1}}) \\
&= x_{\sigma(k_\sigma)} + \left(x_{\sigma(n)} - x_{\sigma(k_\sigma)}\right) F\left(1_{\{\sigma(k_\sigma+1),...,\sigma(n)\}}, 0_{\{\sigma(1),...,\sigma(k_\sigma)\}}\right) \\
&= x_{\sigma(k_\sigma)} + \left(x_{\sigma(n)} - x_{\sigma(k_\sigma)}\right) \theta_\sigma(k_\sigma + 1) = x_{\sigma(k_\sigma)}
\end{aligned}
$$

since $\theta_\sigma(k_\sigma + 1) = 0$. On the other hand, by **(In)**,

$$
\begin{aligned}
F(x) &= F\left([x_{\sigma(1)}, \dots, x_{\sigma(n)}]_{\sigma^{-1}}\right) \\
&\geq F([\underbrace{x_{\sigma(1)}, \dots, x_{\sigma(1)}}_{(k_\sigma-1) \text{ times}}, \underbrace{x_{\sigma(k_\sigma)}, \dots, x_{\sigma(k_\sigma)}}_{(n-k_\sigma+1) \text{ times}}]_{\sigma^{-1}}) \\
&= x_{\sigma(1)} + \left(x_{\sigma(k_\sigma)} - x_{\sigma(1)}\right) F\left(1_{\{\sigma(k_\sigma),...,\sigma(n)\}}, 0_{\{\sigma(1),...,\sigma(k_\sigma-1)\}}\right) \\
&= x_{\sigma(1)} + \left(x_{\sigma(k_\sigma)} - x_{\sigma(1)}\right) \theta_\sigma(k_\sigma) = x_{\sigma(k_\sigma)}
\end{aligned}
$$

since $\theta_\sigma(k_\sigma) = 1$. Hence we have

$$F(x) = x_{\sigma(k_\sigma)} .$$

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Let us note that for the fuzzy measure μ defined by $\mu(S) = F(1_S, 0_{-S})$, we have

$$C_\mu(x) = \sum_{i=1}^{n} x_{\sigma(i)} \left[\mu(\{\sigma(i), \ldots, \sigma(n)\}) - \mu(\{\sigma(i+1), \ldots, \sigma(n)\}) \right] \ .$$

Moreover,

$$
\begin{aligned}
\mu(\{\sigma(i), \ldots, \sigma(n)\}) &= F\left(1_{\{\sigma(i),\ldots,\sigma(n)\}}, 0_{\{\sigma(1),\ldots,\sigma(i-1)\}}\right) \\
&= \theta_\sigma(i) = \begin{cases} 1 \text{ if } i \in \{1, \ldots, k_\sigma\} \\ 0 \text{ if } i \in \{k_\sigma + 1, \ldots, n\} \end{cases}
\end{aligned}
$$

Hence

$$C_\mu(x) = x_{\sigma(k_\sigma)} = F(x) \ .$$

∎

In Lemma 6, **(In)** and **(weak SPL)** are sufficient to obtain that F is a Choquet integral only because the range of $F(1_A, 0_{-A})$ belongs to $\{0, 1\}$. This is no more true in the general case.

Theorem 1 *F_μ satisfies* **(LM)**, **(In)**, **(PW)** *and* **(weak SPL)** *if and only if $F_\mu \equiv C_\mu$ in $\mathbb{R}^n$.*

Proof : Clearly, the Choquet integral satisfies **(LM)**, **(In)**, **(PW)** and **(weak SPL)**.

Consider now F_μ satisfying **(LM)**, **(In)**, **(PW)** and **(weak SPL)**. We write $\mu(A) = \sum_{B \subset N} m(B) u_B(A)$, where m is the Möbius transform $m(B) = \sum_{C \subset B} (-1)^{|B|-|C|} \mu(C)$, and u_B is the unanimity game, that is to say $u_B(A)$ equals 1 if $B \subset A$ and 0 otherwise. Since $m(\emptyset) = 0$, we have $\mu(A) = \sum_{B \subset N, B \neq \emptyset} m(B) u_B(A)$ Thanks to **(LM)**,

$$F_\mu(x) = \sum_{B \subset N, B \neq \emptyset} m(B) F_{u_B}(x) \ ,$$

where u_B is a $\{0, 1\}$ valued fuzzy measure. By lemma 6, for $B \neq \emptyset$, there exists a fuzzy measure v such that $F_{u_B}(x) = C_v(x)$ for all $x \in \mathbb{R}^n$. Clearly, by **(PW)**, we have for $B \neq \emptyset$

$$v(A) = F_{u_B}(1_A, 0_{-A}) = u_B(A) \ .$$

Hence

$$F_{u_B}(x) = C_{u_B}(x) \ \forall x \in \mathbb{R}^n \ .$$

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By linearity of the Choquet integral with respect to the fuzzy measure,

$$\forall x \in \mathbb{R}^n, F_\mu(x) = \sum_{B \subset N,\, B \neq \emptyset} m(B) C_{u_B}(x) = C_{\sum_{B \subset N,\, B \neq \emptyset} m(B) u_B}(x) = C_\mu(x) \ .$$

∎

This theorem is a generalization of a result of J.L. Marichal [15]. J.L. Marichal proved in [15] that the Choquet integral is the only aggregation function satisfying **(LM)**, **(In)**, **(PW)** and **(SPL)**.

Let us sum up the result we have shown. From the construction of the u_i's (conditions **(Intra$_a$)**, **(Intra$_b$)**, **(Intra$_c$)**, **(Intra$_d$)**), **(Intra$_e$)**) and μ (conditions **(Inter$_a$)**, **(Inter$_b$)**, **(Inter$_c$)**, **(Inter$_d$)**), **(Inter$_e$)**), and adding the two assumptions (4) and (5), we obtained axioms **(PW)** and **(weak SPL)** (see Lemma 5), and axiom **(LM)** was partially obtained (see Lemma 3). Moreover, we saw that axiom **(In)** is a very natural requirement. Theorem 1 shows that there is only one model that fits with previous information. This is the Choquet integral. Therefore, the Choquet integral is the only model that is suitable with our construction.

6 Discussion on the practical construction of the u_i's and μ

Some practical issues concerning the construction of u_i and μ are considered in this section.

6.1 Intra-criterion information

Let us investigate here how to construct in practice the utility function u_i. We follow the Macbeth methodology [1, 2]. The goal of this section is not to give very precise details concerning the Macbeth approach. We refer to references [1, 2] for a more detailed explanation of the Macbeth approach.

For the sake of simplicity, assume that attribute X_i has a finite set of values : $X_i = \left\{ a_1^i, \ldots, a_{p_i}^i \right\}$ with $a_1^i = \mathbf{0}_i$ et $a_{p_i}^i = \mathbf{1}_i$. We aim at determining $u_i(a_j^i)$ for all $j \in \{1, \ldots, p_i\}$.

From a theoretical standpoint, the construction of u_i from **(Intra$_a$)**, **(Intra$_b$)** and **(Intra$_c$)** is straightforward if the data is consistent. Indeed, applying **(Intra$_b$)** with $x_i = a_j^i$, $y_i = z_i = \mathbf{0}_i$ and $w_i = \mathbf{1}_i$, together with **(Intra$_c$)** yields

$$u_i(a_j^i) = k(a_j^i, \mathbf{0}_i, \mathbf{1}_i, \mathbf{0}_i)$$

where k is given by (**Intra**$_b$). Proceeding in this way ensures uniqueness of u_i.

Of course, it is not reasonable to ask a DM to give directly the value of k as a real number. The idea of the Macbeth methodology is to ask an information of an ordinal nature to the DM. It is well-known from psychological studies that human beings can handle at most 5 plus or minus 2 items at the same time. The Macbeth methodology proposes to ask to the DM a satisfaction level belonging to an ordinal scale composed of 6 elements: {*very small, small, mean, large, very large, extreme*}=: $\mathcal{E}$. If the DM assesses the value of $u_i(a_j^i)$ directly in this scale, the utility functions will be very rough.

In order to cope with the finiteness of the satisfaction scale $\mathcal{E}$, we ask the DM to give much more information than just assessing the $u_i(a_j^i)$ for $j \in \{1, \ldots, p_i\}$. In fact, the DM is asked to assess (giving a value in the scale $\mathcal{E}$) the difference of satisfaction $u_i(a_j^i) - u_i(a_k^i)$ between two values a_j^i and a_k^i, for any $j \neq k$ such that $(\mathbf{0}_{-i}, a_j^i) \succ (\mathbf{0}_{-i}, a_k^i)$. The information asked in practice is thus quite similar to (**Intra**$_b$). The advantage of asking $u_i(a_j^i) - u_i(a_k^i)$ is that it leads to a redundant information that will enable to compute accurate but non-unique values of $u_i(a_l^i)$ for $l \in \{1, \ldots, p_i\}$. The second advantage is that it is easier for a human being to give some relative information regarding a difference (for instance $u_i(a_j^i) - u_i(a_k^i)$) than to give some absolute information (for instance $u_i(a_j^i)$).

There is no unique utility function u_i corresponding to the data composed of the values $u_i(a_j^i) - u_i(a_k^i)$ (for $j \neq k$) assessed in the scale $\mathcal{E}$. All possible solutions are consistent with the given information. In practice, one utility function is chosen among all possible ones [1].

The drawback of asking redundant information is that some inconsistencies may be introduced by the DM. For instance, for $a_{j_1}^i$, $a_{j_2}^i$ and $a_{j_3}^i$ such that $(\mathbf{0}_{-i}, a_{j_1}^i) \succ (\mathbf{0}_{-i}, a_{j_2}^i) \succ (\mathbf{0}_{-i}, a_{j_3}^i)$, here is an example of inconsistency : the difference of satisfaction degree between $a_{j_1}^i$ and $a_{j_2}^i$ is judged *very small* by the DM, the one between $a_{j_2}^i$ and $a_{j_3}^i$ is also judged *very small*, and the one between $a_{j_1}^i$ and $a_{j_3}^i$ is judged *extreme*. C.A. Bana e Costa, and J.C. Vansnick showed that inconsistencies are related to cyclones in the preference relation structure [1]. This property enables to detect and explain all possible inconsistencies [1].

6.2 Inter-criteria information

We could imagine proceeding as for u_i to obtain μ. The DM would be asked about the difference of satisfaction between the alternatives $(\mathbf{1}_A, \mathbf{0}_{-A})$ and $(\mathbf{1}_B, \mathbf{0}_{-B})$ (for $A \neq B$). Actually, even if this way is possible, it is not

generally used because of the following two reasons. The first one is that it may not be natural for a DM to give his preferences on the prototypical acts $(1_A, 0_{-A})$. The second one is that it enforces the DM to construct a ratio scale over 2^n alternatives. This requires roughly 4^n questions to be asked to the DM. This is too much in practice.

Conditions $(\mathbf{Inter}_a)$, $(\mathbf{Inter}_b)$ and $(\mathbf{Inter}_c)$ were introduced to show the practicality of the use of the Choquet integral to model the preference relation $\succ$ on the Cartesian product of the attributes.

In practice, $(\mathbf{Inter}_a)$, $(\mathbf{Inter}_b)$ and $(\mathbf{Inter}_c)$ are replaced by any classical method designed to deduce the fuzzy measure from information on acts given in $[0, 1]^n$. These acts are directly described by satisfaction degrees in $[0, 1]$ over all criteria. The methods mentioned in the introduction are well-suited for performing this step: we can cite a linear method [13, 8], a quadratic method [4, 6] and an heuristic-based method [5].

7 Conclusion

We give in this paper a result that provides some information concerning the preferences of the DM over each attribute and the aggregation of criteria, and a priori assumptions leading to the Choquet model. The difficulty of the intra-criterion step is that one has to determine the utility functions (which have a meaning only through the overall model and thus through the aggregation function) without a precise knowledge of the aggregation function (that is determined in the next step). Let us recall the main property that makes this possible. It is in fact a consequence of the **(weak SPL)** property :

$$\frac{u(x_i, \mathbf{0}_{-i})}{u(\mathbf{1}_i, \mathbf{0}_{-i})} = \frac{F_\mu(u_i(x_i), 0_{-i})}{F_\mu(1_i, 0_{-i})} = u_i(x_i) \ .$$

Hence, $u_i(x_i)$ is proportional to $u(x_i, \mathbf{0}_{-i})$ whatever the value of μ may be.

Some questions about the practicality of the information asked to the DM are now raised.

Note that regarding the questions to be asked to the DM, psychological aspects must be taken into consideration. For the construction of u_i, we propose here to use the acts $(x_i, \mathbf{0}_{-i})$. Let us recall that $\mathbf{0}$ corresponds to the *completely unacceptable* absolute level. As a consequence, since the act $(x_i, \mathbf{0}_{-i})$ is unacceptable almost everywhere, one could argue that this act can be considered as almost unacceptable, so that the DM will have some

troubles giving his feeling about the differences between some $(x_i, \mathbf{0}_{-i})$ and $(y_i, \mathbf{0}_{-i})$. If this happens, it would be reasonable to replace the space $X\rfloor_i$ by

$$X\rfloor_i' := \{(x_i, \mathbf{1}_{-i}) \ , \ x_i \in X_i\} \ ,$$

and thus condition $(\mathbf{Intra}_a)$ by

$(\mathbf{Intra}_a\text{'})$ $\forall x_i, y_i \in X_i, \ u_i(x_i) \geq u_i(y_i) \Leftrightarrow (x_i, \mathbf{1}_{-i}) \succeq (y_i, \mathbf{1}_{-i})$.

Then (6) shall be replaced by

$$\frac{F_{\gamma\mu}\left(\alpha a_i + \beta, (\alpha + \beta)_{-i}\right) - F_{\gamma\mu}\left(\alpha b_i + \beta, (\alpha + \beta)_{-i}\right)}{F_{\gamma\mu}\left(\alpha c_i + \beta, (\alpha + \beta)_{-i}\right) - F_{\gamma\mu}\left(\alpha d_i + \beta, (\alpha + \beta)_{-i}\right)}$$
$$= \frac{a_i - b_i}{c_i - d_i}$$

for all $a_i, b_i, c_i, d_i \in \mathbb{R}^+$, and for all $\alpha, \gamma > 0, \ \beta \in \mathbb{R}$. It can be checked that the rest of the paper remains unchanged, leading also to theorem 1.

If the same phenomenon occurs with $X\rfloor_i'$, the DM satisfies strongly to the *saturation effect.* In this case, it may be more appropriate to model the DM with the help of a purely ordinal aggregator, like the Sugeno operator. A construction of a multicriteria decision model in a purely ordinal context, based on ordinal scales and the Sugeno integral is presented in [9].

Acknowledgements

References

[1] C.A. Bana e Costa, and J.C. Vansnick. A theoretical framework for Measuring Attractiveness by a Categorical Based Evaluation TecHnique (Macbeth). In: Proc. XIth Int. Conf. on MultiCriteria Decision Making, pp 15-24, Portugal, August 1994.

[2] C.A. Bana e Costa, and J.C. Vansnick. Preference relations and MCDM. In: MultiCriteria Decision Making: advances in MCDM models, algorithms, theory and applications. T. Gal, T. Steward, T. Hanne Eds. Kluwer, 1999.

[3] J.C. Fodor and M. Roubens. Fuzzy preferences modeling and multicriteria decision aid. Kluwer Academic Publisher, 1994.

[4] M. Grabisch, J.M. Nicolas, Classification by fuzzy integral – performance and tests. Fuzzy Sets & Systems, Special issue on Pattern Recognition, No 65, pp 255-271, 1994.

[5] M. Grabisch, A new algorithm for identifying fuzzy measures and its application to pattern recognition, In: Int. Joint Conf. of the 4th IEEE Int. Conf. on Fuzzy Systems and the 2nd Int. Fuzzy Engineering Symposium, pp 145-150, Yokohama, Japan, March 1995.

[6] M. Grabisch, The application of fuzzy integrals in multicriteria decision making, European Journal of Operational Research, No 89, pp 445-456, 1996.

[7] M. Grabisch and T. Roblin, Aggregation of ordinal information, in. EUROFUSE, pp 496-501, Budapest, Hungary, 1999.

[8] M. Grabisch, T. Murofushi and M. Sugeno (Eds.), Fuzzy measures and integrals, Physica-Verlag, Heidelberg, New York, 2000.

[9] M. Grabisch, S. Dia and Ch. Labreuche, A multicriteria decision making framework in ordinal context based on Sugeno integral, In Joint 9th IFSA World Congress and 20th NAFIPS Int. Conf., Vancouver, Canada, July 2001.

[10] M. Grabisch, Ch. Labreuche & J.C. Vansnick. On the Extension of Pseudo-Boolean Functions for the Aggregation of Interacting Criteria. Submitted to European Journal of Operationnal Research.

[11] D.H. Krantz, R.D. Luce, P. Suppes and A. Tversky. Foundations of measurement, vol 1: Additive and Polynomial Representations. Academic Press, 1971.

[12] C. Labreuche and M. Grabisch. The Choquet integral as a way to aggregate scales of differences in multicriteria decision making, In EUROFUSE workshop on preference modelling and applications, pp. 147-151, Granada, Spain, 2001.

[13] J.L. Marichal & M. Roubens, Dependence between criteria and multiple criteria decision aid, In 2nd Int. Workshop on Preferences and Decisions, pp. 69-75, Trento, Italy, 1998.

[14] J.L. Marichal, Aggregation operators for multicriteria decision aid. Ph.D. thesis. Univ. of Liège, Belgium. 1998.

[15] J.L. Marichal, An axiomatic approach of the discrete Choquet integral as a tool to aggregate interacting criteria, IEEE Trans. on Fuzzy Systems, Vol. 8, No 6, pp 800-807, 2000.

[16] A. Rico, M. Grabisch, C. Labreuche, Représentation des préférences dans un cadre ordinal pour l'intégrale de Sugeno, in LFA (rencontres francophones de la Logique Floue et ses Applications) 2000, pp 53-59, La Rochelle, France, oct. 2000.

[17] A. Rico, M. Grabisch, C. Labreuche, A. Chateauneuf, Preference Modelling on Totally Ordered Sets by the Sugeno Integral, Submitted to Discrete Applied Math.