A k-factor GIGARCH process: Estimation and Application on electricity market spot prices.

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***Abstract*— Some crucial time series of market data, such as electricity spot price, exhibit long-memory, in the sense of slowly-decaying correlations, combined with heteroskedasticity. To be able to modelize such a behaviour, we consider the k-factor GIGARCH process. The related parameter estimation problem is addressed using an extension of Whittle estimation. We develop the corresponding asymptotic theory for estimation. We apply this approach to electricity prices (spot prices) from the German energy market (European Energy eXchange). For these data, we propose two models of k-factor GIGARCH process. To conclude, the forecasting performances of these models are analysed in detail.**

I. Introduction

However, Long-range dependence is now a phenomenon which have attracted the attention of more and more statisticians and economists since the works of Granger and Joyeux (1980), Hosking (1981). The long memory parameter d of the FARIMA(p, d, q) model is generally estimated in the spectral domain by considering a least squares procedure based on the periodogram, see for instance Geweke and Porter-Hudak (1983), Robinson (1994, 1995) or Giraitis and *al*. (1998) or frequency-domain maximum likelihood (ML) estimation procedure proposed by Fox and Taqqu (1986), Sowell (1992) or Cheung and Diebold (1993). For a review of estimation techniques in long memory models, we refer to Beran (1994), Guégan (1994), Bisaglia and Guégan (1998) or Bisaglia (1998).

many empirical time series often possess a persistent periodic behaviour, especially when dealing with hourly, daily or monthly frequencies. For intance, the hourly spot prices of Germany electricity market shows a strong seasonality of daily and weekly. Unfortunately, the FARIMA process does not allow to take into account a cyclical or periodic behaviour. Therefore, many authors proposed different long memory processes, fo which the common point is to have a spectral density with one or more singularities, not restrict at the origin, somewhere on the interval $[0, \pi]$. For instance, we refer to the works of Gray and *al.* (1989), Chung (1996(a), 1996(b)) or Yajima (1996) for models with periodogram having a single peak, see the paper of Giraitis and Leipus (1995), Hosoya (1997) or Woodward and *al.* (1998) for models with periodogram having more than one peak. Some authors proposed a two-steps procedure to estimate the parameter the Gegenbauer autoregressive moving-average (GARMA) model. In the first step, they estimate the frequency in which the periodogram becomes unbounded by using a grid-search procedure, see Gray and *al.* (1989) or Chung (1996(a), 1996(b)) for more details, or by taking the maximum of the periodogram (Yajima (1996)). Then, in a second step, the memory parameter is estimated by using a classical parametric or semi-parametric methods of the long memory domain. Later, a simultaneous Whittle's method is proposed. For more details, we refer to Férrara and Guégan (2001), Férrara (2000) or Giraistis and Leipus (1995). All of the aforementioned works assume that the conditional variance of time series is a constant over time, however.

Time series models with a time-varying conditional variance (ARCH) was first propsed by Engle (1982). This class of models has important applications, particulary in finance and economics, see Bollerslev (1986), Bollerslev, Engle and Woodridge (1988). An extension of this model, considering the long memory behaviour, has been introduced by Baillie and *al.* (1996): it is the so-called FIGARCH models.

Considering the behaviour of very complex data, such that the electricity spot prices, it seems natural to consider time series both taking into account persistence with quasi periodic behaviour and heteroskedasticity. Guégan (2000) introduced the k-factor GIGARCH process and studied later in (2003) the properties of stationarity and invertibility of the process. Our aim is to study in this article, the parameters estimation of the k-factor GIGARCH process and the asymptotic properties of the estimators. Two estimations procedures were proposed: the first is based on the conditional sum of squares method (CSS) and the second approach is an extension of Whittle estimation procedure. The article is organized as follow. Section 2 gives the models definition. Section 3 discusses the two estimations procedures and the asymptotic properties of the estimators. Section 4 applies the k-factor GIGARCH process to the hourly spot prices of German electricity market (August, 15^{th} 2000 to December, 31^{th} 2002).

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II. GIGARCH PROCESSES

Assume that $(\xi_t)_{t\in Z}$ is a white noise process with unit variance and let the polynomials $\phi(B)$ and $\theta(B)$ denote the ARMA operators. Let B denote the backshift operator and $0 < d_i < \frac{1}{2}$ if $|\nu_i| < 1$ or $0 < d_i < \frac{1}{4}$ if $|\nu_i| = 1$ for $i = 1, \cdots, k$. We define a centered k-factor GIGARCH process $(X_t)_{t\in Z}$ by, $\forall\, t$

$$\phi(B)\prod_{i=1}^{k}\left(I - 2\nu_i B + B^2\right)^{d_i} X_t = \theta(B)\,\varepsilon_t, \tag{1}$$

where

$$\varepsilon_t = \sqrt{h_t}\xi_t, \text{ with } h_t = a_0 + \sum_{i=1}^{r} a_i\varepsilon_{t-i} + \sum_{i=1}^{s} b_i h_{t-i}. \tag{2}$$

For $i = 1, \cdots, k$, the frequencies $\lambda_i = \arccos(\nu_i)$ are called the Gegenbauer frequencies (or G-frequencies). The process defined by the equations (**??**)-(**??**) was introduced by Guégan (2000, 2003) generalizing fractionally integrated generalized autoregressive conditional heteroskedasticity process (FIGARCH) introduced by Baillie, Bollerslev and Mikkelsen (1996).

We recall that the Gegenbauer polynomials, often used in applied mathematics because of their orthogonality and recursion properties, are defined by:

$$\left(I - 2\nu z + z^2\right)^{-d} = \sum_{j\geq 0} C_j(d,\nu)\, z^j, \tag{3}$$

where $|z| \leq 1$ and $|\nu| \leq 1$.

The coefficients $(C_j(d,\nu))_{j\in Z}$ of this development can be computed in many different ways. For example, Rainville (1960) shows that:

$$C_j(d,\nu) = \sum_{k=0}^{\left[\frac{j}{2}\right]} \frac{(-1)^k\,\Gamma(d+j-k)\,(2\nu)^{j-2k}}{\Gamma(d)\,\Gamma(k+1)\,\Gamma(j-2k+1)}, \tag{4}$$

where $\left[\dfrac{j}{2}\right]$ is the integer part of $\dfrac{j}{2}$. An easiest way to compute the Gegenbauer polynomials $(C_j(d,\nu))_{j\in Z}$ is based on the following recursion formula $\forall\, j > 1$:

$$\begin{aligned} C_j(d,\nu) &= 2\nu\left(\frac{d-1}{j}+1\right)C_{j-1}(d,\nu) \\ &\quad - \left(2\frac{d-1}{j}+1\right)C_{j-2}(d,\nu), \end{aligned} \tag{5}$$

where $C_0(d,\nu) = 1$ and $C_1(d,\nu) = 2d\nu$.

The properties of stationarity and invertibility of a k-factor GIGARCH process are established and proved by Guégan (2003).

In this section, we provide some results related to the asymptotic properties of the k-factor GIGARCH process estimators, obtained by two different methods: the conditional sum of squares (CSS) and the Whittle approach.

III. ESTIMATION OF GIGARCH MODELS

Given a stationary k-factor GIGARCH process $\{X_t\}_{t=1}^{T}$ defined by the equations (**??**)-(**??**). We denote $\gamma = (\phi_1, \cdots, \phi_p, \theta_1, \cdots, \theta_q, d_1, \cdots, d_k)$, $\delta = (a_0, a_1, \cdots, a_r, b_1, \cdots, b_s)$ and $\omega = (\gamma, \delta)$ its parameters. We assume that $\omega_0 = (\gamma_0, \delta_0)$ is the true value of ω and is in the interior of the compact set $\Theta \subseteq R^{p+q+k+r+s+1}$.

A. Conditional Sum of Squares estimation

The conditional sum of squares estimator $\hat{\omega}_T$ of ω in Θ maximizes the conditional logarithmic likelihood $L(\omega)$ on F_0, where F_t is the σ-algebra generated by $(X_s, s \leq t)$. If we assume that the innovations $(\varepsilon_t)_{t\in Z}$ have a conditional Gaussian distribution then the conditional log-likelihood is defined by:

$$L(\omega) = \frac{1}{T}\sum_{t=1}^{T}\ell_t, \quad \ell_t = -\frac{1}{2}\log(h_t) - \frac{\varepsilon_t^2}{2h_t}. \tag{6}$$

Now, if we assume that the innovations $(\varepsilon_t)_{t\in Z}$ have a conditional Student distribution with l degrees of freedom, then the CSS estimator $\hat{\omega}_T$ maximizes the likelihood function $L(\omega)$ defined by

$$\begin{aligned} L(\omega) &= T\left[\log\Gamma\left\{\frac{(l+1)}{2}\right\} - \log\Gamma\left(\frac{l}{2}\right) - \frac{1}{2}\log(l-2)\right] \\ &\quad - \frac{1}{2}\sum_{t=1}^{T}\left\{\log(h_t) + (l+1)\left[\log\Big(1 + \frac{\varepsilon_t^2}{h_t(l-2)}\Big)\right]\right\} \end{aligned} \tag{7}$$

In the following Theorem, $L(\omega)$ represents the log likelihood introduced in (**??**) or in (**??**).

Theorem 3.1: Suppose that the process $(X_t)_{t\in Z}$ is generated by equations (**??**)-(**??**). Assume that $a_0 > 0, a_1, \cdots, a_r, b_1, \cdots, b_s \geq 0$, $\sum_{i=1}^{r} a_i + \sum_{i=1}^{s} b_i < 1$, $E\left(\varepsilon_t^4\right) < \infty$, $0 < d_i < \frac{1}{2}$ if $|\nu_i| < 1$ or $0 < d_i < \frac{1}{4}$ if $|\nu_i| = 1$ for $i = 1, \cdots, k$ and all roots of the polynomials $\phi(B)$ and $\theta(B)$ lie outside the unit circle. If we assume that all the G-frequencies are known. Then

1) There exists a CSS estimator $\hat{\omega}_T$ that satisfies $\frac{\partial L(\omega)}{\partial\omega} = 0$ and $\hat{\omega}_T \xrightarrow{P} \omega_0$ as $T \to \infty$.

2) $\sqrt{T}\,(\hat{\omega}_T - \omega_0) \xrightarrow{D} N\left(0, \Omega_0^{-1}\right)$ as $T \to \infty$, where $\xrightarrow{P}$ and $\xrightarrow{D}$ denotes respectively the convergence in probability

and in distribution. Furthermore, $\Omega_0 = diag\left(\Omega_{\gamma_0}, \Omega_{\delta_0}\right)$ and Ω_{γ_0} and Ω_{δ_0} are values of Ω_γ and Ω_δ at $\omega = \omega_0$, with $\Omega_\gamma = E\left(\left[\frac{1}{h_t}\frac{\partial\varepsilon_t}{\partial\gamma}\frac{\partial\varepsilon_t}{\partial\gamma^T} + \frac{1}{2h_t^2}\frac{\partial h_t}{\partial\gamma}\frac{\partial h_t}{\partial\gamma^T}\right]\right)$ and $\Omega_\delta = E\left(\left[\frac{1}{2h_t^2}\frac{\partial h_t}{\partial\delta}\frac{\partial h_t}{\partial\delta^T}\right]\right)$.

3) The information matrices Ω_γ and Ω_δ can be estimated consistently by

$$\hat{\Omega}_\gamma = \frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{h_t}\frac{\partial\varepsilon_t}{\partial\gamma}\frac{\partial\varepsilon_t}{\partial\gamma^T} + \frac{1}{2h_t^2}\frac{\partial h_t}{\partial\gamma}\frac{\partial h_t}{\partial\gamma^T}\right], \quad (8)$$

and

$$\hat{\Omega}_\delta = \frac{1}{T}\sum_{t=1}^{T}\left(\left[\frac{1}{2h_t^2}\frac{\partial h_t}{\partial\delta}\frac{\partial h_t}{\partial\delta^T}\right]\right). \quad (9)$$

The proof is given in for the Gaussian case (Diongue and *al.* (2003)). Note that, if the innovations $(\varepsilon_t)_{t\in Z}$ have a conditional Student distribution with l degrees of freedom, then the proof can easily done using the same steps as in previous case.

B. Whittle estimation

In this paragraph, we investigate the Whittle's method to estimate all parameters of the k-factor GIGARCH process defined by equations (**??**)-(**??**). The first step consists to estimate the long-memory parameters $d = (d_1, \cdots, d_k)$ and the ARMA(p,q) parameters $\alpha = (\phi_1, \cdots, \phi_p, \theta_1, \cdots, \theta_q)$ using the Whittle's approach (for more details, see Chung (1996(a), 1996(b))) or Férrara and Guégan (2001)). In the second step, the GARCH(r,s) parameters $\delta = (a_0, a_1, \cdots, a_r, b_1, \cdots, b_s)$ are estimated using Whittle's method applied to the residuals of the long-memory process (see Giraitis and Robinson (2001) for more details).

Theorem 3.2: Let $\{X_t\}_{t=1}^T$ be a k-factor GIGARCH process defined by equations (**??**)-(**??**). Let us assume that the same hypothesis given in Theorem (**??**) are verified. Then

1) $\hat{\gamma}_T \overset{a.s}{\to} \gamma_0$ as $T \to \infty$.

2) Furthermore: $\sqrt{T}\left(\hat{\alpha}_T - \alpha_0\right) \overset{D}{\to} N\left(0, 4\pi V\left(\alpha_0\right)^{-1}\right)$, as $T \to \infty$, Where,

$$V(\alpha)_{ij} = \int_{-\pi}^{\pi} g^2(\lambda,\omega)\frac{\partial g^{-1}(\lambda,\omega)}{\partial\alpha_i}\frac{\partial g^{-1}(\lambda,\omega)}{\partial\alpha_j}d\lambda. \quad (10)$$

Here $g(\lambda,\omega)$ denotes the spectral density of the process $(X_t)_{t\in Z}$.

3) Moreover $\sqrt{T}\left(\hat{d}_T - d\right) \overset{D}{\to} N\left(0, 4\pi V(d)^{-1}\right)$, with

$$V(d)_{ij} = \int_{-\pi}^{\pi}\log\left|4\sin\left[\frac{(\lambda-\lambda_i)}{2}\right]\sin\left[\frac{(\lambda+\lambda_i)}{2}\right]\right| \times \log\left|4\sin\left[\frac{(\lambda-\lambda_j)}{2}\right]\sin\left[\frac{(\lambda+\lambda_j)}{2}\right]\right| d\lambda. \quad (12)$$

The Theorem **??** follows from the proof of Hosoya's Theorem 2.3 (1997).

To estimate the GARCH(r,s) parameters δ, we consider the process $\left(\varepsilon_t^2\right)_{t\in Z}$ in its ARMA representation. This means that we can rewritte (**??**) as: $\varepsilon_t^2 - \sum_{i=1}^{max(r,s)}(a_i+b_i)\varepsilon_{t-i}^2 = a_0 + \upsilon_t - \sum_{j=1}^{s} b_j\upsilon_{t-j}$, where $b_i = 0$ if $i \in (s,r]$ and $a_i = 0$ if $i \in (r,s]$. The process $(\upsilon_t)_{t\in Z}$ defined by $\upsilon_t = \varepsilon_t^2 - h_t$ constitutes a white noise sequence with mean zero and variance σ^2. We introduce now some complementary assumptions to get the consistency and asymptotic normality of $\hat{\delta}_T$:

(H_0). For $t = 0, \pm1, \ldots$, the process $(\xi_t)_{t\in Z}$ introduced in equation (**??**), is strictly stationary, ergodic with finite Jth moment and $E(\xi_t|F_{t-1}) = 0$, $E(\xi_t^2|F_{t-1}) = 1$, and $E(\xi_t^{2j}|F_{t-1}) = \upsilon_{2j}$ almost-surely, with $j = 2, \cdots, \frac{J}{2}$, where υ_{2j} are constants such that $|\upsilon_J|^{\frac{2}{J}}\left(\sum_{i=1}^{r} a_i + \sum_{i=1}^{s} b_i\right) < 1$.

(H_1).

1) $\int_{-\pi}^{\pi}\log f(\lambda,\delta)\,d\lambda = 0$, for all δ, with $f(\lambda,\delta)$ the spectral density of the process $(\varepsilon_t)_{t\in Z}$.

2) $f(\lambda,\delta)^{-1}$ is continuous in $(\lambda,\delta) \in [-\pi,\pi]\times\Lambda$, where $\Lambda \subset R^{r+s+1}$ is a compact.

3) $\mu_L(\{\lambda; f(\lambda,\delta) \neq f(\lambda,\delta_0)\}) \geq 0$, for $\delta \in \Lambda$ with μ_L the Lebesque measure.

(H_2).

1) δ_0 is an interior point of Λ and in a neighborhood of δ_0, $\frac{\partial f(\lambda,\delta)^{-1}}{\partial\delta}$ and $\frac{\partial^2 f(\lambda,\delta)^{-1}}{\partial\delta\partial\delta^T}$ exist and are continuous in λ and δ.

2) $\frac{\partial f(\lambda,\delta_0)^{-1}}{\partial\delta}$ is K-Lipchitzienne with $K > \frac{1}{2}$.

3) The matrix W given by

$$W = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\partial\log f(\lambda,\delta_0)}{\partial\delta}\frac{\partial\log f(\lambda,\delta_0)}{\partial\delta^T}d\lambda \quad (13)$$

is non singular.

Theorem 3.3: Let $(X_t)_{t\in Z}$ be a stationary, causal and invertible process defined by the equations (**??**)-(**??**).

1) Under (H_0) with $J = 4$ and (H_1), $\hat{\delta}_T \overset{P}{\to} \delta_0$, as $T \to \infty$.
2) Under (H_0) with $J = 8$, (H_1) and (H_2), $\sqrt{T}\left(\hat{\delta}_T - \delta_0\right) \to N\left(0, 2W^{-1} + W^{-1}VW^{-1}\right)$, as $T \to \infty$. Here V is given by

$$V = \frac{2\pi}{\sigma^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\partial f(\lambda,\delta_0)^{-1}}{\partial\delta}\frac{\partial f(\omega,\delta_0)^{-1}}{\partial\delta^T}h(\lambda,-\omega,\omega)\,d\lambda d\omega, \quad (14)$$

TABLE I

GENERAL DESCRIPTIVE STATISTICS.

Statistics	Value
Mean	2.97
Variance	0.375
Skewness	-2.6
Kurtosis	29.67

with $h(\lambda,\omega,\upsilon) = \frac{1}{8\pi^3} \sum_{j,k,l=-\infty}^{+\infty} e^{ij\lambda - ik\omega - il\upsilon} Cum(\varepsilon_0, \varepsilon_j, \varepsilon_k, \varepsilon_l)$,

and Cum is the order four's cumulant for the process $(\varepsilon_t)_{t\in Z}$.

The proof of Theorem **??** is similar to the proofs of Theorem 2.1 and Theorem 2.2 given in Giraitis and Robinson's (2001).

IV. APPLICATION

In this section, we provide an application of k-factor GIGARCH process, which points out the interest of such model to analyze and to forecast time series with long-range dependence, especially when the forecast horizon is wide. We consider here the hourly series $\{S_t\}_{t=1}^{20856}$ of the spot prices of German electricity market from August, 15^{th} 2000 to December, 31^{th} 2003, and we note $Y_t = \log(S_t)$ (**??**) this series of 20856 points. As shows in Figure **??**, the serie studied seems to be stationary in mean but we observe the presence of peaks around December 2001 and July 2001. Let us look at the ACF (AutoCorrelation Function), represented in Figure **??**. The ACF of the series Y_t seems to exhibit a slow decay pattern for the 2500 lags and a strong seasonality of twenty four hours and one hundred sixty eight hours is observed. The periodogram (Figure **??**) of the series is unbounded for relatively three frequencies. Note that the frequencies coincid respectively to the period of one week, one day and one half-day.

Summary statistics of the data over the sample period are presented in Table **??**. The skewness is non positive, indicating that the series have long left hand side tails, and the kurtosis statistic is greater than 3, signifying that the series have fat tails.

Our aim is to forecast from January, 1^{th} 2003 to January, 30^{th} 2003. To reach our goal, we consider two different approachs of modelling and we compare them according to their predictive ability. Note that all the approachs are long memory approach. We assess the predictive ability of each model by considering the RMSE forecast criterion defined by:

$$RMSE(f) = \sqrt{\frac{1}{h}\sum_{k=1}^{h}\left(X_{t+k} - \hat{X}_t(k)\right)^2},$$

where h is the forecast horizon and $\hat{X}_t(k)$ is the predict value of X_{t+k}.

First approach:

When dealing with time series having a strong seasonality, denote S, a first idea is to remove this seasonality by using the $(I - B^S)$ filter. Let us denoted $(Z_t)_t$ the seasonally filterd series such that:

$$\left(I - B^{168}\right)\left(Y_t - \bar{Y}\right) = Z_t.$$

The ACF of the series $(Z_t)_t$ (Figure **??**) seems to exhibit a relatively slow decay pattern for the 2500 first lags. Moreover, the periodogram of the series $(Z_t)_t$ (Figure **??**) clearly possesses a peak for a frequency located very close, but not equal, to zero. We assume here that the spectral density of this series is unbounded for only one frequency. Furthermore, we have detect great variability in the variance of this series (Figure **??**).

The parameter estimation is done using the Whittle's method and the values are resumed in Table **??**. Using Yajima (1996), the periodogram is unbounded at the frequency $\lambda = 0.001$.

TABLE II

PARAMETER ESTIMATES FOR THE 1-FACTOR GIGARCH PROCESS WITH CONDITIONAL STUDENT DISTRIBUTION WITH 3 DEGREES OF FREEDOM.

Parameter	Estimate	St. dev.
d	0.38	0.017
ϕ_{24}	0.874	0.342
θ_{24}	0.706	0.342
θ_{168}	0.851	0.303
a_0	0.0167	3.36 10^{-4}
a_1	0.594	0.072

Second approach:

In this approach, we assume that the periodogram of the series $(Y_t)_t$ is unbounded for three frequencies. Therefore, we fit a 3-factor GIGARCH process to the series (Y_t), considering the frequencies $\lambda_1 = 0.2658$, $\lambda_2 = 0.0374$, $\lambda_3 = 0.5236$. The parameter estimation is resumed in Table **??**.

TABLE III

PARAMETER ESTIMATES FOR THE 3-FACTOR GIGARCH PROCESS WITH CONDITIONAL STUDENT DISTRIBUTION WITH 3 DEGREES OF FREEDOM.

Parameter	Estimate	St. dev.
d_1	0.135	0.009
d_2	0.214	0.006
d_3	0.141	0.015
ϕ_{24}	0.863	0.179
ϕ_{168}	0.951	0.221
θ_{24}	0.720	0.331
θ_{168}	0.794	0.298
a_0	0.0141	3.39 10^{-4}
a_1	0.756	0.365

V. CONCLUSION

The conclusion goes here.

[Evolution] [ACF] [Periodogram]

Fig. 1. Evolution (top), estimation of the ACF (center) and the spectral density (buttom) of the hourly log spot prices of German elctricity market

[Evolution] [ACF] [Periodogram]

Fig. 2. Evolution (top), estimation of the ACF (center) and the spectral density (buttom) of the hourly log spot prices of German elctricity market

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